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GIVENTAL'S NON-LINEAR MASLOV INDEX ON LENS SPACES

GUSTAVO GRANJA, Yael Karshon, Milena Pabiniak, and Sheila Sandon

ABSTRACT. Givental's *non-linear Maslov index*, constructed in 1990, is a quasimorphism on the universal cover of the identity component of the contactomorphism group of real projective space. This invariant was used by several authors to prove contact rigidity phenomena such as orderability, unboundedness of the discriminant and oscillation metrics, and a contact geometric version of the Arnold conjecture. In this article we give an analogue for lens spaces of Givental's construction and its applications.

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1. INTRODUCTION

A *discriminant point* of a contactomorphism ϕ of a co-oriented contact manifold (V, ξ) is a point p of V such that $\phi(p) = p$ and $(\phi^*\alpha)_p = \alpha_p$ for some (hence any) contact form α for ξ ; the *discriminant* of (V, ξ) is the space of those contactomorphisms that have at least one discriminant point. Givental's non-linear Maslov index [Gi90] assigns to every contact isotopy $\{\phi_t\}$ of real projective space \mathbb{RP}^{2n-1} with its standard contact structure an integer $\mu(\{\phi_t\})$ that is defined using generating functions and can be interpreted as an intersection index of the path $\{\phi_t\}$ in the contactomorphism group with (a certain subspace of) the discriminant. This number only depends on the homotopy class of $\{\phi_t\}$ with fixed endpoints, and thus defines a map

$$\mu: \widetilde{\text{Cont}}_0(\mathbb{RP}^{2n-1}) \rightarrow \mathbb{Z}$$

on the universal cover of the identity component of the contactomorphism group. It follows from [Gi90, Theorem 9.1] that μ is a quasimorphism, i.e. a homomorphism up to a bounded error (cf. Ben Simon [BeS07]). While quasimorphisms on Hamiltonian groups were studied by several authors, starting with Biran, Entov and Polterovich [BEP04] and Entov and Polterovich [EnP03], Givental's non-linear Maslov index and its reductions studied by Borman and Zapolsky [BZ15] are the only known non-trivial quasimorphisms on contactomorphism groups.

In [Gi90] Givental also studied intersections with the discriminant in two other related settings. One is a space of Legendrian submanifolds of \mathbb{RP}^{2n-1} , with discriminant given by those Legendrians

that intersect a fixed one. The second setting (that was also studied by Th  ret [Th98]) is the Hamiltonian group of complex projective space \mathbb{CP}^{n-1} with the Fubini–Study symplectic form; in this case the discriminant is formed by Hamiltonian diffeomorphisms of \mathbb{CP}^{n-1} that lift to contactomorphisms of S^{2n-1} having discriminant points. The applications of the non-linear Maslov index that were discussed already in [Gi90] include a proof of the Arnold conjecture for fixed points of Hamiltonian diffeomorphisms and for Lagrangian intersections in \mathbb{CP}^{n-1} , results about existence of Reeb chords between Legendrians in \mathbb{RP}^{2n-1} that are Legendrian isotopic to each other, and a proof of the chord and Weinstein conjectures for \mathbb{RP}^{2n-1} .

After the work of Givental, discriminant points appeared again more recently in proofs (based on generating functions) of other contact rigidity results. In [Bh01] Bhupal used the rigidity of discriminant points to define a partial order on the identity component of the group of compactly supported contactomorphisms of the standard contact Euclidean space \mathbb{R}^{2n+1} . Elaborating on the work of Bhupal, the fourth author obtained a new proof of the contact non-squeezing theorem of Eliashberg, Kim and Polterovich [EKP06], and a construction of an integer-valued bi-invariant metric on the identity component of the group of compactly supported contactomorphisms of $\mathbb{R}^{2n} \times S^1$ ([Sa11a] and [Sa10] respectively). In these works the role played by discriminant and translated points is made more explicit, and appears to be similar to the one described in [Gi90]. Recall from [Sa11a, Sa12] that a point p of a contact manifold (V, ξ) is said to be a *translated point* of a contactomorphism ϕ with respect to a contact form α for ξ if p is a discriminant point of $\varphi_{-\eta}^\alpha \circ \phi$ for some real number η (called the *time-shift*), where φ_t^α denotes the Reeb flow. Consider the contact product $(V \times V \times \mathbb{R}, \ker(e^\theta \alpha_1 - \alpha_2))$, where θ is the \mathbb{R} -coordinate and α_1 and α_2 the pullbacks of α by the projection of $V \times V \times \mathbb{R}$ on the first and second factor respectively. Denote by $\text{gr}(\phi)$ the Legendrian graph of ϕ , i.e. the Legendrian submanifold of $V \times V \times \mathbb{R}$ defined by

$$\text{gr}(\phi) = \{(p, \phi(p), g(p)) \mid p \in V\}$$

where g is the function determined by the relation $\phi^* \alpha = e^g \alpha$. Discriminant points of ϕ correspond to intersections between $\text{gr}(\phi)$ and the diagonal $\Delta \times \{0\} = \text{gr}(\text{id})$; since the Reeb vector field of $e^\theta \alpha_1 - \alpha_2$ is $(0, R_\alpha, 0)$, where R_α denotes the Reeb vector field of α , translated points of ϕ correspond to Reeb chords between $\text{gr}(\phi)$ and the diagonal $\Delta \times \{0\}$. By Weinstein’s theorem, if ϕ is \mathcal{C}^1 -close to the identity then $\text{gr}(\phi)$ can be identified with the 1-jet $j^1 f \subset J^1(\Delta \times \{0\})$ of a function f on $V \cong \Delta \times \{0\}$. Critical points of f correspond to Reeb chords in $J^1(\Delta \times \{0\})$ between the zero section and $j^1 f$, hence¹ to Reeb chords in $V \times V \times \mathbb{R}$ between $\Delta \times \{0\}$ and $\text{gr}(\phi)$, hence to translated points of ϕ . Moreover, critical points of critical value zero correspond to discriminant points. It follows from this local description that for such a ϕ translated points always exists, while discriminant points can be removed by a small perturbation. On the other hand, Givental’s non-linear Maslov index for \mathbb{RP}^{2n-1} and the bi-invariant metric for $\mathbb{R}^{2n} \times S^1$ defined in [Sa10] show that there exist contact isotopies that must intersect the discriminant at least a certain number of times, and these intersections cannot be removed by perturbing the contact isotopy in the same homotopy class with fixed endpoints. This rigidity of discriminant points is also the main ingredient in [Sa11a] for the construction of a contact capacity for domains of $\mathbb{R}^{2n} \times S^1$, and for the proof of the contact non-squeezing theorem. Elaborating on this idea, the fourth author, in collaboration with Colin, defined [CS12] bi-invariant (pseudo)metrics (the *discriminant metric* and the *oscillation pseudometric*) on the universal cover of the identity component of the contactomorphism group of any compact contact manifold, and, using results from [Gi90], proved that both are unbounded in the case of \mathbb{RP}^{2n-1} . Previously, Givental’s non-linear Maslov index was also used by Eliashberg and Polterovich [EP00] to show that \mathbb{RP}^{2n-1} is *orderable*, i.e. it does not admit any positive contractible loop of contactomorphisms², and by the fourth author [Sa11c]

¹Recall that this correspondence is in general not 1–1. Every Reeb chord in $J^1(\Delta \times \{0\})$ between the zero section and $j^1 f$ corresponds to a Reeb chord in $V \times V \times \mathbb{R}$ between $\Delta \times \{0\}$ and $\text{gr}(\phi)$, but there might be Reeb chords in $V \times V \times \mathbb{R}$ between $\Delta \times \{0\}$ and $\text{gr}(\phi)$ that are not contained in the Weinstein neighborhood of $\Delta \times \{0\}$, and thus do not correspond to a Reeb chord in $J^1(\Delta \times \{0\})$ between the zero section and $j^1 f$.

²Recall from [EP00] that a contact isotopy of a co-oriented contact manifold is said to be *positive* (*non-negative*) if it moves every point in a direction positively transverse (or tangent) to the contact distribution, and that a compact

to prove that any contactomorphism of \mathbb{RP}^{2n-1} isotopic to the identity has at least $2n$ translated points.

In the present article we give an analogue for lens spaces of the construction of Givental's non-linear Maslov index and its applications. This is the first step of a more general program: study the contact rigidity phenomena mentioned above in the case of prequantizations of symplectic toric manifolds, by extending to the contact case techniques from [Gi95]. See also [Te18] for work in this direction, and [AF12] for other work related to the non-linear Maslov index.

Although \mathbb{RP}^{2n-1} and S^{2n-1} are both prequantizations of \mathbb{CP}^{n-1} , and the former is the quotient of the latter by the antipodal \mathbb{Z}_2 -action, S^{2n-1} (for $n > 1$) is not orderable [EKP06] and does not admit non-trivial quasimorphisms and unbounded bi-invariant metrics [FPR18]. In this work we prove that lens spaces (in particular those that are prequantizations of \mathbb{CP}^{n-1}) behave rather as \mathbb{RP}^{2n-1} : in spite of the fact that the ring structure of the cohomology of general lens spaces is different from that of projective space, which affects the proofs of some of the key properties of the topological invariant that is used in the construction, we show that it is still possible to define the non-linear Maslov index and use it to extend to lens spaces the applications described above.

Let $k \geq 2$ be an integer and $\underline{w} = (w_1, \dots, w_n)$ an n -tuple of positive integers that are relatively prime to k . The lens space $L_k^{2n-1}(\underline{w})$ is the quotient of the unit sphere S^{2n-1} in \mathbb{C}^n by the free \mathbb{Z}_k -action generated by the map

$$(1.1) \quad (z_1, \dots, z_n) \mapsto (e^{\frac{2\pi i}{k} \cdot w_1} z_1, \dots, e^{\frac{2\pi i}{k} \cdot w_n} z_n).$$

We equip the lens space $L_k^{2n-1}(\underline{w})$ with its standard contact structure, i.e. the kernel of the contact form whose pullback to S^{2n-1} is equal to the pullback from \mathbb{R}^{2n} of the 1-form $\sum_{j=1}^n (x_j dy_j - y_j dx_j)$. We denote by $\{r_t\}$ the Reeb flow on L_k^{2n-1} with respect to this contact form.

Throughout the article, when the weights are not relevant in the discussion we denote the lens space $L_k^{2n-1}(\underline{w})$ simply by L_k^{2n-1} . As usual, we see the universal cover $\widetilde{\text{Cont}_0}(L_k^{2n-1})$ of the identity component of the contactomorphism group as the space of contact isotopies starting at the identity modulo smooth 1-parameter families with fixed endpoints; the group operation is given by $[\{\phi_t\}] \cdot [\{\psi_t\}] = [\{\phi_t \circ \psi_t\}]$.

Our main result is the following theorem.

1.2. Theorem. *For any lens space L_k^{2n-1} with its standard contact structure there is a map*

$$\mu: \widetilde{\text{Cont}_0}(L_k^{2n-1}) \rightarrow \mathbb{Z}$$

such that $\mu([\{r_{2\pi l t}\}_{t \in [0,1]}]) = 2nl$ for every integer l , and with the following properties:

- (i) (Quasimorphism.) *For any two elements $[\{\phi_t\}]$ and $[\{\psi_t\}]$ of $\widetilde{\text{Cont}_0}(L_k^{2n-1})$ we have*

$$|\mu([\{\phi_t\}] \cdot [\{\psi_t\}]) - \mu([\{\phi_t\}]) - \mu([\{\psi_t\}])| \leq 2n + 1.$$
- (ii) (Positivity.) *If $\{\phi_t\}$ is a non-negative contact isotopy then $\mu([\{\phi_t\}]) \geq 0$. If $\{\phi_t\}$ is a positive contact isotopy then $\mu([\{\phi_t\}]) > 0$.*
- (iii) (Relation with discriminant points.) *Let $\{\phi_t\}_{t \in [0,1]}$ be a contact isotopy of L_k^{2n-1} , and $[t_0, t_1]$ a subinterval of $[0, 1]$. If $\mu([\{\phi_t\}_{t \in [0, t_0]}]) \neq \mu([\{\phi_t\}_{t \in [0, t_1]}])$ then there is $\underline{t} \in [t_0, t_1]$ such that $\phi_{\underline{t}}$ belongs to the discriminant. If there is only one such \underline{t} then the following holds: if $\phi_{\underline{t}}$ has only finitely many discriminant points then*

$$|\mu([\{\phi_t\}_{t \in [0, t_0]}]) - \mu([\{\phi_t\}_{t \in [0, t_1]}])| \leq 2;$$

co-oriented contact manifold (V, ξ) is said to be *orderable* if the relation \leq on $\widetilde{\text{Cont}_0}(V, \xi)$ defined by posing

$$[\{\phi_t\}] \leq [\{\psi_t\}] \text{ if } [\{\psi_t\}] \cdot [\{\phi_t\}]^{-1} \text{ can be represented by a non-negative contact isotopy}$$

is a partial order. By [EP00, Criterion 1.2.C], this is equivalent to asking that (V, ξ) does not admit any positive contractible loop of contactomorphisms. Recall also that the oscillation pseudometric on a compact co-oriented contact manifold (V, ξ) is a metric if and only if (V, ξ) is orderable [CS12].

if all discriminant points of ϕ_t are non-degenerate³ then

$$|\mu([\{\phi_t\}_{t \in [0, t_0]}]) - \mu([\{\phi_t\}_{t \in [0, t_1]}])| \leq 1.$$

The map $\mu: \widetilde{\text{Cont}}_0(L_k^{2n-1}) \rightarrow \mathbb{Z}$ (the *non-linear Maslov index*) is defined at the beginning of Section 4, using the material that is developed in Sections 2 and 3. The quasimorphism property is proved in Proposition 4.8, positivity in Proposition 4.21 and the relation with discriminant points in Proposition 4.15. The calculation for the Reeb flow is presented in Example 4.13.

Theorem 1.2 allows to extend the applications of the non-linear Maslov index to the case of lens spaces, giving the following results (see Section 5).

1.3. Corollary. *Consider a lens space L_k^{2n-1} with its standard contact structure. Then:*

- (i) L_k^{2n-1} is orderable.
- (ii) The discriminant and oscillation metrics on $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ are unbounded.
- (iii) Any contactomorphism of L_k^{2n-1} contact isotopic to the identity has at least n translated points with respect to the standard contact form. Moreover, if all translated points are non-degenerate then their number is at least $2n$.
- (iv) Any contact form on L_k^{2n-1} defining the standard contact structure has at least one closed Reeb orbit.

Orderability of lens spaces was also proved with different methods by Milin [Mi08] and by the fourth author [Sa11b]. Regarding part (iii), this proves for the standard contact form of lens spaces two statements in direction of the following conjecture: if (V, ξ) is a compact contact manifold then any contactomorphism ϕ contact isotopic to the identity should have at least as many translated points (with respect to any contact form for ξ) as the minimal number of critical points of a smooth function on V . This conjecture, formulated in [Sa11c], can be thought of as a contact analogue of the Arnold conjecture on fixed points of Hamiltonian diffeomorphisms. As in the Hamiltonian case, one can consider weaker versions obtained by replacing the lower bound on translated points by the Lusternik–Schnirelmann category or (even weaker) the cup length, or the version where the lower bound is the sum of the Betti numbers if all translated points are assumed to be non-degenerate. Working with \mathbb{Z}_k -coefficients (with k prime), for any lens space L_k^{2n-1} the sum of the Betti numbers is $2n$, while the cuplength is $2n$ if $k = 2$ (i.e. for \mathbb{RP}^{2n-1}) and $n + 1$ if $k > 2$. While our bound in the general case is just n , the one obtained in [Sa11c] in the case of \mathbb{RP}^{2n-1} is $2n$. On the other hand, since the Lusternik–Schnirelmann category of L_k^{2n-1} is $2n$ for all k , we should still have at least $2n$ translated points for all L_k^{2n-1} also in the degenerate case. It might be possible to prove this using Massey products (similarly to [Vi97]), but this goes beyond the scope of the present article (see also Remark 4.20).

1.4. Remark. In the case $k = 2$ our arguments prove, as in [Gi90], the following stronger form of Theorem 1.2: in (i) the bound is $2n$, rather than $2n + 1$, and in (iii) the last bound holds also in the degenerate case. Using this one recovers (see Section 5) the stronger bound in Corollary 1.3(iii) that holds in the case of \mathbb{RP}^{2n-1} : any contactomorphism contact isotopic to the identity has at least $2n$ translated points with respect to the standard contact form. It is enough to prove Theorem 1.2 in the case when k is prime. Indeed, for any multiple k' of k one then obtains a quasimorphism on $\widetilde{\text{Cont}}_0(L_{k'}^{2n-1})$ with the required properties by pulling back μ by the natural map $\widetilde{\text{Cont}}_0(L_{k'}^{2n-1}) \rightarrow \widetilde{\text{Cont}}_0(L_k^{2n-1})$. Because of this, if k is even then Theorem 1.2 and Corollary 1.3 hold in the same stronger form as in the case $k = 2$. By a similar method (i.e. pulling back μ via the natural map induced by taking a quotient in stages), one also obtains quasimorphisms

³Recall from [Sa11c] that a discriminant point p of a contactomorphism ϕ of a contact manifold $(V, \xi = \ker(\alpha))$ is said to be *non-degenerate* if there are no vectors $X \in T_p V$ such that $\phi_* X = X$ and $dg(X) = 0$, where g is the function defined by $\phi^* \alpha = e^g \alpha$. A translated point of ϕ of time-shift η is said to be *non-degenerate* if it is a non-degenerate discriminant point of $\varphi_{-\eta}^\alpha \circ \phi$ (where φ_t^α denotes the Reeb flow).

for quotients of S^{2n-1} by a finite non-cyclic subgroup G of $U(n)$ that acts freely on S^{2n-1} , as any such G contains a non-trivial cyclic subgroup. \diamond

Note that there exist contactomorphisms of L_k^{2n-1} that have exactly $2n$ translated points. Indeed, as above, if a contactomorphism ϕ of L_k^{2n-1} is sufficiently \mathcal{C}^1 -close to the identity then its graph $\text{gr}(\phi)$ in the contact product $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$ is contained in a Weinstein neighborhood of the diagonal $\Delta \times \{0\}$, and can be identified to the 1-jet of a function f on $\Delta \times \{0\} \cong L_k^{2n-1}$. Critical points of f correspond to Reeb chords in $J^1(\Delta \times \{0\})$ between the zero section and $j^1 f$, hence to Reeb chords in $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$ between $\Delta \times \{0\}$ and $\text{gr}(\phi)$, hence to translated points of ϕ . This correspondence is now 1–1. Indeed, if p is a translated point of ϕ with respect to the standard contact form α_0 on L_k^{2n-1} then there exists a Reeb chord $\gamma : [0, 1] \rightarrow L_k^{2n-1}$ of α_0 with $\gamma(0) = p$ and $\gamma(1) = \phi(p)$ so that all points $(p, \gamma(t), 0)$ are in the considered Weinstein neighborhood. Since the Reeb flow on $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$ is given by $\text{Id} \times \varphi_t^\alpha \times \text{Id}$ it follows that for every translated point of ϕ there is a Reeb chord in $L_k^{2n-1} \times L_k^{2n-1} \times \mathbb{R}$ between $(p, p, 0) \in \Delta \times \{0\}$ and $(p, \phi(p), 0) \in \text{gr}(\phi)$ that is contained in the Weinstein neighborhood, and thus any translated point of ϕ corresponds to a critical point of f . Choosing a function f on L_k^{2n-1} with exactly $2n$ critical points we thus obtain a contactomorphism with exactly $2n$ translated points.

Since the Reeb flow of the standard contact form α_0 on L_k^{2n-1} is 1-periodic, if p is a translated point with respect to α_0 of a contactomorphism ϕ then there are infinitely many Reeb chords connecting p and $\phi(p)$, hence the translated point p has infinitely many time-shifts. Thus Corollary 1.3(iii) implies that any contactomorphism of L_k^{2n-1} contact isotopic to the identity has infinitely many pairs (translated point, time-shift) with respect to α_0 . Using the properties of the non-linear Maslov index that are listed in Theorem 1.2, we will prove in Section 5 that this result remains true for any contact form on L_k^{2n-1} defining the standard contact structure.

1.5. Corollary. *Let α be any contact form on L_k^{2n-1} defining the standard contact structure. For any contactomorphism ϕ of L_k^{2n-1} contact isotopic to the identity there are infinitely many distinct real numbers that are time-shifts of translated points of ϕ with respect to α . In particular, ϕ has at least one translated point with respect to α .*

As in the case of projective space, it follows from Theorem 1.2 that the *asymptotic non-linear Maslov index*

$$\bar{\mu} : \widetilde{\text{Cont}}_0(L_k^{2n-1}) \rightarrow \mathbb{R}, \quad \bar{\mu}([\{\phi_t\}]) = \lim_{m \rightarrow \infty} \frac{\mu([\{\phi_t\}]^m)}{m}$$

is *monotone*, i.e. $\mu([\{\phi_t\}]) \leq \mu([\{\psi_t\}])$ whenever $[\{\phi_t\}] \leq [\{\psi_t\}]$, and has the *vanishing property*, i.e. it vanishes on any element that can be represented by a contact isotopy supported in a displaceable set (see Propositions 4.22 and 4.23). In [BZ15] Borman and Zolinsky showed that, in analogy with the symplectic case [Bo13], in certain situations monotone quasimorphisms descend under contact reduction; starting from Givental's non-linear Maslov index on $\mathbb{R}P^{2n-1}$ they thus obtained induced quasimorphisms on those contact toric manifolds that can be written in a certain way as contact reductions of $\mathbb{R}P^{2n-1}$. Moreover, it is proved in [BZ15] that if a contact manifold admits a non-trivial monotone quasimorphism then it is orderable, and if the prequantization of a symplectic toric manifold admits a non-trivial monotone quasimorphism with the vanishing property (a property that is preserved under contact reduction) then it has a non-displaceable pre-Lagrangian toric fibre. As already observed in [BZ15, Remark 1.5], our generalization to lens spaces of Givental's non-linear Maslov index allows us to extend the class of contact toric manifolds that inherit, by contact reduction, a quasimorphism. We thus obtain the following result (see Section 5).

1.6. Corollary. *Let (W^{2n}, ω) be a compact monotone symplectic toric manifold. Write the moment polytope as $\Delta = \{x \in \mathfrak{t}^*, \langle \nu_j, x \rangle + \lambda \geq 0 \text{ for } j = 1, \dots, d\}$, where d is the number of facets and $\nu_j \in \mathfrak{t}$ are vectors normal to the facets and primitive in the integer lattice $\mathfrak{t}_{\mathbb{Z}} = \ker(\exp : \mathfrak{t} \rightarrow \mathbb{T}^n)$. Suppose that, for some $k \in \mathbb{N}$, $\sum_{j=1}^d \nu_j \in k \mathfrak{t}_{\mathbb{Z}}$. Then there is a rescaling ω' of the symplectic form*

such that the prequantization of (W, ω) admits a non-trivial monotone quasimorphism with the vanishing property, and so is orderable and contains a non-displaceable pre-Lagrangian toric fibre.

Note however that the link with discriminant points seems to be lost in the reduction process. Therefore, it is not clear if it is possible to use these induced quasimorphisms to obtain the applications (other than orderability) listed in Corollary 1.3.

The original idea of the construction of Givental's non-linear Maslov index in $\mathbb{R}P^{2n-1}$ is as follows. Given a contact isotopy $\{\phi_t\}_{t \in [0,1]}$ of $\mathbb{R}P^{2n-1}$, one associates to it a 1-parameter family of functions $f_t: \mathbb{R}P^{2M-1} \rightarrow \mathbb{R}$, for some large M , so that critical points of f_t of critical value zero correspond to discriminant points of ϕ_t . In order to detect intersections of $\{\phi_t\}$ with the discriminant one then analyzes the changes in topology of the sublevel sets $A_t = \{f_t \leq 0\}$. This is done by studying the *cohomological index* of these sets, i.e. the dimension of the image of the homomorphism $H^*(\mathbb{R}P^{2M-1}; \mathbb{Z}_2) \rightarrow H^*(A_t; \mathbb{Z}_2)$ induced by the inclusion $A_t \hookrightarrow \mathbb{R}P^{2M-1}$. The non-linear Maslov index of $\{\phi_t\}_{t \in [0,1]}$ is then defined to be the difference between the cohomological indices of A_0 and A_1 . The key difference in the construction for lens spaces is in the properties of the cohomological index. In the case of projective space the cohomological index satisfies *subadditivity* (the cohomological index of a union $A \cup B$ is not more than the sum of the cohomological indices of A and B) and *join additivity* (the cohomological index of an *equivariant join* is equal to the sum of the cohomological indices of the factors). The proofs of both properties use the fact that the cohomology ring of projective space is generated by the class in degree one, and thus they do not go through in the case of lens spaces. However we show that weaker versions of the subadditivity and join additivity properties also hold in the case of general lens spaces, and are enough to define the non-linear Maslov index and prove the properties listed in Theorem 1.2.

In the case $k = 2$, Givental's proof of the join additivity property uses an equivariant Künneth formula [Gi90, Proposition A.1]. A crucial ingredient in the proof is the fact that the Künneth short exact sequence (of modules over the equivariant cohomology of a point) splits, but it is not clear to us why this fact should be true. In any case, for $k > 2$ the Künneth (or Eilenberg–Moore) spectral sequence does not collapse in general (due to the fact that for $k > 2$ the \mathbb{Z}_k -equivariant cohomology of a point has zero divisors) and so we do not even have a Künneth short exact sequence for the equivariant cohomology of a product. We thus present a different proof (even for the case $k = 2$). Our proof is based on the study of an equivariant join operation in homology, which is defined in the same way for all k . When $k = 2$ the properties of this operation imply Givental's join additivity, while for $k > 2$, as the join of two even dimensional generators of the equivariant homology of a point is zero, we only obtain a weaker *join quasi-additivity property* (Proposition 3.9 (v)). As mentioned above, this property is still strong enough to imply the applications we are interested in.

Finally, the construction of generating functions that we use in this article is not the one from [Gi90]⁴, but is an adaptation to our setting of the work of Théret [Th98] (cf. Remark A.6).

The article is organized as follows. In Section 2 we explain how to construct 1-parameter families of generating functions for contact isotopies of lens spaces, and discuss some properties. In Section 3 we describe the topological invariant (the cohomological index) that is used to analyze the changes in topology of the sublevel sets of generating functions, deferring to Appendix B the most technical part of the proof of the join quasi-additivity property of this invariant. In Section 4 we put these ingredients together to define the non-linear Maslov index of a contact isotopy, and prove the properties listed in Theorem 1.2. In Section 5 we use the non-linear Maslov index to prove Corollaries 1.3, 1.5 and 1.6, mostly following the corresponding arguments in the case of projective space. In Appendix A we discuss several interpretations of the composition formula that is used in the construction of generating functions.

⁴Because of this, strictly speaking we do not know whether in the special case of projective space our quasimorphism actually coincides with Givental's. However, all the features and properties are the same.

Throughout the article, when we say that we follow Givental [Gi90] or Th  ret [Th98] we mean that their arguments, developed for \mathbb{RP}^{2n-1} and \mathbb{CP}^{n-1} , can be repeated for lens spaces without any modification other than replacing the action of \mathbb{Z}_2 or S^1 by the \mathbb{Z}_k -action (1.1).

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2. GENERATING FUNCTIONS

In this section we explain how, given a contact isotopy $\{\phi_t\}_{t \in [0,1]}$ of a lens space L_k^{2n-1} starting at the identity, one can define a 1-parameter family of functions f_t on a higher dimensional lens space L_k^{2M-1} so that critical points of f_t of critical value zero correspond to discriminant points of ϕ_t . We also discuss some properties of such families of generating functions that are important in the applications: uniqueness, monotonicity and quasi-additivity.

Generating functions for symplectomorphisms of \mathbb{R}^{2n} . We start by recalling the definition of generating functions of Lagrangian submanifolds in cotangent bundles. Observe first that any Lagrangian section of the cotangent bundle T^*B of a manifold B is the graph of a closed 1-form on B ; if this 1-form is exact, given by the differential of a function F , then we say that F is a generating function for the Lagrangian. Generalizing this idea, one can associate a generating function to a larger class of Lagrangian submanifolds of T^*B by the following construction, which goes back to H  rmander [Ho71]. Consider a function $F: E \rightarrow \mathbb{R}$ defined on the total space of a fibre bundle $p: E \rightarrow B$. Let N_E^* be the fibre conormal bundle, i.e. the space of points (e, η) of T^*E such that η vanishes on the kernel of $dp|_e$. We say that F is a *generating function* if $dF: E \rightarrow T^*E$ is transverse to N_E^* . If this condition is satisfied then the set of fibre critical points $\Sigma_F = (dF)^{-1}(N_E^*)$ is a smooth submanifold of E , and the map

$$i_F: \Sigma_F \rightarrow T^*B, \quad e \mapsto (p(e), v^*(e))$$

defined by posing $v^*(e)(X) = dF(\hat{X})$ for $X \in T_{p(e)}B$, where \hat{X} is any vector in T_eE such that $p_*(\hat{X}) = X$, is a Lagrangian immersion. If i_F is an embedding then we say that F is a generating function for the Lagrangian submanifold $L_F := i_F(\Sigma_F)$ of T^*B . In this case, critical points of F are in 1–1 correspondence with intersections of L_F with the zero section.

In our applications, generating functions are always defined on trivial bundles of the form $E = \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}^{2n}$. Denoting the coordinates by (ζ, ν) with $\zeta \in \mathbb{R}^{2n}$ and $\nu \in \mathbb{R}^{2nN}$, we then have that dF is transverse to N_E^* if and only if zero is a regular value of the vertical derivative $\frac{\partial F}{\partial \nu}: E \rightarrow (\mathbb{R}^{2nN})^*$; moreover, $\Sigma_F = (\frac{\partial F}{\partial \nu})^{-1}(0)$ and $i_F(\zeta, \nu) = (\zeta, \frac{\partial F}{\partial \zeta}(\zeta, \nu))$.

For a symplectomorphism Φ of \mathbb{R}^{2n} , as in [Vi92] we consider the Lagrangian submanifold Γ_Φ of $T^*\mathbb{R}^{2n}$ that is the image of the graph of Φ under the symplectomorphism $\tau: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow T^*\mathbb{R}^{2n}$ given by

$$\tau(x, y, X, Y) = \left(\frac{x+X}{2}, \frac{y+Y}{2}, Y-y, x-X \right)$$

or, in complex notation,

$$(2.1) \quad \tau(z, Z) = \left(\frac{z+Z}{2}, i(z-Z) \right).$$

We say that a function F is a generating function for Φ if it is a generating function for Γ_Φ . Since τ sends the diagonal onto the zero section, critical points of a generating function F of Φ are in 1–1 correspondence with fixed points of Φ . Note also that if F is a generating function for Φ then $-F$ is a generating function for Φ^{-1} .

Any Hamiltonian symplectomorphism Φ of \mathbb{R}^{2n} such that Γ_Φ is a section of $T^*\mathbb{R}^{2n}$ has a generating function $F: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. In order to obtain a generating function for more general Hamiltonian symplectomorphisms we use the following composition formula⁵.

2.2. Proposition (Composition formula). *If $F_1: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \rightarrow \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R}$ are, respectively, generating functions for symplectomorphisms $\Phi^{(1)}$ and $\Phi^{(2)}$ of \mathbb{R}^{2n} , then the function $F_1 \# F_2: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$ defined by*

$$F_1 \# F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

(where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on \mathbb{R}^{2n}) is a generating function for the composition $\Phi = \Phi^{(2)} \circ \Phi^{(1)}$.

In Appendix A we discuss two interpretations of the composition formula in terms of symplectic reduction, a generalization to any even number of factors and the relation to the method of broken trajectories of Chaperon, Laudenbach and Sikorav [Ch84, LS85, Si85, Si87] and to the construction in Givental [Gi90]. Below we present a direct proof.

Proof. Step 1: Criterion for fibre critical points.

The vertical derivative of $F_1 \# F_2$ is

$$(2.3) \quad (q; \zeta_1, \zeta_2, \nu_1, \nu_2) \mapsto \left(\frac{\partial F_1}{\partial \zeta_1} + 2i(\zeta_2 - q), \frac{\partial F_2}{\partial \zeta_2} - 2i(\zeta_1 - q), \frac{\partial F_1}{\partial \nu_1}, \frac{\partial F_2}{\partial \nu_2} \right),$$

thus a point $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$ is a fibre critical point of $F_1 \# F_2$ if and only if (ζ_j, ν_j) is a fibre critical point of F_j ($j = 1, 2$) and

$$(2.4) \quad \begin{cases} \frac{\partial F_1}{\partial \zeta_1} = -2i(\zeta_2 - q) \\ \frac{\partial F_2}{\partial \zeta_2} = 2i(\zeta_1 - q). \end{cases}$$

Since $F_j: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_j} \rightarrow \mathbb{R}$ is a generating function for $\Phi^{(j)}$ ($j = 1, 2$), the map

$$(2.5) \quad \Sigma_{F_j} \rightarrow \mathbb{R}^{2n}, (\zeta_j, \nu_j) \mapsto z_j$$

given by the composition

$$\Sigma_{F_j} \xrightarrow{i_{F_j}} \Gamma_{\Phi^{(j)}} \xrightarrow{\tau^{-1}|_{\Gamma_{\Phi^{(j)}}}} \text{gr}(\Phi^{(j)}) \subset \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \xrightarrow{(z, Z) \mapsto z} \mathbb{R}^{2n}$$

⁵Th  ret's composition formula in [Th98] is

$$F_1 \# F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(q + \zeta_2, \nu_1) + F_2(\zeta_1 + \zeta_2, \nu_2) + 2 \langle q - \zeta_1, i\zeta_2 \rangle.$$

Although it differs from ours just by a change of variables, in [Th98] it is proved to hold only under the assumption that F_1 or F_2 has no fibre variables. This is not sufficient for our purposes: in the proof of the quasimorphism property of the non-linear Maslov index (Proposition 4.8) we need to allow fibre variables in both factors.

is a diffeomorphism. Under the change of variables (2.5) the equations (2.4) become

$$\begin{cases} i(z_1 - \Phi^{(1)}(z_1)) = -2i\left(\frac{z_2 + \Phi^{(2)}(z_2)}{2} - q\right) \\ i(z_2 - \Phi^{(2)}(z_2)) = 2i\left(\frac{z_1 + \Phi^{(1)}(z_1)}{2} - q\right) \end{cases}$$

i.e.

$$(2.6) \quad \begin{cases} z_2 = \Phi^{(1)}(z_1) \\ q = \frac{z_1 + \Phi^{(2)}(z_2)}{2} \end{cases}.$$

Step 2: $F_1 \sharp F_2$ is a generating function.

In order to prove that $F_1 \sharp F_2$ is a generating function we need to show that zero is a regular value of the vertical derivative of $F_1 \sharp F_2$. This can be seen as follows. The diffeomorphism (2.5) is the restriction to $\Sigma_{F_j} \subset \mathbb{R}^{2n} \times \mathbb{R}^{2nN_j}$ of the map

$$(2.7) \quad \mathbb{R}^{2n} \times \mathbb{R}^{2nN_j} \rightarrow \mathbb{R}^{2n}, \quad (\zeta_j, \nu_j) \mapsto \zeta_j + \frac{1}{2i} \frac{\partial F_j}{\partial \zeta_j} \Big|_{(\zeta_j, \nu_j)}.$$

Thus, for every $(\zeta_j, \nu_j) \in \Sigma_{F_j} = (\frac{\partial F_j}{\partial \nu_j})^{-1}(0)$, the restriction to $T_{(\zeta_j, \nu_j)} \Sigma_{F_j} = \ker \left(d\left(\frac{\partial F_j}{\partial \nu_j}\right) \Big|_{(\zeta_j, \nu_j)} \right)$ of the differential at (ζ_j, ν_j) of (2.7) is bijective. As, moreover, $d\left(\frac{\partial F_j}{\partial \nu_j}\right) \Big|_{(\zeta_j, \nu_j)} : \mathbb{R}^{2n} \times \mathbb{R}^{2nN_j} \rightarrow \mathbb{R}^{2nN_j}$ is surjective, this implies that the matrix

$$M_j = \begin{pmatrix} \frac{1}{2i} \frac{\partial^2 F_j}{\partial \zeta_j^2} + I & \frac{1}{2i} \frac{\partial^2 F_j}{\partial \nu_j \partial \zeta_j} \\ \frac{\partial^2 F_j}{\partial \zeta_j \partial \nu_j} & \frac{\partial^2 F_j}{\partial \nu_j^2} \end{pmatrix}$$

is invertible at every (ζ_j, ν_j) that is fibre critical for F_j . The differential of (2.3) at $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$ is given by the matrix

$$\left(\begin{array}{cc|cc} -2iI & \frac{\partial^2 F_1}{\partial \zeta_1^2} & 2iI & \frac{\partial^2 F_1}{\partial \nu_1 \partial \zeta_1} & 0 \\ 2iI & -2iI & \frac{\partial^2 F_2}{\partial \zeta_2^2} & 0 & \frac{\partial^2 F_2}{\partial \nu_2 \partial \zeta_2} \\ \hline 0 & \frac{\partial^2 F_1}{\partial \zeta_1 \partial \nu_1} & 0 & \frac{\partial^2 F_1}{\partial \nu_1^2} & 0 \\ 0 & 0 & \frac{\partial^2 F_2}{\partial \zeta_2 \partial \nu_2} & 0 & \frac{\partial^2 F_2}{\partial \nu_2^2} \end{array} \right)$$

which, by elementary row and column operations, can be brought to the form

$$\left(\begin{array}{c|cc} \star & M_1 & 0 \\ \star & \star & M_2 \end{array} \right).$$

Since each M_j is invertible, the columns of this matrix span all of $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}$, proving that zero is a regular value of the vertical derivative of $F_1 \sharp F_2$.

Step 3: $F_1 \sharp F_2$ is a generating function for Φ .

We need to show that the Lagrangian immersion $i_{F_1 \sharp F_2} : \Sigma_{F_1 \sharp F_2} \rightarrow T^*\mathbb{R}^{2n}$ induces a diffeomorphism between $\Sigma_{F_1 \sharp F_2}$ and Γ_Φ . The relation (2.6) for fibre critical points and the fact that the maps (2.5) are diffeomorphisms imply that the map

$$(2.8) \quad \Sigma_{F_1 \sharp F_2} \rightarrow \mathbb{R}^{2n}, \quad (q; \zeta_1, \zeta_2, \nu_1, \nu_2) \mapsto z_1$$

is a diffeomorphism. For a fibre critical point $(q; \zeta, \nu)$ we have

$$\frac{\partial(F_1 \sharp F_2)}{\partial q}(q; \zeta, \nu) = 2i(\zeta_1 - \zeta_2) = i(z_1 + \Phi^{(1)}(z_1) - z_2 - \Phi^{(2)}(z_2)) = i(z_1 - \Phi(z_1))$$

and so

$$i_{F_1 \sharp F_2}(q; \zeta, \nu) = \left(q, \frac{\partial(F_1 \sharp F_2)}{\partial q}(q, \zeta, \nu) \right) = \left(\frac{z_1 + \Phi(z_1)}{2}, i(z_1 - \Phi(z_1)) \right) = \tau(z_1, \Phi(z_1)).$$

In other words, $i_{F_1 \# F_2} : \Sigma_{F_1 \# F_2} \rightarrow T^*\mathbb{R}^{2n}$ is the composition of the diffeomorphism (2.8) with the embedding $\mathbb{R}^{2n} \rightarrow T^*\mathbb{R}^{2n}$, $z_1 \mapsto \tau(z_1, \Phi(z_1))$, and so it induces a diffeomorphism between $\Sigma_{F_1 \# F_2}$ and Γ_Φ . \square

Proposition 2.2 (as well as the analogous constructions in [Ch84, LS85, Si85, Si87, Th98]) can be used to show that every compactly supported Hamiltonian diffeomorphism of \mathbb{R}^{2n} has a generating function quadratic at infinity. This class of generating functions is used for instance in the work of Viterbo [Vi92] and Traynor [Tr94]. As we now explain, in our case (similarly to [Gi90] and [Th98]) we use Proposition 2.2 to produce instead *conical generating functions* for Hamiltonian diffeomorphisms of \mathbb{R}^{2n} that lift contactomorphisms of lens spaces.

Generating functions for contact isotopies of lens spaces. Throughout our discussion, we fix the vector of weights \underline{w} that defines the action (1.1) of \mathbb{Z}_k on \mathbb{R}^{2n} , and denote the lens space $L_k^{2n-1}(\underline{w})$ by L_k^{2n-1} . On products $(\mathbb{R}^{2n})^N$ that occur as the domains of definition of various generating functions we always take the diagonal action of \mathbb{Z}_k that is given by this same \underline{w} on each factor, and denote the corresponding lens space by L_k^{2nN-1} . The $\mathbb{R}_{>0}$ -action on \mathbb{R}^{2n} , or on products of \mathbb{R}^{2n} , is always the radial action.

We say that a contact isotopy is a *based contact isotopy* if it starts at the identity.

Given a based contact isotopy $\{\phi_t\}$ of L_k^{2n-1} , we obtain a Hamiltonian isotopy $\{\Phi_t\}$ of $\mathbb{R}^{2n} \setminus \{0\}$ by first lifting $\{\phi_t\}$ to a \mathbb{Z}_k -equivariant based contact isotopy of S^{2n-1} and then lifting this to a Hamiltonian isotopy of the symplectization of the sphere, which we identify with $\mathbb{R}^{2n} \setminus \{0\}$. We now explain this procedure in more detail. Recall that the symplectization of a co-oriented contact manifold (V, ξ) is the symplectic submanifold SV of T^*V that consists of those covectors that vanish on the contact distribution and are positive with respect to the given co-orientation. Given a contactomorphism of V , its lift to the cotangent bundle restricts to a symplectomorphism of SV ; the lift to SV of a contact isotopy of V is a Hamiltonian isotopy. A choice of a contact form α for ξ gives a diffeomorphism $\mathbb{R} \times V \rightarrow SV$, defined by $(\theta, u) \mapsto e^\theta \alpha|_u$. In the special case of $V = S^{2n-1}$ and $\alpha = \sum_{j=1}^n (x_j dy_j - y_j dx_j)$ we further identify $\mathbb{R} \times V$ with $\mathbb{R}^{2n} \setminus \{0\}$ by the map $(\theta, u) \mapsto \frac{1}{2} e^\theta u$. The lift of a contactomorphism $\phi : S^{2n-1} \rightarrow S^{2n-1}$ is then the map $\Phi : \mathbb{R}^{2n} \setminus \{0\} \rightarrow \mathbb{R}^{2n} \setminus \{0\}$ given by the formula

$$\Phi(z) = \frac{|z|}{e^{g(\frac{z}{|z|})}} \phi\left(\frac{z}{|z|}\right),$$

where $\phi^* \alpha = e^g \alpha$, and we extend Φ continuously to \mathbb{R}^{2n} by setting $\Phi(0) = 0$. For any based contact isotopy $\{\phi_t\}$ of L_k^{2n-1} the resulting homeomorphisms Φ_t of \mathbb{R}^{2n} are $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -equivariant, and are smooth symplectomorphisms on $\mathbb{R}^{2n} \setminus \{0\}$. We call such maps *conical symplectomorphisms* of \mathbb{R}^{2n} .

As in [Th98], in order to work with conical symplectomorphisms we must relax the smoothness assumption on our generating functions. Notice first that if Φ is a conical symplectomorphism of \mathbb{R}^{2n} such that Γ_Φ is a section of $T^*\mathbb{R}^{2n}$ then its generating function $F : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is \mathbb{Z}_k -invariant, homogeneous of degree 2, smooth on $\mathbb{R}^{2n} \setminus \{0\}$ and \mathcal{C}^1 with Lipschitz differential everywhere⁶. More generally, we say that $F : E \rightarrow \mathbb{R}$, where $E = \mathbb{R}^{2n} \times \mathbb{R}^{2nN}$, is a *conical function* if it is \mathcal{C}^1 with Lipschitz differential, \mathbb{Z}_k -invariant and homogeneous of degree 2. Moreover we say that such an $F : E \rightarrow \mathbb{R}$ is a *conical generating function* if it is smooth near its fibre critical points other than the origin $(0, 0) \in \mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ and $dF : E \rightarrow T^*E$ is transverse to the fibre conormal bundle N_E^* except possibly at the origin. If this condition is satisfied then the set Σ_F of fibre critical points is a smooth $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -invariant submanifold except possibly at the origin, and the corresponding map $i_F : \Sigma_F \rightarrow T^*\mathbb{R}^{2n}$ is continuous, $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -equivariant, and is a smooth Lagrangian immersion outside the origin. If i_F is a homeomorphism between Σ_F and Γ_Φ for a conical symplectomorphism Φ then we say that F is a conical generating function for Φ and for

⁶Indeed, the first partial derivatives of $F|_{\mathbb{R}^{2n} \setminus \{0\}}$ are $\mathbb{R}_{>0}$ -homogeneous of degree one, hence extend continuously to \mathbb{R}^{2n} , and the second partial derivatives of $F|_{\mathbb{R}^{2n} \setminus \{0\}}$ are $\mathbb{R}_{>0}$ -invariant, hence bounded.

the induced contactomorphism ϕ of L_k^{2n-1} . By a *family of conical generating functions* we mean a collection of conical generating functions $F_s: E \rightarrow \mathbb{R}$, parametrized by $s \in S$ for some manifold with corners S (for example $[0, 1]$ or $[0, 1] \times [0, 1]$), such that the map $(s, x) \mapsto F_s(x)$ is \mathcal{C}^1 with locally Lipschitz differential and is smooth near (s, x) whenever x is a fibre critical point of F_s other than the origin.

2.9. Proposition. *If $F_t^{(1)}$ and $F_t^{(2)}$ are families of conical generating functions for contact isotopies $\{\phi_t^{(1)}\}$ and $\{\phi_t^{(2)}\}$ of L_k^{2n-1} , then the functions $F_t^{(1)} \# F_t^{(2)}$ defined as in Proposition 2.2 form a family of conical generating functions for the composition $\{\phi_t^{(2)} \circ \phi_t^{(1)}\}$.*

Proof. We first show that $F_t^{(1)} \# F_t^{(2)}$ is a family of conical generating functions. It is immediate to see that each $F_t^{(1)} \# F_t^{(2)}$ is \mathbb{Z}_k -invariant and homogeneous of degree 2, and that the family is \mathcal{C}^1 with locally Lipschitz differential. The property of being smooth at fibre critical points other than the origin is also preserved by the composition formula, as we now explain. Let $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$ be a fibre critical point of $F_t^{(1)} \# F_t^{(2)}$ other than the origin. From (2.3) we see that (ζ_j, ν_j) is a fibre critical point of $F_t^{(j)}$ for $j = 1, 2$. Moreover, (ζ_j, ν_j) is the origin if and only if the point $z_j \in \mathbb{R}^{2n}$ that corresponds to it by the bijection (2.5) is also the origin. If this happens for one of $j = 1$ or $j = 2$ then (2.6) implies that q and both z_1 and z_2 are the origin and thus, using the bijection (2.5) once more, that $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$ is the origin, contrary to our assumptions. Therefore both (ζ_1, ν_1) and (ζ_2, ν_2) must be different from the origin, and so $F_t^{(1)} \# F_t^{(2)}$ is smooth at $(q; \zeta_1, \zeta_2, \nu_1, \nu_2)$. The proof now continues by repeating the proof of Proposition 2.2; notice that derivatives of order higher than one are taken only at fibre critical points. \square

Using Proposition 2.9 we now show that any based contact isotopy of L_k^{2n-1} has what we call a based family of conical generating functions. Later in this section we show that for a given based contact isotopy such a based family of conical generating functions is essentially unique.

We say that a homeomorphism of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ is a *fibre preserving conical homeomorphism* if it takes each fibre $\{z\} \times \mathbb{R}^{2nN}$ to itself and is $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -equivariant. The *stabilization* of a conical generating function $F: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ by a non-degenerate \mathbb{Z}_k -invariant quadratic form $Q: \mathbb{R}^{2nN'} \rightarrow \mathbb{R}$ is the conical generating function

$$F \oplus Q: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \times \mathbb{R}^{2nN'} \rightarrow \mathbb{R}.$$

On the set of conical generating functions we consider the smallest equivalence relation under which two such functions are equivalent if they differ by stabilization or by a fibre preserving conical homeomorphism that restricts to a diffeomorphism between neighborhoods of fibre critical points other than the origin.

2.10. Example. The zero function $0: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is equivalent to $0 \# 0: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \rightarrow \mathbb{R}$; indeed, $0 \# 0$ differs from 0 by a stabilization followed by the fibre preserving conical homeomorphism $(q; \zeta_1, \zeta_2) \mapsto (q; \zeta_1 - q, \zeta_2 - q)$. \checkmark

2.11. Remark. If $F^{(1)}$ and $F^{(2)}$ are equivalent respectively to $G^{(1)}$ and $G^{(2)}$ then $F^{(1)} \# F^{(2)}$ is equivalent to $G^{(1)} \# G^{(2)}$. Moreover⁷, if $F^{(j)}$ and $G^{(j)}$ differ by a fibre preserving conical homeomorphism for each $j = 1, 2$ then so do $F^{(1)} \# F^{(2)}$ and $G^{(1)} \# G^{(2)}$. \diamond

2.12. Remark. If conical generating functions F and G are equivalent then there exist a stabilization of F and a stabilization of G that differ by a fibre preserving conical homeomorphism that restricts to a diffeomorphism between neighborhoods of the fibre critical points. \diamond

⁷This fact is needed in the proof of Lemma 4.2.

We say that a family $F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ of conical generating functions is a *based family of conical generating functions* if F_0 is equivalent to the zero function $\mathbb{R}^{2n} \rightarrow \mathbb{R}$.

2.13. *Remark.* By Example 2.10 and Remark 2.11, if $F_t^{(1)}$ and $F_t^{(2)}$ are based families of conical generating functions then so is $F_t^{(1)} \# F_t^{(2)}$. \diamond

2.14. **Proposition** (Existence of generating functions). *Any based contact isotopy $\{\phi_t\}_{t \in [0,1]}$ of L_k^{2n-1} has a based family $F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ of conical generating functions.*

Proof. For N big enough we can write $\phi_t = \phi_t^{(N)} \circ \dots \circ \phi_t^{(1)}$ for based contact isotopies $\{\phi_t^{(j)}\}$ having (based) families of conical generating functions $F_t^{(j)} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$. (For instance, we can take $\phi_t^{(j)} = \phi_{\frac{j}{N}t} \circ (\phi_{\frac{j-1}{N}t})^{-1}$.) By Proposition 2.9 and Remark 2.13, a family of the form $F_t := F_t^{(1)} \# \dots \# F_t^{(N)}$ (for any choice of parenthetization of the factors) is then a based family of conical generating functions for $\{\phi_t\}$. \square

In Section 4 we use generating functions to define the non-linear Maslov index of a contact isotopy of L_k^{2n-1} . In order to show that the index is well defined we use the fact that generating functions are in some sense unique. The following discussion leads to this result, which we state and prove in Proposition 2.19 below.

By a *family of fibre preserving conical homeomorphisms* we mean a collection of fibre preserving conical homeomorphism θ_s , parametrized by $s \in S$ for some manifold with corners S (for example $[0, 1]$ or $[0, 1] \times [0, 1]$), such that $(s, x) \mapsto \theta_s(x)$ is continuous. The *stabilization* of a 1-parameter family $F_t : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ of conical generating functions by a non-degenerate \mathbb{Z}_k -invariant quadratic form $Q : \mathbb{R}^{2nN'} \rightarrow \mathbb{R}$ is the 1-parameter family of conical generating functions

$$F_t \oplus Q : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \times \mathbb{R}^{2nN'} \rightarrow \mathbb{R}.$$

On the set of 1-parameter families of conical generating functions we consider the smallest equivalence relation under which two such families are equivalent if they differ by stabilization or by a 1-parameter family of fibre preserving conical homeomorphisms that restrict to diffeomorphisms on neighborhoods of the fibre critical points other than the origin.

2.15. *Remark.* If two families $F_t^{(1)}$ and $F_t^{(2)}$ are equivalent respectively to $G_t^{(1)}$ and $G_t^{(2)}$ then $F_t^{(1)} \# F_t^{(2)}$ is equivalent to $G_t^{(1)} \# G_t^{(2)}$. \diamond

Equivalent 1-parameter families of conical generating functions generate the same contact isotopy. In Proposition 2.19 we prove (mostly following Théret's proof of [Th98, Proposition 4.7]) a partial converse: any two *based* families of conical generating functions for a given contact isotopy of L_k^{2n-1} are equivalent. The main ingredient is [Th98, Lemma 4.8], whose proof holds also in our situation and gives the following result.

2.16. **Lemma.** *Let $\{\phi_t\}_{t \in [0,1]}$ be a based contact isotopy of L_k^{2n-1} . Suppose that $F_{s,t} : \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$, for $(s, t) \in [0, 1] \times [0, 1]$, is a family of conical generating functions such that for every (s, t) the function $F_{s,t}$ is a conical generating function for ϕ_t . Then there is a family $\theta_{s,t}$ of fibre preserving conical homeomorphisms of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ such that $F_{s,t} \circ \theta_{s,t} = F_{0,t}$, and such that each homeomorphism $\theta_{s,t}$ restricts to a diffeomorphism from a neighborhood of the set of fibre critical points of $F_{0,t}$ other than the origin to a neighborhood of the set of fibre critical points of $F_{s,t}$ other than the origin. In particular, the 1-parameter families $F_{0,t}$ and $F_{1,t}$ are equivalent.*

Proof. Write $E := \mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ and $\mathcal{E} = [0, 1] \times [0, 1] \times E$. We seek a locally Lipschitz two-parameter family of homogeneous of degree 1, \mathbb{Z}_k -invariant, vertical vector fields $X_{s,t}$ on $E \setminus \{(0, 0)\}$ so that

$$(2.17) \quad \frac{\partial F_{s,t}}{\partial s}(q, \zeta) + \frac{\partial F_{s,t}}{\partial \zeta}(q, \zeta) X_{s,t}(q, \zeta) = 0.$$

Assuming we have such a family of vector fields, for each t the local flow of the time-dependent vector field $s \mapsto X_{s,t}(q, \zeta)$ on $E \setminus \{(0, 0)\}$ exists thanks to the locally Lipschitz assumption. By homogeneity, the local flows fit into a family $s \mapsto \theta_{s,t}(q, \zeta)$ of global flows, which we extend to a family of fibre preserving conical homeomorphisms by demanding that $\theta_{s,t}(0, 0) = (0, 0)$. The relation (2.17) then implies that $\frac{\partial}{\partial s} F_{s,t}(\theta_{s,t}(q, \zeta)) = 0$, and so $F_{s,t}(\theta_{s,t}(q, \zeta)) = F_{0,t}(q, \zeta)$.

Writing $F : \mathcal{E} \rightarrow \mathbb{R}$, $F(s, t, q, \zeta) = F_{s,t}(q, \zeta)$, let

$$\Sigma = \left\{ \frac{\partial F}{\partial \zeta} = 0 \right\} = \bigsqcup_{s,t} \{s\} \times \{t\} \times \Sigma_{F_{s,t}}.$$

Moreover denote

$$\mathcal{S} = \{(s, t, q, \zeta) \in \mathcal{E} \mid \|q\|^2 + \|\zeta\|^2 = 1\}$$

and

$$\mathcal{E}_{\neq 0} = [0, 1] \times [0, 1] \times (E \setminus \{(0, 0)\}).$$

Notice that the subset Σ of \mathcal{E} is $\mathbb{R}_{>0}$ -invariant, and its intersection with $\mathcal{E}_{\neq 0}$ is a smooth submanifold (as it is the preimage of a regular value of $\frac{\partial F}{\partial \zeta} : \mathcal{E}_{\neq 0} \rightarrow \mathbb{R}^{2nN}$).

The vector field X is constructed following the steps below.

Step 1: Solving (2.17) in a neighborhood of $\mathcal{S} \setminus \mathcal{S} \cap \Sigma$. In a neighborhood of $\mathcal{S} \setminus \mathcal{S} \cap \Sigma$ the family of vertical vector fields

$$X_{s,t}^1(q, \zeta) = \frac{-\frac{\partial F_{s,t}}{\partial s}(q, \zeta)}{|\frac{\partial F_{s,t}}{\partial \zeta}|^2} \sum_k \frac{\partial F_{s,t}}{\partial \zeta_k}(q, \zeta) \frac{\partial}{\partial \zeta_k}$$

is well defined and solves (2.17). It is locally Lipschitz, because so are $\frac{\partial F_{s,t}}{\partial s}$ and $\frac{\partial F_{s,t}}{\partial \zeta}$.

Step 2: Solving (2.17) in a neighborhood of $\mathcal{S} \cap \Sigma$ in \mathcal{S} . Let U be an open neighborhood of $\mathcal{S} \cap \Sigma$ in \mathcal{E} on which F is smooth. Let G_1 denote the restriction of $\frac{\partial F}{\partial \zeta}$ to U , so $U \cap \Sigma = G_1^{-1}(0)$. The set $G_1^{-1}(0)$ consists only of regular points of G_1 . Let $G_2 : U \rightarrow \mathbb{R}$ denote the restriction of $-\frac{\partial F}{\partial s}$ to U . Then G_2 is smooth on U and we claim that it vanishes on Σ .

Indeed, since the image of the family of Lagrangian embeddings $\Sigma_{F_{s,t}} \rightarrow T^*\mathbb{R}^{2n}$ is independent of s , around each point $(s, t, q, \zeta) \in \Sigma \cap \mathcal{E}_{\neq 0}$, we may pick coordinates (s, t, x, y) such that $\Sigma = \{(s, t, x, y) : y = 0\}$ and $x = x(t, q, \zeta)$ is independent of s (we take $y = \frac{\partial F}{\partial \zeta}$).

As the Lagrangian embedding of $\Sigma_{F_{s,t}}$ is given by the expression $(q, \zeta) \mapsto (q, \frac{\partial F_{s,t}}{\partial q}) \in T^*(\mathbb{R}^{2n})$ it follows that $\frac{\partial}{\partial s} \left(\frac{\partial F}{\partial q}(s, t, x, 0) \right) = 0$. Clearly we also have $\frac{\partial}{\partial s} \left(\frac{\partial F}{\partial \zeta}(s, t, x, 0) \right) = 0$. It is easy to check that the operator $\frac{\partial}{\partial s}$ is the same in both systems of coordinates, hence $\frac{\partial F}{\partial s}$ has vanishing derivatives with respect to q and ζ on $\Sigma \cap \mathcal{E}_{\neq 0}$. As $\frac{\partial F}{\partial s}$ is homogeneous of degree 2 in (q, ζ) it must vanish on Σ , as required.

Hadamard's Lemma (see [Th95, Théorème 92]) now provides a smooth $B : U \rightarrow \mathbb{R}^{2nN}$ such that in some neighborhood of $U \cap \Sigma$ we have $G_2 = \langle B, G_1 \rangle$. We use the coordinates of the map B to define a vertical vector field

$$X^2 = \sum_{k=1}^{2nN} B_k \frac{\partial}{\partial \zeta_k}$$

on the above neighborhood of $U \cap \Sigma$. Then

$$-\frac{\partial F}{\partial s} = G_2 = \langle B, G_1 \rangle = \sum_{k=1}^{2nN} X_k^2 \frac{\partial F}{\partial \zeta_k},$$

and so (2.17) holds. Note that this vector field is smooth, thus also locally Lipschitz.

Step 3: Patch these vector fields together. We use a smooth bump function to obtain a locally Lipschitz, vertical vector field that is defined in a neighborhood of \mathcal{S} and solves (2.17).

Step 4: Average and extend. Average by the \mathbb{Z}_k -action to obtain a \mathbb{Z}_k -invariant, locally Lipschitz, vertical vector field \tilde{X} . Then restrict \tilde{X} to \mathcal{S} and extend to the whole $\mathcal{E}_{\neq 0}$ by homogeneity, i.e. define

$$X(s, t, q, \zeta) = |(q, \zeta)| \tilde{X}\left(s, t, \frac{q}{|(q, \zeta)|}, \frac{\zeta}{|(q, \zeta)|}\right)$$

for $(q, \zeta) \neq 0$ and $X(s, t, 0, 0) = 0$. Then X is homogeneous of degree 1, \mathbb{Z}_k -invariant, vertical and locally Lipschitz. Moreover, its flow restricts to a diffeomorphism from a neighborhood of the set of fibre critical points of $F_{0,t}$ other than the origin to a neighborhood of the set of fibre critical points of $F_{s,t}$ other than the origin. \square

Using Lemma 2.16 we now prove the following statement.

2.18. Lemma (0 is neutral). *Let $F_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ be a family of conical generating functions for a contact isotopy $\{\phi_t\}$, and let $0: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ be the zero function. Then $F_t \# 0$ and $0 \# F_t$ are equivalent to F_t .*

Proof. We prove that $F_t \# 0$ is equivalent to F_t (the case of $0 \# F_t$ is similar). For each $s \in [0, 1]$ we define a 1-parameter family $F_{s,t}: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN}) \rightarrow \mathbb{R}$ by

$$F_{s,t}(q; \zeta_1, \zeta_2, \nu) = F_t(s\zeta_1 + (1-s)q, \nu) - 2\langle \zeta_2 - q, i(\zeta_1 - q) \rangle.$$

Then $F_{1,t} = F_t \# 0$, and $F_{0,t}(q; \zeta_1, \zeta_2, \nu) = F_t(q, \nu) - 2\langle \zeta_2 - q, i(\zeta_1 - q) \rangle$ is equivalent to F_t since it differs from it by a stabilization followed by the fibre preserving conical homeomorphism $(q; \zeta_1, \zeta_2, \nu) \mapsto (q; \zeta_1 - q, \zeta_2 - q, \nu)$. By arguments as in the proof of Proposition 2.2 one sees that for each $s \in [0, 1]$ the 1-parameter family $F_{s,t}$ generates $\{\phi_t\}$. By Lemma 2.16 we conclude that $F_{0,t}$ and $F_{1,t}$ are equivalent, and thus so are F_t and $F_t \# 0$. \square

We now put these ingredients together to prove the following result.

2.19. Proposition (Uniqueness of generating functions). *Any two based families of conical generating functions for the same based contact isotopy of L_k^{2n-1} are equivalent.*

Proof. Let F_t and G_t be two based families of conical generating functions for a based contact isotopy $\{\phi_t\}$. By Proposition 2.9, for every $s \in [0, 1]$ the family $F_{st} \# ((-G_{st}) \# G_t)$ generates $\{\phi_t\} = \{(\phi_t \circ \phi_{st}^{-1}) \circ \phi_{st}\}$, and the family $(-G_{st}) \# G_{st}$ generates $\{\text{id}\} = \{\phi_{st} \circ \phi_{st}^{-1}\}$. By Lemma 2.16 we thus have that $F_0 \# ((-G_0) \# G_t)$ is equivalent to $F_t \# ((-G_t) \# G_t)$ and that $(-G_0) \# G_0$ is equivalent to $(-G_t) \# G_t$. Denoting equivalence of families of generating functions by \sim , Remark 2.15, Lemma 2.18, Example 2.10 and the fact that the families F_t and G_t are based then imply

$$G_t \sim 0 \# (0 \# G_t) \sim F_0 \# ((-G_0) \# G_t) \sim F_t \# ((-G_t) \# G_t) \sim F_t \# ((-G_0) \# G_0) \sim F_t \# 0 \sim F_t.$$

\square

The next result is used in Section 4 to obtain that the non-linear Maslov index descends to a map on the universal cover of the identity component of the contactomorphism group.

2.20. Proposition (Uniqueness for a homotopy class). *Suppose that $\{\phi_{0,t}\}$ and $\{\phi_{1,t}\}$ are two contact isotopies from the identity to the same contactomorphism ϕ that are smoothly homotopic with fixed endpoints. Then there are based families $F_{0,t}$ and $F_{1,t}$ of conical generating functions for $\phi_{0,t}$ and $\phi_{1,t}$ respectively such that $F_{0,0} = F_{1,0}$ and $F_{0,1}$ and $F_{1,1}$ differ by a fibre preserving conical homeomorphism.*

Proof. Let $\{\phi_{s,t}\}$ be a smooth homotopy with fixed endpoints from $\{\phi_{0,t}\}$ to $\{\phi_{1,t}\}$, and for N big enough write $\phi_{s,t} = \phi_{s,t}^{(N)} \circ \dots \circ \phi_{s,t}^{(1)}$, with each family $\{\phi_{s,t}^{(j)}\}_{s \in [0,1]}$ depending smoothly on s and t , for based contact isotopies $\{\phi_{s,t}^{(j)}\}_{[0,1]}$ having families of conical generating function $F_{s,t}^{(j)}: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. (For instance, we can take $\phi_{s,t}^{(j)} = \phi_{s, \frac{j}{N}t} \circ (\phi_{s, \frac{j-1}{N}t})^{-1}$.) For every s consider the based family

$F_{s,t} = F_{s,t}^{(1)} \# \cdots \# F_{s,t}^{(N)}$. Then $F_{s,0} = 0 \# \cdots \# 0$ and $F_{s,1}$ generates ϕ . In particular $F_{0,0} = F_{1,0}$ and, by Lemma 2.16, $F_{0,1}$ and $F_{1,1}$ differ by a fibre preserving conical homeomorphism. \square

Let $\{\phi_t\}_{t \in [0,1]}$ be a contact isotopy of L_k^{2n-1} , and consider a (based) family $F_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ of conical generating functions. Since all F_t are homogeneous of degree 2, they are determined by their restrictions $\bar{f}_t: S^{2n(N+1)-1} \rightarrow \mathbb{R}$. Moreover, as the F_t are invariant by the \mathbb{Z}_k -action, the \bar{f}_t are also \mathbb{Z}_k -invariant and so they descend to a family of functions

$$f_t: L_k^{2n(N+1)-1} \rightarrow \mathbb{R}$$

that are \mathcal{C}^1 with Lipschitz differential. We also say that f_t (as well as the corresponding F_t) is a (based) family of generating functions for the contact isotopy $\{\phi_t\}$.

2.21. Proposition. *Let $f_t: L_k^{2n(N+1)-1} \rightarrow \mathbb{R}$ be a family of generating functions for a based contact isotopy $\{\phi_t\}$ of L_k^{2n-1} . Then for every t there is a 1–1 correspondence between discriminant points of ϕ_t whose preimage in S^{2n-1} is a \mathbb{Z}_k -orbit of discriminant points for the lift and critical points of f_t of critical value zero, that takes non-degenerate discriminant points to non-degenerate critical points, and that is given by the restriction of a map $L_k^{2n-1} \rightarrow L_k^{2n(N+1)-1}$ that is isotopic to the standard inclusion $[z] \mapsto [z, 0]$.*

Proof. For every t , denote by F_t the conical function on $\mathbb{R}^{2n(N+1)}$ inducing f_t . Since F_t is homogeneous of degree 2 and \mathbb{Z}_k -invariant, its critical points come in $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -orbits and all have critical value zero. Such $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -orbits are in 1–1 correspondence with critical points of f_t of critical value zero. On the other hand, discriminant points of ϕ_t that are also discriminant points of the lift to S^{2n-1} correspond to $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -orbits of fixed points of Φ_t . We then use the fact that (non-degenerate) discriminant points of ϕ_t correspond to (non-degenerate) critical points of F_t (see for example [Sa11c, Lemma 3.5]). We now show that the 1–1 correspondence is given by the restriction of a map $L_k^{2n-1} \rightarrow L_k^{2n(N+1)-1}$ that is isotopic to the standard inclusion. Recall that, for every $t \in [0, 1]$, the conical map

$$\mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow T^*\mathbb{R}^{2n}, (\zeta, \nu) \mapsto (\zeta, \frac{\partial F_t}{\partial \zeta}(\zeta, \nu))$$

restricts to a homeomorphism i_{F_t} between the set Σ_{F_t} of fibre critical points and Γ_{Φ_t} . Fix now $t \in [0, 1]$. The required isotopy is the map induced on the quotient by the $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -action by the composition

$$\begin{aligned} [0, t] \times \mathbb{R}^{2n} &\rightarrow \bigcup_{s \in [0, t]} \{s\} \times \Gamma_{\Phi_s} \rightarrow \bigcup_{s \in [0, t]} \{s\} \times \Sigma_{F_s} \subset [0, t] \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \\ (s, \zeta) &\mapsto (s, \tau(\zeta, \Phi_s(\zeta))) \mapsto (s, (i_{F_s})^{-1}(\tau(\zeta, \Phi_s(\zeta)))) . \end{aligned}$$

\square

Monotonicity and quasi-additivity of generating functions. We start with the following monotonicity property for generating function, which is used to show that lens spaces are orderable.

2.22. Proposition (Monotonicity of generating functions). *Let $\{\phi_t\}_{t \in [0,1]}$ be a based contact isotopy of L_k^{2n-1} .*

- (i) *Assume that $\{\phi_t\}_{t \in [0,1]}$ has a family $f_t: L_k^{2n-1} \rightarrow \mathbb{R}$ of generating functions without fibre variable, and that $\{\phi_t\}_{t \in [0,1]}$ is non-negative (respectively non-positive, positive, negative). Then $\frac{\partial f_t}{\partial t} \geq 0$ (respectively $\frac{\partial f_t}{\partial t} \leq 0$, $\frac{\partial f_t}{\partial t} > 0$, $\frac{\partial f_t}{\partial t} < 0$) for all $t \in [0, 1]$.*
- (ii) *For a general $\{\phi_t\}_{t \in [0,1]}$, assume that, for a subinterval $[a, b]$ of $[0, 1]$, $\{\phi_t\}_{t \in [a, b]}$ is non-negative (respectively non-positive). Then there exist*
 - *a based contact isotopy $\{\psi_t\}_{t \in [0,1]}$ representing the same element of $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ as $\{\phi_t\}_{t \in [0,1]}$ and such that the restriction of $\{\phi_t\}$ and $\{\psi_t\}$ to $[0, a]$, $[a, b]$ and $[b, 1]$ are homotopic with fixed endpoints and $\{\psi_t\}_{t \in [a, b]}$ is non-negative (respectively non-positive);*

- a based family $f_t : L_k^{2M-1} \rightarrow \mathbb{R}$ of generating functions for $\{\psi_t\}_{t \in [0,1]}$ such that $\frac{\partial f_t}{\partial t} \geq 0$ (respectively $\frac{\partial f_t}{\partial t} \leq 0$) for all $t \in [a, b]$.

As in [Sa11c, Lemma 3.6], the main ingredient for the proof is the classical Hamilton–Jacobi equation for generating functions.

2.23. Lemma (Hamilton–Jacobi equation). *Let $F_t : B \rightarrow \mathbb{R}$ be a family of functions with $F_0 \equiv 0$. For each t , denote by $L_t \subset T^*B$ the graph of the differential of F_t . Consider a Hamiltonian isotopy $\{\varphi_t\}$ of T^*B , generated by a Hamiltonian function $H_t : T^*B \rightarrow \mathbb{R}$, such that, for every t , L_t is the image of the zero section by φ_t . For $x \in B$ and $t \in [0, 1]$ let $(q_t(x), p_t(x)) = \varphi_t(x, 0)$. Then*

$$H_t(q_t(x), p_t(x)) = \frac{\partial F_t}{\partial t}(q_t(x)) + c(t)$$

for some function of time $c(t)$.

Proof. We have

$$\begin{aligned} \frac{\partial}{\partial x} H_t(q_t(x), p_t(x)) &= \frac{\partial H_t}{\partial q} \frac{\partial q_t}{\partial x} + \frac{\partial H_t}{\partial p} \frac{\partial p_t}{\partial x} = \dot{p}_t \frac{\partial q_t}{\partial x} - \dot{q}_t \frac{\partial p_t}{\partial x} \\ &= \left(\frac{\partial^2 F_t}{\partial t \partial q} + \frac{\partial^2 F_t}{\partial q^2} \dot{q}_t \right) \frac{\partial q_t}{\partial x} - \dot{q}_t \left(\frac{\partial^2 F_t}{\partial q^2} \frac{\partial q_t}{\partial x} \right) \\ &= \frac{\partial^2 F_t}{\partial q \partial t} \frac{\partial q_t}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\partial F_t}{\partial t}(q_t(x)) \right) \end{aligned}$$

where the second equality follows from Hamilton’s equations for H_t , and the third from differentiating the relation $\frac{\partial F_t}{\partial q}(q_t(x)) = p_t(x)$ with respect to t and with respect to x . \square

The proof of Proposition 2.22 also uses concatenation of contact isotopies. The following remark allows us to represent elements of $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ by concatenations.

2.24. Remark (Concatenation). Let $\{\phi_t\}_{t \in [0,1]}$ and $\{\psi_t\}_{t \in [0,1]}$ be based contact isotopies. For any smooth map $\rho : [0, 1] \rightarrow [0, 1]$ such that $\rho(0) = 0$ and $\rho(1) = 1$ and whose derivatives of orders ≥ 1 all vanish at the endpoints 0 and 1, the concatenation $\{\phi_{\rho(t)}\}_{t \in [0,1]} \sqcup \{\psi_{\rho(t)} \circ \phi_{\rho(1)}\}_{t \in [0,1]}$ is smooth and it represents the same element of $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ as the composition $\{\psi_t \circ \phi_t\}_{t \in [0,1]}$. We can choose ρ to be strictly monotone. Then, for any closed subinterval I of $[0, 1]$, if $\{\phi_t\}_{t \in I}$ is respectively embedded, non-negative or non-positive then so is $\{\phi_{\rho(t)}\}_{t \in \rho^{-1}(I)}$. \diamond

Proof of Proposition 2.22. Suppose first that $\{\phi_t\}$ has a family $f_t : L_k^{2n-1} \rightarrow \mathbb{R}$ of generating functions without fibre variables. Let $\{\Phi_t\}$ be the corresponding conical Hamiltonian isotopy of \mathbb{R}^{2n} , and consider the Hamiltonian isotopy $\{\tau \circ (\text{id} \times \Phi_t) \circ \tau^{-1}\}$ of $T^*\mathbb{R}^{2n}$, where as usual τ denotes the identification (2.1). For each t , the image of the zero section by $\tau \circ (\text{id} \times \Phi_t) \circ \tau^{-1}$ is the Lagrangian $\Gamma_{\Phi_t} = \tau(\text{gr}(\Phi_t))$. If the lift of $\{\phi_t\}$ to S^{2n-1} is generated by the contact Hamiltonian function $h_t : S^{2n-1} \rightarrow \mathbb{R}$ then on $\mathbb{R}^{2n} \setminus \{0\}$ the isotopy $\{\Phi_t\}$ is generated by the Hamiltonian function $H_t : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ defined by $H_t(z) = |z|^2 h_t(\frac{z}{|z|})$, and on $\tau(\overline{\mathbb{R}^{2n}} \times (\mathbb{R}^{2n} \setminus \{0\}))$ the isotopy $\{\tau \circ (\text{id} \times \Phi_t) \circ \tau^{-1}\}$ is generated by the Hamiltonian function $\overline{H}_t \circ \tau^{-1}$, where $\overline{H}_t : \overline{\mathbb{R}^{2n}} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}$ is defined by $\overline{H}_t(z, z') = H_t(z')$. Thus, if the contact isotopy $\{\phi_t\}$ is non-negative (respectively non-positive, positive, negative) then

$$\overline{H}_t(z, \Phi_t(z)) = H_t(\Phi_t(z)) \geq 0$$

(respectively ≤ 0 , > 0 , < 0) for $z \neq 0$. Since all our functions are conical the Hamilton–Jacobi equation of Lemma 2.23 holds with $c(t) \equiv 0$. We thus obtain

$$\frac{\partial F_t}{\partial t} \left(\frac{z + \Phi_t(z)}{2} \right) = \overline{H}_t(z, \Phi_t(z)) \geq 0$$

(respectively ≤ 0 , > 0 , < 0) for $z \neq 0$, where F_t is the conical function induced by f_t . Since $z \mapsto \frac{z + \Phi_t(z)}{2}$ is onto \mathbb{R}^{2n} , we conclude that $\frac{\partial f_t}{\partial t} \geq 0$ (respectively ≤ 0 , > 0 , < 0). This proves (i).

In order to prove (ii), take a sufficiently fine partition $0 = t_0 < t_1 < \dots < t_N = 1$ such that $\{t_0, \dots, t_N\}$ contains the endpoints of the interval I . For $j = 1, \dots, N$ let $\psi_t^{(j)}$ be a smooth reparametrization of

$$\widehat{\psi}_t^{(j)} = \begin{cases} \text{id} & \text{if } t \in [0, t_{j-1}] \\ \phi_t \circ (\phi_{t_{j-1}})^{-1} & \text{if } t \in [t_{j-1}, t_j] \\ \phi_{t_j} \circ (\phi_{t_{j-1}})^{-1} & \text{if } t \in [t_j, 1] \end{cases}$$

so that the $\psi_t^{(j)}$ have a family of generating functions without fibre variable and are non-negative (respectively non-positive) for $t \in I$, and so that $\{\psi_t := \psi_t^{(N)} \circ \dots \circ \psi_t^{(1)}\}$ is a reparametrization of $\{\phi_t\}$. We then conclude using (i) and the fact that if we apply the composition formula (Proposition 2.2) to two non-decreasing (respectively non-increasing) families of functions then the resulting family is also non-decreasing (respectively non-increasing). \square

We now note that the generating function for a composition that is given by Proposition 2.2 agrees in codimension $2n$ with the direct sum of the generating functions of the factors.

2.25. Proposition (Quasiadditivity of generating functions). *Suppose that $F_1: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \rightarrow \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R}$ are (conical) generating functions for the (conical) symplectomorphisms $\Phi^{(1)}$ and $\Phi^{(2)}$ respectively. Then there is a linear $(\mathbb{Z}_k \times \mathbb{R}_{>0})$ -equivariant injection*

$$\iota: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2})$$

such that $(F_1 \# F_2) \circ \iota = F_1 \oplus F_2$.

Proof. We have

$$F_1 \oplus F_2(\zeta_1, \zeta_2; \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2)$$

and

$$F_1 \# F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle.$$

Thus for the injection $\iota(\zeta_1, \zeta_2; \nu_1, \nu_2) = (\zeta_1; \zeta_1, \zeta_2, \nu_1, \nu_2)$ we have $(F_1 \# F_2) \circ \iota = F_1 \oplus F_2$. \square

The above quasiadditivity property is crucial in proving that the non-linear Maslov index is a quasimorphism (Proposition 4.8). If generating functions had been additive, not just quasiadditive, then at least for real projective spaces the non-linear Maslov index would have been a homomorphism. Since no non-trivial homomorphisms exist on contactomorphism groups [Ts08, Ry10], this shows that the lack of additivity of the composition formula is not a technical failure but something essential.

3. THE COHOMOLOGICAL INDEX

As already outlined in the introduction, the value of the non-linear Maslov index of a contact isotopy $\{\phi_t\}$ of a lens space L_k^{2n-1} depends on the changes in topology of the sublevel sets of a based family $f_t: L_k^{2M-1} \rightarrow \mathbb{R}$ of conical generating functions. As in [Gi90] and [Th98], the topological invariant that we use to analyze these changes is the cohomological index. In this section we review the definition of this invariant and describe some of its properties in the case of lens spaces. Cohomological indices have also been studied in a more general context by Fadell and Rabinowitz [FR78] (see also Remark 3.7).

Continuity of the cohomological index (Proposition 3.9 (ii) below) is important for our applications. Since the sets that we need to consider (sublevel sets of generating functions) might not be locally contractible, in order to guarantee continuity we work with Čech cohomology (as for instance in [FR78]). Note that Čech cohomology agrees with singular cohomology on spaces that are paracompact and locally contractible (see [Sp66, Corollary 6.8.8 and Theorem 6.9.1]), in particular

on manifolds or, more generally, on CW-complexes. Recall also that the Čech cohomology of a compact subset A of a manifold can be computed in terms of singular cohomology as

$$(3.1) \quad \check{H}^*(A) = \varinjlim_j H^*(U_j)$$

where U_j is any decreasing sequence of open sets having A as their intersection. Indeed, $\check{H}^*(A) = \varinjlim_j \check{H}^*(U_j)$ (see [Sp66, Theorem 6.6.2]) and, since the U_j are open, their Čech cohomology agrees with their singular cohomology.

We now assume that k is prime (cf. Remark 1.4).

3.2. Definition. Let A be a paracompact Hausdorff topological space and $\pi: \tilde{A} \rightarrow A$ a principal \mathbb{Z}_k -bundle with classifying map $g: A \rightarrow B\mathbb{Z}_k = L_k^\infty$. The *cohomological index* of $\pi: \tilde{A} \rightarrow A$ is the dimension over \mathbb{Z}_k of the image of the induced map $g^*: \check{H}^*(L_k^\infty; \mathbb{Z}_k) \rightarrow \check{H}^*(A; \mathbb{Z}_k)$. If A is a subset of a lens space⁸ $L_k^{2M-1}(\underline{w})$ then its cohomological index, denoted by $\text{ind}(A)$, is defined to be the cohomological index of the restriction $\pi: \tilde{A} \rightarrow A$ of the principal \mathbb{Z}_k -bundle⁹ $\pi: S^{2M-1}(\underline{w}) \rightarrow L_k^{2M-1}(\underline{w})$. \diamond

We now specialize to the case when the prime k is different from 2, leaving to the reader the task of adapting the discussion to the (easier) case of $k = 2$ (cf. Remarks 3.5 and 3.7).

A principal \mathbb{Z}_k -bundle $\tilde{A} \rightarrow A$ is determined by the Čech cohomology class $\alpha \in \check{H}^1(A; \mathbb{Z}_k)$ that is represented by the transition functions for a choice of local trivializations. The Bockstein homomorphism $\mathcal{B}: \check{H}^q(A) \rightarrow \check{H}^{q+1}(A)$ (see [Ha02, Section 3.E]) is a derivation whose square is zero, so, setting¹⁰ $\beta = \mathcal{B}(\alpha) \in \check{H}^2(A; \mathbb{Z}_k)$, we have

$$\mathcal{B}(\alpha\beta^j) = \beta^{j+1} \quad \text{and} \quad \mathcal{B}(\beta^j) = 0 \quad \text{for all } j \geq 0.$$

A map of principal \mathbb{Z}_k -bundles $\tilde{A} \rightarrow \tilde{B}$ pulls back the classes α, β on the base B to the classes α, β on the base A .

3.3. Lemma. *For any M -tuple of weights \underline{w} and $0 \leq j \leq 2M - 1$,*

$$(3.4) \quad H^j(L_k^{2M-1}(\underline{w}); \mathbb{Z}_k) \text{ is generated by } \begin{cases} \beta^i & \text{for } j = 2i \\ \alpha\beta^i & \text{for } j = 2i + 1. \end{cases}$$

For a subset A of $L_k^{2M-1}(\underline{w})$,

$$\text{ind}(A) = \dim_{\mathbb{Z}_k}(\text{im } \iota^*)$$

where $\iota^*: \check{H}^*(L_k^{2M-1}(\underline{w}); \mathbb{Z}_k) \rightarrow \check{H}^*(A; \mathbb{Z}_k)$ is the map on Čech cohomology that is induced by the inclusion $\iota: A \hookrightarrow L_k^{2M-1}(\underline{w})$. Moreover,

$$\text{im } \iota^* \cap \check{H}^j(A; \mathbb{Z}_k) \cong \begin{cases} \mathbb{Z}_k & \text{if } 0 \leq j < \text{ind}(A) \\ 0 & \text{if } j \geq \text{ind}(A). \end{cases}$$

Proof. For (3.4), see [Ha02, Example 3E.2]. The equality $\text{ind}(A) = \dim_{\mathbb{Z}_k}(\text{im } \iota^*)$ follows from the facts that the classifying map $g(\underline{w}): L_k^{2M-1}(\underline{w}) \rightarrow L_k^\infty$ induces a surjection in cohomology (by (3.4)) and that $g(\underline{w}) \circ \iota$ is a classifying map for A . The ring structure and the action of the Bockstein homomorphism imply that if $x \in H^*(L_k^{2M-1}(\underline{w}))$ is non-zero and $\iota^*(x) = 0$ then $\iota^*(y) = 0$ for all y with $\deg(y) \geq \deg(x)$; this implies the last statement. \square

⁸In our applications M is a multiple of n and the M -tuple of weights on L_k^{2M-1} has the form $\underline{w} = (\underline{w}', \dots, \underline{w}')$ for an n -tuple of weights \underline{w}' on L_k^{2n-1} . However in this section \underline{w} can be any tuple of weights.

⁹We write $S^{2M-1}(\underline{w})$ or $\mathbb{C}^M(\underline{w})$ when we wish to specify the \mathbb{Z}_k -action.

¹⁰If $\{\psi_{ij}: U_i \cap U_j \rightarrow \mathbb{Z}\}$ are lifts of the transition functions $\{\alpha_{ij}: U_i \cap U_j \rightarrow \mathbb{Z}_k\}$, then β is represented by the Čech 2-cocycle $\{\frac{1}{k}(\psi_{ij} + \psi_{j\ell} + \psi_{\ell i}) \mod k\}$.

It follows from Lemma 3.3 that if A is a subset of $L_k^{2M-1}(\underline{w})$ then $0 \leq \text{ind}(A) \leq 2M$. Also note that $\text{ind}(\emptyset) = 0$, and that $\text{ind}(A) = 1$ if A is finite and non-empty.

A *lens subspace* of $L_k^{2M-1}(\underline{w})$ is the \mathbb{Z}_k -quotient of the intersection of $S^{2M-1}(\underline{w})$ with a \mathbb{Z}_k -invariant complex linear subspace of $\mathbb{C}^M(\underline{w})$. Lemma 3.3 implies that the cohomological index of a $2r - 1$ dimensional lens subspace is $2r$.

3.5. *Remark.* For a subset A of a real projective space \mathbb{RP}^M ,

$$(3.6) \quad \text{ind}(A) = \min\{j \in \mathbb{N} \mid \iota^*(x^j) = 0\}$$

where x is the generator of $\check{H}^1(\mathbb{RP}^M; \mathbb{Z}_2)$. Similarly one defines the cohomological index for subsets of complex projective spaces and for principal S^1 -bundles; in this case the analogue of (3.6) holds for x a generator of degree two (cf. [Th98]).

3.7. *Remark.* For a compact Lie group G , let $\pi: \tilde{A} \rightarrow A$ be a principal G -bundle over a paracompact Hausdorff topological space with classifying map $g: A \rightarrow BG$. For any non-zero class $\eta \in \check{H}^*(BG)$ Fadell and Rabinowitz [FR78] define the η -index as the maximal $j \in \mathbb{N}$ such that $g^*(\eta^j) \neq 0$. If $G = \mathbb{Z}_2$ and η is the generator of $\check{H}^1(\mathbb{RP}^\infty; \mathbb{Z}_2)$ then the η -index just differs by 1 from the index of Definition 3.2. If however $G = \mathbb{Z}_k$ with $k \neq 2$ and β is a generator of $\check{H}^2(L_k^\infty; \mathbb{Z}_k)$ then the β -index is equal to $\lfloor \frac{\text{ind}(A)-1}{2} \rfloor$. We use the index from Definition 3.2 rather than the β -index in order to obtain a better bound on the number of translated points (see Section 5): using the β -index we would only prove existence of n translated points on L_k^{2n-1} , even in the non-degenerate case. \diamond

Given subsets A of $L_k^{2M-1}(\underline{w})$ and B of $L_k^{2M'-1}(\underline{w}')$ with preimages $\tilde{A} \subset S^{2M-1}(\underline{w})$ and $\tilde{B} \subset S^{2M'-1}(\underline{w}')$, their \mathbb{Z}_k -join is the subset

$$A *_{\mathbb{Z}_k} B \subset L_k^{2(M+M')-1}(\underline{w}, \underline{w}')$$

defined by

$$(3.8) \quad A *_{\mathbb{Z}_k} B = \left\{ [\sqrt{t}a, \sqrt{1-t}b] \mid a \in \tilde{A}, b \in \tilde{B}, 0 \leq t \leq 1 \right\}$$

if A and B are non-empty. If B is empty, we define $A *_{\mathbb{Z}_k} \emptyset$ to be the image of A under the natural embedding $a \mapsto [a, 0]$ of $L_k^{2M-1}(\underline{w})$ into $L_k^{2(M+M')-1}(\underline{w}, \underline{w}')$. We define $\emptyset *_{\mathbb{Z}_k} B$ similarly. Finally, the \mathbb{Z}_k -join of the empty sets is empty.

We now describe the properties of the cohomological index that we need for our applications. The proofs of properties (i)-(iv) are easy adaptations of the corresponding proofs in [Gi90, Th98, FR78] and are included for the convenience of the reader. In (v), the lower bound on $\text{ind}(A *_{\mathbb{Z}_k} B)$ requires a more involved proof, which we postpone to Appendix B.

3.9. **Proposition.** *The cohomological index of subsets of lens spaces has the following properties:*

- (i) (Monotonicity) *If $A \subset B \subset L_k^{2M-1}(\underline{w})$ then $\text{ind}(A) \leq \text{ind}(B)$.*
- (ii) (Continuity) *Every closed subset A of $L_k^{2M-1}(\underline{w})$ has a neighborhood U such that if $A \subset V \subset U$ then $\text{ind}(V) = \text{ind}(A)$.*
- (iii) (Lefschetz property) *Let A be a closed subset of $L_k^{2M-1}(\underline{w})$, and let $A' = A \cap H$ where $H \subset L_k^{2M-1}(\underline{w})$ is a lens subspace of codimension 2. Then $\text{ind}(A') \geq \text{ind}(A) - 2$.*
- (iv) (Subadditivity) *For closed subsets A and B of $L_k^{2M-1}(\underline{w})$ we have*

$$\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B) + 1$$

and

$$\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B) \quad \text{if } \text{ind}(A) \text{ is even or } \text{ind}(B) \text{ is even.}$$

- (v) (Join quasi-additivity) *For closed subsets A of $L_k^{2M-1}(\underline{w})$ and B of $L_k^{2M'-1}(\underline{w}')$ we have*

$$|\text{ind}(A *_{\mathbb{Z}_k} B) - \text{ind}(A) - \text{ind}(B)| \leq 1$$

and

$$\operatorname{ind}(A *_{\mathbb{Z}_k} B) = \operatorname{ind}(A) + \operatorname{ind}(B) \quad \text{if } \operatorname{ind}(A) \text{ is even or } \operatorname{ind}(B) \text{ is even.}$$

In particular (Join stability),

$$\operatorname{ind}(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) = \operatorname{ind}(A) + 2K.$$

3.10. *Remark.* The above properties (ii), (iii), (iv), and (v) are stated for closed subspaces of lens spaces but they hold (and are proved) also for open subsets of lens spaces.

Proof Proposition 3.9. (i) Let \tilde{A} and \tilde{B} be the preimages in $S_k^{2n-1}(\underline{w})$ of A and B . The result follows from the fact that the restriction to A of the classifying map of $\tilde{B} \rightarrow B$ is a classifying map for $\tilde{A} \rightarrow A$.

(ii) Let x be a generator of $H^{\operatorname{ind}(A)}(L_k^{2M-1}(\underline{w}); \mathbb{Z}_k)$. Then $i_A^*(x) = 0$. By (3.1), there exists an open neighborhood U of A such that $i_U^*(x) = 0$, where $i_U: U \rightarrow M$ is the inclusion map. By Lemma 3.3, $\operatorname{ind}(U) \leq \operatorname{ind}(A)$. By monotonicity, $\operatorname{ind}(U) \geq \operatorname{ind}(A)$.

(iii) Assume that $\operatorname{ind}(A) \geq 3$, otherwise the inequality is trivial. By continuity of the cohomological index, there exist open neighborhoods U of A and V of H such that $\operatorname{ind}(U) = \operatorname{ind}(A)$, $\operatorname{ind}(V) = \operatorname{ind}(H)$, and $\operatorname{ind}(U \cap V) = \operatorname{ind}(A \cap H)$. We have the following commuting diagram, where D denotes the Poincaré duality isomorphism and \bullet the homology intersection product (see [Do95, VIII.13.5]):

$$\begin{array}{ccc} H_*(U) \otimes H_*(V) & \xrightarrow{\bullet} & H_*(U \cap V) \\ \downarrow i_{U*} \otimes i_{V*} & & \downarrow i_{U \cap V*} \\ H_*(L_k^{2M-1}(\underline{w})) \otimes H_*(L_k^{2M-1}(\underline{w})) & & H_*(L_k^{2M-1}(\underline{w})) \\ \downarrow D \otimes D & & \downarrow D \\ H^*(L_k^{2M-1}(\underline{w})) \otimes H^*(L_k^{2M-1}(\underline{w})) & \xrightarrow{\cup} & H^*(L_k^{2M-1}(\underline{w})). \end{array}$$

Let x be a class in $H_{\operatorname{ind}(A)-1}(U)$ such that $i_{U*}(x) \neq 0$ (this exists, as homology and cohomology with field coefficients are dually paired) and similarly let y be a class in $H_{2M-3}(V)$ with $i_{V*}(y) \neq 0$. Since $D(i_{U*}(x))$ is a non-zero class in $H^{\leq 2M-3}(L_k^{2M-1}(\underline{w}))$ and $D(i_{V*}(y))$ is a non-zero multiple of β , we have $D(i_{U*}(x)) \cup D(i_{V*}(y)) \neq 0$. It follows that $i_{U \cap V*}(x \bullet y) \neq 0$, which shows that $\operatorname{ind}(A \cap H) = \operatorname{ind}(U \cap V) \geq \operatorname{ind}(A) - 2$.

(iv) Assume that $\operatorname{ind}(A) + \operatorname{ind}(B) < 2M$, otherwise the inequality is trivial. By continuity, there exist open neighborhoods U of A and V of B such that $\operatorname{ind}(U) = \operatorname{ind}(A)$, $\operatorname{ind}(V) = \operatorname{ind}(B)$, and $\operatorname{ind}(U \cup V) = \operatorname{ind}(A \cup B)$. By the exact cohomology sequence of the pair

$$\dots \rightarrow H^*(L_k^{2M-1}(\underline{w}), U) \xrightarrow{j_U^*} H^*(L_k^{2M-1}(\underline{w})) \xrightarrow{i_U^*} H^*(U) \rightarrow \dots$$

and Lemma 3.3, the index of U is the lowest degree of a non-zero class in the image of j_U^* . A similar statement holds for V . Consider the commutative diagram

$$\begin{array}{ccc} H^*(L_k^{2M-1}(\underline{w}), U) \otimes H^*(L_k^{2M-1}(\underline{w}), V) & \xrightarrow{\cup} & H^*(L_k^{2M-1}(\underline{w}), U \cup V) \\ \downarrow j_U^* \otimes j_V^* & & \downarrow j_{U \cup V}^* \\ H^*(L_k^{2M-1}(\underline{w})) \otimes H^*(L_k^{2M-1}(\underline{w})) & \xrightarrow{\cup} & H^*(L_k^{2M-1}(\underline{w})). \end{array}$$

Assume first that one of the indices is even, for instance that of A . Let x be a class in $H^{\operatorname{ind}(A)}(L_k^{2M-1}(\underline{w}), U)$ such that $j_U^*(x) \neq 0$, and y a class in $H^{\operatorname{ind}(B)}(L_k^{2M-1}(\underline{w}), V)$ such that $j_V^*(y) \neq 0$. Since $j_U^*(x)$ is a non-zero multiple of $\beta^{\operatorname{ind}(A)/2}$ and $\operatorname{ind}(A) + \operatorname{ind}(B) < 2M$, it follows that $j_{U \cup V}^*(x \cup y) = j_U^*(x) \cup j_V^*(y)$ is non-zero and so $\operatorname{ind}(A \cup B) \leq \operatorname{ind}(A) + \operatorname{ind}(B)$. If both $\operatorname{ind}(A)$ and $\operatorname{ind}(B)$ are odd, replace x in the above argument with a class x' in $H^{\operatorname{ind}(A)+1}(L_k^{2M-1}(\underline{w}), U)$ such that $j_U^*(x') \neq 0$, to obtain $\operatorname{ind}(A \cup B) \leq \operatorname{ind}(A) + \operatorname{ind}(B) + 1$.

- (v) The subset $A' = (A *_{\mathbb{Z}_k} B) \setminus B$ deformation retracts to A , and the subset $B' = (A *_{\mathbb{Z}_k} B) \setminus A$ deformation retracts to B . Since $A *_{\mathbb{Z}_k} B = A' \cup B'$, the subadditivity property (iv) implies that $\text{ind}(A *_{\mathbb{Z}_k} B) \leq \text{ind}(A) + \text{ind}(B) + 1$ and $\text{ind}(A *_{\mathbb{Z}_k} B) \leq \text{ind}(A) + \text{ind}(B)$ if at least one of the indices is even. The reverse inequalities are proved in Appendix B.

□

3.11. *Remark.* In the case of real projective spaces the cohomology ring is generated by the generator in degree one, and the above arguments can be adapted to show that properties (iii) and (iv) of Proposition 3.9 hold in the following stronger form:

- (iii') If A is a closed subset of \mathbb{RP}^M , and $A' = A \cap H$ where $H \subset \mathbb{RP}^M$ is a real projective subspace of codimension one, then $\text{ind}(A') \geq \text{ind}(A) - 1$.
 (iv') For closed subsets A and B of \mathbb{RP}^M we have

$$\text{ind}(A \cup B) \leq \text{ind}(A) + \text{ind}(B).$$

Moreover, we also have the following result (see Remark B.23 or [Gi90]):

- (v') (*Join additivity*) For closed subsets A of \mathbb{RP}^M and B of $\mathbb{RP}^{M'}$ we have

$$\text{ind}(A *_{\mathbb{Z}_2} B) = \text{ind}(A) + \text{ind}(B).$$

Analogous properties hold for the cohomological index of subsets of complex projective spaces (see [Th98]). As we will see in Sections 4 and 5, the weaker properties that we have in the case of lens spaces still suffice to define a non-linear Maslov index and recover the applications we are interested in. ◇

3.12. *Remark.* Rafael Gomes and the first author [GG] have shown that for k an odd prime there exist subsets A and B of L_k^{2M-1} such that $\text{ind}(A *_{\mathbb{Z}_k} B) = \text{ind}(A) + \text{ind}(B) + 1$, so join additivity does not hold in general. ◇

We define the *index* of a conical function $F: \mathbb{R}^{2M} \rightarrow \mathbb{R}$ by

$$\text{ind}(F) = \text{ind}(\{f \leq 0\})$$

where f is the function on $L_k^{2M-1}(\underline{w})$ induced by F .

3.13. *Remark.* If Q is a \mathbb{Z}_k -invariant quadratic form on \mathbb{R}^{2M} then $\text{ind}(Q)$ coincides with $i(Q)$, the maximal dimension of a subspace on which Q is negative semi-definite. In particular, \mathbb{Z}_k -invariance implies (if $k > 2$) that in this case $\text{ind}(Q)$ is even. ◇

Given functions f and g on $L_k^{2M-1}(\underline{w})$ and $L_k^{2M'-1}(\underline{w}')$ respectively, we write

$$f \oplus g: L_k^{2(M+M')-1}(\underline{w}, \underline{w}') \rightarrow \mathbb{R}$$

for the function induced by the sum $F \oplus G: \mathbb{R}^{2(M+M')} \rightarrow \mathbb{R}$, where F and G are the conical functions on \mathbb{R}^{2M} and $\mathbb{R}^{2M'}$ associated to f and g .

3.14. **Proposition.** *Let $f: L_k^{2M-1}(\underline{w}) \rightarrow \mathbb{R}$ and $g: L_k^{2M'-1}(\underline{w}') \rightarrow \mathbb{R}$ be continuous functions. Then*

$$\text{ind}(\{f \oplus g \leq 0\}) = \text{ind}(\{f \leq 0\} *_{\mathbb{Z}_k} \{g \leq 0\}).$$

Proof. By continuity (Proposition 3.9(ii)) there is a neighborhood U of $\{f \leq 0\} *_{\mathbb{Z}_k} \{g \leq 0\}$ with $\text{ind}(U) = \text{ind}(\{f \leq 0\} *_{\mathbb{Z}_k} \{g \leq 0\})$. Consider the diagram

$$\begin{array}{ccc}
 \{f \leq 0\} *_{\mathbb{Z}_k} \{g \leq 0\} & \xrightarrow{\quad} & \{f \oplus g \leq 0\} \\
 \downarrow & \nearrow r & \uparrow \\
 U & & \\
 \downarrow & \nearrow j & \\
 L^{2M+2M'-1}(\underline{w}, \underline{w}') & &
 \end{array}$$

By monotonicity, it suffices to show that the inclusion j can be deformed into a map r with image contained in U . This can be done as follows. We work \mathbb{Z}_k -equivariantly on the preimages in $S^{M+M'-1}(\underline{w}, \underline{w}')$. To simplify the formulas, given $x \in S^{2M-1}(\underline{w})$ and $y \in S^{2M'-1}(\underline{w}')$ we write $tx + (1-t)y$ for $(\sqrt{t}x, \sqrt{1-t}y) \in S^{2M+2M'-1}(\underline{w}, \underline{w}')$. Moreover we still write $f: S^{2M-1}(\underline{w}) \rightarrow \mathbb{R}$, $g: S^{2M'-1}(\underline{w}') \rightarrow \mathbb{R}$, and $f \oplus g: S^{2(M+M')-1}(\underline{w}, \underline{w}') \rightarrow \mathbb{R}$ for the composition of the original functions f , g , and $f \oplus g$ with the projections from spheres to lens spaces. With this notation we have

$$(f \oplus g)(tx + (1-t)y) = tf(x) + (1-t)g(y).$$

The rough idea for constructing the map r is as follows. If a point $tx + (1-t)y$ is in $\{f \oplus g \leq 0\}$ then at least one of $f(x)$ and $g(y)$ is non-positive. The map r will act as the identity on points $tx + (1-t)y$ with $f(x)$ and $g(y)$ both non-positive. If $f(x)$ is positive, and thus $g(y)$ is negative, r will move the point $tx + (1-t)y$ to a point $(1-s)(tx + (1-t)y) + sy$, with $s \in [0, 1]$ big enough so that this point is in the chosen neighborhood U of $\{f \leq 0\} *_{\mathbb{Z}_k} \{g \leq 0\}$. Similarly if $g(y)$ is positive. However, one needs to interpolate between these deformations in order to ensure that the resulting map is continuous. Here are the details. For each $\delta > 0$ consider the map

$$R_\delta: \{f \oplus g \leq 0\} \times [0, 1] \rightarrow \{f \oplus g \leq 0\}$$

defined by the expression

$$R_\delta(tx + (1-t)y, s) = \begin{cases} (1-s)(tx + (1-t)y) + sy & \text{if } f(x) \geq \delta \\ (1 - s \frac{f(x)}{\delta})(tx + (1-t)y) + s \frac{f(x)}{\delta} y & \text{if } 0 \leq f(x) \leq \delta \\ tx + (1-t)y & \text{if } f(x) \leq 0 \text{ and } g(y) \leq 0 \\ (1 - s \frac{g(y)}{\delta})(tx + (1-t)y) + s \frac{g(y)}{\delta} x & \text{if } 0 \leq g(y) \leq \delta \\ (1-s)(tx + (1-t)y) + sx & \text{if } g(y) \geq \delta. \end{cases}$$

Thus R_δ moves a point $tx + (1-t)y$ such that $f(x) > 0$ along the segment

$$s \mapsto (1-s)(tx + (1-t)y) + sy$$

a portion of the way towards y (note that $g(y)$ must be negative). To ensure continuity, the portion depends on the value of f at x . The pasting lemma guarantees the continuity of R_δ . By continuity of $(t, x, y) \mapsto (tx + (1-t)y)$ and compactness of its domain, for δ small enough the set $\{f \oplus g \leq 0\} \cap \{tx + (1-t)y \mid f(x) \leq \delta \text{ or } g(y) \leq \delta\}$ is contained in U . For such δ , the image of $R_\delta(\cdot, 1)$ is contained in the preimage of U in $S^{2(M+M')-1}(\underline{w}, \underline{w}')$. We take r to be the map induced by $R_\delta(\cdot, 1)$ on the \mathbb{Z}_k -orbits. \square

Proposition 3.14 and Proposition 3.9 (v) imply the following result.

3.15. Corollary. *For conical functions $F: \mathbb{R}^{2M} \rightarrow \mathbb{R}$ and $G: \mathbb{R}^{2M'} \rightarrow \mathbb{R}$ we have*

$$|\text{ind}(F \oplus G) - \text{ind}(F) - \text{ind}(G)| \leq 1.$$

Moreover, if either F or G has even index, in particular if either F or G is a \mathbb{Z}_k -invariant quadratic form (for $k > 2$), then

$$\text{ind}(F \oplus G) = \text{ind}(F) + \text{ind}(G).$$

4. THE NON-LINEAR MASLOV INDEX

Using the construction of generating functions given in Section 2 and the definition and properties of the cohomological index discussed in Section 3, we now define the non-linear Maslov index $\mu: \widetilde{\text{Cont}}_0(L_k^{2n-1}) \rightarrow \mathbb{Z}$ on the universal cover of the identity component of the contactomorphism group of L_k^{2n-1} , and describe the properties that are used in the applications.

Definition and quasimorphism property. As before, L_k^{2n-1} denotes a lens space with any vector of weights. The *non-linear Maslov index* of a based contact isotopy $\{\phi_t\}_{t \in [0,1]}$ of L_k^{2n-1} is defined by

$$\mu(\{\phi_t\}) = \text{ind}(F_0) - \text{ind}(F_1)$$

where $F_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ is a based family of conical generating functions for $\{\phi_t\}$. Existence of such a family is given by Proposition 2.14; by Proposition 2.19 and Corollary 3.15, μ does not depend on the choice. Moreover, Proposition 2.20 implies that $\mu(\{\phi_t\})$ only depends on the smooth homotopy class of $\{\phi_t\}$ with fixed endpoints, and thus μ descends to a map

$$\mu: \widetilde{\text{Cont}}_0(L_k^{2n-1}) \rightarrow \mathbb{Z}.$$

4.1. *Example.* We have $\mu(\{\text{id}\}_{t \in [0,1]}) = 0$. Note also that if a based contact isotopy $\{\phi_t\}_{t \in [0,1]}$ has a family of generating functions with no fibre variable then $0 \leq \mu(\{\phi_t\}) \leq 2n$; by Proposition 2.22(i), if moreover $\{\phi_t\}$ is positive then $\mu(\{\phi_t\}) = 2n$, and if it is non-positive then $\mu(\{\phi_t\}) = 0$. \square

We now prove that the non-linear Maslov index is a quasimorphism (in the case of projective spaces, see also [BeS07] and [Gi90, Theorem 9.1]). We start with the following lemma.

4.2. **Lemma.** *If two conical generating functions F and G are equivalent to the zero function then*

$$\text{ind}(F \# G) = \text{ind}(F) + \text{ind}(G).$$

Proof. By Remark 2.12, there are stabilizations $F \oplus P_F$ and $G \oplus P_G$ of F and G that are composition by fibre preserving conical homeomorphisms of stabilizations $0 \oplus Q_F$ and $0 \oplus Q_G$ of the zero function. Using Corollary 3.15 we thus have

$$(4.3) \quad \text{ind}(F) + \text{ind}(P_F) = \text{ind}(F \oplus P_F) = \text{ind}(0 \oplus Q_F) = 2n + \text{ind}(Q_F)$$

and similarly for G ; moreover, since $(F \oplus P_F) \# (G \oplus P_G)$ and $(F \# G) \oplus P_F \oplus P_G$ just differ by a permutation of the homogeneous coordinates we have

$$(4.4) \quad \text{ind}((F \oplus P_F) \# (G \oplus P_G)) = \text{ind}(F \# G) + \text{ind}(P_F) + \text{ind}(P_G).$$

By Remark 2.11, $(F \oplus P_F) \# (G \oplus P_G)$ differs from $(0 \oplus Q_F) \# (0 \oplus Q_G)$ by a fibre preserving conical homeomorphism, and so

$$(4.5) \quad \text{ind}((F \oplus P_F) \# (G \oplus P_G)) = \text{ind}((0 \oplus Q_F) \# (0 \oplus Q_G)).$$

Since

$$(4.6) \quad (0 \oplus Q_F) \# (0 \oplus Q_G) = (0 \# 0) \oplus Q_F \oplus Q_G,$$

we have

$$(4.7) \quad \text{ind}((0 \oplus Q_F) \# (0 \oplus Q_G)) = 4n + \text{ind}(Q_F) + \text{ind}(Q_G).$$

By (4.4), (4.5), (4.7) and (4.3) we thus have $\text{ind}(F \# G) = \text{ind}(F) + \text{ind}(G)$. \square

4.8. **Proposition** (Quasimorphism property). *For elements $[\{\phi_t\}]$ and $[\{\psi_t\}]$ of $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ we have*

$$|\mu([\{\phi_t\}] \cdot [\{\psi_t\}]) - \mu([\{\phi_t\}]) - \mu([\{\psi_t\}])| \leq 2n + 1.$$

Proof. By Proposition 2.9 and Remark 2.13, if $F_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \rightarrow \mathbb{R}$ and $G_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R}$ are based families of conical generating functions for $\{\phi_t\}$ and $\{\psi_t\}$ respectively then

$$G_t \# F_t: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$$

is a based family of conical generating functions for $\{\phi_t \circ \psi_t\}$. Since $[\{\phi_t\}] \cdot [\{\psi_t\}] = [\{\phi_t \circ \psi_t\}]$ we have $\mu([\{\phi_t\}] \cdot [\{\psi_t\}]) = \text{ind}(G_0 \# F_0) - \text{ind}(G_1 \# F_1)$ and thus

$$\begin{aligned} & \left| \mu([\{\phi_t\}] \cdot [\{\psi_t\}]) - \mu([\{\phi_t\}]) - \mu([\{\psi_t\}]) \right| \\ &= \left| \text{ind}(G_0 \# F_0) - \text{ind}(G_1 \# F_1) - \text{ind}(F_0) + \text{ind}(F_1) - \text{ind}(G_0) + \text{ind}(G_1) \right| \\ &\leq \left| \text{ind}(G_0 \# F_0) - \text{ind}(G_0) - \text{ind}(F_0) \right| + \left| \text{ind}(G_1 \# F_1) - \text{ind}(G_1) - \text{ind}(F_1) \right| \\ &= \left| \text{ind}(G_1 \# F_1) - \text{ind}(G_1) - \text{ind}(F_1) \right| \end{aligned}$$

where the last equality follows from Lemma 4.2. By Proposition 2.25, $G_1 \# F_1$ coincides with $G_1 \oplus F_1$ in codimension $2n$. Therefore, using the Lefschetz property from Proposition 3.9 we get

$$\left| \text{ind}(G_1 \# F_1) - \text{ind}(G_1 \oplus F_1) \right| \leq 2n.$$

On the other hand, by Corollary 3.15 we have

$$\left| \text{ind}(G_1 \oplus F_1) - \text{ind}(G_1) - \text{ind}(F_1) \right| \leq 1.$$

Thus we obtain

$$\left| \text{ind}(G_1 \# F_1) - \text{ind}(G_1) - \text{ind}(F_1) \right| \leq 2n + 1$$

and so

$$\left| \mu([\{\phi_t\}] \cdot [\{\psi_t\}]) - \mu([\{\phi_t\}]) - \mu([\{\psi_t\}]) \right| \leq 2n + 1.$$

□

Recall that a quasimorphism $\nu: G \rightarrow \mathbb{R}$ is said to be *homogeneous* if $\nu(x^m) = m\nu(x)$ for all $x \in G$ and $m \in \mathbb{Z}$. Any quasimorphism $\nu: G \rightarrow \mathbb{R}$ has an associated homogeneous quasimorphism, defined by

$$\bar{\nu}(x) = \lim_{m \rightarrow \infty} \frac{\nu(x^m)}{m}$$

(see for instance [Ca09, Section 2.2.2]). The homogeneous quasimorphism $\bar{\mu}: \widetilde{\text{Cont}}_0(L_k^{2n-1}) \rightarrow \mathbb{R}$ associated to the non-linear Maslov index is called the *asymptotic non-linear Maslov index*¹¹.

The linear case. We now show that if the lift to \mathbb{R}^{2n} of a based contact isotopy $\{\phi_t\}$ of L_k^{2n-1} is a loop $\{\Phi_t\}$ in $\text{Sp}(2n; \mathbb{R})$ then $\mu(\{\phi_t\})$ is equal to the linear Maslov index of $\{\Phi_t\}$. If $\{\Phi_t\}$ is a path in $\text{Sp}(2n; \mathbb{R})$ starting at the identity then the construction of Section 2 gives a family $Q_t: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ of generating functions so that each Q_t is a \mathbb{Z}_k -invariant quadratic form. As in [Th95, Th99], we define (cf. Remark 3.13)

$$\nu(\{\Phi_t\}) = \text{ind}(Q_0) - \text{ind}(Q_1) = i(Q_0) - i(Q_1)$$

where i denotes the maximal dimension of a subspace on which a quadratic form is negative semi-definite. By the arguments in Section 2, the integer $\nu(\{\Phi_t\})$ is well defined and depends only on the homotopy class (with fixed endpoints) of the path $\{\Phi_t\}$. Moreover, if $\{\Phi_t\}$ is the lift of a contact isotopy $\{\phi_t\}$ of L_k^{2n-1} then $\nu(\{\Phi_t\}) = \mu(\{\phi_t\})$.

4.9. Proposition. *The induced map $\nu: \pi_1(\text{Sp}(2n; \mathbb{R})) \rightarrow \mathbb{Z}$ is a group homomorphism, and agrees with the linear Maslov index.*

The proof of this result is based on the following lemma, which is taken from [Th95, Proposition 35] and whose equivariant version is also used in Section 5 to prove the contact Arnold conjecture.

¹¹In [Gi90, Section 9] the asymptotic non-linear Maslov index of a contact isotopy $\{\phi_t\}_{t \in [0, \infty)}$ starting at the identity is defined as $\bar{\mu}_{\text{Giv}}(\{\phi_t\}_{t \in [0, \infty)}) = \lim_{T \rightarrow \infty} \frac{\mu(\{\phi_t\}_{t \in [0, T]})}{T}$. Given a contact isotopy $\{\phi_t\}_{t \in [0, 1]}$ we can extend it to a contact isotopy defined for $t \in [0, \infty)$ by posing, for $t = l + s$ with $l \in \mathbb{N}$ and $s \in (0, 1)$, $\phi_t = \phi_s \circ (\phi_1)^l$. Then $\bar{\mu}_{\text{Giv}}(\{\phi_t\}_{t \in [0, \infty)}) = \bar{\mu}(\{\phi_t\}_{t \in [0, 1]})$.

4.10. Lemma. *If $Q: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ is a quadratic form generating the identity then there is an isotopy $\{\Psi_s\}_{s \in [0,1]}$ of fibre preserving linear diffeomorphisms of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ such that Ψ_0 is the identity and $Q \circ \Psi_1$ is a quadratic form that only depends on the fibre variable. Moreover, if Q is \mathbb{Z}_k -invariant then $\{\Psi_s\}_{s \in [0,1]}$ can be chosen to be \mathbb{Z}_k -equivariant.*

Proof. Write $Q(z) = \frac{1}{2} \langle z, Bz \rangle$ for a symmetric matrix $B = \begin{bmatrix} a & b \\ b^T & c \end{bmatrix}$. Since Q generates the zero section of $T^*\mathbb{R}^{2n}$ we have that c is invertible and $a - b c^{-1} b^T = 0$. Then

$$\Psi_s(\zeta, \nu) = (\zeta, \nu - s c^{-1} b^T \zeta)$$

is an isotopy of fibre preserving linear diffeomorphisms of $\mathbb{R}^{2n} \times \mathbb{R}^{2nN}$ such that $Q \circ \Psi_1$ only depends on the fibre variable, as $Q \circ \Psi_1(\zeta, \nu) = \frac{1}{2} \nu^T c \nu$. If Q is \mathbb{Z}_k -invariant then $\{\Psi_s\}$ is \mathbb{Z}_k -equivariant. \square

Proof of Proposition 4.9. Let $\{\Phi_t^{(1)}\}$ and $\{\Phi_t^{(2)}\}$ be loops in $\mathrm{Sp}(2n; \mathbb{R})$, based at the identity. If $Q_t^{(1)}$ and $Q_t^{(2)}$ are based families of generating quadratic forms for $\{\Phi_t^{(1)}\}$ and $\{\Phi_t^{(2)}\}$ respectively then $Q_t^{(1)} \# Q_t^{(2)}$ is a based family of generating quadratic forms for $\{\Phi_t^{(2)} \circ \Phi_t^{(1)}\}$, and so

$$(4.11) \quad \nu([\{\Phi_t^{(1)}\}] \cdot [\{\Phi_t^{(2)}\}]) = \mathrm{ind}(Q_0^{(1)} \# Q_0^{(2)}) - \mathrm{ind}(Q_1^{(1)} \# Q_1^{(2)}).$$

By Lemma 4.2,

$$(4.12) \quad \mathrm{ind}(Q_0^{(1)} \# Q_0^{(2)}) = \mathrm{ind}(Q_0^{(1)}) + \mathrm{ind}(Q_0^{(2)}).$$

By Lemma 4.10, for $j = 1, 2$ there is an isotopy $\{\Psi_s^{(j)}\}_{s \in [0,1]}$ of fibre preserving linear diffeomorphisms such that $\Psi_0^{(j)}$ is the identity and $Q_1^{(j)} \circ \Psi_1^{(j)}$ is a quadratic form that does not depend on the base variable, and so is equal to a quadratic form $\underline{Q}_1^{(j)}$ on the fibre. Using (4.6) we therefore obtain

$$(Q_1^{(1)} \circ \Psi_1^{(1)}) \# (Q_1^{(2)} \circ \Psi_1^{(2)}) = (0 \# 0) \oplus \underline{Q}_1^{(1)} \oplus \underline{Q}_1^{(2)},$$

and so

$$\mathrm{ind}(Q_1^{(1)} \# Q_1^{(2)}) = 4n + \mathrm{ind}(\underline{Q}_1^{(1)}) + \mathrm{ind}(\underline{Q}_1^{(2)}) = \mathrm{ind}(Q_1^{(1)}) + \mathrm{ind}(Q_1^{(2)}).$$

Together with (4.11) and (4.12) this gives

$$\nu([\{\Phi_t^{(1)}\}] \cdot [\{\Phi_t^{(2)}\}]) = \nu([\{\Phi_t^{(1)}\}]) + \nu([\{\Phi_t^{(2)}\}]),$$

and thus $\nu: \pi_1(\mathrm{Sp}(2n; \mathbb{R})) \rightarrow \mathbb{Z}$ is a homomorphism.

In order to show that ν agrees with the linear Maslov index, it now suffices to check that it takes the value 2 on the standard loop $t \mapsto \Phi_t = e^{2\pi i t} \in \mathrm{Sp}(2; \mathbb{R})$. For $0 \leq t \leq \frac{1}{3}$ we have the family of generating quadratic forms $Q_t(z) = \frac{\sin(2\pi t)}{1 + \cos(2\pi t)} \langle z, z \rangle$. Applying the composition formula twice we obtain a family of generating quadratic forms on \mathbb{R}^{10} determined by the family of 10×10 symmetric matrices

$$A_t = \begin{bmatrix} 0 & J & -J & 0 & 0 \\ -J & 0 & J & J & -J \\ J & -J & \lambda_t & 0 & 0 \\ 0 & -J & 0 & \lambda_t & J \\ 0 & J & 0 & -J & \lambda_t \end{bmatrix}, \quad \text{with } \lambda_t = \frac{\sin\left(\frac{2\pi t}{3}\right)}{1 + \cos\left(\frac{2\pi t}{3}\right)} \mathrm{Id}_{2 \times 2}$$

where we denote by J the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. It follows that

$$\nu(\{\Phi_t\}) = i(A_0) - i(A_1) = 6 - 4 = 2$$

as required. \square

We can now prove the formula at the beginning of Theorem 1.2.

4.13. *Example.* Recall that we denote by $\{r_t\}$ the Reeb flow on L_k^{2n-1} with respect to the contact form whose pullback to S^{2n-1} is equal to the pullback from \mathbb{R}^{2n} of the 1-form $\sum_{j=1}^n (x_j dy_j - y_j dx_j)$. Since the lift to \mathbb{R}^{2n} of $\{r_{2\pi t}\}_{t \in [0,1]}$ is a loop in $\text{Sp}(2n; \mathbb{R})$ of linear Maslov index $2n$, it follows from Proposition 4.9 that

$$\mu(\{r_{2\pi t}\}_{t \in [0,1]}) = \bar{\mu}(\{r_{2\pi t}\}_{t \in [0,1]}) = 2nl$$

for every integer l . □

Relation with discriminant points. We now show that the way the non-linear Maslov index of a contact isotopy changes for t varying in a subinterval of $[0, 1]$ is related to the changes in the topology of the set of discriminant points. Before stating the results we recall the following fact.

4.14. **Lemma.** *Let V be a compact manifold and $f_t: V \rightarrow \mathbb{R}$, for $t \in [0, 1]$, a family of functions such that the total map $f: V \times [0, 1] \rightarrow \mathbb{R}$ is \mathcal{C}^1 with Lipschitz differential. Suppose that $a \in \mathbb{R}$ is a regular value of f_t for every $t \in [0, 1]$. Then there is an isotopy θ_t of V such that $\theta_t(\{f_0 \leq a\}) = \{f_t \leq a\}$.*

Proof. Consider the open subset

$$\mathcal{U} := \{(x, t) \mid df_t|_x \neq 0\}$$

of $V \times [0, 1]$. By assumption, the set $\{(x, t) \mid f_t(x) = a\}$ is contained in \mathcal{U} . Fix a Riemannian metric on V . Since the functions f_t are \mathcal{C}^1 with Lipschitz differential, their gradient flow is well defined and enjoys the usual properties. Note also that the gradient ∇f_t is non-zero at a point x exactly when $(x, t) \in \mathcal{U}$. For every $t \in [0, 1]$ define a vector field u_t on $\mathcal{U}_t := \{x \in V \mid df_t|_x \neq 0\}$ by $u_t = \nabla f_t / \|\nabla f_t\|^2$. Then $df_t(u_t) \equiv 1$ on \mathcal{U} . Take $\varepsilon > 0$ small enough so that the closed neighborhood

$$\mathcal{W} := \{(x, t) \in V \times [0, 1] \mid |f(x, t) - a| \leq \varepsilon\}$$

of $\{(x, t) \mid f_t(x) = a\}$ is contained in \mathcal{U} . Let $\rho: V \times [0, 1] \rightarrow \mathbb{R}$ be a smooth function that is supported in \mathcal{U} and is equal to 1 on \mathcal{W} , and consider the time-dependent vector field $\{X_t\}_{t \in [0, 1]}$ on V that is given by

$$X_t(x) = -\rho(x, t) \dot{f}_t(x) u_t(x)$$

for (x, t) in \mathcal{U} and that vanishes for (x, t) outside of \mathcal{U} . Its flow is an isotopy θ_t of V with the required properties. Indeed

$$\frac{d}{dt} f_t(\theta_t(x)) = \dot{f}_t(\theta_t(x)) + df_t(X_t)(\theta_t(x)) = (1 - \rho(\theta_t(x), t)) \dot{f}_t(\theta_t(x))$$

thus $\frac{d}{dt} f_t(\theta_t(x)) = 0$ if $(\theta_t(x), t) \in \mathcal{W}$, and so each θ_t sends $\{f_0 = a\}$ onto $\{f_t = a\}$. Since θ_0 is the identity, by continuity the isotopy θ_t sends $\{f_0 \leq a\}$ onto $\{f_t \leq a\}$ for all t . □

We can now prove that the non-linear Maslov index detects discriminant points, as described in Theorem 1.2(iii). More precisely we show the following result.

4.15. **Proposition** (Relation with discriminant points). *Let $\{\phi_t\}_{t \in [0, 1]}$ be a based contact isotopy of L_k^{2n-1} , and let $[t_0, t_1]$ be a subinterval of $[0, 1]$.*

(i) *If there are no values of $t \in [t_0, t_1]$ for which ϕ_t belongs to the discriminant then*

$$\mu([\{\phi_t\}_{t \in [0, t_0]}]) = \mu([\{\phi_t\}_{t \in [0, t_1]}])$$

(ii) *If \underline{t} is the only value of $t \in [t_0, t_1]$ for which ϕ_t belongs to the discriminant then*

$$|\mu([\{\phi_t\}_{t \in [0, t_1]}]) - \mu([\{\phi_t\}_{t \in [0, t_0]}])| \leq \text{ind}(\Delta(\phi_{\underline{t}})) + 1$$

where $\Delta(\phi_{\underline{t}}) \subset L_k^{2n-1}$ is the set of discriminant points of $\phi_{\underline{t}}$. Consequently,

$$|\mu([\{\phi_t\}_{t \in [0, t_1]}]) - \mu([\{\phi_t\}_{t \in [0, t_0]}])| \leq 2n + 1$$

and if $\phi_{\underline{t}}$ has only finitely many discriminant points then

$$|\mu([\{\phi_t\}_{t \in [0, t_1]}]) - \mu([\{\phi_t\}_{t \in [0, t_0]}])| \leq 2.$$

(iii) If \underline{t} is the only value of $t \in [t_0, t_1]$ for which ϕ_t belongs to the discriminant, and moreover all discriminant points of $\phi_{\underline{t}}$ are non-degenerate, then

$$|\mu([\{\phi_t\}_{t \in [0, t_1]}) - \mu([\{\phi_t\}_{t \in [0, t_0]})| \leq 1.$$

Proof. Let $f_t: L_k^{2M-1} \rightarrow \mathbb{R}$ be a based family of generating functions for ϕ_t . If there are no values of $t \in [t_0, t_1]$ for which ϕ_t belongs to the discriminant then, by Proposition 2.21, zero is a regular value of f_t for all $t \in [t_0, t_1]$. Hence, (i) follows from Lemma 4.14.

Suppose now that \underline{t} is the unique value of $t \in [t_0, t_1]$ for which ϕ_t belongs to the discriminant. For any $\epsilon > 0$ we have

$$(4.16) \quad |\mu([\{\phi_t\}_{t \in [0, t_1]}) - \mu([\{\phi_t\}_{t \in [0, t_0]})| \leq \text{ind}(\{f_{\underline{t}} \leq \epsilon\}) - \text{ind}(\{f_{\underline{t}} \leq -\epsilon\}).$$

This is a consequence of (i), monotonicity of the index, and the fact that, since $f: L_k^{2M-1} \times [0, 1] \rightarrow \mathbb{R}$ is continuous, for every $\epsilon > 0$ and $a \in \mathbb{R}$ there exists $\delta > 0$ such that for all t, t' with $|t - t'| < \delta$ we have $\{f_t(x) \leq a\} \subset \{f_{t'}(x) \leq a + \epsilon\}$.

Let C be the set of critical points of $f_{\underline{t}}$ with critical value 0. By continuity of the index, there is an open subset W in L_k^{2M-1} that contains C and has the same index. For sufficiently small $\epsilon > 0$ we have

$$(4.17) \quad \text{ind}(\{f_{\underline{t}} \leq \epsilon\}) \leq \text{ind}(\{f_{\underline{t}} \leq -\epsilon\} \cup W).$$

This follows from monotonicity of the index and the fact that, as we now explain, $\{f_{\underline{t}} \leq \epsilon\}$ deformation retracts into $\{f_{\underline{t}} \leq -\epsilon\} \cup W$ (cf. [Vi97, p. 548]). Pick $\delta > 0$ such that if $df_{\underline{t}}(x) = 0$ and $|f_{\underline{t}}(x)| \leq \delta$ then $x \in W$. Consider the disjoint closed sets

$$V_0 = \{x \in L_k^{2M-1} \mid df_{\underline{t}}(x) = 0 \text{ or } |f_{\underline{t}}(x)| \geq 2\delta\}$$

and

$$V_1 = f_{\underline{t}}^{-1}([- \delta, \delta]) \setminus W,$$

and let $\rho: L_k^{2M-1} \rightarrow [0, 1]$ be a smooth function that vanishes in a neighborhood of V_0 and is constant equal to 1 in a neighborhood of V_1 .

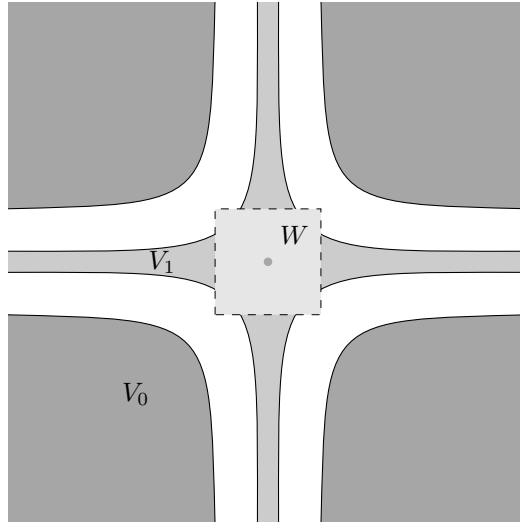


FIGURE 1. The sets V_0 , V_1 and W near a non-degenerate critical point of index one on a surface.

Fix a metric on L_k^{2M-1} and consider the vector field $X = -\rho \frac{\nabla f_{\underline{t}}}{\|\nabla f_{\underline{t}}\|^2}$. Writing θ_t for the flow of X we have

$$(4.18) \quad \frac{d}{dt} f_{\underline{t}}(\theta_t(x)) = -\rho(\theta_t(x)).$$

Let $m = \max \{ \|X(x)\| \mid x \in L_k^{2M-1} \}$ and $d = \text{dist} \left(\overline{\{\rho(x) < 1\}} \cap f_{\underline{t}}^{-1}([- \delta, \delta]), W^c \right)$. Note that $d > 0$. For $\epsilon < \min\{\delta, \frac{d}{2m}\}$ we now prove that

$$(4.19) \quad \theta_{2\epsilon}(\{f_{\underline{t}} \leq \epsilon\}) \subset \{f_{\underline{t}} \leq -\epsilon\} \cup W.$$

Given $x \in \{f_{\underline{t}} \leq \epsilon\}$ set

$$s(x) = \inf \{ t \in [0, 2\epsilon] \mid \rho(\theta_t(x)) < 1 \text{ or } t = 2\epsilon \}.$$

If $s(x) = 2\epsilon$ then $\rho(\theta_t(x)) = 1$ for all $t \in [0, 2\epsilon]$ and, by (4.18), $f_{\underline{t}}(\theta_{2\epsilon}(x)) = f_{\underline{t}}(x) - 2\epsilon \leq -\epsilon$. If $s(x) < 2\epsilon$ then $\theta_{s(x)}(x) \in \{\rho < 1\} \cap f_{\underline{t}}^{-1}([- \delta, \delta])$ (as $\epsilon < \delta$). Then we must have $\text{dist}(\theta_{s(x)}, W^c) \geq d$ and our bound on ϵ ensures that the path $\{\theta_t(x) \mid t \in [s(x), 2\epsilon]\}$ is entirely contained in W . In particular, $\theta_{2\epsilon}(x) \in W$. This completes the proof of (4.19) and hence of (4.17).

By (4.17), subadditivity of the cohomological index and Proposition 2.21 we have

$$\begin{aligned} \text{ind}(\{f_{\underline{t}} \leq \epsilon\}) &\leq \text{ind}(\{f_{\underline{t}} \leq -\epsilon\} \cup W) \leq \text{ind}(\{f_{\underline{t}} \leq -\epsilon\}) + \text{ind}(W) + 1 \\ &= \text{ind}(\{f_{\underline{t}} \leq -\epsilon\}) + \text{ind}(C) + 1 \\ &= \text{ind}(\{f_{\underline{t}} \leq -\epsilon\}) + \text{ind}(\Delta(\phi_{\underline{t}})) + 1. \end{aligned}$$

Together with (4.16), this implies (ii).

As for (iii), if all discriminant points of $\phi_{\underline{t}}$ are non-degenerate then (by Proposition 2.21) all critical points of $f_{\underline{t}}$ of critical value zero are non-degenerate. Thus $f_{\underline{t}}$ has only finitely many critical points with critical value zero, and zero is an isolated critical value of $f_{\underline{t}}$. We can choose W so that $\{f_{\underline{t}} \leq -\epsilon\} \cup W$ can be obtained from $\{f_{\underline{t}} \leq -\epsilon\}$ by attaching a finite number of disjoint handles. If H is a handle and $A \subset L_k^{2M-1}$ then $\text{ind}(A \cup H) \leq \text{ind}(A) + 1$ (as the sum of the Betti numbers of $A \cup H$ is at most one more than the sum of Betti numbers of A) and, unless the index of the handle H is equal to $\text{ind}(A) + 1$, we have $\text{ind}(A \cup H) = \text{ind}(A)$. Attaching the handles in W sequentially, starting with those of highest index, we therefore obtain

$$\text{ind}(\{f_{\underline{t}} \leq -\epsilon\} \cup W) \leq \text{ind}(\{f_{\underline{t}} \leq -\epsilon\}) + 1,$$

which, together with (4.16) and (4.17), concludes the proof of (iii). \square

4.20. Remark. In order to prove the version of the contact Arnold conjecture with the bound given by the Lusternik–Schnirelmann category (cf. Section 5) we would need to know that the conclusion of Proposition 4.15(iii) holds also in the degenerate case. Using Massey products similarly to [Vi97] it is possible to prove (at least if $k = 3$) that this is the case if, in the notation of the proof above, $\text{ind}(\{f_{\underline{t}} \leq \epsilon\}) = 1$. It is not clear to us whether such arguments can be pushed further to improve this result. \diamond

Further properties. We now prove the positivity property from Theorem 1.2, and the fact that the asymptotic non-linear Maslov index is monotone and has the vanishing property.

4.21. Proposition (Positivity). *If $\{\phi_t\}$ is a non-negative (respectively non-positive) contact isotopy then $\mu([\{\phi_t\}]) \geq 0$ (respectively $\mu([\{\phi_t\}]) \leq 0$). Moreover, if $\{\phi_t\}$ is positive then $\mu([\{\phi_t\}]) > 0$.*

Proof. It follows from monotonicity of generating functions (Proposition 2.22) and monotonicity of the cohomological index (Proposition 3.9(i)) that if $\{\phi_t\}$ is non-negative (respectively non-positive) then $\mu([\{\phi_t\}]) \geq 0$ (respectively $\mu([\{\phi_t\}]) \leq 0$). By Example 4.1, if $\{\phi_t\}$ is a small positive contact isotopy then $\mu([\{\phi_t\}]) = 2n > 0$. Since (by Proposition 2.22 and Proposition 3.9(i)) the cohomological index does not decrease along a positive contact isotopy, we conclude that $\mu([\{\phi_t\}]) > 0$ for any positive contact isotopy $\{\phi_t\}$. \square

We now show that the asymptotic non-linear Maslov index satisfies the following stronger property. Recall from [EP00] that for a contact manifold (V, ξ) the relation \leq on $\widetilde{\text{Cont}}_0(V, \xi)$ is defined by posing $[\{\phi_t\}] \leq [\{\psi_t\}]$ if $[\{\psi_t\}] \cdot [\{\phi_t\}]^{-1}$ can be represented by a non-negative contact isotopy.

As in [BZ15] we say that a quasimorphism ν on $\widetilde{\text{Cont}}_0(V, \xi)$ is *monotone* if $\nu([\{\phi_t\}]) \leq \nu([\{\psi_t\}])$ whenever $[\{\phi_t\}] \leq [\{\psi_t\}]$. The proof of the following result is a direct imitation of the proof of the similar statement for real projective spaces that is given in [BZ15].

4.22. Proposition. *The asymptotic non-linear Maslov index $\bar{\mu}$ on $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ is monotone.*

Proof. Suppose that $[\{\phi_t\}] \leq [\{\psi_t\}]$. Since the set of non-negative elements of $\widetilde{\text{Cont}}_0(L_k^{2n-1})$ is conjugation invariant and closed under multiplication, $[\{\phi_t\}]^m \leq [\{\psi_t\}]^m$ for any $m \in \mathbb{Z}_{>0}$. By Proposition 4.21 we thus have $\mu([\{\psi_t\}]^m \cdot [\{\phi_t\}]^{-m}) \geq 0$, and so

$$\begin{aligned} \mu([\{\psi_t\}]^m) - \mu([\{\phi_t\}]^m) &= \mu([\{\psi_t\}]^m) + \mu([\{\phi_t\}]^{-m}) \\ &\geq \mu([\{\psi_t\}]^m \cdot [\{\phi_t\}]^{-m}) - D \geq -D, \end{aligned}$$

where D is the error of the quasimorphism. Dividing by m and taking the limit when $m \rightarrow \infty$ we obtain that $\bar{\mu}([\{\phi_t\}]) \leq \bar{\mu}([\{\psi_t\}])$. \square

For the next result we also follow [BZ15]. Recall that a subset \mathcal{U} of a contact manifold (V, ξ) is said to be *displaceable* if there exists a contactomorphism ψ contact isotopic to the identity such that $\bar{\mathcal{U}} \cap \psi(\mathcal{U}) = \emptyset$. A quasimorphism ν on $\widetilde{\text{Cont}}_0(V, \xi)$ is said to have the *vanishing property* if $\nu([\{\phi_t\}]) = 0$ for any contact isotopy $\{\phi_t\}_{t \in [0,1]}$ that is supported in $[0,1] \times \mathcal{U}$ for a displaceable set \mathcal{U} .

4.23. Proposition. *The asymptotic non-linear Maslov index has the vanishing property.*

Proof. Suppose that a subset \mathcal{U} of L_k^{2n-1} is displaceable by a contactomorphism ψ contact isotopic to the identity. After taking a \mathcal{C}^0 -perturbation, we can assume that ψ has no discriminant points. Let $\{\psi_t\}_{t \in [0,1]}$ be a contact isotopy from the identity to $\psi_1 = \psi$, and $\{\phi_t\}_{t \in [0,1]}$ a contact isotopy supported in $[0,1] \times \mathcal{U}$. We need to show that $\bar{\mu}([\{\phi_t\}]) = 0$. Observe first that, for every $m \in \mathbb{N}$ and $t \in [0,1]$, the contactomorphism $\psi \circ \phi_t^m$ has no discriminant points. Indeed, assume by contradiction that p is a discriminant point of $\psi \circ \phi_t^m$. Since ϕ_t^m is supported in \mathcal{U} and ψ has no discriminant points we must have $p \in \mathcal{U}$. But then $\phi_t^m(p) \in \mathcal{U}$ and so $\psi \circ \phi_t^m(p) \in \mathcal{U} \cap \psi(\mathcal{U})$ contradicting the hypothesis that ψ displaces \mathcal{U} . Since $\psi \circ \phi_t^m$ has no discriminant points for all $t \in [0,1]$, it follows from Proposition 4.15(i) that for the concatenation $\{\psi_t\} \sqcup \{\psi \circ \phi_t^m\}$ we have

$$\mu(\{\psi_t\} \sqcup \{\psi \circ \phi_t^m\}) = \mu(\{\psi_t\}).$$

Since $\{\psi_t\} \sqcup \{\psi \circ \phi_t^m\}$ and $\{\psi_t \circ \phi_t^m\}$ are homotopic, the quasimorphism property (Proposition 4.8) implies that $|\mu(\{\phi_t^m\})| \leq 2n+1$. As this holds for every $m \in \mathbb{N}$, we conclude that $\bar{\mu}([\{\phi_t\}]) = 0$. \square

5. APPLICATIONS

In this section we use the properties of the non-linear Maslov index to prove the applications listed in Corollaries 1.3, 1.5 and 1.6. Most of the arguments are taken from [EP00, CS12, Sa11c, BZ15] with only minor changes to adapt them to the case of lens spaces, and are included here for the sake of completeness.

Orderability. As in the case of projective spaces discussed in [EP00], orderability of lens spaces follows from positivity of the non-linear Maslov index and the fact that the non-linear Maslov index is well defined on the universal cover of the contactomorphism group. Indeed, suppose by contradiction that a lens space L_k^{2n-1} is not orderable, and thus admits a positive contractible loop $\{\phi_t\}_{t \in [0,1]}$. Since $\{\phi_t\}_{t \in [0,1]}$ is contractible we have that $\mu([\{\phi_t\}_{t \in [0,1]}]) = 0$. On the other hand, since $\{\phi_t\}$ is positive, Proposition 4.21 implies that $\mu([\{\phi_t\}]) > 0$, giving a contradiction.

Unboundedness of the discriminant and oscillation metrics. Recall that any bi-invariant (pseudo)metric $d : G \times G \rightarrow \mathbb{R}$ on a group G defines a conjugation-invariant (pseudo)norm $\|\cdot\| : G \rightarrow \mathbb{R}$ by posing $\|g\| = d(g, \text{id})$; conversely, any conjugation-invariant (pseudo)norm $\|\cdot\| : G \rightarrow \mathbb{R}$ defines a bi-invariant (pseudo)metric by posing $d(g_1, g_2) = \|g_1 g_2^{-1}\|$. The discriminant and oscillation metrics on $\widetilde{\text{Cont}}_0(V, \xi)$ for a compact co-oriented contact manifold (V, ξ) are defined as follows [CS12]. As proved in [CS12], any non-trivial element in $\widetilde{\text{Cont}}_0(V, \xi)$ has a representative $\{\phi_t\}_{t \in [0,1]}$ that can be written as the concatenation of a finite number of pieces $\{\phi_t\}_{t \in [t_{j-1}, t_j]}$, $j = 1, \dots, L$, such that each piece is *embedded*, i.e. for every two distinct t and t' in $[t_{j-1}, t_j]$ the composition $\phi_t \circ \phi_{t'}^{-1}$ does not have any discriminant point. The discriminant norm of a non-trivial element $[\{\phi_t\}]$ is then defined to be the minimal number of pieces in such decompositions (in other words, the discriminant norm is the word norm with respect to the generating set formed by elements of $\widetilde{\text{Cont}}_0(V, \xi)$ that can be represented by an embedded contact isotopy). Moreover, any non-trivial element in $\widetilde{\text{Cont}}_0(V, \xi)$ has a representative $\{\phi_t\}_{t \in [0,1]}$ that can be written as the concatenation of a finite number of embedded pieces such that each piece is either non-negative or non-positive. Let L_+ and L_- be respectively the minimal number of non-negative and of non-positive pieces in such decompositions; the oscillation pseudonorm of $[\{\phi_t\}]$ is then defined to be $L_+ + L_-$. The oscillation pseudometric is non-degenerate, hence a metric, if and only if (V, ξ) is orderable.

In [CS12] the non-linear Maslov index has been used to show that the discriminant and oscillation metrics for real projective space are unbounded, hence not equivalent to the trivial metric. The argument, applied to lens spaces, is as follows.

Consider the Reeb flow $\{r_t\}$ on L_k^{2n-1} with respect to the contact form whose pullback to S^{2n-1} is equal to the pullback from \mathbb{R}^{2n} of the 1-form $\sum_{j=1}^n (x_j dy_j - y_j dx_j)$. We first show that the discriminant norm of the $3l$ -th iteration $\{r_{6\pi lt}\}_{t \in [0,1]}$ of the loop $\{r_{2\pi t}\}_{t \in [0,1]}$ is at least $l + 1$. By Example 4.13 we know that $\mu([\{r_{6\pi lt}\}_{t \in [0,1]}]) = 6nl$. Let $\{\phi_t\}_{t \in [0,1]}$ be a contact isotopy that represents $[\{r_{6\pi lt}\}_{t \in [0,1]}]$, is a concatenation of embedded pieces and minimizes the number of such pieces. Then $\mu([\{\phi_t\}_{t \in [0,1]}]) = 6nl$. If $l > 0$ then, by Example 4.1, for a sufficiently small $\epsilon > 0$ we have that $\mu([\{\phi_t\}_{t \in [0, \epsilon]}])$ is $(\leq 2n, \text{ hence})$ different from $6nl$. By Proposition 4.15(i) this implies that $\{\phi_t\}_{t \in [0,1]}$ must intersect the discriminant, and so $\{\phi_t\}_{t \in [0,1]}$ has at least two embedded pieces. Suppose now that $l > 1$, and write $\{\phi_t\}_{t \in [0,1]}$ as a concatenation of L embedded pieces $\{\phi_t\}_{t \in [t_{j-1}, t_j]}$, $j = 1, \dots, L$. For each j , since $\{\phi_t\}_{t \in [t_{j-1}, t_j]}$ is embedded we have in particular that $\phi_t \circ \phi_{t_{j-1}}^{-1}$ does not have any discriminant point for every $t \in (t_{j-1}, t_j]$. Consider a value of time $\underline{t} \in (t_{j-1}, t_j]$ such that $\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, \underline{t}]}$ is C^1 -small. By Proposition 4.15(i) and Example 4.1 we have

$$\mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, t_j]}]) = \mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, \underline{t}]}]) \leq 2n.$$

Suppose now by contradiction that $L < l + 1$. Then

$$(5.1) \quad \sum_{j=1}^L \mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, t_j]}]) \leq 2nL < 2n(l + 1).$$

On the other hand, by the quasimorphism property (Proposition 4.8) we have

$$\left| \mu([\{\phi_t\}_{t \in [0,1]}]) - \sum_{j=1}^L \mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, t_j]}]) \right| \leq (L - 1)(2n + 1) < l(2n + 1),$$

and thus

$$\sum_{j=1}^L \mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, t_j]}]) > 4nl - l.$$

This contradicts (5.1), and thus concludes the proof that the discriminant norm of $\{r_{6\pi lt}\}_{t \in [0,1]}$ is at least $l + 1$.

The fact that the oscillation metric is unbounded can be seen by combining the above argument with positivity of the non-linear Maslov index, as follows. We show that the oscillation norm of the class $[\{r_{20\pi lt}\}_{t \in [0,1]}]$ of $10l$ -th iteration of the loop $\{r_{2\pi t}\}_{t \in [0,1]}$ is at least $l + 1$. For $[\{r_{20\pi lt}\}_{t \in [0,1]}]$, the minimal number L_- of non-positive embedded pieces is zero, because any subdivision of the representative $\{r_{20\pi lt}\}_{t \in [0,1]}$ has no non-positive pieces. So we need to show that the minimal number L_+ of embedded non-negative pieces is at least $l + 1$. Let $\{\phi_t\}_{t \in [0,1]}$ be a contact isotopy that represents $[\{r_{20\pi lt}\}_{t \in [0,1]}]$, is a concatenation of non-negative or non-positive embedded pieces and minimizes the number of non-negative ones. Then, by Example 4.13, $\mu([\{\phi_t\}_{t \in [0,1]}]) = 20nl$. Regarding adjacent embedded pieces of the same sign as a single non-negative or non-positive isotopy, let K_+ and K_- respectively be the number of non-negative and of non-positive isotopies in the decomposition. If $K_+ \geq l + 1$ then the number L_+ of non-negative embedded pieces is also at least $l + 1$ and so we are done. Assume thus that $K_+ < l + 1$, and so (since $K_- \leq K_+ + 1$) $K := K_+ + K_- < 2l + 2$. By the quasimorphism property we then have

$$\left| \mu([\{\phi_t\}_{t \in [0,1]}]) - \sum_{j=1}^K \mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, t_j]}) \right| \leq (K-1)(2n+1) < (2l+1)(2n+1).$$

Write $\sum_{j=1}^K \mu([\{\phi_t \circ \phi_{t_{j-1}}^{-1}\}_{t \in [t_{j-1}, t_j]}) = \mu_+ + \mu_-$, where μ_+ and μ_- are respectively the sum of the non-linear Maslov indices of the non-negative and of the non-positive pieces. By Proposition 4.21 we have $\mu_- \leq 0$, and thus we conclude that

$$(5.2) \quad \mu_+ > 20nl - (2l+1)(2n+1).$$

On the other hand, if the number L_+ of non-negative embedded pieces was less than $l + 1$ then we would have $\mu_+ < (4n+1)(l+1)$, contradicting (5.2). This concludes the proof.

Contact Arnold conjecture. Adapting to our case the argument given for \mathbb{RP}^{2n-1} in [Sa11c] (which in turn is an adaptation of the proof of the Hamiltonian Arnold conjecture for \mathbb{CP}^n given in [Th98] and [Gi90]) we show that for any contactomorphism of L_k^{2n-1} that is contact isotopic to the identity the number of translated points (with respect to the standard contact form) is at least n , and at least $2n$ if all translated points are assumed to be non-degenerate.

Recall that we denote by $\{r_t\}$ the Reeb flow on L_k^{2n-1} . The set of translated points of a contactomorphism ϕ of L_k^{2n-1} contact isotopic to the identity is equal to the union for $t \in (0, 1]$ of the sets of those discriminant points of $r_{2\pi t} \circ \phi$ that correspond to \mathbb{Z}_k -orbits of discriminant points of the lift to the sphere. Without loss of generality we can assume that ϕ has no discriminant points. Indeed, either ϕ has infinitely many translated points or, for some value \underline{t} of $t \in [0, 1]$, the composition $r_{2\pi \underline{t}} \circ \phi$ has no discriminant points; we can then replace ϕ with $r_{2\pi \underline{t}} \circ \phi$, since the translated points of $r_{2\pi \underline{t}} \circ \phi$ are in 1-1 correspondence with those of ϕ . Assume thus that ϕ has no discriminant points, and let $\{\phi_t\}_{t \in [0,1]}$ be a contact isotopy from the identity to ϕ . We first prove that for the concatenation $\{\phi_t\}_{t \in [0,1]} \sqcup \{r_{2\pi t} \circ \phi\}_{t \in [0,1]}$ we have

$$(5.3) \quad \mu(\{\phi_t\}_{t \in [0,1]} \sqcup \{r_{2\pi t} \circ \phi\}_{t \in [0,1]}) - \mu(\{\phi_t\}_{t \in [0,1]}) = 2n.$$

Let F_t be a based family of generating functions for $\{\phi_t\}_{t \in [0,1]}$, and Q_t a based family of \mathbb{Z}_k -invariant generating quadratic forms for $\{r_{2\pi t}\}_{t \in [0,1]}$. Then $F_t \# Q_t$ is a based family of generating functions for $\{r_{2\pi t} \circ \phi_t\}$, which is homotopic to $\{\phi_t\} \sqcup \{r_{2\pi t} \circ \phi\}$, and $F_t \# Q_0$ is a based family of generating functions for $\{\phi_t\}$. Thus

$$\mu(\{\phi_t\}_{t \in [0,1]} \sqcup \{r_{2\pi t} \circ \phi\}_{t \in [0,1]}) - \mu(\{\phi_t\}_{t \in [0,1]}) = \text{ind}(F_1 \# Q_0) - \text{ind}(F_1 \# Q_1).$$

By Lemma 4.10 there are isotopies starting from the identity $\{\Psi_s^0\}_{s \in [0,1]}$ and $\{\Psi_s^1\}_{s \in [0,1]}$ of \mathbb{Z}_k -equivariant fibre preserving linear diffeomorphisms such that $Q_0 \circ \Psi_1^0$ and $Q_1 \circ \Psi_1^1$ are independent of the base variable, and so are equal to quadratic forms \overline{Q}_0 and \overline{Q}_1 on the fibre. For $j = 0, 1$ we have $F_1 \# (Q_j \circ \Psi_1^j) = (F_1 \# 0) \oplus \overline{Q}_j$, and so

$$\text{ind}(F_1 \# Q_j) = \text{ind}(F_1 \# (Q_j \circ \Psi_1^j)) = \text{ind}((F_1 \# 0) \oplus \overline{Q}_j) = \text{ind}(F_1 \# 0) + \text{ind}(\overline{Q}_j)$$

where the last equality follows from Corollary 3.15. By Example 4.13 we thus have

$$\operatorname{ind}(F_1 \sharp Q_0) - \operatorname{ind}(F_1 \sharp Q_1) = \operatorname{ind}(\overline{Q}_0) - \operatorname{ind}(\overline{Q}_1) = 2n,$$

hence (5.3). Knowing this, Proposition 4.15(ii) and the fact that ϕ has no discriminant points imply that either there are at least n distinct values of $t \in (0, 1)$ at which μ jumps, and so ϕ has at least n translated points (which are necessarily all distinct, since their lifts to the sphere have different time-shifts), or there is at least one value of t at which μ jumps by more than 2, and thus ϕ has infinitely many translated points. Moreover, if all translated points of ϕ are non-degenerate then Proposition 4.15(iii) implies that there must be at least $2n$ of them (all distinct).

Existence of translated points with respect to an arbitrary contact form that induces the standard contact structure. Let α be a contact form on L_k^{2n-1} that induces the standard contact structure, and consider a contactomorphism ϕ that is contact isotopic to the identity. We now prove that there are infinitely many distinct real numbers that are time-shifts of translated points of ϕ with respect to α , and thus in particular that ϕ has at least one translated point with respect to α . Denote by α_0 the standard contact form, and by f the function such that $\alpha = e^f \alpha_0$. Then $h = e^{-f}$ is the contact Hamiltonian function (with respect to α_0) of the Reeb flow $\{\varphi_t^\alpha\}$ of α . Let

$$\epsilon := \min h > 0.$$

For any based contact isotopy $\{\phi_t\}_{t \in [0,1]}$ with $\phi_1 = \phi$ we prove that

$$(5.4) \quad \mu\left([\{\phi_t\}_{t \in [0,1]} \sqcup \{\varphi_{tT}^\alpha \circ \phi\}_{t \in [0,1]}\right) \geq \mu([\{\phi_t\}_{t \in [0,1]}) + 2n m_T - 6n - 3,$$

where $m_T = \lfloor \frac{\epsilon T}{2\pi} \rfloor$. For this, note first that

$$[\{\phi_t\}_{t \in [0,1]} \sqcup \{\varphi_{tT}^\alpha \circ \phi\}_{t \in [0,1]}] \geq [\{\phi_t\}_{t \in [0,1]} \sqcup \{r_{t\epsilon T} \circ \phi\}_{t \in [0,1]}],$$

where, as above, $\{r_t\}_{t \in \mathbb{R}}$ denotes the Reeb flow of α_0 . Indeed,

$$[\{\phi_t\}_{t \in [0,1]} \sqcup \{\varphi_{tT}^\alpha \circ \phi\}_{t \in [0,1]}] \cdot [\{\phi_t\}_{t \in [0,1]} \sqcup \{r_{t\epsilon T} \circ \phi\}_{t \in [0,1]}]^{-1}$$

is represented by the contact isotopy $\{\operatorname{Id}\}_{t \in [0,1]} \sqcup \{\varphi_{tT}^\alpha \circ r_{t\epsilon T}^{-1}\}_{t \in [0,1]}$, and the contact Hamiltonian function of $\{\varphi_{tT}^\alpha \circ r_{t\epsilon T}^{-1}\}$ with respect to α_0 is

$$Th \left(1 - \frac{\epsilon}{h \circ (\varphi_{tT}^\alpha)^{-1}}\right)$$

which is non-negative by our choice of ϵ . By the quasimorphism property (Proposition 4.8) and positivity (Proposition 4.21), it follows that

$$(5.5) \quad \mu([\{\phi_t\}_{t \in [0,1]}] \cdot [\{\varphi_{tT}^\alpha\}_{t \in [0,1]}]) \geq \mu([\{\phi_t\}_{t \in [0,1]}] \cdot [\{r_{t\epsilon T}\}_{t \in [0,1]}]) - 2n - 1.$$

By Proposition 4.21, $\mu([\{r_{t\tau}\}_{t \in [0,1]}]) \geq 0$ for all $\tau \in [0, 2\pi]$; by the quasimorphism property we thus have

$$(5.6) \quad \mu([\{r_{t\epsilon T}\}_{t \in [0,1]}]) \geq \mu([\{r_{2\pi m_T t}\}_{t \in [0,1]}]) - 2n - 1.$$

Using (5.5), the quasimorphism property, (5.6) and Example 4.13 we obtain

$$\begin{aligned} \mu\left([\{\phi_t\}_{t \in [0,1]} \sqcup \{\varphi_{tT}^\alpha \circ \phi\}_{t \in [0,1]}\right) &= \mu([\{\phi_t\}_{t \in [0,1]}] \cdot [\{\varphi_{tT}^\alpha\}_{t \in [0,1]}]) \\ &\geq \mu([\{\phi_t\}_{t \in [0,1]}] \cdot [\{r_{t\epsilon T}\}_{t \in [0,1]}]) - 2n - 1 \\ &\geq \mu([\{\phi_t\}_{t \in [0,1]}]) + \mu([\{r_{t\epsilon T}\}_{t \in [0,1]}]) - 4n - 2 \\ &\geq \mu([\{\phi_t\}_{t \in [0,1]}]) + \mu([\{r_{2\pi m_T t}\}_{t \in [0,1]}]) - 6n - 3 \\ &= \mu([\{\phi_t\}_{t \in [0,1]}]) + 2n m_T - 6n - 3. \end{aligned}$$

Thus, inequality (5.4) holds. This inequality in turn implies that $|\mu([\{\phi_t\}_{t \in [0,1]} \sqcup \{\varphi_{tT}^\alpha \circ \phi\}_{t \in [0,1]}])|$ approaches ∞ as $T \rightarrow \infty$. By Proposition 4.15(i) we thus conclude that there are infinitely many distinct real numbers that are time-shifts of translated points of ϕ with respect to α , and so in particular ϕ has at least one translated point with respect to α .

Weinstein conjecture. Following Givental [Gi90], we now prove that any contact form α on L_k^{2n-1} defining the standard contact structure has closed Reeb orbits. As above, denote by α_0 the standard contact form, and by f the function such that $\alpha = e^f \alpha_0$. Then $h = e^{-f}$ is the contact Hamiltonian function (with respect to α_0) of the Reeb flow $\{\varphi_t^\alpha\}$ of α and, for every $m \in \mathbb{N}$, mh is the contact Hamiltonian function of $\{\varphi_{mt}^\alpha\}$. Discriminant points of φ_{mt}^α correspond to closed Reeb orbits of α of period mt . Since $h > 0$, there is $m \in \mathbb{N}$ such that $mh \geq 2\pi$. The constant function 2π is the contact Hamiltonian of $\{r_{2\pi t}\}_{t \in [0,1]}$. By Proposition 4.22 and Example 4.13 we then have $\bar{\mu}(\{\varphi_{mt}^\alpha\}_{t \in [0,1]}) \geq \bar{\mu}(\{r_{2\pi t}\}_{t \in [0,1]}) = 2n > 0$. It thus follows from Proposition 4.15 that α has at least one closed Reeb orbit.

Constructing new quasimorphisms via contact reduction, and more applications to orderability and non-displaceability. In [BZ15] Borman and Zolotarev explain how in certain situations quasimorphisms descend under contact reduction, and use this to show that Givental's asymptotic non-linear Maslov index on projective spaces induces quasimorphisms on certain prequantizations of symplectic toric manifolds. Moreover they obtain applications to orderability and existence of non-displaceable pre-Lagrangian toric fibres. As already observed in [BZ15, Remark 1.5], our extension of Givental's non-linear Maslov index to lens spaces allows us to enlarge the class of spaces to which the results of [BZ15] apply.

Consider a closed contact manifold (V, ξ) , and suppose that it is equipped with a non-trivial monotone quasimorphism $\nu: \widetilde{\text{Cont}}_0(V, \xi) \rightarrow \mathbb{R}$. Following [BZ15] we say that a subset Y of V is *subheavy* with respect to ν if ν vanishes on all elements that can be represented by a contact isotopy generated by an autonomous Hamiltonian that vanishes on Y . Suppose now that (V, ξ) is also equipped with a contact \mathbb{T}^n -action, and denote by $f_\alpha: V \rightarrow \mathbb{R}^n$ the moment map with respect to a \mathbb{T}^n -invariant contact form α for ξ . Recall that if \mathbb{T}^n acts freely on the level set $f_\alpha^{-1}(0)$ then α induces a contact form α' on the quotient $V' = f_\alpha^{-1}(0)/\mathbb{T}^n$ (see for instance [Ge08, Theorem 7.7.5]). The contact manifold $(V', \xi' = \ker(\alpha'))$ is said to be the contact reduction of (V, ξ) at the level $f_\alpha^{-1}(0)$. By [BZ15, Theorem 1.8], if $f_\alpha^{-1}(0)$ is subheavy with respect to the non-trivial monotone quasimorphism ν then ν naturally descend to a non-trivial monotone quasimorphism $\nu': \widetilde{\text{Cont}}_0(V', \xi') \rightarrow \mathbb{R}$. Moreover, if ν has the vanishing property then so does ν' . By [BZ15, Theorem 1.3], if a monotone symplectic toric manifold (W, ω) is *even*, i.e. the sum of the normals of the moment polytope $\Delta \subset \mathfrak{t}^*$ is in $2\mathfrak{t}_\mathbb{Z}$, then there is a rescaling $a\omega$ of the symplectic form such that the prequantization (V, ξ) of $(W, a\omega)$ can be written as contact reduction of a projective space \mathbb{RP}^{2n-1} at a level $f_\alpha^{-1}(0)$ containing the torus

$$T_{\mathbb{RP}^{2n-1}} = \{[z] \in \mathbb{RP}^{2n-1} \mid |z_1|^2 = \dots = |z_n|^2\}.$$

By [BZ15, Lemma 1.22 and Theorem 1.11 (i)], $T_{\mathbb{RP}^{2n-1}}$ is subheavy with respect to the asymptotic non-linear Maslov index $\bar{\mu}$ and so, by [BZ15, Proposition 1.10(ii)], $\widetilde{f_\alpha^{-1}(0)}$ is also subheavy. It follows that $\bar{\mu}$ descends to a non-trivial monotone quasimorphism on $\widetilde{\text{Cont}}_0(V, \xi)$ with the vanishing property. By [BZ15, Theorem 1.28], if a contact manifold (V, ξ) admits a non-trivial monotone quasimorphism $\nu: \widetilde{\text{Cont}}_0(V, \xi) \rightarrow \mathbb{R}$ then it is orderable; by [BZ15, Theorem 1.17], if moreover (V, ξ) is the prequantization of a symplectic toric manifold and the quasimorphism ν also has the vanishing property then V has a non-displaceable pre-Lagrangian toric fibre. The conclusion is thus that any monotone even symplectic toric manifold has a prequantization that is orderable and has a non-displaceable pre-Lagrangian toric fibre.

In the case of lens spaces, repeating the proof of [BZ15, Lemma 1.22] one sees that

$$T_{L_k^{2n-1}} = \{[z] \in L_k^{2n-1} \mid |z_1|^2 = \dots = |z_n|^2\} \subset L_k^{2n-1}(1, \dots, 1)$$

is subheavy with respect to the asymptotic non-linear Maslov index on $L_k^{2n-1}(1, \dots, 1)$. Consider now a compact monotone symplectic toric manifold (W^{2n}, ω) . Write the moment polytope as $\Delta = \{x \in \mathfrak{t}^* \mid \langle \nu_j, x \rangle + \lambda \geq 0 \text{ for } j = 1, \dots, d\}$, where d is the number of facets and $\nu_j \in \mathfrak{t}$ are vectors normal to the facets and primitive in the integer lattice $\mathfrak{t}_\mathbb{Z} = \ker(\exp: \mathfrak{t} \rightarrow \mathbb{T}^n)$. Suppose

that, for some integer $k \geq 2$,

$$(5.7) \quad \sum_{j=1}^d \nu_j \in k \mathfrak{t}_{\mathbb{Z}}.$$

Then the same argument as in the proof of [BZ15, Theorem 1.3] shows that there is a rescaling¹² $a\omega$ of the symplectic form such that the prequantization of $(W, a\omega)$ can be written as contact reduction of $L_k^{2n-1}(1, \dots, 1)$ at a level $f_{\alpha}^{-1}(0)$ containing $T_{L_k^{2n-1}}$. Therefore, such a prequantization admits a non-trivial monotone quasimorphism with the vanishing property, and so it is orderable and it contains a non-displaceable pre-Lagrangian toric fibre.

5.8. Example. A compact monotone symplectic toric manifold (W^{2n}, ω) satisfying condition (5.7) can be obtained by the following generalization of [BZ15, Example (i) of page 385]. Consider the \mathbb{CP}^1 -bundle over \mathbb{CP}^n obtained, for $1 \leq k \leq n$, as the projectivization $\mathbb{P}(1 \oplus \mathcal{O}(k))$ of the direct sum of a trivial line bundle with the bundle $\mathcal{O}(k)$ over \mathbb{CP}^n . This manifold can be equipped with a monotone symplectic structure, and the inward normals of the corresponding moment polytope are $e_1, \dots, e_n, e_{n+1}, -e_{n+1}, ke_{n+1} - e_1 - \dots - e_n$. \square

5.9. Remark. One can show that for any compact monotone symplectic toric manifold (W, ω) the prequantization of W with appropriately rescaled ω can be written as a contact reduction of $L_k^{2n-1}(\underline{w})$ at a level containing $T_{L_k^{2n-1}(\underline{w})}$. However, if $\underline{w} \neq (1, \dots, 1)$ we do not know whether $T_{L_k^{2n-1}(\underline{w})}$ is subheavy and so we cannot conclude that this prequantization has an induced quasimorphism. In order to prove that $T_{L_k^{2n-1}} \subset L_k^{2n-1}(1, \dots, 1)$ is subheavy one uses the fact that the Clifford torus in $\mathbb{CP}^{n-1} = \mathbb{CP}^{n-1}(1, \dots, 1)$ is the unique non-displaceable orbit of the standard torus action. A similar statement is not true in general for weighted projective spaces: for example $\mathbb{CP}(1, 3, 5)$ contains a 2-dimensional family of non-displaceable Lagrangian toric fibres [WW13].

APPENDIX A. ON THE CONSTRUCTION OF GENERATING FUNCTIONS

In Proposition 2.2 we proved that if $\Phi^{(1)}$ and $\Phi^{(2)}$ are Hamiltonian symplectomorphisms of \mathbb{R}^{2n} with generating functions $F_1: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \rightarrow \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \times \mathbb{R}^{2nN_2} \rightarrow \mathbb{R}$ respectively then the function $F_1 \sharp F_2: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$ defined by

$$F_1 \sharp F_2(q; \zeta_1, \zeta_2, \nu_1, \nu_2) = F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

is a generating function for the composition $\Phi = \Phi^{(2)} \circ \Phi^{(1)}$. Here we present two alternative proofs of this fact in terms of symplectic reduction, we generalize the composition formula to the case of any even number of factors and discuss its relation with the method of broken trajectories by Chaperon, Laudenbach and Sikorav [Ch84, LS85, Si85, Si87].

Recall that if V is a coisotropic submanifold of a symplectic manifold (W, ω) then the kernel of the restriction of ω to V is an integrable distribution. If the space of leaves V/\sim is a manifold then it inherits a symplectic form ω , and is said to be the symplectic reduction of (W, ω) along V . If L is a Lagrangian submanifold of W that is transverse to V then the restriction to $L \cap V$ of the projection $V \rightarrow V/\sim$ is a Lagrangian immersion. For instance, consider a fibre bundle $p: E \rightarrow B$. The fibre conormal bundle N_E^* is a coisotropic submanifold of T^*E , and the symplectic reduction can be identified with T^*B . If $F: E \rightarrow \mathbb{R}$ is a generating function then the Lagrangian immersion $i_F: \Sigma_F \rightarrow T^*B$ described in Section 2 is the reduction of $dF \subset T^*E$ with respect to N_E^* .

¹²If ω is rescaled so that $[\omega] = c_1(TW)$ then we can take $a = k$. Indeed, the moment polytope for $(W, k\omega)$ can be written as $\Delta = \{x \in \mathfrak{t}^*, \langle \nu_j, x \rangle + k \geq 0 \text{ for } j = 1, \dots, d\}$ and the prequantization (V, ξ) of $(W, k\omega)$ corresponds to a cone in \mathbb{R}^{n+1} with primitive inward normals $(\nu_j, k) \in \mathfrak{t} \times \mathbb{R}$, $j = 1, \dots, d$. The contact toric manifold (V, ξ) is a contact reduction of $(S^{2d-1}, \ker \alpha_{std})$ by a subgroup K of T^d containing $[\frac{1}{k}, \dots, \frac{1}{k}] \in \mathbb{R}^d / \mathbb{Z}^d = T^d$ (because $\sum \frac{1}{k}(\nu_j, k)$ is in the lattice of $\mathfrak{t} \times \mathbb{R}$; see [L02]). In fact it would be enough to take $\omega = k\eta$ where η denotes the primitive integral class in the direction of $c_1(TW)$.

First interpretation. The first interpretation of Proposition 2.2 in terms of symplectic reduction that we present is an adaptation to our composition formula of the discussion in Th  ret [Th95, Section I.3]. We use three basic properties of generating functions (Lemmas A.1, A.2 and A.3) whose verification is immediate and therefore left to the reader.

A.1. Lemma. *If $L_1 \subset T^*B_1$ has generating function $F_1: B_1 \times \mathbb{R}^{N_1} \rightarrow \mathbb{R}$ and $L_2 \subset T^*B_2$ has generating function $F_2: B_2 \times \mathbb{R}^{N_2} \rightarrow \mathbb{R}$ then the product $L_1 \times L_2 \subset T^*B_1 \times T^*B_2 \equiv T^*(B_1 \times B_2)$ has generating function $F_1 \oplus F_2: (B_1 \times B_2) \times (\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}) \rightarrow \mathbb{R}$.*

A.2. Lemma. *Suppose that $L \subset T^*B$ has generating function $F: B \times \mathbb{R}^N \rightarrow \mathbb{R}$, and consider a symplectomorphism A_h of T^*B of the form $A_h(q, p) = (q, p + dh(q))$ for some function $h: B \rightarrow \mathbb{R}$. Then $A_h(L) \subset T^*B$ has generating function $F + h$.*

A.3. Lemma. *If a Lagrangian submanifold L of $T^*(\mathbb{R}^n \times \mathbb{R}^m)$ has a generating function $F: (\mathbb{R}^n \times \mathbb{R}^m) \times \mathbb{R}^N \rightarrow \mathbb{R}$, then the reduction $\bar{L} \subset T^*\mathbb{R}^n$ of L with respect to the coisotropic submanifold $V = \mathbb{R}^n \times \mathbb{R}^m \times (\mathbb{R}^N)^* \times 0$ of $T^*(\mathbb{R}^n \times \mathbb{R}^m)$ has a generating function $\bar{F}: \mathbb{R}^n \times (\mathbb{R}^m \times \mathbb{R}^N) \rightarrow \mathbb{R}$, $\bar{F}(\zeta_1; \zeta_2, \nu) = F(\zeta_1, \zeta_2; \nu)$.*

Since Γ_{id} has generating function $\mathbb{R}^{2n} \rightarrow \mathbb{R}$, $q \mapsto 0$, Lemma A.1 implies that the function $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$ defined by

$$(q, \zeta_1, \zeta_2; \nu_1, \nu_2) \mapsto F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2)$$

is a generating function for $\Gamma_{\text{id}} \times \Gamma_{\Phi^{(1)}} \times \Gamma_{\Phi^{(2)}} \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$. By applying Lemma A.2 with

$$h: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad h(q, \zeta_1, \zeta_2) = -2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

we obtain that the function $\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2nN_1} \times \mathbb{R}^{2nN_2}) \rightarrow \mathbb{R}$ defined by

$$(A.4) \quad (q, \zeta_1, \zeta_2; \nu_1, \nu_2) \mapsto F_1(\zeta_1, \nu_1) + F_2(\zeta_2, \nu_2) - 2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

is a generating function for $A_h(\Gamma_{\text{id}} \times \Gamma_{\Phi^{(1)}} \times \Gamma_{\Phi^{(2)}}) \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$. The function (A.4) is equal to $F_1 \sharp F_2$, except that in the latter ζ_1 and ζ_2 are fibre variables. Thus, it follows from Lemma A.3 that $F_1 \sharp F_2$ is a generating function for the reduction of

$$L := A_h(\Gamma_{\text{id}} \times \Gamma_{\Phi^{(1)}} \times \Gamma_{\Phi^{(2)}}) \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$$

along the coisotropic submanifold

$$V = \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times (\mathbb{R}^{2n})^* \times \{0\} \times \{0\}$$

of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$. We are left to prove that such reduction is equal to Γ_{Φ} . Observe that the reduction $V \rightarrow V/\sim$ sends a point $(q, \zeta_1, \zeta_2, \xi, 0, 0)$ to (q, ξ) . We have

$$\begin{aligned} L &= A_h(\Gamma_{\text{id}} \times \Gamma_{\Phi^{(1)}} \times \Gamma_{\Phi^{(2)}}) \\ &= A_h\left(\left\{\left(q, \frac{z_1 + \Phi^{(1)}(z_1)}{2}, \frac{z_2 + \Phi^{(2)}(z_2)}{2}, 0, i(z_1 - \Phi^{(1)}(z_1)), i(z_2 - \Phi^{(2)}(z_2))\right)\right\}\right) \\ &= \left\{\left(q, \frac{z_1 + \Phi^{(1)}(z_1)}{2}, \frac{z_2 + \Phi^{(2)}(z_2)}{2}, 0, i(z_1 - \Phi^{(1)}(z_1)), i(z_2 - \Phi^{(2)}(z_2))\right)\right. \\ &\quad \left.+ dh\left(q, \frac{z_1 + \Phi^{(1)}(z_1)}{2}, \frac{z_2 + \Phi^{(2)}(z_2)}{2}\right)\right\} \\ &= \left\{\left(q, \frac{z_1 + \Phi^{(1)}(z_1)}{2}, \frac{z_2 + \Phi^{(2)}(z_2)}{2}, i(z_1 + \Phi^{(1)}(z_1) - z_2 - \Phi^{(2)}(z_2)),\right.\right. \\ &\quad \left.\left.i(z_1 - \Phi^{(1)}(z_1) + z_2 + \Phi^{(2)}(z_2) - 2q), i(z_2 - \Phi^{(2)}(z_2) - z_1 - \Phi^{(1)}(z_1) + 2q)\right)\right\}. \end{aligned}$$

The intersection $L \cap V$ is given by the points in the above set that satisfy

$$\begin{cases} 2q = z_1 - \Phi^{(1)}(z_1) + z_2 + \Phi^{(2)}(z_2) \\ 2q = z_1 + \Phi^{(1)}(z_1) - z_2 + \Phi^{(2)}(z_2) \end{cases}$$

hence $z_2 = \Phi^{(1)}(z_1)$ and $q = \frac{z_1 + \Phi^{(2)}(z_2)}{2} = \frac{z_1 + \Phi(z_1)}{2}$. The reduction is thus given by

$$\begin{aligned} (L \cap V)/\sim &= \left\{ \left(q, i(z_1 + \Phi^{(1)}(z_1) - z_2 - \Phi^{(2)}(z_2)) \right) \text{ with } z_2 = \Phi^{(1)}(z_1) \text{ and } q = \frac{z_1 + \Phi(z_1)}{2} \right\} \\ &= \left\{ \left(\frac{z_1 + \Phi(z_1)}{2}, i(z_1 - \Phi(z_1)) \right) \right\} = \Gamma_\Phi \end{aligned}$$

as we wanted.

Second interpretation. The second alternative proof of Proposition 2.2 that we discuss uses symplectic reduction at the level of graphs and is based on the fact, immediate to verify, that the function

$$h: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}, \quad h(q, \zeta_1, \zeta_2) = -2 \langle \zeta_2 - q, i(\zeta_1 - q) \rangle$$

is a generating function for the symplectomorphism

$$\sigma: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n}, \quad \sigma(z_0, z_1, z_2) = (z_2, z_0, z_1).$$

Proposition 2.2 can be deduced from this fact as follows. For simplicity of notation we assume that the generating functions of $\Phi^{(1)}$ and $\Phi^{(2)}$ have no fibre variables (the general case does not present any additional difficulty). Suppose thus that $\Phi^{(1)}$ and $\Phi^{(2)}$ are Hamiltonian symplectomorphisms of \mathbb{R}^{2n} with generating functions $F_1: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ and $F_2: \mathbb{R}^{2n} \rightarrow \mathbb{R}$ respectively, and consider the function $F = F_1 \# F_2: \mathbb{R}^{2n} \times (\mathbb{R}^{2n} \times \mathbb{R}^{2n}) \rightarrow \mathbb{R}$, $F(q; \zeta_1, \zeta_2) = F_1(\zeta_1) + F_2(\zeta_2) + h(q, \zeta_1, \zeta_2)$. Denote the coordinates of $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ by $(q, \zeta_1, \zeta_2; p_0, p_1, p_2)$, and recall that, by the definition of generating function, the Lagrangian submanifold L_F of $T^*\mathbb{R}^{2n}$ generated by F is the symplectic reduction of $dF \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2n} \times \mathbb{R}^{2n})$ with respect to the fibre conormal bundle $V = \{p_1 = p_2 = 0\}$. The submanifold

$$V_\Phi := \left\{ (q, \zeta_1, \zeta_2, p_0, -\frac{\partial F_1}{\partial \zeta_1}(\zeta_1), -\frac{\partial F_2}{\partial \zeta_2}(\zeta_2)) \right\}$$

is also coisotropic (it is $T^*(\mathbb{R}^{2n}) \times (-dF_1) \times (-dF_2)$ hence the product of one symplectic and two Lagrangian factors). The projection $V_\Phi \rightarrow V_\Phi / \sim$ forgets the Lagrangian factors and is therefore given by $(q, \zeta_1, \zeta_2, p_0, p_1, p_2) \mapsto (q, p_0)$.

The plane dh intersects V_Φ at the points defined by the conditions

$$p_0 = 2i(\zeta_1 - \zeta_2); \quad -\frac{\partial F_1}{\partial \zeta_1} = 2i(\zeta_2 - q); \quad -\frac{\partial F_2}{\partial \zeta_2} = -2i(\zeta_1 - q).$$

These are equivalent to the conditions that (q, ζ_1, ζ_2) is a fibre critical point and $p_0 = \frac{\partial F}{\partial q}$. It follows that L_F also equals the symplectic reduction of dh with respect to V_Φ .

Since $\tau^{-1}(dh) = \text{gr}(\sigma)$, our problem is reduced to proving that the reduction of $\text{gr}(\sigma)$ along

$$\tau^{-1}(V_\Phi) = \{ (z_0, z_1, z_2; Z_0, Z_1, Z_2) \mid Z_1 = (\Phi^{(1)})^{-1}(z_1) \text{ and } Z_2 = (\Phi^{(2)})^{-1}(z_2) \}$$

is equal to the graph of Φ . But, the projection $\tau^{-1}(V_\Phi) \rightarrow \tau^{-1}(V_\Phi) / \sim$ is given by

$$(z_0, z_1, z_2; Z_0, Z_1, Z_2) \mapsto (z_0, Z_0)$$

and $\text{gr}(\sigma) \cap \tau^{-1}(V_\Phi)$ is the set of points $(z_0, z_1, z_2, z_2, z_0, z_1)$ such that $z_0 = (\Phi^{(1)})^{-1}(z_1)$ and $z_1 = (\Phi^{(1)})^{-1}(z_2)$. The projection sends such a point to $(z_0, z_2) = (z_0, \Phi(z_0))$.

Generalization to any even number of factors. The second interpretation of Proposition 2.2 in terms of symplectic reduction permits to easily generalize the composition formula to the case of any even number of factors (obtaining an alternative proof of Proposition 2.14). Assume that N is even, and consider the symplectomorphism σ of $\mathbb{R}^{2n} \times (\mathbb{R}^{2n})^N$ defined by

$$\sigma(z_0, z_1, \dots, z_N) = (z_N, z_0, z_1, \dots, z_{N-1}).$$

A straightforward calculation shows that the function $h: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ given by

$$h(q, \zeta_1, \dots, \zeta_N) = 2 \sum_{1 \leq j \leq N} (-1)^j \langle \zeta_j, iq \rangle + 2 \sum_{1 \leq j < \ell \leq N} (-1)^{j+\ell-1} \langle \zeta_j, i\zeta_\ell \rangle$$

is a generating function for σ .

A.5. Proposition. *Suppose that, for each $j = 1, \dots, N$, $\Phi^{(j)}$ is a Hamiltonian diffeomorphism of \mathbb{R}^{2n} with generating function $F_j: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. Then the function $F: \mathbb{R}^{2n} \times \mathbb{R}^{2nN} \rightarrow \mathbb{R}$ defined by*

$$F(q; \zeta_1, \dots, \zeta_N) = F_1(\zeta_1) + \dots + F_N(\zeta_N) + h(q, \zeta_1, \dots, \zeta_N)$$

is a generating function for the composition $\Phi = \Phi^{(N)} \circ \dots \circ \Phi^{(1)}$.

Proof. Denote the coordinates on $T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2nN})$ by $(q, \zeta_1, \dots, \zeta_N; p_0, p_1, \dots, p_N)$. The Lagrangian submanifold L_F of $T^*\mathbb{R}^{2n}$ generated by F is the symplectic reduction of $dh \subset T^*(\mathbb{R}^{2n} \times \mathbb{R}^{2nN})$ with respect to the coisotropic submanifold

$$V_\Phi = \left\{ p_1 = -\frac{\partial F_1}{\partial \zeta_1}(\zeta_1), \dots, p_N = -\frac{\partial F_N}{\partial \zeta_N}(\zeta_N) \right\}.$$

Since $\tau^{-1}(dh) = \text{gr}(\sigma)$, our problem is reduced to proving that the reduction of $\text{gr}(\sigma)$ along

$$\tau^{-1}(V_\Phi) = \{ (z_0, z_1, \dots, z_N; Z_0, Z_1, \dots, Z_N) \mid Z_1 = (\Phi^{(1)})^{-1}(z_1), \dots, Z_N = (\Phi^{(N)})^{-1}(z_N) \}$$

is equal to the graph of Φ . But, the projection $\tau^{-1}(V_\Phi) \rightarrow \tau^{-1}(V_\Phi) / \sim$ is given by

$$(z_0, z_1, \dots, z_N; Z_0, Z_1, \dots, Z_N) \mapsto (z_0, Z_0)$$

and $\text{gr}(\sigma) \cap \tau^{-1}(V_\Phi)$ is the set of points $(z_0, z_1, \dots, z_N, z_N, z_0, z_1, \dots, z_{N-1})$ such that $z_{j-1} = (\Phi^{(j)})^{-1}(z_j)$ for $j = 1, \dots, N$. The projection sends such a point to $(z_0, \Phi(z_0))$. \square

A.6. Remark. In the case of \mathbb{RP}^{2n-1} Givental does not use directly the generating function given by Proposition A.5. Instead he studies a path $-\Phi_t$ starting at $-\text{id}$ by looking at a family of generating functions F_t of the path Φ_t (starting at the identity). For fibre critical points of F_t we have $q = \frac{z_1 + \Phi_t(z_1)}{2}$. Thus critical points of the restriction of F_t to the fibre over $q = 0$ correspond to fixed points of $-\Phi_t$. So, instead of looking at the whole function F_t , Givental only considers the restriction of F_t to the fibre over $q = 0$. \diamond

Relation with the method of broken trajectories. We now discuss the relation between the composition formula of Proposition 2.2 and the construction of generating functions via the method of broken trajectories, due to Chaperon, Laudenbach and Sikorav [Ch84, LS85, Si85, Si87].

The method of broken trajectories is used to construct generating functions for Lagrangian submanifolds of a cotangent bundle T^*B that are Hamiltonian isotopic to the zero section. The idea is to first interpret the symplectic action functional on a space of paths in T^*B as a generating function with infinite dimensional domain, and then to construct a finite dimensional approximation. Recall that the symplectic action functional associated to a time-dependent Hamiltonian function H_t on an exact symplectic manifold $(W, \omega = -d\lambda)$ is the functional \mathcal{A}_H on the space of paths $\gamma: [0, 1] \rightarrow W$ which is defined by

$$\mathcal{A}_H(\gamma) = \int_0^1 \lambda\left(\frac{\partial \gamma}{\partial t}\right) - H_t(\gamma(t)) dt.$$

A path γ is a critical point of \mathcal{A}_H with respect to variations with fixed endpoints if and only if it is a trajectory of the Hamiltonian flow of H_t . Consider now the case where W is a cotangent bundle T^*B . Let E be the space of paths $\gamma: [0, 1] \rightarrow T^*B$ that begin at the zero section, and see it as the total space of a fibre bundle over B with projection $p: E \rightarrow B$ given by $p(\gamma) = \pi(\gamma(1))$, where π is the projection of T^*B into B . Given a time-dependent Hamiltonian function $H_t: T^*B \rightarrow \mathbb{R}$, consider the functional $F: E \rightarrow \mathbb{R}$ defined by

$$F(\gamma) = \mathcal{A}_H(\gamma).$$

The fibre critical points of $F: E \rightarrow \mathbb{R}$ are the trajectories of the Hamiltonian flow of H_t , and the covector $v^*(\gamma)$ associated to a fibre critical point γ is $\gamma(1)$. Thus, F generates the image of the zero section by the time-1 map of the Hamiltonian flow of H_t . Although F is not a generating function in the usual sense, because its domain is infinite dimensional, a finite dimensional reduction can be

obtained as follows. Let N be an integer. Consider the direct sum $\bigoplus_{i=1}^{N-1} TB \oplus T^*B$, and denote its elements by expressions of the form $e = (q, X, P)$, where q is a point of B , $X = (X_1, \dots, X_{N-1})$ is an $(N-1)$ -tuple of vectors $X_j \in T_q B$ and $P = (P_1, \dots, P_{N-1})$ is an $(N-1)$ -tuple of covectors $P_j \in T_q^* B$. Let \mathcal{U} be a neighborhood of the zero section of TB , and consider the subspace E_N of $\bigoplus_{i=1}^{N-1} TB \oplus T^*B$ that is formed by those elements $e = (q, X, P)$ such that all X_j belong to \mathcal{U} . If \mathcal{U} is sufficiently small then an element $e = (q, X, P)$ of E_N can be interpreted as a broken Hamiltonian trajectory of H_t , with N smooth pieces and $N-1$ jumps, as follows. The first smooth piece γ_1 is obtained by following the Hamiltonian flow of H_t for $t \in [0, \frac{1}{N}]$ from the point $(q, 0)$ to a point of T^*B that we denote by η_1^- . The second smooth piece γ_2 starts from a point η_1^+ , which is uniquely determined by η_1^- , X_1 and P_1 in a way that we describe later, and follows the flow of H_t for $t \in [\frac{1}{N}, \frac{2}{N}]$ to a point η_2^- . We continue in this way to obtain the whole broken trajectory. In order to describe the jumps we fix a Riemannian metric on B , and consider the associated Levi-Civita connections on TB and T^*B . The point $\eta_1^+ = (q_1^+, p_1^+)$ is determined by $\eta_1^- = (q_1^-, p_1^-)$, X_1 and P_1 in the following way. Denote by $\bar{X}_1 \in T_{q_1^-} B$ and $\bar{P}_1 \in T_{q_1^-}^* B$ the vector and the covector obtained by parallel transport of X_1 and P_1 along the projection to B of the path γ_1 . Since X_1 is in \mathcal{U} , which is assumed to be sufficiently small, we can then define $q_1^+ = \exp_{q_1^-}(\bar{X}_1)$ and $p_1^+ = ((d\exp_{q_1^-}|_{\bar{X}_1})^*)^{-1}(P_1)$. The other jumps are defined similarly. Consider the projection $p: E_N \rightarrow B$ that sends a point $e = (q, X, P)$ to the projection q_N^- to B of the endpoint η_N^- of the broken Hamiltonian trajectory associated to e . Define a function $F: E_N \rightarrow \mathbb{R}$ by

$$(A.7) \quad F(e) = \sum_{j=1}^N \mathcal{A}_H(\gamma_j) + \sum_{j=1}^{N-1} P_j(X_j).$$

Denote the flow of H_t by $\{\varphi_t\}_{t \in [0,1]}$, and assume that, for all $j = 1, \dots, N$, the symplectomorphism $\varphi_{\frac{j}{N}} \circ (\varphi_{\frac{j-1}{N}})^{-1}$ is sufficiently \mathcal{C}^1 -small. Then [Si85] the fibre critical points of F are the unbroken trajectories, and the covector $v^*(e)$ associated to a fibre critical point e is given by $v^*(e) = p_N^-$. Thus, $F: E_N \rightarrow \mathbb{R}$ is a generating function for the image of the zero section by φ_1 .

If φ_1 is already sufficiently \mathcal{C}^1 -small then the above construction (for $N = 1$) reduces to the following. The space E_1 can be identified with B , by associating to a point q of B the Hamiltonian trajectory γ_q of H_t starting at q . We see E_1 as the total space of a fibre bundle over B by the diffeomorphism

$$(A.8) \quad E_1 \rightarrow B \quad , \quad q \mapsto \pi(\gamma_q(1)).$$

Then the function $F: E_1 \rightarrow \mathbb{R}$, $F(q) = \mathcal{A}_H(\gamma_q)$ is a generating function, with respect to the projection (A.8), for the image of the zero section by φ_1 . In other words, the function on B obtained by precomposing F with the inverse of (A.8) is a generating function (with respect to the projection $B \rightarrow B$ given by the identity) for the image of the zero section by φ_1 .

Returning to a general N , in the case when B is \mathbb{R}^n with the Euclidean metric we can identify E_N with the product $\mathbb{R}^n \times (\mathbb{R}^n)^{N-1} \times (\mathbb{R}^n)^{N-1}$. For an element $e = (q, X_1, \dots, X_{N-1}, P_1, \dots, P_{N-1})$ we then have $q_j^+ = q_j^- + X_j$, $p_j^+ = P_j$ and $P_j(X_j) = \langle P_j, X_j \rangle$.

We now discuss how the composition formula of Proposition 2.2 is related to this construction. We are interested in Lagrangians of $T^*\mathbb{R}^{2n}$ of the form $\Gamma_\Phi = \tau(\text{gr}(\Phi))$, where $\tau: \mathbb{R}^{2n} \times \mathbb{R}^{2n} \rightarrow T^*\mathbb{R}^{2n}$ is the identification (2.1). Any such Lagrangian is Hamiltonian isotopic to the zero section by the Hamiltonian isotopy of $T^*\mathbb{R}^{2n}$ that corresponds under τ to a Hamiltonian isotopy of the form $\text{id} \times \Phi_t$. The Hamiltonian function thus satisfies

$$(A.9) \quad H_t(\tau(z, Z)) = H_t(\tau(z + a, Z)) = H_t(\tau(z, Z) + (\frac{a}{2}, ia)) \quad \text{for all } a \in \mathbb{R}^{2n} \equiv \mathbb{C}^n.$$

Suppose now that the flow $\{\Psi_t\}_{t \in [0,1]}$ of H_t is the concatenation of pieces $\{\Psi_t\}_{t \in [\frac{j-1}{N}, \frac{j}{N}]}$ so that each $\Psi^{(j)} := \Psi_{\frac{j}{N}} \circ \Psi_{\frac{j-1}{N}}^{-1}$ has a generating function $F_j: \mathbb{R}^{2n} \rightarrow \mathbb{R}$. Using (A.9) we can relate the action terms in (A.7) with the action of Hamiltonian trajectories starting at the zero section, and thus with the functions F_j . Observe first that $\mathcal{A}_H(\gamma_1) = F_1(\zeta_1)$ where $\zeta_1 := q_1^- = \frac{q + \Psi^{(1)}(q)}{2}$. For

$j = 2, \dots, N$ set

$$\tilde{\gamma}_j(t) = \gamma_j(t) + \left(\frac{iP_{j-1}}{2}, -P_{j-1}\right).$$

The $\tilde{\gamma}_j$ are Hamiltonian trajectories starting at $(q_{j-1}^- + X_{j-1} + \frac{iP_{j-1}}{2}, 0)$, and

$$\mathcal{A}_H(\tilde{\gamma}_j) - \mathcal{A}_H(\gamma_j) = \int_0^1 \lambda\left(\frac{\partial \tilde{\gamma}_j}{\partial t}\right) - \lambda\left(\frac{\partial \gamma_j}{\partial t}\right) dt = -\langle P_{j-1}, q_j^- - (q_{j-1}^- + X_{j-1}) \rangle.$$

Set

$$\zeta_j = \frac{u_j + \Psi^{(j)}(u_j)}{2}$$

where $u_j = q_{j-1}^- + X_{j-1} + \frac{iP_{j-1}}{2}$. Then $\mathcal{A}_H(\tilde{\gamma}_j) = F_j(\zeta_j)$, and so the function (A.7) reduces to

$$\begin{aligned} F(e) &= F_1(\zeta_1) + \dots + F_N(\zeta_N) + \sum_{j=2}^N \langle P_{j-1}, q_j^- - (q_{j-1}^- + X_{j-1}) \rangle + \sum_{j=1}^{N-1} \langle P_j, X_j \rangle \\ &= F_1(\zeta_1) + \dots + F_N(\zeta_N) + \sum_{j=1}^{N-1} \langle P_j, q_{j+1}^- - q_j^- \rangle. \end{aligned}$$

Set $q_j := q_{j+1}^- = \zeta_{j+1} - \frac{iP_j}{2}$ for $j = 1, \dots, N-1$, and consider the change of variables

$$e = (q, X_1, \dots, X_{N-1}, P_1, \dots, P_{N-1}) \mapsto (\zeta_1, \dots, \zeta_N, q_1, \dots, q_{N-1}).$$

Then

$$\begin{aligned} \sum_{j=1}^{N-1} \langle P_j, q_{j+1}^- - q_j^- \rangle &= \langle -2i(\zeta_2 - q_1), q_1 - \zeta_1 \rangle + \sum_{j=2}^{N-1} \langle -2i(\zeta_{j+1} - q_j), q_j - q_{j-1} \rangle \\ &= -2 \langle \zeta_2 - q_1, i(q_1 - \zeta_1) \rangle - 2 \sum_{j=2}^{N-1} \langle \zeta_{j+1} - q_j, i(q_j - q_{j-1}) \rangle \end{aligned}$$

and so the function (A.7) reduces to the function $((F_1 \sharp F_2) \sharp \dots) \sharp F_N$ obtained by iteratively applying the composition formula of Proposition 2.2.

APPENDIX B. THE HOMOLOGY JOIN

Recall that the \mathbb{Z}_k -join $A *_{\mathbb{Z}_k} B$ of subsets A of $L_k^{2M-1}(\underline{w})$ and B of $L_k^{2M'-1}(\underline{w}')$ is the subset of $L_k^{2(M+M')-1}(\underline{w}, \underline{w}')$ defined by (3.8). In this section we complete the proof of the join quasi-additivity property of the cohomological index (Part (v) of Proposition 3.9) by proving the following lower bounds on the index of the equivariant join: if A and B are closed, then

$$(B.1) \quad \text{ind}(A *_{\mathbb{Z}_k} B) \geq \begin{cases} \text{ind}(A) + \text{ind}(B) & \text{if at least one of the indices is even} \\ \text{ind}(A) + \text{ind}(B) - 1 & \text{if both indices are odd.} \end{cases}$$

To prove this lower bound we develop an equivariant join operation on homology.

The join stability property of Proposition 3.9(v),

$$(B.2) \quad \text{ind}(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) = \text{ind}(A) + 2K,$$

is a special case of the join quasi-additivity property, but since several of our applications of the non-linear Maslov index only need this special case, we also give a short direct proof of it.

B.3. Remark. By continuity of the index (Proposition 3.9(i)), and since for every neighborhood \mathcal{O} of $A *_{\mathbb{Z}_k} B$ there exist neighborhoods U of A and V of B such that \mathcal{O} contains $U *_{\mathbb{Z}_k} V$, it is enough to prove (B.1) and (B.2) when A and B are *open* subsets of lens spaces.

B.4. Lemma. *Let A be an open subset of a lens space $L_k^{2M-1}(\underline{w})$. Then the map*

$$\iota_*: H_j(A; \mathbb{Z}_k) \rightarrow H_j(L_k^{2M-1}(\underline{w}); \mathbb{Z}_k)$$

that is induced by the inclusion $\iota: A \hookrightarrow L_k^{2M-1}(\underline{w})$ is surjective (with image $\cong \mathbb{Z}_k$) for all $j < \text{ind}(A)$ and is the zero map for all $j \geq \text{ind}(A)$.

Proof. Because A is a manifold, its Čech cohomology agrees with its singular cohomology. The result then follows from Lemma 3.3 by the duality between homology and cohomology with field coefficients. \square

Proof of the join stability property (B.2).

Recall that the Thom space of a real vector bundle $\pi: E \rightarrow X$ is the space $\text{Th}(\pi) = D(E)/S(E)$, where $S(E)$ and $D(E)$ denote the total spaces of the corresponding unit sphere bundle and closed disk bundle. An orientation of E gives rise to the Thom isomorphism

$$T: \widetilde{H}_{j+m}(\text{Th}(\pi)) \rightarrow H_j(X) \quad \text{for } j \geq 0,$$

where the tilde denotes reduced homology, and m is the rank of $\pi: E \rightarrow X$ (see for instance [Sp66, Theorem 5.7.10]).

By Remark B.3, we can assume that A is an *open* subset of $L_k^{2M-1}(\underline{w})$. The preimage \widetilde{A} of A in $S_k^{2M-1}(\underline{w}')$ is a principal \mathbb{Z}_k -bundle. Consider the associated vector bundle

$$(B.5) \quad \pi: (\widetilde{A} \times \mathbb{R}^{2K}(\underline{w}')) / \mathbb{Z}_k \rightarrow A.$$

We claim that there is a cofibre sequence

$$(B.6) \quad L_k^{2K-1}(\underline{w}') \hookrightarrow A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}') \xrightarrow{q} \text{Th}(\pi),$$

where the first map is the canonical embedding into the join. Indeed, there is a natural \mathbb{Z}_k -equivariant homeomorphism

$$\psi: (\widetilde{A} * S^{2K-1}(\underline{w}')) \setminus S^{2K-1}(\underline{w}') \rightarrow \widetilde{A} \times \text{int}(D^{2K}(\underline{w}'))$$

where $\text{int}(D^{2K}(\underline{w}')) \subset \mathbb{C}^k(\underline{w}')$ is the open unit disc, and \mathbb{Z}_k acts on $\widetilde{A} \times \text{int}(D^{2K}(\underline{w}'))$ diagonally. The map that ψ induces on the quotient spaces identifies $(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) \setminus L_k^{2K-1}(\underline{w}')$ with the total space of the open disk bundle of the vector bundle (B.5) and extends to a homeomorphism

$$(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) / L_k^{2K-1}(\underline{w}') \rightarrow \text{Th}(\pi),$$

which expresses (B.6) as a cofibre sequence.

The canonical embedding $L_k^{2K-1}(\underline{w}') \hookrightarrow A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')$ is injective in homology (for example, because its composition with the classifying map of $A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')$ is a classifying map for $L_k^{2K-1}(\underline{w}')$, which is injective in homology). It follows that the long exact sequence associated to (B.6) splits into short exact sequences

$$0 \rightarrow H_j(L_k^{2K-1}(\underline{w}')) \rightarrow H_j(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) \rightarrow H_j(\text{Th}(\pi)) \rightarrow 0,$$

and thus the collapse map

$$q: A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}') \rightarrow \text{Th}(\pi)$$

induces an isomorphism in homology in degrees $\geq 2K$. Consider the isomorphism $*\ell$ defined by

$$*\ell: H_j(A) \xrightarrow{T^{-1}} H_{j+2K}(\text{Th}(\pi)) \xrightarrow{(q_*)^{-1}} H_{j+2K}(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) \quad \text{for } j \geq 0.$$

Since $*\ell$ is natural, by applying it to A and to $L_k^{2M-1}(\underline{w})$ we obtain a commuting diagram

$$(B.7) \quad \begin{array}{ccc} H_j(A) & \xrightarrow{*\ell} & H_{j+2K}(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) \\ \downarrow \iota_* & & \downarrow (\iota *_{\mathbb{Z}_k} \text{id})_* \\ H_j(L_k^{2M-1}(\underline{w})) & \xrightarrow{*\ell} & H_{j+2K}(L_k^{2(M+K)-1}(\underline{w}, \underline{w}')) \end{array}$$

Since $\text{ind}(A)$ and $\text{ind}(A *_{\mathbb{Z}_k} L_k^{2K}(\underline{w}'))$ are the lowest degrees of the homology groups on which the corresponding vertical arrows of (B.7) are the zero maps (by Lemma B.4), and since the horizontal arrows of (B.7) are isomorphisms that increase the degree by $2K$, we conclude that $\text{ind}(A *_{\mathbb{Z}_k} L_k^{2K-1}(\underline{w}')) = \text{ind}(A) + 2K$.

The join and the equivariant join. In order to construct a join operation on singular homology we first need to discuss in some detail the join operation on topological spaces. Here, all topological spaces are assumed to be Hausdorff. Let

$$\Delta^m = \{(t_0, \dots, t_m) \in \mathbb{R}_{\geq 0}^{m+1} \mid t_0 + \dots + t_m = 1\}$$

be the standard m -simplex ($m \geq 0$). The *join* of two topological spaces X and Y is defined to be the quotient

$$X * Y = (X \times Y \times \Delta^1) / \sim$$

where the only non-trivial relations are $(x, y, (0, 1)) \sim (x', y, (0, 1))$ and $(x, y, (1, 0)) \sim (x, y', (1, 0))$ for all $x, x' \in X$ and $y, y' \in Y$. We denote the equivalence class of $(x, y, (t_0, t_1))$ in $X * Y$ by $t_0x + t_1y$. Similarly, the join of $(m+1)$ topological spaces X_0, \dots, X_m is defined by

$$X_0 * \dots * X_m = (X_0 \times \dots \times X_m \times \Delta^m) / \sim,$$

with the class of $(x_0, \dots, x_m, (t_0, \dots, t_m))$ denoted by $t_0x_0 + \dots + t_mx_m$, and with $t_0x_0 + \dots + t_mx_m = t'_0x'_0 + \dots + t'_mx'_m$ if and only if $t_j = t'_j$ for all j and $x_j = x'_j$ whenever $t_j \neq 0$.

The join operation is natural, in the sense that continuous functions $f_j: X_j \rightarrow X'_j$ induce a continuous function $f_0 * \dots * f_m: X_0 * \dots * X_m \rightarrow X'_0 * \dots * X'_m$, so that $(g_0 \circ f_0) * \dots * (g_m \circ f_m) = (g_0 * \dots * g_m) \circ (f_0 * \dots * f_m)$. In particular, when the spaces X_j are equipped with an action of a group G , their join acquires the G -action given by $a \cdot (t_0x_0 + \dots + t_mx_m) = t_0(a \cdot x_0) + \dots + t_m(a \cdot x_m)$. The (equivariant) join of principal G -bundles is a principal G -bundle. If X and Y are spaces equipped with principal G -bundles $\tilde{X} \rightarrow X$ and $\tilde{Y} \rightarrow Y$, we define their G -join to be the space $X *_G Y := (\tilde{X} * \tilde{Y})/G$ equipped with the principal bundle $\tilde{X} * \tilde{Y} \rightarrow X *_G Y$.

The natural maps

$$(B.8) \quad (X * Y) * Z \xrightarrow{\varphi_1} X * Y * Z \xleftarrow{\varphi_2} X * (Y * Z)$$

given by $\varphi_1: s_0(t_0x + t_1y) + s_1z \mapsto (s_0t_0)x + (s_0t_1)y + s_1z$ and similarly for φ_2 are homeomorphisms (because they are continuous proper bijections of Hausdorff spaces). We have similar maps for any parenthetization of any number of factors. We also have the map

$$(B.9) \quad \tau: X * Y \rightarrow Y * X, \quad t_0x + t_1y \mapsto t_1y + t_0x.$$

These maps are natural, in the sense that for any continuous maps $f: X \rightarrow X'$, $g: Y \rightarrow Y'$ and $h: Z \rightarrow Z'$ the diagrams

$$(B.10) \quad \begin{array}{ccc} (X * Y) * Z & \xrightarrow{\varphi_1} & X * Y * Z \\ \downarrow (f * g) * h & & \downarrow f * g * h \\ (X' * Y') * Z' & \xrightarrow{\varphi_1} & X' * Y' * Z' \end{array} \quad \text{and} \quad \begin{array}{ccc} X * Y & \xrightarrow{\tau} & Y * X \\ \downarrow f * g & & \downarrow g * f \\ X' * Y' & \xrightarrow{\tau} & Y' * X' \end{array}$$

commute, and similarly for φ_2 . For principal G -bundles over X, Y, Z , the associativity and commutativity homeomorphisms (B.8) and (B.9) pass to the quotients and yield homeomorphisms

$$(X *_G Y) *_G Z \rightarrow X *_G Y *_G Z \leftarrow X *_G (Y *_G Z) \quad \text{and} \quad X *_G Y \rightarrow Y *_G X$$

that are natural in the sense analogous to (B.10).

Joins of standard simplices, spheres, and lens spaces. For the standard unit spheres, we have identifications

$$\psi_{M,M'}: S^{2M-1} * S^{2M'-1} \xrightarrow{\cong} S^{2(M+M')-1}$$

given by

$$t(z_0, \dots, z_M) + t'(z'_0, \dots, z'_{M'}) \rightarrow (\sqrt{t}z_0, \dots, \sqrt{t}z_M, \sqrt{t'}z'_0, \dots, \sqrt{t'}z'_{M'})$$

(these maps are continuous proper bijections, hence homeomorphisms). These identifications descend to lens spaces, where we use the same notation

$$(B.11) \quad \psi_{M,M'}: L_k^{2M-1}(\underline{w}) *_{\mathbb{Z}_k} L_k^{2M'-1}(\underline{w}') \xrightarrow{\cong} L_k^{2(M+M')-1}(\underline{w}, \underline{w}').$$

We have similar identifications for multiple joins.

These identifications are consistent, in the sense that the diagrams

$$(B.12) \quad \begin{array}{ccc} L_k^{2M-1} *_{\mathbb{Z}_k} L_k^{2M'-1} *_{\mathbb{Z}_k} L_k^{2M''-1} & \xrightarrow{\psi_{M,M',M''}} & L_k^{2(M+M'+M'')-1} \\ \downarrow (\varphi_1)^{-1} & & \uparrow \psi_{M+M',M''} \\ (L_k^{2M-1} *_{\mathbb{Z}_k} L_k^{2M'-1}) *_{\mathbb{Z}_k} L_k^{2M''-1} & \xrightarrow{\psi_{M,M'} *_{\mathbb{Z}_k} \text{Id}} & L_k^{2(M+M')-1} *_{\mathbb{Z}_k} L_k^{2M''-1} \end{array}$$

commute, and similarly for the other parenthetization, where to simplify notation we omitted the weights. By induction we get consistent identifications of iterated multiple joins of lens spaces with higher lens spaces.

Given subsets $A \subset L_k^{2M-1}(\underline{w})$ and $B \subset L_k^{2M'-1}(\underline{w}')$, identifying $A *_{\mathbb{Z}_k} B$ with a subset of $L_k^{2M-1}(\underline{w}) *_{\mathbb{Z}_k} L_k^{2M'-1}(\underline{w}')$ by naturality and further with a subset of $L_k^{2(M+M')-1}(\underline{w}, \underline{w}')$ by (B.11), we get the same subset of $L_k^{2(M+M')-1}(\underline{w}, \underline{w}')$ that was described in (3.8).

B.13. Remark. If f and g are smooth maps between spheres or lens spaces, then $f * g$ (viewed as a map between higher dimensional spheres or lens spaces) might not be smooth.

For the standard simplices, the identifications

$$(B.14) \quad \Delta^l * \Delta^m \xrightarrow{\cong} \Delta^{l+m+1}$$

given by

$$u_0(t_0, \dots, t_l) + u_1(s_0, \dots, s_m) \mapsto (u_0 t_0, \dots, u_0 t_l, u_1 s_0, \dots, u_1 s_m)$$

are consistent, in a sense similar to (B.12). In particular, the composition

$$(B.15) \quad \Delta^{l+m+n+2} \cong \Delta^l * \Delta^m * \Delta^n \cong (\Delta^l * \Delta^m) * \Delta^n \cong \Delta^{l+m+1} * \Delta^n \cong \Delta^{l+m+n+2}$$

is the identity map, and similarly with the other parenthetization.

The join operation on homology. Given singular simplices $\sigma: \Delta^l \rightarrow X$ and $\mu: \Delta^m \rightarrow Y$ on X and Y , the identification (B.14) makes their join into a singular $(l+m+1)$ -simplex

$$\sigma * \mu: \Delta^{l+m+1} \rightarrow X * Y.$$

Extending bilinearly, we obtain a map of singular chains

$$(B.16) \quad *: C_l(X) \otimes C_m(Y) \rightarrow C_{l+m+1}(X * Y).$$

Similarly, the triple join gives a map of singular chains

$$C_j(X) \otimes C_l(Y) \otimes C_m(Z) \rightarrow C_{j+l+m+2}(X * Y * Z).$$

The definitions of the join and boundary operations $*$ and ∂ directly imply that for any two singular simplices $\sigma: \Delta^l \rightarrow X$ and $\mu: \Delta^m \rightarrow Y$ we have

$$(B.17) \quad \partial(\sigma * \mu) = \begin{cases} (\partial\sigma) * \mu + (-1)^{l+1} \sigma * \partial\mu & \text{if } l, m > 0, \\ (\partial\sigma) * \mu + (-1)^{l+1} \sigma & \text{if } l > 0, m = 0, \\ \mu - \sigma * (\partial\mu) & \text{if } l = 0, m > 0 \\ \mu - \sigma & \text{if } l = m = 0. \end{cases}$$

It follows from (B.17) that the chain join (B.16) defines an operation

$$*: H_l(X) \otimes H_m(Y) \rightarrow H_{l+m+1}(X * Y) \quad \text{when } l > 0 \text{ and } m > 0.$$

We now show that the same considerations go through also for principal G -bundles for a finite group G , and that moreover if we consider \mathbb{Z}_k -coefficients with k dividing the order of G then the induced operation in homology is defined in all degrees.

Let $\tilde{X} \rightarrow X$ be a principal G -bundle and $C_*(\tilde{X})^G$ the complex of G -invariant chains on \tilde{X} . There is a canonical isomorphism of complexes

$$\varphi: C_*(X) \xrightarrow{\cong} C_*(\tilde{X})^G$$

which on a singular simplex $\sigma: \Delta^m \rightarrow X$ is given by $\varphi(\sigma) = \sum_{g \in G} g \cdot \tilde{\sigma}$, where $\tilde{\sigma}: \Delta^m \rightarrow \tilde{X}$ is any lift of σ . The join operation sends G -invariant chains on \tilde{X} and \tilde{Y} to G -invariant chains on $\tilde{X} * \tilde{Y}$. Defining

$$\sigma *_G \mu := \varphi^{-1}(\varphi(\sigma) * \varphi(\mu))$$

on simplices, and extending bilinearly, we obtain an equivariant join operation on chains,

$$(B.18) \quad *_G: C_l(X) \otimes C_m(Y) \rightarrow C_{l+m+1}(X *_G Y).$$

The definitions of the join operation on chains (B.16) and on equivariant chains (B.18) make sense with arbitrary ring coefficients. We now consider \mathbb{Z}_k -coefficients, for k dividing the order of G .

B.19. Lemma. *Let $\tilde{X} \rightarrow X$ and $\tilde{Y} \rightarrow Y$ be principal G -bundles. Assume that k divides the order of G . Then the equivariant join operation on chains (B.18) satisfies*

$$\partial(\sigma *_G \mu) = (\partial\sigma) *_G \mu + (-1)^{l+1} \sigma *_G \partial\mu.$$

Proof. By bilinearity, it is enough to consider the case of simplices (rather than chains) $\sigma: \Delta^l \rightarrow X$ and $\mu: \Delta^m \rightarrow Y$. If $l > 0$ and $m > 0$ the result follows from (B.17). Assume now that $l > 0$ and $m = 0$ (the remaining cases are similar). Let $\tilde{\sigma}: \Delta^l \rightarrow \tilde{X}$ and $\tilde{\mu}: \Delta^m \rightarrow \tilde{Y}$ be any lifts of σ and μ . Since we use \mathbb{Z}_k -coefficients, we have

$$\begin{aligned} \partial(\sigma *_G \mu) &= \partial\left(\varphi^{-1}(\varphi(\sigma) * \varphi(\mu))\right) \\ &= \varphi^{-1}\left(\partial\left(\sum_{g, h \in G} (g \cdot \tilde{\sigma}) * (h \cdot \tilde{\mu})\right)\right) \\ &= \varphi^{-1}\left(\sum_{g, h \in G} \left((\partial(g \cdot \tilde{\sigma})) * (h \cdot \tilde{\mu}) + (-1)^{l+1} g \cdot \tilde{\sigma}\right)\right) \\ &= \varphi^{-1}\left(\varphi(\partial\sigma) * \varphi(\mu)\right) + \varphi^{-1}\left(|G|(-1)^{l+1} \varphi(\sigma)\right) \\ &= (\partial\sigma) *_G \mu. \end{aligned}$$

□

Lemma B.19 implies that if k divides the order of G then the equivariant join operation on chains induces an operation on homology:

$$*_G: H_l(X; \mathbb{Z}_k) \otimes H_m(Y; \mathbb{Z}_k) \rightarrow H_{l+m+1}(X *_G Y; \mathbb{Z}_k) \quad \text{for all } l \geq 0 \text{ and } m \geq 0.$$

Moreover, the naturality of the chain level formula provides join operations on relative homology

$$*_G: H_l(X, A; \mathbb{Z}_k) \otimes H_m(Y, B; \mathbb{Z}_k) \rightarrow H_{l+m+1}(X *_G Y, (X *_G B) \cup (A *_G Y); \mathbb{Z}_k)$$

such that $\overline{x_l *_G y_m} = \overline{x_l} *_G \overline{y_m}$ for any $x_l \in H_l(X; \mathbb{Z}_k)$ and $y_m \in H_m(Y; \mathbb{Z}_k)$, where $\overline{x_l}$, $\overline{y_m}$ and $\overline{x_l *_G y_m}$ denote the images of x_l , y_m and $x_l *_G y_m$ in relative homology.

It follows from the consistency of the identifications of the standard simplices (specifically, from (B.15) being the identity map) that the join operation on chains (B.16) is associative, in the following sense. For any three singular simplices $\sigma: \Delta^j \rightarrow X$, $\mu: \Delta^l \rightarrow Y$ and $\nu: \Delta^m \rightarrow Z$ we have

$$(B.20) \quad \varphi_1 \circ ((\sigma * \mu) * \nu) = \sigma * \mu * \nu = \varphi_2 \circ (\sigma * (\mu * \nu)),$$

where φ_1 and φ_2 are the homeomorphisms of (B.8). This further implies that the equivariant join operation on homology is associative, in the following sense. For homology classes α, β, γ on spaces X, Y, Z equipped with principal G -bundles with k dividing the order of G , we have $(\varphi_1)_*((\alpha * \beta) * \gamma) = \alpha * \beta * \gamma = (\varphi_2)_*(\alpha * (\beta * \gamma))$.

The join operation in homology also satisfies a commutativity property. We postpone this result to at the end of this appendix (Proposition B.24); we do not need it for our applications.

Computations for lens spaces. Lemma B.21 and Proposition B.22 contain computations of equivariant joins for lens spaces. For our applications, we only need the “if” direction of Proposition B.22 and we don’t need Lemma B.21.

B.21. Lemma. *Let $x_0 \in H_0(L_k^1; \mathbb{Z}_k)$ be the homology class of a point. Then $x_0 *_k x_0 = 0$.*

Proof. The class $x_0 \in H_0(L_k^1; \mathbb{Z}_k)$ is represented by the singular simplex sending Δ^0 to $[1] \in L_k^1$. The class $x_0 *_k x_0 \in H_1(L_k^3; \mathbb{Z}_k)$ is represented by $\sum_{j=0}^{k-1} \sigma_j$, where, for $j = 0, \dots, k-1$, $\sigma_j: \Delta^1 \rightarrow [0, 1] \rightarrow L_k^3$ is the singular 1-simplex given by

$$\sigma_j(t) = \left[\sqrt{1-t}, \sqrt{t} e^{j \frac{2\pi i}{k}} \right].$$

The paths σ_j all have initial point $[1, 0]$ and end point $[0, 1]$ in L_k^3 . The concatenation $\overline{\sigma_0} \sigma_j$ is a loop in L_k^3 in the homology class $[\sigma_j - \sigma_0]$. As the lift of $\overline{\sigma_0} \sigma_j$ to S^3 starting at $(0, 1)$ ends at $(0, e^{2\pi i j/k})$ we see that, for $1 \leq j \leq k-1$, the loop $\overline{\sigma_0} \sigma_j$ generates $\pi_1(L_k^3)$ and $\overline{\sigma_0} \sigma_j = (\overline{\sigma_0} \sigma_1)^j$ in $\pi_1(L_k^3)$. In $H_1(L_k^3)$ we have $[\sigma_j - \sigma_0] = j[\sigma_1 - \sigma_0]$. So

$$x_0 *_k x_0 = \left[\sum_{j=0}^{k-1} \sigma_j \right] = \left[\sum_{j=0}^{k-1} (\sigma_j - \sigma_0) \right] = \sum_{j=0}^{k-1} j[\sigma_1 - \sigma_0] = \frac{k(k-1)}{2} [\sigma_1 - \sigma_0] = 0.$$

□

B.22. Proposition. *Suppose that x_m and $x_{m'}$ are non-zero elements of $H_m(L_k^{2M-1}(\underline{w}); \mathbb{Z}_k)$ and $H_{m'}(L_k^{2M'-1}(\underline{w}'); \mathbb{Z}_k)$. Then the join $x_m *_k x_{m'}$ is non-zero if and only if m or m' is odd.*

Proof. By functoriality, it suffices to consider the case when the weights \underline{w} and \underline{w}' are of the form $(1, \dots, 1)$. Indeed, for any \underline{w} (and similarly for \underline{w}'), the classifying map of $L_k^{2M-1}(\underline{w})$ induces an injection in homology (even over \mathbb{Z}) and can be obtained as the composition of a map $L_k^{2M-1}(\underline{w}) \rightarrow L_k^{2K-1} := L_k^{2K-1}(1, \dots, 1)$ with the classifying map $L_k^{2K-1} \rightarrow L_k^\infty$ for some sufficiently large K .

Let $\sigma_0: \Delta^0 \rightarrow L_k^1$ be the simplex $1 \mapsto [1]$ and let $\sigma_1: \Delta^1 \rightarrow L_k^1$ be the simplex $(t_0, t_1) \mapsto [e^{2\pi i t_1/k}]$. As chains, σ_0 and σ_1 are closed; denote their homology classes by $y_0 = [\sigma_0]$ and $y_1 = [\sigma_1]$.

The standard cell decomposition of L_k^{2M+1} can be described as follows (see for instance [Ha02, Example 2.43]). There is one cell e^j in each dimension $0 \leq j \leq 2M+1$. The standard inclusion $L_k^{2M-1} \rightarrow L_k^{2M+1}$, $[z] \mapsto [z, 0]$ takes the j th cell of L_k^{2M-1} to the j th cell of L_k^{2M+1} for all $0 \leq j \leq 2M-1$, and is injective in homology. Moreover, under the identification $L_k^{2M+1} = L_k^{2M-1} *_k L_k^1$ of (B.11), we have that $e^{2M} = e^{2M-1} *_k e^0$ and $e^{2M+1} = e^{2M-1} *_k e^1$.

When j is odd, the j th skeleton of the cellular decomposition of L_k^{2M+1} is the lens subspace L_k^j , embedded by the standard inclusion $[z] \mapsto [z, 0]$. We denote the j th skeleton of the cell complex by L_k^j even when j is even.

Let

$$\sigma_{2M-1}: (\Delta^{2M-1}, \partial\Delta^{2M-1}) \rightarrow (L_k^{2M-1}, L_k^{2M-2})$$

be a characteristic map of e^{2M-1} . Then $[\sigma_{2M-1}]$ is a generator of $H_{2M-1}(L_k^{2M-1}, L_k^{2M-2}; \mathbb{Z}_k)$. Let y_{2M-1} be the generator of $H_{2M-1}(L_k^{2M-1})$ that maps to $[\sigma_{2M-1}]$ under the isomorphism

$$H_{2M-1}(L_k^{2M-1}; \mathbb{Z}_k) \rightarrow H_{2M-1}(L_k^{2M-1}, L_k^{2M-2}; \mathbb{Z}_k).$$

For $j = 0, 1$ the chains $\sigma_{2M-1} *_{\mathbb{Z}_k} \sigma_j$ are triangulations of the cells e^{2M+j} relative to their boundary, so they represent generators of $H_{2M+j}(L_k^{2M+j}, L_k^{2M+j-1}; \mathbb{Z}_k)$. By the naturality of the equivariant join, $y_{2M-1} *_{\mathbb{Z}_k} y_j$ maps to $[\sigma_{2M-1} *_{\mathbb{Z}_k} \sigma_j]$ under the isomorphism

$$H_{2M+j}(L_k^{2M+j}; \mathbb{Z}_k) \rightarrow H_{2M+j}(L_k^{2M+j}, L_k^{2M+j-1}; \mathbb{Z}_k).$$

It follows that $y_{2M-1} *_{\mathbb{Z}_k} y_j$ is a generator of $H_{2M+j}(L_k^{2M+1}; \mathbb{Z}_k)$.

Taking iterations, and using associativity to remove the brackets, we conclude that each of the classes $y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1$, $y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1 *_{\mathbb{Z}_k} y_0$ and (by a similar argument) $y_0 *_{\mathbb{Z}_k} y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1$ is a generator of the homology group in the appropriate dimension.

Let $x_m \in H_m(L_k^{2M-1}; \mathbb{Z}_k)$ and $x_{m'} \in H_{m'}(L_k^{2M'-1}; \mathbb{Z}_k)$ be non-zero classes. Suppose that m and m' are not both even. Expressing x_m as a non-zero scalar multiple of $y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1$ or $y_0 *_{\mathbb{Z}_k} y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1$, and expressing $x_{m'}$ as a non-zero scalar multiple of $y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1$ or $y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1 *_{\mathbb{Z}_k} y_0$, we conclude (by associativity) that $x_m * x_{m'}$ (in which y_0 might occur as a first or last factor but not both) is non-zero.

Since $(y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1 *_{\mathbb{Z}_k} y_0) *_{\mathbb{Z}_k} (y_0 *_{\mathbb{Z}_k} y_1 *_{\mathbb{Z}_k} \dots *_{\mathbb{Z}_k} y_1)$ is zero (by associativity and by Lemma B.21), it similarly follows that if m and m' are both even then $x_m *_{\mathbb{Z}_k} x_{m'}$ is zero. \square

Proof of the lower bounds (B.1) on the index of a join. By Remark B.3, we can assume that A and B are open subsets of $L_k^{2M-1}(\underline{w})$ and of $L_k^{2M'-1}(\underline{w}')$.

First, suppose that $\text{ind}(A)$ or $\text{ind}(B)$ is even. By Lemma B.4, there exist classes $\alpha \in H_{\text{ind}(A)-1}(A)$ and $\beta \in H_{\text{ind}(B)-1}(B)$ whose images in $H_{\text{ind}(A)-1}(L_k^{2M-1}(\underline{w}))$ and in $H_{\text{ind}(B)-1}(L_k^{2M'-1}(\underline{w}'))$ are non-zero. By Lemma B.22 and the naturality of the equivariant join, $\alpha *_{\mathbb{Z}_k} \beta$ is a class in $H_{\text{ind}(A)+\text{ind}(B)-1}(A *_{\mathbb{Z}_k} B)$ whose image in $H_{\text{ind}(A)+\text{ind}(B)-1}(L_k^{2(M+M')-1}(\underline{w}, \underline{w}'))$ is non-zero. By Lemma B.4, this shows that $\text{ind}(A *_{\mathbb{Z}_k} B) \geq \text{ind}(A) + \text{ind}(B)$.

If $\text{ind}(A)$ or $\text{ind}(B)$ are both odd, we apply a similar argument to classes $\alpha \in H_{\text{ind}(A)-1}(A)$ and $\beta' \in H_{\text{ind}(B)-2}(B)$ to conclude that $\text{ind}(A *_{\mathbb{Z}_k} B) \geq \text{ind}(A) + \text{ind}(B) - 1$.

B.23. Remark. In the case of projective space, any cell of the standard cellular decomposition is the equivariant join of the cell in the previous degree with a 0-cell. Therefore the proof of Proposition B.22 shows that in this case the join of two generators in any degree is non-zero. It follows from this argument and property (iv') in Remark 3.11 that in the case of projective spaces the cohomological index is join additive: for closed subsets A of \mathbb{RP}^M and B of $\mathbb{RP}^{M'}$ we have $\text{ind}(A *_{\mathbb{Z}_2} B) = \text{ind}(A) + \text{ind}(B)$. \diamond

Commutativity of the homology join. We complete our discussion of the join operation on homology with a commutativity property of this operation.

B.24. Proposition. Let $\tilde{X} \rightarrow X$ and $\tilde{Y} \rightarrow Y$ be principal G -bundles and suppose that k divides the order of G . Let $\tau: X *_G Y \rightarrow Y *_G X$ denote the homeomorphism (B.9). Then for $\alpha \in H_l(X; \mathbb{Z}_k)$ and $\beta \in H_m(Y; \mathbb{Z}_k)$ we have

$$\tau_*(\alpha *_G \beta) = (-1)^{(l+1)(m+1)} \beta *_G \alpha.$$

Proof. We use the geometric interpretation of singular cycles that appears in [Ha02, p. 108–109]. Let $x \in C_l(X; \mathbb{Z}_k)$ be a cycle, and write $x = \sum_i x_i \sigma_i$ with $\sigma_i: \Delta^l \rightarrow X$ and $x_i \neq 0$. Let

$$K_x = \left(\coprod \Delta_i^l \right) / \sim$$

be the disjoint union of one l -simplex Δ_i^l for each σ_i , quotiented by the identification of the facets of the Δ_i^l s that give rise (via the maps σ_i) to the same singular $(l-1)$ -simplex. Then the singular simplices σ_i 's assemble to give a map $\sigma: K_x \rightarrow X$. We denote by $\bar{\sigma}_i: \Delta_i^l \rightarrow K_x$ the inclusion in the coproduct followed by quotient. Then

$$\bar{x} := \sum_i x_i \bar{\sigma}_i \in C_l(K_x; \mathbb{Z}_k)$$

is a cycle and $\sigma_*(\bar{x}) = x$. Now let $x = \sum_i x_i \sigma_i \in C_l(X; \mathbb{Z}_k)$ and $y = \sum_j y_j \mu_j \in C_m(Y; \mathbb{Z}_k)$ be cycles representing the homology classes α and β respectively. Then

$$x *_G y = \varphi^{-1} \left(\sum_{g, h \in G} \sum_{i, j} x_i y_j (g \cdot \tilde{\sigma}_i) * (h \cdot \tilde{\mu}_j) \right)$$

and

$$y *_G x = \varphi^{-1} \left(\sum_{g, h \in G} \sum_{i, j} x_i y_j (h \cdot \tilde{\mu}_j) * (g \cdot \tilde{\sigma}_i) \right)$$

where $\tilde{\sigma}_i$ and $\tilde{\mu}_j$ are some lifts of σ_i and μ_j . Let $\nu: K_{x *_G y} \rightarrow X *_G Y$ and $\eta: K_{y *_G x} \rightarrow Y *_G X$ be the maps geometrically realizing the cycles $x *_G y$ and $y *_G x$ via the procedure described above, and $\bar{x} *_G \bar{y} \in C_{l+m+1}(K_{x *_G y}; \mathbb{Z}_k)$ and $\bar{y} *_G \bar{x} \in C_{l+m+1}(K_{y *_G x}; \mathbb{Z}_k)$ be the corresponding cycles. Then the diagram

$$\begin{array}{ccc} K_{x *_G y} & \xrightarrow{\nu} & X *_G Y \\ T \downarrow & & \downarrow \tau \\ K_{y *_G x} & \xrightarrow{\eta} & Y *_G X \end{array}$$

commutes, where T is the homeomorphism induced by the canonical homeomorphisms

$$\Delta^{l+m+1} = \Delta^l * \Delta^m \xrightarrow{\tau} \Delta^m * \Delta^l = \Delta^{l+m+1}$$

between the top cells of $K_{x *_G y}$ and $K_{y *_G x}$. Since for cellular homology with \mathbb{Z}_k -coefficients we have $T_*([\bar{x} *_G \bar{y}]) = (-1)^{(m+1)(l+1)} [\bar{y} *_G \bar{x}]$, the same holds in singular homology. Hence

$$\begin{aligned} \tau_*(\alpha *_G \beta) &= \tau_* \nu_*([\bar{x} *_G \bar{y}]) = \eta_* T_*([\bar{x} *_G \bar{y}]) \\ &= \eta_*((-1)^{(l+1)(m+1)} [\bar{y} *_G \bar{x}]) = (-1)^{(l+1)(m+1)} \beta *_G \alpha. \end{aligned}$$

□

REFERENCES

- [AF12] P. Albers and U. Frauenfelder, A variational approach to Givental's nonlinear Maslov index. *Geom. Funct. Anal.* **22** (2012), 1033–1050.
- [BeS07] G. Ben Simon, The nonlinear Maslov index and the Calabi homomorphism. *Commun. Contemp. Math.* **9** (2007), 769–780.
- [Bh01] M. Bhupal, A partial order on the group of contactomorphisms of \mathbb{R}^{2n+1} via generating functions. *Turkish J. Math.* **25** (2001), 125–135.
- [BEP04] P. Biran, M. Entov and L. Polterovich, Calabi quasimorphisms for the symplectic ball. *Commun. Contemp. Math.* **6** (2004), 793–802.
- [Bo13] M. S. Borman, Quasi-states, quasi-morphisms, and the moment map. *Int. Math. Res. Not.* **11** (2013), 2497–2533.
- [BZ15] M. S. Borman and F. Zapolsky, Quasimorphisms on contactomorphism groups and contact rigidity. *Geom. Topol.* **19** (2015), 365–411.
- [Ca09] D. Calegari, *scl. MSJ Memoirs*, 20. Mathematical Society of Japan, Tokyo, 2009.
- [Ch84] M. Chaperon, Une idée du type “géodésiques brisées” pour les systèmes hamiltoniens. *C. R. Acad. Sci. Paris, Sér. I Math.* **298** (1984), 293–296.
- [CS12] V. Colin and S. Sandon, The discriminant and oscillation lengths for contact and Legendrian isotopies. *J. Eur. Math. Soc.* **17** (2015), 1657–1685.

- [Do95] A. Dold, *Lectures on Algebraic Topology*. Classics in Mathematics, Springer-Verlag, 1995.
- [EKP06] Y. Eliashberg, S.S. Kim and L. Polterovich, Geometry of contact transformations and domains: orderability vs squeezing. *Geom. and Topol.* **10** (2006), 1635–1747.
- [EP00] Y. Eliashberg and L. Polterovich, Partially ordered groups and geometry of contact transformations. *Geom. Funct. Anal.* **10** (2000), 1448–1476.
- [EnP03] M. Entov and L. Polterovich, Calabi quasimorphism and quantum homology. *Int. Math. Res. Not.* **30** (2003), 1635–1676.
- [EnP06] M. Entov and L. Polterovich, Quasistates and symplectic intersections. *Comment. Math. Helv.* **81** (2006), 75–99.
- [FR78] E. Fadell and P. H. Rabinowitz, Generalized cohomological index theories for Lie group actions with an application to bifurcation questions for Hamiltonian systems. *Invent. Math.* **45** (1978), 139–174.
- [FPR18] M. Fraser, L. Polterovich and D. Rosen, On Sandon-type metrics for contactomorphism groups. *Ann. Math. Qué.* **42** (2018), 191–214.
- [Ge08] H. Geiges, *An introduction to contact topology*. Cambridge Studies in Advanced Mathematics, 109. Cambridge University Press, Cambridge, 2008.
- [Gi89] A. B. Givental, Periodic mappings in symplectic topology. *Funktsional. Anal. I Prilozhen.* **23** (1989), 37–52, 96; translation in *Funct. Anal. Appl.* **23** (1989), 287–300.
- [Gi90] A.B. Givental, Nonlinear generalization of the Maslov index. *Theory of singularities and its applications*, 71–103, *Adv. Soviet Math.* **1**, Amer. Math. Soc., Providence, RI, 1990.
- [Gi95] A.B. Givental, A symplectic fixed point theorem for toric manifolds. *The Floer memorial volume*, 445–481, *Progr. Math.*, 133, Birkhäuser, Basel, 1995.
- [GG] R. Gomes and G. Granja, work in progress.
- [Ha02] A. Hatcher, *Algebraic Topology*, Cambridge University Press, Cambridge, 2002.
- [Ho71] L. Hörmander, Fourier integral operators I. *Acta Math.* **127** (1971), 79–183.
- [LS85] F. Laudenbach and J.C. Sikorav, Persistance d’intersection avec la section nulle au cours d’une isotopie hamiltonienne dans un fibré cotangent. *Invent. Math.* **82** (1985), 349–357.
- [L02] E. Lerman, Contact toric manifolds. *Journal of Symplectic Geometry* **1** (2002), no. 4, 785–828.
- [Mi08] I. Milin, Orderability of contactomorphism groups of lens spaces. *Ph.D. Thesis, Stanford University* (2008).
- [Ry10] T. Rybicki, Commutators of contactomorphisms. *Adv. Math.*, **225** (2010), 3291–3326.
- [Sa10] S. Sandon, An integer valued bi-invariant metric on the group of contactomorphisms of $\mathbb{R}^{2n} \times S^1$. *J. Topol. Anal.* **2** (2010), 327–339.
- [Sa11a] S. Sandon, Contact homology, capacity and non-squeezing in $\mathbb{R}^{2n} \times S^1$ via generating functions. *Ann. Inst. Fourier (Grenoble)* **61** (2011), 145–185.
- [Sa11b] S. Sandon, Equivariant homology of generating functions and orderability of lens spaces. *J. Symplectic Geom.* **9** (2011) 123–146.
- [Sa11c] S. Sandon, A Morse estimate for translated points of contactomorphisms of spheres and projective spaces. *Geom. Dedicata* **165** (2013), 95–110.
- [Sa12] S. Sandon, On iterated translated points for contactomorphisms of \mathbb{R}^{2n+1} and $\mathbb{R}^{2n} \times S^1$. *Internat. J. Math.* **23** (2012), 1250042, 14 pp.
- [Si85] J.-C. Sikorav, Sur les immersions lagrangiennes dans un fibré cotangent admettant une phase génératrice globale. *C. R. Acad. Sci. Paris Sér. I Math.* **302** (1986), 119–122.
- [Si87] J.-C. Sikorav, Problèmes d’intersections et de points fixes en géométrie hamiltonienne. *Comment. Math. Helv.* **62** (1987), 62–73.
- [Sp66] E. Spanier, *Algebraic Topology*, Springer-Verlag, New York, 1966.
- [Te18] B. Tervil, Translated points for prequantization spaces over monotone toric manifolds. *arXiv:1811.09984*.
- [Th95] D. Théret, Utilisation des fonctions génératrices en géométrie symplectique globale. *Ph.D. Thesis, Université Denis Diderot (Paris 7)* (1995).
- [Th98] D. Théret, Rotation numbers of Hamiltonian isotopies in complex projective spaces. *Duke Math. J.* **94** (1998), 13–27.
- [Th99] D. Théret, A Lagrangian camel. *Comment. Math. Helv.* **74** (1999), 591–614.
- [Tr94] L. Traynor, Symplectic Homology via generating functions. *Geom. Funct. Anal.* **4** (1994), 718–748.
- [Ts08] T. Tsuboi, On the simplicity of the group of contactomorphisms. *Groups of diffeomorphisms*, 491–504, *Adv. Stud. Pure Math.*, 52, Math. Soc. Japan, Tokyo, 2008.
- [Vi92] C. Viterbo, Symplectic topology as the geometry of generating functions. *Math. Ann.* **292** (1992), 685–710.
- [Vi97] C. Viterbo, Some remarks on Massey products, tied cohomology classes, and the Lusternik-Shnirelman category. *Duke Math. J.* **86** (1997), 547–564.
- [WW13] G. Wilson and C.T. Woodward, Quasimap Floer cohomology for varying symplectic quotients. *Canad. J. Math.* **65** (2013), 467–480.

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