Lewy-Stampacchia's inequality for a pseudomonotone parabolic problem

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Abstract: The main aim of this paper is to extend to the case of a pseudomonotone operator Lewy-Stampacchia's inequality proposed by F. Donati [7] in the framework of monotone operators. For that, an *ad hoc* type of perturbation of the operator is proposed.

Keywords: Variational inequalities, penalization, pseudomonotone operator, Lewy-Stampacchia's inequality

MSC: 35K86, 35R35

1 Introduction

The aim of this paper is to prove the existence of a solution to the parabolic variational inequality

$$\int_0^T \langle \partial_t u, v-u\rangle dt + \int_Q a(t,x,u,\nabla u)\cdot\nabla(v-u)dxdt \geq \int_0^T \langle f, v-u\rangle dt$$

and, especially, to give the associated inequality of Lewy-Stampacchia

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot,\cdot,u,\nabla u)] - f \leq g^- = (f - \partial_t\psi + \operatorname{div}[a(\cdot,\cdot,\psi,\nabla\psi)])^-,$$

where $u \mapsto -\operatorname{div}[a(t,x,u,\nabla u)]$ is a pseudomonotone operator under the constraint $u \geq \psi$.

After the first results of H. Lewy and G. Stampacchia [14] concerning inequalities in the context of superharmonic problems, many authors have been interested in the so-called Lewy-Stampacchia's inequality associated with obstacle problems. Without exhaustiveness, let us cite the monograph of J.F. Rodrigues [21] and the papers of A. Mokrane and F. Murat [18] for pseudomonotone elliptic problems, A. Mokrane and G. Vallet [19] in the context of Sobolev spaces with variable exponents, A. Pinamonti and E. Valdinoci [20] in the framework of Heisenberg group, R. Servadei and E. Valdinoci [24] for nonlocal operators or N. Gigli and S. Mosconi [11] concerning an abstract presentation.

The literature on Lewy-Stampacchia's inequality is mainly aimed at elliptic problems, or close to elliptic problems and fewer papers are concerned with other type of problems. Let us cite J. F. Rodrigues [22] for hyperbolic problems, F. Donati [7] for parabolic problems with a monotone operator or L. Mastroeni and M. Matzeu [17] in the case of a double obstacle.

There is a large literature on parabolic problems with constraints. To cite some recent ones, consider [6, 13] where the main operator is monotone, associated with a nonlinear, possible graph, reaction term.

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Concerning Lewy-Stampacchia's inequality, to the best of the author's knowledge, F. Donati's work [7] has not been extended to pseudomonotone parabolic problems with a Leray Lions operator. In this paper, we propose such a result, with very general assumptions on the Carathéodory function a, by using a method of penalization of the constraint associated with a suitable perturbation of the operator. As proposed *e.g.* by [12, p.102] and [4] for sub/super solutions to obstacle quasilinear elliptic problems, this perturbation is one of the main new point of the proof. Indeed, without it, one is usually only concerned by Lewy-Stampacchia's inequality in the elliptic case, and one needs to assume, as in [18], some additional, now useless, Hölder-continuity assumptions for a with respect to u and ∇u. Thus, this perturbation allows us on the one hand to prove Lewy-Stampacchia's inequality in the pseudomonotone parabolic case, and on the other hand to reduce significantly the list of assumptions. Let us mention also that, with this method, one is to revisit Lewy-Stampacchia's inequality proposed in [18, 19] by assuming only basic assumptions. The second essential result is an extension of the formula of time-integration by parts of Mignot-Bamberger[2] & Alt-Luckhaus[1] to non-classical situations. Some information are given too about the time-continuity of an element u when u and $\partial_t u$ are not in spaces in duality relation.

The paper is organized in the following way: after giving the hypotheses and the main result (Theorem 2.2) in Section 2, Section 3 is devoted to the proof of this result. A first step is devoted to the existence of a solution to the penalized/perturbed problem associated with a parameter ε; then, some *a priori* estimates and passage to the limit with respect to ε are considered when g^- is a regular non-negative element. A first proof of Lewy-Stampacchia's inequality is given when g^- is still regular; finally, the proof of Lewy-Stampacchia's inequality is extended to the general case. A last part, Section 4, presents an annex containing technical results used in the proofs, in particular the time-integration by part and the time-continuity mentioned above.

2 Notation, hypotheses and main result

Let us denote by $\Omega \subset \mathbb{R}^d$ a Lipschitz bounded domain, for any $T > 0$, by $Q = (0, T) \times \Omega$ and by $p \in (1, +\infty)$. As usual, p' denotes the conjugate exponent of p, $V = W_0^{1,p}(\Omega)$ if $p \geq 2$ and $V = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ with the graph-norm else. Then, the corresponding dual spaces are $V' = W^{-1,p'}(\Omega)$ if $p \geq 2$ and $V' = W^{-1,p'}(\Omega) + L^2(\Omega)$ else (cf. *e.g.* [10, p.24]).
In this situation, the Lions-Gelfand triple [23, §. 7.2]

$$V \underset{dense}{\hookrightarrow} L^2(\Omega) \underset{dense}{\hookrightarrow} V'$$

holds and one denotes, as usually, by

$$W(0, T) = \{u \in L^p(0, T; V),\ \partial_t u \in L^{p'}(0, T; V')\}.$$

Assume in the sequel the following:

H_1 : A is a Leray-Lions pseudomonotone operator of the form

$$v \mapsto A(v) = -\mathrm{div}\Big[a(t, x, v, \nabla v)\Big],$$

which acts from $W^{1,p}(\Omega)$ into $W^{-1,p'}(\Omega)$ where

$\mathrm{H}_{1,1}$ $a : (t, x, u, \vec{\xi}) \in Q \times \mathbb{R} \times \mathbb{R}^d \mapsto a(t, x, u, \vec{\xi}) \in \mathbb{R}^d$ is a Carathéodory function on $Q \times \mathbb{R}^{d+1}$,

$\mathrm{H}_{1,2}$ a is strictly monotone with respect to its last argument:

$$\forall (t, x) \in Q \text{ a.e., } \forall u \in \mathbb{R},\ \forall \vec{\xi}, \vec{\eta} \in \mathbb{R}^d,$$
$$\vec{\xi} \neq \vec{\eta} \Rightarrow [a(t, x, u, \vec{\xi}) - a(t, x, u, \vec{\eta})] \cdot (\vec{\xi} - \vec{\eta}) > 0.$$

$\mathrm{H}_{1,3}$ a is coercive and bounded: there exist constants $\bar{\alpha} > 0$, $\bar{\beta} > 0$ and $\bar{y} \geq 0$, a function $\bar{h}$ in $L^1(Q)$ and a function $\bar{k}$ in $L^p(Q)$ and two exponents $q, r < p$ such that, for a.e. $(t, x) \in Q$, for all

$u \in \mathbb{R}$ and for all $\vec{\xi} \in \mathbb{R}^d$,

$$a(t,x,u,\vec{\xi})\cdot\vec{\xi} \geq \bar{\alpha}|\vec{\xi}|^p - \Big[\bar{y}|u|^q + |\bar{h}(t,x)|\Big], \tag{1}$$

$$|a(t,x,u,\vec{\xi})| \leq \bar{\beta}\Big[|\bar{k}(t,x)| + |u|^{r/p} + |\vec{\xi}|\Big]^{p-1}. \tag{2}$$

H_2 : assume that the obstacle ψ belongs to $L^p(0,T;W^{1,p}(\Omega)) \cap L^p(0,T;L^2(\Omega))$; that $\partial_t\psi$ belongs to $L^{p'}(0,T;V')$ and $\psi \leq 0$ on $\partial\Omega$ (See Section 4.4 for some comments on the time regularity of such elements).

H_3 : the right hand side f, which is assumed to be such that

$$g = f - \partial_t\psi - A(\psi) = g^+ - g^-$$

belongs to the order dual

$$L^p(0,T;V)^\star = \{T = T_1 - T_2,\ T_i \in (L^{p'}(0,T;V'))^+, i = 1,2\}$$

where $(L^{p'}(0,T;V'))^+$ denotes the non-negative elements of $L^{p'}(0,T;V')$.

H_4 : $u_0 \in L^2(\Omega)$ satisfies the constraint, *i.e.* $u_0 \geq \psi(0)$.

As usual concerning obstacle problems one denotes by

$$\mathcal{K}(\psi) := \{u \in W(0,T),\ u \geq \psi\}.$$

Remark 2.1. *$\mathcal{K}(\psi)$ is a not empty convex set.*

Proof. Indeed, if one denotes by $v^\star$, the solution in $W(0,T)$ to

$$\partial_t v^\star - \Delta_p v^\star = \partial_t\psi - \Delta_p\psi \in L^{p'}(0,T;V'), \qquad v^\star(t=0) = \psi(0).$$

$-(v^\star-\psi)^- \in L^p(0,T;V)$ is an admissible test-function and one has that

$$0 = -\langle\partial_t(v^\star-\psi),(v^\star-\psi)^-\rangle + \int_\Omega 1_{\{v^\star-\psi<0\}}[|\nabla v^\star|^{p-2}\nabla v^\star - |\nabla\psi|^{p-2}\nabla\psi]\cdot\nabla(v^\star-\psi)dx.$$

Then, Corollary 4.5 with $\beta = 1$ and $\alpha = 1$ yields for any $t \in (0,T)$

$$0 \geq -\int_0^t \langle\partial_t(v^\star-\psi),(v^\star-\psi)^-\rangle ds = -\int_\Omega \int_0^{(v^\star-\psi)(t)} s^- ds dx + \int_\Omega \int_0^{(v^\star-\psi)(0)} s^- ds dx$$
$$= \frac{1}{2}\|(v^\star-\psi)^-(t)\|_{L^2}^2$$

since $(v^\star-\psi)^-(0) = 0$. As a consequence, $v^\star \geq \psi$ and $v^\star \in \mathcal{K}(\psi)$. □

Our aim is to prove the following result.

Theorem 2.2. *Under the above assumptions (H_1)-(H_4), there exists at least u in $\mathcal{K}(\psi)$ with $u(t=0) = u_0$ and such that, for any $v \in L^p(0,T;V)$, $v \geq \psi$ implies that*

$$\int_0^T \langle\partial_t u, v-u\rangle dt + \int_Q a(t,x,u,\nabla u)\cdot\nabla(v-u)dxdt \geq \int_0^T \langle f, v-u\rangle dt.$$

Moreover, the following Lewy-Stampacchia's inequality holds

$$0 \leq \partial_t u - \mathrm{div}[a(\cdot,\cdot,u,\nabla u)] - f \leq g^- = (f - \partial_t\psi + \mathrm{div}[a(\cdot,\cdot,\psi,\nabla\psi)])^-.$$

3 Proof of Theorem 2.2

Theorem 2.2 will be proved in four steps.
In a first part, we establish the existence of a solution to a problem where the constraint $u \geq \psi$ is penalized. Moreover, the crucial point in the method developed in the present paper is to replace $a(\cdot, \cdot, u, \vec{\xi})$ by $a(\cdot, \cdot, \max(u, \psi), \vec{\xi})$. The aim of this additional perturbation is to ensure, formally, a monotone behavior of the operator when u violates the constraint. This is the aim of Theorem 3.2.

For technical reasons, some *a priori* estimates and the passage to the limit will be obtained firstly by assuming that g^- is regular. This is the object of Lemmas 3.3, 3.4, 3.5 and Theorem 3.7. Then a proof of Lewy-Stampacchia's inequality, still with a regular g^-, will be presented in Lemma 3.9.

Finally, one will be able to prove Lewy-Stampacchia's inequality in the general case.

3.1 Penalization

Denote by $\tilde{q} = \min(p, 2)$ and let us define the function Θ

$$\Theta : \mathbb{R} \to \mathbb{R}, \quad x \mapsto -[x^-]^{\tilde{q}-1},$$

and the perturbed operator

$$\begin{aligned} \tilde{a}(t, x, u, \xi) : & Q \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d \\ & (x, t, u, \vec{\xi}) \mapsto \tilde{a}(t, x, u, \vec{\xi}) = a(t, x, \max(u, \psi(t, x)), \vec{\xi}). \end{aligned} \tag{3}$$

Remark 3.1. *We wish to draw the reader's attention to the fact that with the proposed perturbation: $\tilde{a}(t, x, u, \vec{\xi}) = a(t, x, \max(u, \psi), \vec{\xi})$, the idea is to make formally the operator monotone and not pseudomonotone any more on the free-set where the constraint is violated.*

We define $\mathcal{A} : L^p(0, T; V) \to L^{p'}(0, T; V')$ such that $[\mathcal{A}(u)](t) := \tilde{A}(u(t)) = -\mathrm{div}[\tilde{a}(t, x, u, \nabla u)]$. Note that, the above assumption H_1 still holds. Indeed,

$$\tilde{a}(t, x, u, \vec{\xi}) \cdot \vec{\xi} \geq \bar{\alpha}|\vec{\xi}|^p - \left[\bar{y}|\max(u, \psi)|^q + |\bar{h}(t, x)|\right], \tag{4}$$

$$|\tilde{a}(t, x, u, \vec{\xi})| \leq \bar{\beta}\left[|\bar{k}(t, x)| + |\max(u, \psi)|^{r/p} + |\vec{\xi}|\right]^{p-1}. \tag{5}$$

Since $|\max(u, \psi)|^q \leq |u|^q + |\psi|^q$, $|\max(u, \psi)|^{r/p} \leq |u|^{r/p} + |\psi|^{r/p}$, (1) and (2) are satisfied by replacing $\bar{h}$ by $\bar{h} + \bar{y}|\psi|^q$ and $\bar{k}$ by $\bar{k} + |\psi|^{r/p}$.

For any positive ε, a cosmetic modification of [23, Section 8.4] (see also [15, Chap. 3]) yields the following result.

Theorem 3.2. *There exists $u \in W(0, T)$ such that $u(t = 0) = u_0$ and*

$$\begin{aligned} & \partial_t u - \mathrm{div}\left[\tilde{a}(t, x, u, \nabla u)\right] + \frac{1}{\varepsilon}\Theta(u - \psi) = f, \\ \textit{i.e.} \quad & \partial_t u - \mathrm{div}\left[a(t, x, \max[u, \psi], \nabla u)\right] + \frac{1}{\varepsilon}\Theta(u - \psi) = f. \end{aligned} \tag{6}$$

3.2 The regular case: $g^- \in L^{\tilde{q}'}(Q) \hookrightarrow L^{p'}(Q)$

Following Assumption H_3 let us recall that $f - \partial_t \psi - A\psi = g = g^+ - g^-$ belongs to the order dual $L^p(0, T; V)^\star$. In this subsection we impose an additional regularity on g^-, namely $0 \leq g^- \in L^{\tilde{q}'}(Q) \hookrightarrow L^{p'}(Q)$.

3.2.1 *A priori* estimates with respect to ε

Let us test the penalized problem (6) with $u_\varepsilon - v^\star$,

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon - v^\star\|^2_{L^2(\Omega)} + \int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx + \frac{1}{\varepsilon}\int_\Omega \Theta(u_\varepsilon-\psi)(u_\varepsilon - v^\star)dx$$
$$= \langle f - \partial_t v^\star, u_\varepsilon - v^\star\rangle + \int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v^\star dx.$$

Thus, by using (1), for any positive δ_1, there exists C_{δ_1} depending on δ_1 and Ω such that

$$\int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla u_\varepsilon dx \geq \int_\Omega \bar{\alpha}|\nabla u_\varepsilon|^p - \bar{y}|\max(u_\varepsilon,\psi)|^q - |\bar{h}|dx$$
$$\geq \bar{\alpha}\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)} - \bar{y}\|u_\varepsilon\|^q_{L^q(\Omega)} - \bar{y}\|\psi\|^q_{L^q(\Omega)} - \|\bar{h}\|_{L^1(\Omega)}$$
$$\geq \bar{\alpha}\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)} - \delta_1\|u_\varepsilon\|^p_{L^p(\Omega)} - \bar{y}\|\psi\|^q_{L^q(\Omega)} - \|\bar{h}\|_{L^1(\Omega)} - C_{\delta_1}.$$

For the third term, $\Theta \leq 0$ and $v^\star \geq \psi$ yield

$$\frac{1}{\varepsilon}\int_\Omega \Theta(u_\varepsilon-\psi)(u_\varepsilon - v^\star)dx \geq \frac{1}{\varepsilon}\int_\Omega \Theta(u_\varepsilon-\psi)(u_\varepsilon - \psi)dx.$$

By using (2), for any positive δ_2, there exists C_{δ_2} depending on δ_2 and Ω such that

$$\int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla v^\star dx \leq \int_\Omega \bar{\beta}\Big[|\bar{k}| + |\max(u_\varepsilon,\psi)|^{r/p} + |\nabla u_\varepsilon|\Big]^{p-1}|\nabla v^\star|dx$$
$$\leq C_{\delta_2}\|v^\star\|^p_{W^{1,p}_0(\Omega)} + \delta_2\Big[\|\bar{k}\|^p_{L^p(\Omega)} + \|\psi\|^r_{L^r(\Omega)} + \|u_\varepsilon\|^r_{L^r(\Omega)} + \|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)}\Big]$$
$$\leq \delta_2\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)} + \delta_2\frac{r}{p}\|u_\varepsilon\|^p_{L^p(\Omega)} + C_{\delta_2}\|v^\star\|^p_{W^{1,p}_0(\Omega)} + \delta_2\|\bar{k}\|^p_{L^p(\Omega)} + C_{\delta_2}.$$

Finally, for any positive δ_3, there exists C_{δ_3} depending on δ_3 and Ω such that

$$\langle f - \partial_t v^\star, u_\varepsilon - v^\star\rangle \leq \delta_3[\|u_\varepsilon\|^p_V + \|v^\star\|^p_V] + C_{\delta_3}\|f - \partial_t v^\star\|^{p'}_{V'}.$$

In conclusion we have

$$\frac{1}{2}\frac{d}{dt}\|u_\varepsilon - v^\star\|^2_{L^2(\Omega)} + \bar{\alpha}\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)} + \frac{1}{\varepsilon}\int_\Omega \Theta(u_\varepsilon-\psi)(u_\varepsilon-\psi)dx$$
$$\leq \delta_1\|u_\varepsilon\|^p_{L^p(\Omega)} + \delta_2\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)} + \delta_2\frac{r}{p}\|u_\varepsilon\|^p_{L^p(\Omega)} + \delta_3\|u_\varepsilon\|^p_V$$
$$+ \bar{y}\|\psi\|^q_{L^q(\Omega)} + C_{\delta_2}\|v^\star\|^p_{W^{1,p}_0(\Omega)} + \delta_3\|v^\star\|^p_V$$
$$+ C_{\delta_3}\|f - \partial_t v^\star\|^{p'}_{V'} + \|\bar{h}\|_{L^1(\Omega)} + \delta_2\|\bar{k}\|^p_{L^p(\Omega)} + C_{\delta_1} + C_{\delta_2}\|\psi\|^r_{L^r(\Omega)}.$$

Then, using Young's inequality and a convenient choice of the parameters $\delta_1, \delta_2, \delta_3$ yield that for any positive δ there exists C depending on the listed parameters such that

$$\sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + \|u_\varepsilon\|^p_{L^p(0,T;W^{1,p}_0(\Omega))} + \frac{1}{\varepsilon}\|\Theta(u_\varepsilon-\psi)(u_\varepsilon-\psi)^-\|_{L^1(Q)}$$
$$\leq C(\delta, \|v^\star\|_{W(0,T)}, \|\psi\|_{L^p(0,T;V)}, \|\bar{k}\|_{L^p(\Omega)}, \|\bar{h}\|_{L^1(Q)}, \|f\|_{L^{p'}(0,T;V')}) + \delta\|u_\varepsilon\|^p_{L^p(0,T;V)}. \quad (7)$$

Lemma 3.3. *There exists a constant C_1 depending on $\|v^\star\|_{W(0,T)}$, $\|\psi\|_{L^p(0,T;V)}$, $\|\bar{k}\|_{L^p(\Omega)}$, $\|\bar{h}\|_{L^1(Q)}$ and $\|f\|_{L^{p'}(0,T;V')}$ such that, for any $\varepsilon > 0$,*

$$\sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + \|u_\varepsilon\|^p_{L^p(0,T;V)} + \frac{1}{\varepsilon}\|(u_\varepsilon-\psi)^-\|^{\tilde{q}}_{L^{\tilde{q}}(Q)} \leq C_1. \quad (8)$$

Proof. If $p \geq 2$, $W_0^{1,p}(\Omega) = V$ so that Lemma 3.3 is a straightforward consequence of (7).

If $p < 2$, it is enough to remark that

$$\begin{aligned}
\sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + \|u_\varepsilon\|^p_{L^p(0,T;V)} &= \sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + \int_0^T [\|u_\varepsilon(t)\|_{L^2(\Omega)} + \|u_\varepsilon(t)\|_{W_0^{1,p}(\Omega)}]^p dt \\
&\leq \sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + 2^{p-1} \int_0^T [\|u_\varepsilon(t)\|^p_{L^2(\Omega)} + \|u_\varepsilon(t)\|^p_{W_0^{1,p}(\Omega)}] dt \\
&\leq \sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + 2^{p-1} \int_0^T [\frac{p}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega)} + \frac{2-p}{2} + \|u_\varepsilon(t)\|^p_{W_0^{1,p}(\Omega)}] dt \\
&\leq (1 + 2^{p-2} pT) \sup_t \|u_\varepsilon\|^2_{L^2(\Omega)}(t) + 2^{p-1} \int_0^T \|u_\varepsilon(t)\|^p_{W_0^{1,p}(\Omega)} dt + 2^{p-2} T(2-p).
\end{aligned}$$

□

It is worth noting that Lemma 3.3 gives that $\frac{1}{\varepsilon}\int_Q ((u_\varepsilon - \psi)^-)^{\tilde{q}} dxdt$ is bounded (with respect to ε) so that we cannot expect to have a bound of the penalized term $\frac{1}{\varepsilon}\Theta(u_\varepsilon - \psi)$ in $L^{p'}(Q)$ nor in $L^{p'}(0, T; V')$.

Using the additional regularity $g^- \in L^{\tilde{q}'}(Q)$ we prove in the following lemma more precise estimates on $(u_\varepsilon - \psi)^-$.

Lemma 3.4. *There exists a constant C_2 depending on C_1 of Lemma 3.3, such that for any $\varepsilon > 0$,*

$$\sup_{t\in(0,T)} \|(u_\varepsilon - \psi)^-(t)\|^2_{L^2(\Omega)} \leq C_2 \|g^-\|_{L^{\tilde{q}'}(Q)} \varepsilon^{1/\tilde{q}}, \tag{9}$$

$$\int_Q \left| \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla\psi) \cdot \nabla(u_\varepsilon - \psi)^- \right| dxds \leq C_2 \|g^-\|_{L^{\tilde{q}'}(Q)} \varepsilon^{1/\tilde{q}}, \tag{10}$$

$$\frac{1}{\varepsilon}\|(u_\varepsilon - \psi)^-\|^{\tilde{q}-1}_{L^{\tilde{q}}(Q)} \leq C_2 \|g^-\|_{L^{\tilde{q}'}(Q)}. \tag{11}$$

Proof. With the admissible test-function $(u_\varepsilon - \psi)^-$, one gets that

$$\begin{aligned}
&-\langle \frac{d}{dt}(u_\varepsilon - \psi), (u_\varepsilon - \psi)^- \rangle - \frac{1}{\varepsilon}\int_\Omega \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi)^- dx \\
&- \int_{\Omega\cap\{u_\varepsilon - \psi < 0\}} \left[\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla\psi)\right] \cdot \nabla(u_\varepsilon - \psi)^- dx \\
&= -\langle f - \partial_t\psi + \operatorname{div}\left[\tilde{a}(t, x, \psi, \nabla\psi)\right], (u_\varepsilon - \psi)^- \rangle dt.
\end{aligned}$$

Then, since $(u_\varepsilon - \psi)^- \in L^p(0, T; V)$ with $(u_\varepsilon - \psi)^-(0) = 0$, Corollary 4.5 yields: for any $t \in (0, T)$,

$$\begin{aligned}
&\frac{1}{2}\|(u_\varepsilon - \psi)^-(t)\|^2_{L^2(\Omega)} - \frac{1}{\varepsilon}\int_0^t \int_\Omega \Theta(u_\varepsilon - \psi)(u_\varepsilon - \psi)^- dxds \\
&+ \int_0^t \int_{\{u_\varepsilon - \psi < 0\}} \left[\tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) - \tilde{a}(t, x, \psi, \nabla\psi)\right] \cdot \nabla(u_\varepsilon - \psi) dxds \\
&= -\int_0^t \langle f - \partial_t\psi + \operatorname{div}\left[\tilde{a}(t, x, \psi, \nabla\psi)\right], (u_\varepsilon - \psi)^- \rangle dt.
\end{aligned}$$

In view of the definition of $\tilde{a}$ we have $\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) = a(t,x,\psi,\nabla u_\varepsilon)$ in the set $\{u_\varepsilon < \psi\}$. Therefore using assumption $H_{1,2}$ we obtain

$$\frac{1}{2}\|(u_\varepsilon-\psi)^-(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \Big|[\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)]\cdot\nabla(u_\varepsilon-\psi)^-\Big|dxds$$
$$+\frac{1}{\varepsilon}\int_0^t \|\Theta(u_\varepsilon-\psi)(u_\varepsilon-\psi)^-\|_{L^1(\Omega)}ds$$
$$\le -\int_0^t\langle g,(u_\varepsilon-\psi)^-\rangle = -\int_0^t\langle g^+,(u_\varepsilon-\psi)^-\rangle + \int_0^T\int_\Omega g^-(u_\varepsilon-\psi)^- dxdt.$$

We recall that Lemma 3.3 yielded $\|(u_\varepsilon-\psi)^-\|^{\tilde{q}}_{L^{\tilde{q}}(Q)} \le C_1\varepsilon$ so that

$$\frac{1}{2}\|(u_\varepsilon-\psi)^-(t)\|^2_{L^2(\Omega)} + \int_0^t\int_\Omega \Big|[\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)]\cdot\nabla(u_\varepsilon-\psi)^-\Big|dxds$$
$$+\frac{1}{\varepsilon}\int_0^t \|(u_\varepsilon-\psi)^-\|^{\tilde{q}}_{L^{\tilde{q}}(\Omega)}ds$$
$$\le -\int_0^t\langle g,(u_\varepsilon-\psi)^-\rangle = -\int_0^t\langle g^+,(u_\varepsilon-\psi)^-\rangle + \int_0^T\int_\Omega g^-(u_\varepsilon-\psi)^- dxdt$$
$$\le \|g^-\|_{L^{\tilde{q}'}(Q)}\|(u_\varepsilon-\psi)^-\|_{L^{\tilde{q}}(Q)} \le \sqrt[\tilde{q}]{C_1\varepsilon}\|g^-\|_{L^{\tilde{q}'}(Q)}$$

and Lemma 3.4 holds. □

Gathering Lemmas 3.3 and 3.4 we prove the following estimates

Lemma 3.5. *There exists a constant C_3 depending on C_1, C_2 and $\|g^-\|_{L^{p'}(Q)}$ such that for any $\varepsilon > 0$*

$$\|\partial_t u_\varepsilon\|_{L^{p'}(0,T;V')} + \|\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\|_{L^{p'}(Q)} + \|\tilde{A}(u_\varepsilon)\|_{L^{p'}(0,T;V')} \le C_3.$$

Proof. The growth condition (5) on $\tilde{a}$ and Lemma 3.3 imply that

$$|\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)|^{p'} = |a(t,x,\max(u_\varepsilon,\psi),\nabla u_\varepsilon)|^{p'}$$
$$\le \tilde{\beta}^{p'}\Big[|\tilde{k}| + |u_\varepsilon|^{r/p} + |\psi|^{r/p} + |\nabla u_\varepsilon|\Big]^p$$
$$\le C\Big[|\tilde{k}|^p + |u_\varepsilon|^p + |\psi|^p + |\nabla u_\varepsilon|^p + 1\Big]$$

and then $\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)$ is bounded in $L^{p'}(Q)^d$. The boundedness of $\|\tilde{A}(u_\varepsilon)\|_{L^{p'}(0,T;V')}$ is a direct consequence of the above inequality.

Recalling that $\partial_t u_\varepsilon = f - \tilde{A}(u_\varepsilon) - \frac{1}{\varepsilon}\Theta(u_\varepsilon-\psi)$ it remains to estimate $\frac{1}{\varepsilon}\Theta(u_\varepsilon-\psi)$ in $L^{p'}(0,T;V')$. We distinguish the two cases $p \ge 2$ and $1 < p < 2$.

If $p \ge 2$ then $\tilde{q} = 2$. From Lemma 3.4 we have $\frac{1}{\varepsilon}\|(u_\varepsilon-\psi)^-\|_{L^2(Q)} \le C$ and since

$$\|\frac{1}{\varepsilon}\Theta(u_\varepsilon-\psi)\|_{L^{p'}(0,T;V')} = \sup_{\|v\|_{L^p(0,T;V)}\le 1}\langle\frac{1}{\varepsilon}\Theta(u-\psi),v\rangle \le \frac{1}{\varepsilon}\|(u_\varepsilon-\psi)^-\|_{L^2(Q)}\|v\|_{L^2(Q)} \le C$$

it follows that $\frac{1}{\varepsilon}\Theta(u_\varepsilon-\psi)$ is bounded in $L^{p'}(0,T;V')$.

If $1 < p < 2$ then $\tilde{q} = p$. From Lemma 3.4 we have $\frac{1}{\varepsilon}\|(u_\varepsilon-\psi)^-\|^{p-1}_{L^p(Q)} \le C$ and we have

$$\|\frac{1}{\varepsilon}\Theta(u_\varepsilon-\psi)\|_{L^{p'}(0,T;V')} = \sup_{\|v\|_{L^p(0,T;V)}\le 1}\langle\frac{1}{\varepsilon}\Theta(u_\varepsilon-\psi),v\rangle \le \frac{1}{\varepsilon}\|(u_\varepsilon-\psi)^-\|^{p-1}_{L^p(Q)} \le C$$

which concludes the proof of Lemma 3.5. □

3.2.2 At the limit when $\varepsilon \to 0$.

The sequence (u_ε) is bounded in $W(0,T)$, therefore, up to a subsequence denoted the same, there exists $u \in W(0,T)$ such that u_ε converges weakly to u in $W(0,T)$. In particular, one gets that $u(t=0) = u_0$.
Then, by classical compactness arguments of type Aubin-Lions-Simon [26], the convergence is strong in $L^p(Q)$, and a.e. in Q[†].

Therefore, $(u_\varepsilon - \psi)^- \to (u-\psi)^-$ in $L^p(Q)$ and thanks to Lemma 3.4, one gets that $(u-\psi)^- = 0$ *i.e.* $u \in \mathcal{K}(\psi)$.

Moreover from Lemma 3.5 there exists $\vec{\xi} \in L^{p'}(Q)^d$ such that

$$\tilde{a}(\cdot,\cdot,u_\varepsilon,\nabla u_\varepsilon) \text{ converges weakly to } \vec{\xi} \text{ in } L^{p'}(Q)^d. \tag{12}$$

By (2), the following estimate holds for any $v \in L^p(0,T;V)$,

$$|\tilde{a}(t,x,u,\nabla v)|^{p'} \le C\Big[1 + |\bar{k}|^p + |u|^p + |\psi|^p + |\nabla v|^p\Big],$$

so that, since $u \in \mathbb{R} \mapsto a(t,x,u,\nabla v)$ is a continuous function, the theory of Nemytskii operators gives that

$$\tilde{a}(t,x,u_\varepsilon,\nabla u) \to \tilde{a}(t,x,u,\nabla u) \text{ in } L^{p'}(Q)^d \tag{13}$$

and

$$\int_Q \tilde{a}(t,x,u_\varepsilon,\nabla u)\cdot\nabla(u_\varepsilon - u)dxdt \to 0. \tag{14}$$

Testing the penalized equation (6) introduced in Theorem 3.2 by $u_\varepsilon - u$ yields

$$\int_0^t \langle \partial_t u_\varepsilon, u_\varepsilon - u\rangle ds + \int_0^t\int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(u_\varepsilon - u)dxds = \int_0^t \langle f, u_\varepsilon - u\rangle ds - \frac{1}{\varepsilon}\int_0^t\int_Q \Theta(u_\varepsilon - \psi)(u_\varepsilon - u)dxds.$$

Since $\int_0^t \langle f, u_\varepsilon - u\rangle ds \to 0$, the following decomposition

$$-\frac{1}{\varepsilon}\int_0^t\int_\Omega \Theta(u_\varepsilon-\psi)(u_\varepsilon-u)dxds = \underbrace{-\frac{1}{\varepsilon}\int_0^t\int_\Omega \Theta(u_\varepsilon-\psi)(u_\varepsilon-\psi)dxds}_{\le 0} \underbrace{-\frac{1}{\varepsilon}\int_0^t\int_\Omega \Theta(u_\varepsilon-\psi)(\psi-u)dxds}_{\le 0}$$

leads to

$$\limsup_\varepsilon \Big[\int_0^t \langle \partial_t u_\varepsilon, u_\varepsilon - u\rangle ds + \int_0^t\int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)\cdot\nabla(u_\varepsilon - u)dxds\Big] \le 0.$$

Using (14) we obtain

$$\limsup_\varepsilon \Big[\int_0^t \langle \partial_t(u_\varepsilon - u), u_\varepsilon - u\rangle ds + \int_0^t\int_\Omega [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\nabla u)]\cdot\nabla(u_\varepsilon - u)dxds\Big] \le 0.$$

The monotone character of the operator $\tilde{a}(x,t,u,\vec{\xi})$ with respect to $\vec{\xi}$ (see Assumption $H_{1,2}$ and (3)) implies

$$\frac{1}{2}\limsup_\varepsilon \|(u_\varepsilon - u)(t)\|^2_{L^2(\Omega)} = \limsup_\varepsilon \int_0^t \langle \partial_t(u_\varepsilon - u), u_\varepsilon - u\rangle ds \le 0$$

† Some arguments are given in Annex 4.2 when $p < 2$.

and

$$\lim_{\varepsilon} \int_0^t \int_\Omega [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\nabla u)] \cdot \nabla(u_\varepsilon - u) dxds = 0. \tag{15}$$

It follows that

$$u_\varepsilon(t) \to u(t) \text{ in } L^2(\Omega) \text{ for any } t \tag{16}$$

and in view of (14)

$$\lim_{\varepsilon} \int_0^t \int_\Omega \tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) \cdot \nabla(u_\varepsilon - u) dxds = 0. \tag{17}$$

Set $\vec{v} \in L^p(Q)^d$. Since

$$\begin{aligned}
0 &\le \int_Q [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\vec{v})] \cdot [\nabla u_\varepsilon - \vec{v}] dxds \\
&= \int_Q [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\vec{v})] \cdot \nabla(u_\varepsilon - u) dxds \\
&\qquad + \int_Q [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\vec{v})] \cdot [\nabla u - \vec{v}] dxds \\
&= \int_Q [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\nabla u)] \cdot \nabla(u_\varepsilon - u) dxds \\
&\qquad + \int_Q [\tilde{a}(t,x,u_\varepsilon,\nabla u) - \tilde{a}(t,x,u_\varepsilon,\vec{v})] \cdot \nabla(u_\varepsilon - u) dxds \\
&\qquad + \int_Q [\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon) - \tilde{a}(t,x,u_\varepsilon,\vec{v})] \cdot [\nabla u - \vec{v}] dxds,
\end{aligned}$$

using (15) and information similar to (14) allow one to pass to the limit and to conclude that

$$0 \le \int_Q [\vec{\xi} - \tilde{a}(t,x,u,\vec{v})] \cdot [\nabla u - \vec{v}] dxds.$$

By the classical Minty's trick, considering $\vec{v} = \nabla u + \lambda \vec{w}$, $\vec{w} \in L^p(Q)^d$ and $\lambda \in \mathbb{R}$, we have necessarily

$$0 = \lim_{\lambda \to 0} \int_Q [\vec{\xi} - \tilde{a}(t,x,u,\nabla u + \lambda \vec{w})] \cdot \vec{w} dxds.$$

Thus, a classical property of radial continuity coming from the assumptions on a yields, for any $\vec{w} \in L^p(Q)^d$,

$$\int_Q \vec{\xi} \cdot \vec{w} dxds = \int_Q \tilde{a}(t,x,u,\nabla u) \cdot \vec{w} dxds = \int_Q a(t,x,u,\nabla u) \cdot \vec{w} dxds,$$

i.e. $\xi = \tilde{a}(t,x,u,\nabla u) = a(t,x,u,\nabla u)$, since $u \ge \psi$.

Remark 3.6. *Note that, following [3, Proof of Lemma 1] ,* (15) *yields the convergence in measure, then the a.e. convergence of* ∇u_ε *to* ∇u *(up to a subsequence if needed), so that this is also a way to identify* $\vec{\xi}$ *has being* $a(t,x,u,\nabla u)$.

We are now in a position to pass to the limit in the penalized problem and to conclude the existence of a solution to the obstacle problem under the additional regularity on g^-.

Let us consider $v \in L^p(0, T; V)$, $v \geq \psi$ as a test function in the penalized problem (6),

$$\int_0^T \langle \partial_t u_\varepsilon, v - u_\varepsilon \rangle + \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(v - u_\varepsilon) dx dt + \frac{1}{\varepsilon} \int_Q \Theta(u_\varepsilon - \psi)(v - u_\varepsilon) dx dt = \int_0^T \langle f, v - u_\varepsilon \rangle dt. \quad (18)$$

In view of (16) we have

$$\begin{aligned} \int_0^T \langle \partial_t u_\varepsilon, v - u_\varepsilon \rangle dt &= \int_0^T \langle \partial_t u_\varepsilon, v \rangle dt - \frac{1}{2} \|u_\varepsilon(T)\|^2_{L^2(\Omega)} + \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} \\ &\to \int_0^T \langle \partial_t u, v \rangle dt - \frac{1}{2} \|u(T)\|^2_{L^2(\Omega)} + \frac{1}{2} \|u_0\|^2_{L^2(\Omega)} = \int_0^T \langle \partial_t u, v - u \rangle dt. \end{aligned}$$

From (17) and the identification $\vec{\xi} = \tilde{a}(t, x, u, \nabla u) = a(t, x, u, \nabla u)$ it follows that

$$\begin{aligned} \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(v - u_\varepsilon) dx dt &= \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(v - u) dx dt + \int_Q \tilde{a}(t, x, u_\varepsilon, \nabla u_\varepsilon) \cdot \nabla(u - u_\varepsilon) dx dt \\ &\to \int_Q \tilde{a}(t, x, u, \nabla u) \cdot \nabla(v - u) dx dt = \int_Q a(t, x, u, \nabla u) \cdot \nabla(v - u) dx dt. \end{aligned}$$

The weak convergence of u_ε to u in $L^p(0, T; V)$ yields that

$$\int_0^T \langle f, v - u_\varepsilon \rangle \to \int_0^T \langle f, v - u \rangle.$$

At last splitting the penalized term in the following way

$$\frac{1}{\varepsilon} \int_Q \Theta(u_\varepsilon - \psi)(v - u_\varepsilon) dx dt = \underbrace{\frac{-1}{\varepsilon} \int_Q [(u_\varepsilon - \psi)^-]^{\tilde{q}-1}(v - \psi) dx dt}_{\leq 0} \underbrace{- \frac{1}{\varepsilon} \|(u_\varepsilon - \psi)^-\|^{\tilde{q}}_{L^{\tilde{q}}(Q)}}_{\to 0 \text{ thanks to (11)}}$$

allows one to pass to the limit in (18). One concludes that a solution exists, *i.e.*

Theorem 3.7. *Assume $H_1 - H_4$, $f - \partial_t \psi - A\psi = g = g^+ - g^- \in L^p(0, T; V)^\star$ where $g^- \in L^{p'}(Q) \cap L^2(Q)$. There exists at least $u \in \mathcal{K}(\psi)$ with $u(t = 0) = u_0$ such that, for any $v \in L^p(0, T; V)$ with $v \geq \psi$,*

$$\int_0^T \langle \partial_t u, v - u \rangle dt + \int_Q a(t, x, u, \nabla u) \cdot \nabla(v - u) dx dt \geq \int_0^T \langle f, v - u \rangle dt.$$

Remark 3.8. *Note that the pseudomonotone assumption of the operator doesn't ensure the uniqueness of the solution. Observe that under additional assumptions on the operator a, namely a local Lipschitz continuity with respect to the third variable, standard arguments allow one to prove the uniqueness of the solution obtained in Theorem 3.7.*

3.3 Lewy-Stampacchia's inequality for a regular g^-.

Note that $\mu_\varepsilon := \partial_t u_\varepsilon - \text{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f = \frac{1}{\varepsilon}[(u_\varepsilon - \psi)^-]^{\tilde{q}-1} \geq 0$, so that the limit $\mu := \partial_t u - \text{div}[\tilde{a}(\cdot, \cdot, u, \nabla u)] - f$ is a non-negative Radon measure which is by Lemma 3.5 an element of $L^{p'}(0, T; V')$.

Using an idea from A. Mokrane and F. Murat [18], denote by $z_\varepsilon := g^- - \frac{1}{\varepsilon}[(u_\varepsilon - \psi)^-]^{\tilde{q}-1}$, we have

$$\partial_t u_\varepsilon + A(u_\varepsilon) + z_\varepsilon = g^+ + \partial_t \psi + A(\psi) \quad i.e. \quad \partial_t(u_\varepsilon - \psi) + A(u_\varepsilon) - A(\psi) + z_\varepsilon = g^+.$$

Observing that

$$\partial_t u_\varepsilon + A(u_\varepsilon) - f = -z_\varepsilon + g^-$$

as in [18] in the elliptic case and under more restrictive assumptions on the operator a, proving that z_ε^- converges to 0 in an appropriate space leads to the Lewy-Stampacchia's inequality. Due to the time variable and the weak assumption on a we have to face to additional difficulties. For technical reasons, we will assume in this section only that, on top of $g^- \in L^{p'}(Q) \cap L^p(0, T; V)$, $g^- \geq 0$, that $\partial_t g^- \in L^{\tilde{q}'}(Q)$. Roughly speaking it allows one to use a test function depending on g^- and together with Lemma 4.3 to perform an integration by part formula and then the convergence analysis of z_ε^-.

The general case will be obtained in the next section by a regularization argument based on Lemma 4.1 of the Annex.

Our aim is now to show the convergence of z_ε^- to 0 in $L^2(Q)$ to prove the following lemma.

Lemma 3.9. *Under the assumptions of Theorem 3.7 and assuming moreover that $g^- \in L^{p'}(Q) \cap L^p(0, T; V)$, $g^- \geq 0$ with $\partial_t g^- \in L^{\tilde{q}'}(Q)$, the solution u satisfies*

$$0 \leq \partial_t u - \operatorname{div}[a(\cdot, \cdot, u, \nabla u)] - f \leq g^- \text{ in } L^{p'}(0, T; V').$$

A priori, following Lemma's 4.3 notations, one should denote by

$$\Psi(t, x, \lambda) = -(g^- - \frac{1}{\varepsilon}[\lambda^-]^{\tilde{q}-1})^- \quad \text{and} \quad \Lambda(t, x, \lambda) = \int_0^\lambda \Psi(t, x, \sigma) d\sigma.$$

For that, we need $\Psi(t, x, u)$ to be a test-function.
Since $x \mapsto [x^-]^{\tilde{q}-1}$ is not *a priori* a Lipschitz-continuous function (*e.g.* if $p < 2$[‡]), therefore, for any positive k, we will denote by $\eta_k(x) = (\tilde{q} - 1)\int_0^{x^+} \min(k, s^{\tilde{q}-2}) ds$, $\Psi_k(t, x, \lambda) = -(g^- - \frac{1}{\varepsilon}\eta_k(\lambda^-))^-$ and $\Lambda_k(t, x, \lambda) = \int_0^\lambda \Psi_k(t, x, \sigma) d\sigma$.
Note that $\Psi_k(t, x, 0) = 0$ and $\partial_t \Psi_k(t, x, \lambda) = \partial_t g^- 1_{\{g^- - \frac{1}{\varepsilon}\eta_k(\lambda^-) < 0\}}$ so that, since $\Psi_k(t, x, u)$ is a test-function, by Lemma 4.3, for any t,

$$\begin{aligned}
&-\int_0^t \int_\Omega \partial_t \Lambda_k(s, x, u_\varepsilon - \psi) dx ds + \int_\Omega \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx - \int_\Omega \Lambda_k(0, x, u_\varepsilon(0) - \psi(0)) dx \\
&- \int_0^t \langle A(u_\varepsilon) - A(\psi), (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- \rangle dt - \int_Q z_\varepsilon (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- dx ds \\
&= -\int_0^t \langle g^+, (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon - \psi)^-])^- \rangle ds \leq 0. \qquad (19)
\end{aligned}$$

We now pass to the limit first as $k \to \infty$ and then as $\varepsilon \to 0$. Since $g^- \geq 0$, one has that $\Psi_k(t, x, \lambda) = 0$ if $\lambda \geq 0$ and as $u_\varepsilon(0) = u_0 \geq \psi(0)$, one gets that

$$\int_\Omega \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx - \int_\Omega \Lambda_k(0, x, u_\varepsilon(0) - \psi(0)) dx = \int_\Omega \Lambda_k(t, x, u_\varepsilon(t) - \psi(t)) dx.$$

‡ $\tilde{q} = \min(2, p)$

Note that $(\Psi_k(t,x,\lambda))_k$ is a non-increasing sequence of functions with non-positive values so that by monotone convergence theorem

$$\int_\Omega \Lambda_k(t,x,u_\varepsilon(t)-\psi(t))dx \underset{k}{\to} -\int_\Omega \int_0^{(u_\varepsilon-\psi)(t)} (g^- - \frac{1}{\varepsilon}[\sigma^-]^{\tilde{q}-1}d\sigma)^- dx \geq 0$$

since the integration holds on the set of negative values of $u_\varepsilon(t)-\psi(t)$.

Due to the definition of z_ε we have

$$\begin{aligned}
&-\int_Q z_\varepsilon(g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-])^- dxdt \\
&= -\int_Q (g^- - \frac{1}{\varepsilon}[(u_\varepsilon-\psi)^-]^{\tilde{q}-1})(g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-])^- dxdt \\
&= \int_Q (g^- - \frac{1}{\varepsilon}[(u_\varepsilon-\psi)^-]^{\tilde{q}-1})^-(g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-])^- dxdt \\
&\quad -\int_Q (g^- - \frac{1}{\varepsilon}[(u_\varepsilon-\psi)^-]^{\tilde{q}-1})^+(g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-])^- dxdt,
\end{aligned}$$

from which it follows using again the monotone convergence theorem

$$-\int_Q z_\varepsilon(g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-])^- dxdt \underset{k}{\longrightarrow} \int_0^T \|z_\varepsilon^-\|^2_{L^2(\Omega)}dt.$$

As far as the first term of (19) is concerned we obtain

$$\begin{aligned}
-\int_Q \partial_t \Lambda_k(t,x,u_\varepsilon-\psi)dxds &= -\int_Q \partial_t g^- \int_0^{u_\varepsilon-\psi} 1_{\{g^- - \frac{1}{\varepsilon}\eta_k(\tau^-)<0\}}d\tau dxds \\
= -\int_Q \partial_t g^- \int_0^{-(u_\varepsilon-\psi)^-} 1_{\{g^- - \frac{1}{\varepsilon}\eta_k(\tau^-)<0\}}d\tau dxds &\geq -\int_Q |\partial_t g^-||(u_\varepsilon-\psi)^-|dxds \underset{\varepsilon}{\longrightarrow} 0.
\end{aligned}$$

For the fourth term of (19) we have the following equality

$$\begin{aligned}
&-\int_0^T \langle A(u_\varepsilon)-A(\psi), (g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-])^-\rangle dt \\
&= \int_Q 1_{\{g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-]<0\}}\Big[\tilde{a}(t,x,u_\varepsilon,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big]\nabla[g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-]]dxdt \\
&= \int_Q 1_{\{g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-]<0\}}\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big]\nabla[g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-]]dxdt,
\end{aligned}$$

since in this situation, the integration holds in the set where $u_\varepsilon \leq \psi$. Thus,

$$\begin{aligned}
&\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big]\nabla[g^- - \frac{1}{\varepsilon}\eta_k[(u_\varepsilon-\psi)^-]] \\
&\geq \frac{1}{\varepsilon}\eta_k'[(u_\varepsilon-\psi)^-]\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon) \\
&\qquad -\tilde{a}(t,x,\psi,\nabla\psi)\Big]\nabla(u_\varepsilon-\psi) - \Big|\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big||\nabla g^-| \\
&\geq -\Big|\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big||\nabla g^-|.
\end{aligned}$$

We now claim that estimate (10) of Lemma 3.4 which gives

$$\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big]\nabla(u_\varepsilon-\psi)^- \underset{\varepsilon}{\longrightarrow} 0 \text{ in } L^1(Q)$$

and Assumptions $H_{1,1}$ to $H_{1,3}$ imply that, up to a subsequence (still denoted by ε), $\nabla(u_\varepsilon-\psi)^-$ converges to 0 a.e. in Q.

Indeed, up to a subsequence (still denoted by ε), u_ε converges to u a.e. in Q with $u \geq \psi$ a.e. and $\Big|\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big||\nabla(u_\varepsilon-\psi)^-| \to 0$ a.e. in Q.

Consider (t,x) such that the above limits hold. Since,

$$\begin{aligned}
-\tilde{a}(t,x,\psi,\nabla u_\varepsilon)\cdot\nabla(u_\varepsilon-\psi)^- &\geq \Big[\bar{\alpha}|\nabla u_\varepsilon|^p-\bar{y}|\psi|^q-|\bar{h}|-\tilde{a}(t,x,\psi,\nabla u_\varepsilon).\nabla\psi\Big]1_{\{u_\varepsilon<\psi\}}\\
&\geq \Big[\bar{\alpha}|\nabla u_\varepsilon|^p-\bar{y}|\psi|^q-|\bar{h}|-\bar{\beta}\Big[|\bar{k}|+|\psi|^{r/p}+|\nabla u_\varepsilon|\Big]^{p-1}|\nabla\psi|\Big]1_{\{u_\varepsilon<\psi\}}\\
&\geq \Big[\bar{\alpha}/2|\nabla u_\varepsilon|^p-C(\psi,\bar{h},\bar{k},\nabla\psi)\Big]1_{\{u_\varepsilon<\psi\}},
\end{aligned}$$

and

$$|\tilde{a}(t,x,\psi,\nabla\psi)\cdot\nabla(u_\varepsilon-\psi)^-| \leq \bar{\beta}\Big[|\bar{k}|+|\psi|^{r/p}+|\nabla\psi|\Big]^{p-1}\Big[|\nabla u_\varepsilon|+|\nabla\psi|\Big]1_{\{u_\varepsilon<\psi\}},$$

one gets that $(\nabla(u_\varepsilon-\psi)^-(t,x))_\varepsilon$ is a bounded sequence.

Since $\nabla(u_\varepsilon-\psi)^-(t,x)=-\nabla(u_\varepsilon-\psi)(t,x)1_{\{u_\varepsilon<\psi\}}(t,x)$, it converges to 0 if $u(t,x)>\psi(t,x)$. Else, at the limit, one has that $u(t,x)=\psi(t,x)$. If one assumes that $\nabla(u_\varepsilon-\psi)^-(t,x)$ is not converging to 0, then there exists a subsequence ε' (depending on (t,x)) and a positive δ such that $\|\nabla(u_{\varepsilon'}-\psi)^-(t,x)\|\geq\delta>0$. Then, necessarily $-\nabla(u_{\varepsilon'}-\psi)^-(t,x)=\nabla(u_{\varepsilon'}-\psi)(t,x)$ and, since it is a bounded sequence in $\mathbb{R}^d$, there exists $\vec{\xi}\in\mathbb{R}^d$ and a new subsequence still labeled ε' such that $\nabla u_{\varepsilon'}(t,x)$ converges to $\vec{\xi}$, with the additional information: $\|\vec{\xi}-\nabla\psi(t,x)\|\geq\delta>0$. Therefore, since $\vec{\xi}\neq\nabla\psi(t,x)$

$$\begin{aligned}
&\Big[\tilde{a}(t,x,\psi,\nabla u_{\varepsilon'}(t,x))-\tilde{a}(t,x,\psi,\nabla\psi(t,x))\Big]\cdot\nabla(u_{\varepsilon'}-\psi)^-(t,x)\\
&=-\Big[\tilde{a}(t,x,\psi,\nabla u_{\varepsilon'}(t,x))-\tilde{a}(t,x,\psi,\nabla\psi(t,x))\Big]\cdot\nabla(u_{\varepsilon'}-\psi)(t,x)\\
&\underset{\varepsilon'}{\longrightarrow}-\Big[\tilde{a}(t,x,\psi,\vec{\xi})-\tilde{a}(t,x,\psi,\nabla\psi(t,x))\Big]\cdot[\vec{\xi}-\nabla\psi(t,x)]<0.
\end{aligned}$$

But, this is in contradiction with the convergence of the same sequence to 0 and the result holds.

Note that $(t,x)\in Q$ a.e.,

$$\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big]1_{\{u_\varepsilon<\psi\}}=\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon 1_{\{u_\varepsilon<\psi\}})-\tilde{a}(t,x,\psi,\nabla\psi 1_{\{u_\varepsilon<\psi\}})\Big]$$

and $\nabla u_\varepsilon 1_{\{u_\varepsilon<\psi\}}-\nabla\psi 1_{\{u_\varepsilon<\psi\}}$ converges to 0 with $\nabla\psi 1_{\{u_\varepsilon<\psi\}}$ bounded. Then, the continuity of $\tilde{a}$ with respect to its fourth argument can be assumed uniform and $\Big[\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big]1_{\{u_\varepsilon<\psi\}}$ converges a.e. to 0.

Since it is bounded in $L^{p'}(Q)$, it converges weakly to 0 in $L^{p'}(Q)$ and

$$\int_Q\Big|\tilde{a}(t,x,\psi,\nabla u_\varepsilon)-\tilde{a}(t,x,\psi,\nabla\psi)\Big||\nabla g^-|1_{\{u_\varepsilon<\psi\}}dxdt\to 0.$$

As a conclusion, z_ε^- converges to 0 in $L^2(Q)$. On the one hand we have

$$\begin{aligned}
&0\leq\mu_\varepsilon=\frac{1}{\varepsilon}\big[(u_\varepsilon-\psi)^-\big]^{\tilde{q}-1}=\partial_t u_\varepsilon-\operatorname{div}[\tilde{a}(\cdot,\cdot,u_\varepsilon,\nabla u_\varepsilon)]-f\\
&\Rightarrow\quad 0\leq\partial_t u-\operatorname{div}[\tilde{a}(\cdot,\cdot,u,\nabla u)]-f;
\end{aligned}$$

On the other hand

$$\begin{aligned}
z_\varepsilon=g^--\frac{1}{\varepsilon}\big[(u_\varepsilon-\psi)^-\big]^{\tilde{q}-1}&\Rightarrow z_\varepsilon^++\partial_t u_\varepsilon-\operatorname{div}[\tilde{a}(\cdot,\cdot,u_\varepsilon,\nabla u_\varepsilon)]-f=g^-+z_\varepsilon^-\\
&\Rightarrow 0\leq\partial_t u-\operatorname{div}[\tilde{a}(\cdot,\cdot,u,\nabla u)]-f\leq g^-.
\end{aligned}$$

Since $\tilde{a}(\cdot,\cdot,u,\nabla u)=a(\cdot,\cdot,u,\nabla u)$, Lemma 3.9 is proved.

Remark 3.10. *Note that, for any $\varphi \in L^p(0, T; V)$,*

$$\begin{aligned}
&\int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi \rangle dt \\
&= \int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi^+ \rangle dt - \int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi^- \rangle dt \\
&\leq \int_0^T \langle \partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f, \varphi^+ \rangle dt \leq \int_0^T \langle g^-, \varphi^+ \rangle dt.
\end{aligned}$$

In such a way, $\|\partial_t u_\varepsilon - \operatorname{div}[\tilde{a}(\cdot, \cdot, u_\varepsilon, \nabla u_\varepsilon)] - f\|_{L^{p'}(0,T;V')} \leq \|g^-\|_{L^{p'}(0,T;V')}$.

3.4 Proof of the main result

In this section, H_3 is assumed and $g = f - \partial_t \psi - A(\psi) = g^+ - g^-$ where $g^+, g^- \in (L^{p'}(0, T; V'))^+$ are non-negative elements of $L^{p'}(0, T; V')$.

Thanks to Lemma 4.1, there exists positives $(g_n^-) \subset L^{p'}(Q)$ such that $g_n^- \to g^-$ in $L^{p'}(0, T; V')$. Then, by a regularization procedure, one can assume that $g_n^- \in L^{p'}(Q) \cap L^p(0, T; V)$, $g_n^- \geq 0$ with $\partial_t g_n^- \in L^{\tilde{q}'}(Q)$. Then, the corresponding sequence f_n converges to f in $L^{p'}(0, T; V')$.

Remark 3.11. *In fact, since $D(Q)^+$ is dense in $L^{p'}(Q)^+$, one can consider g_n^- as regular as needed.*

Associated with g_n^-, Theorem 3.7 and Lemma 3.9 provide the existence of $u_n \in \mathcal{K}(\psi)$ with $u_n(t = 0) = u_0$ and such that, for any $v \in L^p(0, T; V)$, $v \geq \psi$ implies that

$$\int_0^T \langle \partial_t u_n, v - u_n \rangle dt + \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla(v - u_n) dx dt \geq \int_0^T \langle f_n, v - u_n \rangle dt$$

and satisfying the Lewy-Stampacchia's inequality

$$0 \leq \partial_t u_n - \operatorname{div}[a(\cdot, \cdot, u_n, \nabla u_n)] - f_n \leq g_n^-.$$

Since this solution comes from the above penalization method, and as C_1 of Lemma 3.3 can be chosen independent of n, one gets that

$$\sup_t \|u_n\|^2_{L^2(\Omega)}(t) + \|u_n\|^p_{L^p(0,T;V)} \leq C_1.$$

Thus, $a(\cdot, u_n, \nabla u_n)$ is bounded in $L^{p'}(Q)^d$ and, thanks to the above Lewy-Stampacchia's inequality, $\partial_t u_n$ is bounded in $L^{p'}(0, T; V')$.

Up to a subsequence denoted similarly, u_n converges weakly to an element $u \in \mathcal{K}(\psi)$ in $W(0, T)$ and strongly in $L^p(Q)$; and $a(\cdot, u_n, \nabla u_n)$ converges to an element $\vec{\xi}$ in $L^{p'}(Q)^d$.
Finally, the embedding of $W(0, T)$ in $C([0, T], L^2(\Omega))$ yields the weak convergence of $u_n(t)$ to $u(t)$ in $L^2(\Omega)$, for any t.

Since $u \in \mathcal{K}(\psi)$,

$$\int_0^T \langle \partial_t u_n, u - u_n \rangle dt + \int_Q a(t, x, u_n, \nabla u_n) \cdot \nabla(u - u_n) dx dt \geq \int_0^T \langle f_n, u - u_n \rangle dt.$$

Therefore, passing to the limit with respect to n in

$$\int_0^T \langle \partial_t u_n, u\rangle dt + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}$$
$$\geq \int_0^T \langle f_n, u-u_n\rangle dt + \frac{1}{2}\|u_n(T)\|^2_{L^2(\Omega)} + \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt$$

yields

$$\int_0^T \langle \partial_t u, u\rangle dt + \int_Q \vec{\xi}\cdot\nabla u dxdt + \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}$$
$$\geq \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} + \limsup_n \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt.$$

Since $\int_0^T \langle \partial_t u, u\rangle dt = \frac{1}{2}\|u(T)\|^2_{L^2(\Omega)} - \frac{1}{2}\|u_0\|^2_{L^2(\Omega)}$, one gets that

$$\limsup_n \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt \leq \int_Q \vec{\xi}\cdot\nabla u dxdt.$$

Thus, (2) and the continuity property of Nemytskii's operator ensure the following limit argument:

$$0 \leq \int_Q [a(t,x,u_n,\nabla u_n) - a(t,x,u_n,\nabla u)]\cdot\nabla(u_n-u)dxdt$$
$$= \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt - \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u dxdt$$
$$- \int_Q a(t,x,u_n,\nabla u)\cdot\nabla(u_n-u)dxdt,$$

thus

$$0 \leq \liminf_n \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt - \int_Q \vec{\xi}\cdot\nabla u dxdt.$$

Then, $\lim_n \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla u_n dxdt = \int_Q \vec{\xi}\cdot\nabla u dxdt$ and arguments already developed previously based on Minty's trick for the pseudomonotone operator A yield the identification $\vec{\xi} = a(t,x,u,\nabla u)$ and one has

$$\lim_n \int_Q a(t,x,u_n,\nabla u_n)\cdot\nabla(u_n-v)dxdt = \int_Q a(t,x,u,\nabla u)\cdot\nabla(u-v)dxdt.$$

From the weak lower semicontinuity of $|\cdot|_{L^2(\Omega)}$, one has

$$\limsup_n \int_0^T \langle \partial_t u_n, v-u_n\rangle dt \leq \int_0^T \langle \partial_t u, v-u\rangle dt.$$

Since $\int_0^T \langle f_n, v-u_n\rangle dt \to \int_0^T \langle f, v-u\rangle dt$, we deduce the existence result of Theorem 2.2 for general f. At last the Lewy-Stampacchia's inequality is a consequence of passing to the limit in the one satisfied by u_n.

4 Annex

4.1 Positive cones in the dual

Lemma 4.1. *The positive cone of $L^p(0,T;V)\cap L^2(Q)$ is dense in the positive cone of $\mathcal{V}'$, the dual set of $\mathcal{V} = L^p(0,T;V)$.*
By a truncation argument, the same result holds for the positive cone of $L^p(0,T;V)\cap L^{p'}(Q)$ when $p<2$.

Proof. This result is given in [7, Lemma p.593]. We propose here a sketch of a proof following the idea of [18].

Consider $f\in L^{p'}(0,T;V')$ such that $f\geq 0$ in the sense: $\forall\varphi\in L^p(0,T;V),\ \varphi\geq 0\Rightarrow\langle f,\varphi\rangle\geq 0$.
Let us construct a sequence $(f_\varepsilon)\subset L^p(0,T;V)$ with $f_\varepsilon\geq 0$ such that $f_\varepsilon\to f$ in $L^{p'}(0,T;V')$.
Consider the following operator $B: L^p(0,T;V)\to L^{p'}(0,T;V')$ defined, for any $u,v\in L^p(0,T;V)$ by

$$\langle Bu,v\rangle=\int_Q|\nabla u|^{p-2}\nabla u.\nabla v\,dxdt+\int_0^T\|u(t)\|^{p-2}_{L^2(\Omega)}\int_\Omega uv\,dx\,dt. \tag{20}$$

$B=DJ$ where $J: u\mapsto\frac{1}{p}\int_Q|\nabla u|^p\,dxdt+\frac{1}{p}\int_0^T\|u(t)\|^p_{L^2(\Omega)}\,dt$, is a G-differentiable, hemi continuous convex function, and, B is a strictly monotone, bounded, continuous and coercive operator from $L^p(0,T;V)$ into $L^{p'}(0,T;V')$. Then ([23, section 2.1]), denote by v the unique solution to $Bv=f$.

For any $\varepsilon>0$ and $n\in\mathbb{N}$, denote by v_ε^n the solution to $Bv_\varepsilon^n+\frac{1}{\varepsilon}T_n(v_\varepsilon^n-v)=0$ in $L^p(0,T;V)$ where T_n is the truncation at the height n.
Using v_ε^n-v as test function, one has $J(v_\varepsilon^n)+\frac{1}{\varepsilon}\|T_n(v_\varepsilon^n-v)\|^2_{L^2(Q)}\leq J(v)$.
Thus, there exists v_ε, weak limit in $L^p(0,T;V)$ of a subsequence (denoted similarly) of v_ε^n satisfying: $J(v_\varepsilon)+\frac{1}{\varepsilon}\|v_\varepsilon-v\|^2_{L^2(Q)}\leq J(v)$ and, by classical monotony arguments, solution in $L^p(0,T;V)$ to the problem:

$$\langle Bv_\varepsilon,w\rangle+\frac{1}{\varepsilon}\int_Q(v_\varepsilon-v)w\,dxdt=0\quad\forall w\in L^p(0,T;V)\cap L^2(Q). \tag{21}$$

Then, up to a subsequence, $v_\varepsilon-v\to 0$ in $L^2(Q)$, $v_\varepsilon\rightharpoonup v$ in $L^p(0,T;V)$ and

$$\langle Bv_\varepsilon-Bv,(v_\varepsilon-v)^+\rangle+\frac{1}{\varepsilon}\int_Q|(v_\varepsilon-v)^+|^2dxdt=-\langle f,(v_\varepsilon-v)^+\rangle\leq 0, \tag{22}$$

so that $f_\varepsilon=-\frac{1}{\varepsilon}(v_\varepsilon-v)\in L^p(0,T;V)\cap L^2(Q)$ is non-negative.
Finally, as $\limsup_\varepsilon J(v_\varepsilon)\leq J(v)$, an argument of uniform convexity yields the convergence of v_ε to v in $L^p(0,T;V)$ and $-\frac{1}{\varepsilon}(v_\varepsilon-v)=Bv_\varepsilon\to Bv=f$ in $L^{p'}(0,T;V')$. □

4.2 Compactness when $p<2$.

Concerning the compactness argument in $L^p(Q)$ when $p<2$: note that there exists an integer $k\geq 1$ such that $W_0^{k,p}(\Omega)\underset{dense}{\hookrightarrow}L^{p'}(\Omega)$ so that

$$W_0^{k,p}(\Omega)\underset{dense}{\hookrightarrow}V\underset{dense}{\hookrightarrow}L^p(\Omega)\equiv[L^{p'}(\Omega)]'\hookrightarrow W^{-k,p'}(\Omega)\text{ and }V'\hookrightarrow W^{-k,p'}(\Omega).$$

Remark 4.2. *Let us justify that the identification $L^p(\Omega)\equiv[L^{p'}(\Omega)]'$ is possible if $L^2(\Omega)$ is already chosen as the pivot-space.*

Indeed, one has: $L^{p'}(\Omega) \underset{dense}{\hookrightarrow} L^2(\Omega) \underset{dense}{\hookrightarrow} L^p(\Omega)$ with reflexive B-spaces, so that $L^p(\Omega)' \underset{dense}{\hookrightarrow} L^2(\Omega)' \underset{dense}{\hookrightarrow} L^{p'}(\Omega)'$.
Consider $T \in L^{p'}(\Omega)'$ and $T_n \in L^2(\Omega)'$ such that $T_n \to T$ in $L^{p'}(\Omega)'$. Then, by the pivot-space identification, there exists $u_n \in L^2(\Omega)$ such that $T_n = u_n$ in the sense of Riesz-identification.
Then, for any $v \in L^{p'}(\Omega)$ with norm 1,

$$|\int_\Omega u_n v dx| \le \|T\|_{L^{p'}(\Omega)'} + \|T_n - T\|_{L^{p'}(\Omega)'}.$$

By considering $v = Sgn(u_n)\frac{|u_n|^{p-1}}{\|u_n\|_{L^p}^{p-1}}$, one has that the sequence (u_n) is bounded in $L^p(\Omega)$ and that, up to a subsequence if needed, it converges weakly to a given u in $L^p(\Omega)$.
Thus, for any $v \in L^{p'}(\Omega)$,

$$\langle T, v\rangle = \lim\langle T_n, v\rangle = \lim \int_\Omega u_n v dx = \int_\Omega uv dx.$$

Since this element u is unique in its way, the identification holds.

Then, since the embedding of V is compact in $L^p(\Omega)$, by Aubin-Lions-Simon [26] compactness theorems, if a sequence is bounded in $W(0,T)$, it is also bounded in $\{u \in L^p(0,T;V),\ \partial_t u \in L^{p'}(0,T;W^{-k,p'}(\Omega))\}$ and relatively compact in $L^p(Q)$.

4.3 On Mignot-Bamberger -Alt-Luckhaus integration by part formula

We propose in next Lemma a time integration by part formula adapted to our situation. Its proof has been inspired by [9].

Lemma 4.3.
Consider $u \in L^p(0,T;W^{1,p}(\Omega)) \cap L^p(0,T;L^2(\Omega))$ such that $\partial_t u \in L^{p'}(0,T;V')$.
Let $\Psi : Q\times\mathbb{R} \to \mathbb{R}$ be a function such that $(t,x) \mapsto \Psi(t,x,\lambda)$ is measurable, $\lambda \mapsto \Psi(t,x,\lambda)$ is non-decreasing (it can be càdlàg[§] or càglàd[¶]) and denote by $\Lambda : Q\times\mathbb{R} \to \mathbb{R}$, $(t,x,\lambda) \mapsto \int_a^\lambda \Psi(t,x,\tau)d\tau$ where a is any arbitrary real number. Assume moreover that $\partial_t\Psi$ exists with $|\Psi(\lambda=0)| + |\partial_t\Psi| \le h \in L^2(Q)$. If $\Psi(t,x,u) \in L^p(0,T;V)$, then, for any $\beta \in W^{1,\infty}(0,T)$ and any $0 \le s < t \le T$,

$$\int_s^t \langle \partial_t u, \Psi(\sigma,x,u)\rangle \beta d\sigma = \int_\Omega \Lambda(t,x,u(t))\beta(t)dx - \int_\Omega \Lambda(s,x,u(s))\beta(s)dx$$
$$- \int_s^t\int_\Omega \Lambda(\sigma,x,u)\beta' dxd\sigma - \int_s^t\int_\Omega \partial_t\Lambda(\sigma,x,u)\beta dxd\sigma.$$

Proof. Thanks to the assumptions on Ψ, it is a measurable function on $Q\times\mathbb{R}$ and Λ is a Carathéodory function on $Q\times\mathbb{R}$. Moreover,

$$|\Psi(t,x,\lambda)| \le \int_0^t |\partial_t\Psi(s,x,\lambda)|ds \le T.h(t,x),$$
$$|\Lambda(t,x,\lambda)| \le |\lambda - a|T.h(t,x) \le |\lambda|^2 + T^2h^2(t,x)/4 + |a|T.h(t,x),$$

§ continu à droite et limite à gauche: right continuous with left limit
¶ continu à gauche et limite à droite : left continuous with right limit

so that Λ, $\Psi \in L^2_{loc}(\mathbb{R}, L^2(Q))$ and the Nemytskii operator associated with Λ is continuous from $L^2(Q)$ to $L^1(Q)$. Moreover, $\partial_t\Lambda(t,x,\lambda) = \int_a^\lambda \partial_t\Psi(t,x,\tau)d\tau$ and
$\left|\partial_t\Lambda(t,x,\lambda)\right| \le |\lambda - a|h(t,x) \le |\lambda|^2 + h^2(t,x)/4 + |a|h(t,x)$ so that the Nemytskii operator associated with $\partial_t\Lambda$ is also continuous from $L^2(Q)$ to $L^1(Q)$.

By assumption, $u \in C([0,T], L^2(\Omega))$ and one extends u to $\bar{u}$ in $\mathbb{R}$ by $\bar{u}(t) = u_0$ if $t < 0$ and $\bar{u}(t) = u(T)$ si $t > T$. Therefore, if $I_1 := (-1, T+1)$, $\bar{u} \in L^p(I_1, W^{1,p}(\Omega)) \cap C(\bar{I}_1, L^2(\Omega))$ such that $\partial_t\bar{u} \in L^{p'}(I_1, V')$ with $\partial_t\bar{u} = 0$ when $t < 0$ or $t > T$.
Similarly to u, denote by $\bar{\Psi}$ the extension to I_1 of Ψ in the same way and by $\bar{\Lambda}$ the corresponding integral as introduced in the Lemma.

For any fixed $0 < h << 1$, let us denote by

$$v_h : t \mapsto \frac{\bar{u}(t+h) - \bar{u}(t)}{h}, \quad w_h : t \mapsto \frac{\bar{u}(t) - \bar{u}(t-h)}{h}.$$

Consider $\beta \in \mathcal{D}(I_1)$ and h, small enough so that $\mathrm{supp}\beta + [-h, h] \subset I_1$. Then, in $L^2(\Omega)$,

$$\int_{I_1} v_h(t)\beta(t)dt \;\&\; \int_{I_1} w_h(t)\beta(t)dt$$
$$\underset{h}{\to} -\int_{-1}^{T+1} \bar{u}(t)\beta'(t)dt = -\int_0^T u(t)\beta'(t)dt + u(T)\beta(T) - u_0\beta(0).$$

Thus, v_h and w_h converge to $\partial_t\bar{u}$ in $\mathcal{D}'[I_1, L^2(\Omega)]$ and $\mathcal{D}'[I_1, V']$, and to $\partial_t u$ in $\mathcal{D}'[0,T; L^2(\Omega)]$ and $\mathcal{D}'[0,T;V']$. Moreover, by [5, Corollary A.2 p.145], the properties of Bochner integral and since $\partial_t\bar{u} = 0$ outside $(0,T)$,

$$\int_{I_1} \|v_h(t)\|^{p'}_{V'} dt = \int_{I_1} \frac{1}{h^{p'}} \| \int_t^{t+h} \partial_t\bar{u}(s)ds\|^{p'}_{V'} dt$$
$$\le \int_{I_1} \frac{1}{h} \int_t^{t+h} \|\partial_t\bar{u}(s)\|^{p'}_{V'} ds dt \le \int_0^T \|\partial_t u(s)\|^{p'}_{V'} ds.$$

Therefore, v_h converges weakly to $\partial_t\bar{u}$ in $L^{p'}[I_1, V']$ and to $\partial_t u$ in $L^{p'}[0,T;V']$ (as well as for w_h).

For any $\beta \in D(I_1)$, one has $\Psi(\cdot, \bar{u})\beta \in L^p(I_1, V)$, and

$$\int_{I_1\times\Omega} w_h\bar{\Psi}(\cdot, \bar{u}(t))\beta dx dt \;\&\; \int_{I_1\times\Omega} v_h\bar{\Psi}(\cdot, u(t))\beta dx dt \underset{h}{\to} \int_{I_1} \langle \partial_t\bar{u}, \bar{\Psi}(\cdot, \bar{u})\rangle \beta dt.$$

Let us recall that a is a given real and $\bar{\Lambda}(t,x,\lambda) = \int_a^\lambda \bar{\Psi}(t,x,\tau)d\tau$. Since $\bar{\Psi}$ is a non-decreasing function of its third variable, for any real numbers u and v, one has

$$(v-u)\bar{\Psi}(t,x,u) \le \bar{\Lambda}(t,x,v) - \bar{\Lambda}(t,x,u) = \int_u^v \bar{\Psi}(t,x,\tau)d\tau \le (v-u)\bar{\Psi}(t,x,v).$$

Thus, assuming moreover that β is non-negative,

$$[\bar{u}(t+h,x) - \bar{u}(t,x)]\bar{\Psi}(t,x,\bar{u}(t))\beta \le [\bar{\Lambda}(t,x,\bar{u}(t+h)) - \bar{\Lambda}(t,x,\bar{u}(t))]\beta$$
$$\le [\bar{u}(t+h,x) - \bar{u}(t,x)]\bar{\Psi}(t,x,\bar{u}(t+h))\beta,$$
$$[\bar{u}(t,x) - \bar{u}(t-h,x)]\bar{\Psi}(t,x,\bar{u}(t-h))\beta \le [\bar{\Lambda}(t,x,\bar{u}(t)) - \bar{\Lambda}(t,x,\bar{u}(t-h))]\beta$$
$$\le [\bar{u}(t,x) - \bar{u}(t-h,x)]\bar{\Psi}(t,x,\bar{u}(t))\beta.$$

and, for h small enough to have $\mathrm{supp}\beta + [-h, h] \subset I_1$,

$$\begin{aligned}\int_{I_1\times\Omega} v_h\beta\bar{\Psi}(\cdot,u(t))dxdt &\le \int_{I_1\times\Omega} \frac{\bar{\Lambda}(\cdot,\bar{u}(t+h)) - \bar{\Lambda}(\cdot,\bar{u}(t))}{h}\beta dxdt\\ &\le \int_{I_1\times\Omega} v_h\beta\bar{\Psi}(\cdot,\bar{u}(t+h))dxdt,\\ \int_{I_1\times\Omega} w_h\beta\bar{\Psi}(\cdot,\bar{u}(t-h))dxdt &\le \int_{I_1\times\Omega} \frac{\bar{\Lambda}(\cdot,\bar{u}(t)) - \bar{\Lambda}(\cdot,\bar{u}(t-h))}{h}\beta dxdt\\ &\le \int_{I_1\times\Omega} w_h\beta\bar{\Psi}(\cdot,\bar{u}(t))dxdt,\end{aligned}$$

so that

$$\begin{aligned}\liminf_h \int_{I_1\times\Omega} \frac{\bar{\Lambda}(\cdot,\bar{u}(t+h)) - \bar{\Lambda}(\cdot,\bar{u}(t))}{h}\beta dxdt &\ge \int_{I_1} \langle\partial_t\bar{u},\bar{\Psi}(\cdot,\bar{u})\rangle\beta dt = \int_0^T \langle\partial_t u,\Psi(\cdot,u)\rangle\beta dt,\\ \limsup_h \int_{I_1\times\Omega} \frac{\bar{\Lambda}(\cdot,\bar{u}(t)) - \bar{\Lambda}(\cdot,\bar{u}(t-h))}{h}\beta dxdt &\le \int_{I_1} \langle\partial_t\bar{u},\bar{\Psi}(\cdot,\bar{u})\rangle\beta dt = \int_0^T \langle\partial_t u,\Psi(\cdot,u)\rangle\beta dt.\end{aligned}$$

On the other hand,

$$\begin{aligned}&\int_{I_1\times\Omega} \frac{\bar{\Lambda}(t,x,\bar{u}(t+h)) - \bar{\Lambda}(t,x,\bar{u}(t))}{h}\beta(t)dxdt\\ &=\frac{1}{h}\int_{I_1\times\Omega} \bar{\Lambda}(t-h,x,\bar{u}(t))\beta(t-h)dxdt - \frac{1}{h}\int_{I_1\times\Omega} \bar{\Lambda}(t,x,\bar{u}(t))\beta(t)dxdt\\ &= \int_{I_1\times\Omega} \frac{\bar{\Lambda}(t-h,x,\bar{u}(t)) - \bar{\Lambda}(t,x,\bar{u}(t))}{h}\beta(t-h)dxdt + \int_{I_1\times\Omega} \frac{\beta(t-h)-\beta(t)}{h}\bar{\Lambda}(t,x,\bar{u}(t))dxdt\end{aligned}$$

and

$$\begin{aligned}&\int_{I_1\times\Omega} \frac{\bar{\Lambda}(t,x,\bar{u}(t)) - \bar{\Lambda}(t,x,\bar{u}(t-h))}{h}\beta(t)dxdt\\ &= \int_{I_1\times\Omega} \frac{\bar{\Lambda}(t,x,\bar{u}(t)) - \bar{\Lambda}(t+h,x,\bar{u}(t))}{h}\beta(t+h)dxdt + \int_{I_1\times\Omega} \frac{\beta(t)-\beta(t+h)}{h}\bar{\Lambda}(t,x,\bar{u}(t))dxdt.\end{aligned}$$

Then, one gets by passing to the limit when $h \to 0$, and thanks to the time-extension procedure,

$$\begin{aligned}&\liminf_h \int_{I_1\times\Omega} \frac{\bar{\Lambda}(t-h,x,\bar{u}(t)) - \bar{\Lambda}(t,x,\bar{u}(t))}{h}\beta(t-h)dxdt\\ &\ge \int_0^T \langle\partial_t u,\Psi(\cdot,u)\rangle\beta dt + \int_{I_1\times\Omega} \bar{\Lambda}(\cdot,\bar{u})\beta' dt\\ &= \int_0^T \langle\partial_t u,\Psi(\cdot,u)\rangle\beta dt + \int_Q \Lambda(\cdot,u)\beta' dt + \int_\Omega \Lambda(0,x,u_0)\beta(0)dx - \int_\Omega \Lambda(T,x,u(T))\beta(T)dx\\ &\ge \limsup_h \int_{I_1\times\Omega} \frac{\bar{\Lambda}(t,x,\bar{u}(t)) - \bar{\Lambda}(t+h,x,\bar{u}(t))}{h}\beta(t+h)dxdt\end{aligned}$$

Note that

$$\int_{I_1\times\Omega} \frac{\bar\Lambda(t-h,x,\bar u(t))-\bar\Lambda(t,x,\bar u(t))}{h}\beta(t-h)dxdt$$
$$= -\int_{I_1\times\Omega}\frac{1}{h}\int_{t-h}^{t}\partial_t\bar\Lambda(s,x,\bar u(t))\beta(t-h)dsdxdt.$$

Since,

$$|\partial_t\bar\Lambda(s,x,\bar u(t))\beta(t-h)|\le\|\beta\|_\infty|\bar u(t,x)-a|h(s,x)$$

is an integrable function, the properties of the point of Lebesgue (Steklov average) yields

$$\int_{I_1\times\Omega}\frac{\bar\Lambda(t-h,x,\bar u(t))-\bar\Lambda(t,x,\bar u(t))}{h}\beta(t-h)dxdt\underset{h}{\to}-\int_{I_1\times\Omega}\partial_t\bar\Lambda(t,x,\bar u(t))\beta(t)dxdt$$
$$=-\int_Q\partial_t\Lambda(t,x,u(t))\beta(t)dxdt.$$

Since the same holds for $\lim_h \int_{I_1\times\Omega}\frac{\bar\Lambda(t,x,\bar u(t))-\bar\Lambda(t+h,x,\bar u(t))}{h}\beta(t+h)dxdt$, $\forall\beta\in D^+([0,T])$,

$$\int_0^T\langle\partial_t u,\Psi(\cdot,u)\rangle\beta dt$$
$$=\int_\Omega[\Lambda(T,\cdot,u(T))\beta(T)-\Lambda(0,\cdot,u_0)\beta(0)]dx-\int_Q[\Lambda(\cdot,\cdot,u)\beta'+\partial_t\Lambda(\cdot,\cdot,u)\beta]dxdt.$$

Since β is involved in linear integral terms, a classical argument of regularization yields the result for any non-negative elements of $W^{1,\infty}$, then for any elements of $W^{1,\infty}$.

Note that T being arbitrary, the result holds for any t and $s=0$, then for any t and s by subtracting the integral from 0 to s to the one from 0 to t. □

Remark 4.4. *As a consequence,*

$$\frac{d}{dt}\Big[\int_\Omega\Lambda(t,x,u)dx\Big]=\langle\partial_t u,\Psi(t,x,u)\rangle+\int_\Omega\partial_t\Lambda(t,x,u)dx \text{ in } \mathcal{D}'(0,T)$$

and $t\mapsto\int_\Omega\Lambda(t,x,u)dx$ *is absolutely-continuous in* $[0,T]$.

Corollary 4.5. *Consider* $u\in L^p(0,T;W^{1,p}(\Omega))\cap L^\infty(0,T;L^2(\Omega))$ *such that* $\partial_t u\in L^{p'}(0,T;V')$, $\alpha\in L^2(\Omega)$, $\alpha\ge 0$ *and* $\Psi:\mathbb{R}\to\mathbb{R}$ *a given non-decreasing function. Assume that* $\Psi(u)\alpha\in L^p(0,T;V)$, *then, for any* $\beta\in W^{1,\infty}(0,T)$ *and any* $0\le s<t\le T$,

$$\int_s^t\langle\partial_t u,\Psi(u)\alpha\rangle\beta d\sigma=-\int_s^t\int_\Omega\int_a^u\Psi(\tau)d\tau\alpha\beta' dxd\sigma$$
$$+\int_\Omega\int_a^{u(t)}\Psi(\tau)d\tau\alpha\beta(t)dx-\int_\Omega\int_a^{u(s)}\Psi(\tau)d\tau\alpha\beta(s)dx,$$

where a is any arbitrary real number.

Remark 4.6. *Note that, by linearity, the same result holds if* $\alpha=\alpha_1-\alpha_2$ *with* $\Psi(u)\alpha_i\in L^p(0,T;V)$ $(i=1,2)$ *or if* $\Psi=\Psi_1-\Psi_2$ *and* Ψ_i $(i=1,2)$ *satisfies the assumptions.*

4.4 Strong continuity in $L^2(\Omega)$

We consider the following notations in the sequel: $V(\Omega) = W^{1,p}(\Omega) \cap L^2(\Omega)$, $V_0(\Omega) = W_0^{1,p}(\Omega) \cap L^2(\Omega)$ and $V'(\Omega) = W^{-1,p'}(\Omega) + L^2(\Omega)$.
Let us prove in this section the following result of continuity.

Lemma 4.7. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$, then we have*

$$u \in L^p(0, T; V(\Omega)), \partial_t u \in L^{p'}(0, T; V'(\Omega)) \Rightarrow u \in C([0, T], L^2(\Omega)).$$

Remark 4.8. *This result is not the usual one since u and $\partial_t u$ are not in spaces being in duality relation and few words are needed concerning the time-derivative.*
Note that both $V(\Omega)$ and $V_0(\Omega)$ are dense subspaces of the chosen pivot space $L^2(\Omega)$ so that it can be identify to a subspace of $V'(\Omega)$ or $(V(\Omega))'$. Therefore, u, as an element of $L^p(0, T; V(\Omega)) \hookrightarrow L^p(0, T, L^2(\Omega)) \hookrightarrow L^{p'}(0, T; V'(\Omega))$, has a time derivative in the sense of distributions in $\mathcal{D}'(0, T, V'(\Omega))$ and it is assumed to belong to $L^{p'}(0, T; V'(\Omega))$.

Proof. This result is based on a classical method: first in $\mathbb{R}^N$, then in the half-space $\mathbb{R}^N_+$ and finally in Ω thanks to an atlas of charts.

Obviously, if $\Omega = \mathbb{R}^N$, we have $W_0^{1,p}(\mathbb{R}^N) = W^{1,p}(\mathbb{R}^N)$, and the result holds by classical arguments (*e.g.* [25, Prop. 1.2 p. 106]).
If $\Omega = \mathbb{R}^N_{+/resp.-} = \{(x', x_N) \in \mathbb{R}^N;\ \ x_N > 0/resp.\ < 0\}$, the method is based on a suitable extension of u to $\mathbb{R}^N$. Following a recommendation of F. Murat, we consider the following extension, proposed in [16, (12.21-22) p.83] and revisited in [8, p.2]:

$$\tilde{u}(t, x', x_N) = \begin{cases} u(t, x', x_N); & x_N > 0 \\ -3u(t, x', -x_N) + 4u(t, x', -2x_N); & x_N < 0. \end{cases}$$

Note that $\tilde{u} \in L^p(0, T; V(\mathbb{R}^N))$ and, thanks to a change of variables, that for any $\varphi \in \mathcal{C}_c^\infty(]0, T[\times\mathbb{R}^N)$, one gets

$$\begin{aligned} &\int_{(0,T)\times\mathbb{R}^N} \tilde{u}\partial_t\varphi \, dxdt \\ &= \int_{(0,T)\times\mathbb{R}^N_-} [-3u(t, x', -x_N) + 4u(t, x', -2x_N)]\partial_t\varphi(t, x', x_N) \, dxdt + \int_{(0,T)\times\mathbb{R}^N_+} u\partial_t\varphi \, dxdt \\ &= \int_{(0,T)\times\mathbb{R}^N_+} (\partial_t(\varphi(t, x', x_N) - 3\varphi(t, x', -x_N) + 2\varphi(t, x', -\frac{x_N}{2}))u(t, x, x_N) \, dxdt \end{aligned}$$

Then

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \tilde{u}(t, x)\partial_t\varphi(t, x)dxdt \\ &= \int_0^T \int_{\mathbb{R}^N_+} (\partial_t(\varphi(t, x', x_N) - 3\varphi(t, x', -x_N) + 2\varphi(t, x', -\frac{x_N}{2}))u(t, x, x_N)dxdt. \end{aligned}$$

By construction, $\psi(t, x) = \varphi(t, x', x_N) - 3\varphi(t, x', -x_N) + 2\varphi(t, x', -\frac{x_N}{2}) = 0$ if $x_N = 0$, as well as $\partial_t\psi$, and $\psi \in W^{1,\infty}(0, T; V_0(\mathbb{R}^N_+))$ with $\|\psi\|_{L^p(0,T;V_0(\mathbb{R}^N_+))} \le C\|\varphi\|_{L^p(0,T;V(\mathbb{R}^N))}$ for a given constant C.

Therefore, $|\int_0^T \int_{\mathbb{R}^N_+} \tilde{u}\partial_t\varphi dxdt| \le C\|\partial_t u\|_{L^{p'}(0,T;V'(\mathbb{R}^N_+))}\|\varphi\|_{L^p(0,T;V(\mathbb{R}^N))}$, and $\partial_t\tilde{u} \in L^{p'}(0, T; V'(\mathbb{R}^N))$. Then, one concludes that $\tilde{u} \in C([0, T], L^2(\mathbb{R}^N))$ *i.e.* $u \in C([0, T], L^2(\mathbb{R}^N_+))$.
Finally, the result holds in the general case by considering an atlas of charts as proposed *e.g.* in [8, p.3]. □

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