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# *KD* is nullary

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In the ordinary modal language, *KD* is the modal logic determined by the class of all serial frames. In this paper, we demonstrate that *KD* is nullary.

## 1. Introduction

The unification problem in a logical system  $L$  can be defined as follows: given a formula  $\phi(x_1, \dots, x_n)$ , determine whether there exists formulas  $\psi_1, \dots, \psi_n$  such that  $\phi(\psi_1, \dots, \psi_n)$  is in  $L$ . The research on unification was motivated by a closely related and more general decision problem, namely the admissibility problem for rules of inference: given a rule  $\frac{\phi_1(x_1, \dots, x_n), \dots, \phi_m(x_1, \dots, x_n)}{\psi(x_1, \dots, x_n)}$ , decide whether for all formulas  $\chi_1, \dots, \chi_n$ , if  $\phi_1(\chi_1, \dots, \chi_n), \dots, \phi_m(\chi_1, \dots, \chi_n)$  are in  $L$  then  $\psi(\chi_1, \dots, \chi_n)$  is in  $L$ . The admissibility problem for rules was put forward by (Friedman, 1975) who asked whether there exists a decision procedure for deciding whether a given rule preserves validity in intuitionistic logic.

Friedman's problem was solved by (Rybakov, 1984; Rybakov, 1985) who demonstrated that the admissibility problem in intuitionistic logic and modal logic  $S4$  is decidable. See also (Iemhoff, 2001; Rybakov, 2001; Rybakov, Terziler, & Gencer, 1999) for a study of unification and inference rules for modal logics. Later on, (Ghilardi, 1999), proving that intuitionistic logic has a finitary unification type, yielded a new solution of Friedman's problem, seeing that deciding whether a given rule preserves validity in intuitionistic logic is equivalent to checking whether the finitely many maximal unifiers of its premises are unifiers of its conclusion. See also (Ghilardi, & Sacchetti, 2004) for a study of unification and most general unifiers in modal logics. With respect to the complexity issue, (Jerábek, 2007) established the *coNEXPTIME*-completeness of the admissibility problem for several intermediate logics and several  $K4$ -extensions, in contrast with the admissibility problem for modal logics contained in  $K4$  which is undecidable if one considers a language with the universal modality (Wolter, & Zakharyashev, 2008). See also (Gencer, & de Jongh, 2009) for a study of unifiability in extensions of  $K4$ .

Is the situation better if the language is restricted? (Cintula, & Metcalfe, 2010) considered the negation-implication fragment of intuitionistic logic and proved that the associated admissibility problem was *PSPACE*-complete. Unification of concept terms has been introduced by (Baader, & Narendran, 2001) as a tool for detecting

redundancies in knowledge bases. In this respect, (Baader, & Küsters, 2001) established the *EXPTIME*-completeness of the unification problem in the description logic  $\mathcal{FL}_0$  whereas (Baader, & Morawska, 2009; Baader, & Morawska, 2010) established the *NP*-completeness of the unification problem in the description logic  $\mathcal{EL}$ .

Tense logics and epistemic logics provide formalisms for expressing properties about programs, time, knowledge, etc. Within their context, (Dzik, 2007; Dzik, 2011) has studied the relationships between the unification type of a fusion of modal logics and the unification types of the modal logics composing this fusion. The unification type of non-classical logics such as common knowledge logics and linear temporal logics has also been studied by (Babenyshv, & Rybakov, 2011) and (Rybakov, 2002; Rybakov, 2005; Rybakov, 2008).

Nevertheless, very little is known about the unification problem in some of the most important description and modal logics considered in Computer Science and Artificial Intelligence. For example, the decidability of the unification problem for the following description and modal logics remains open: description logic  $\mathcal{ALC}$ , modal logic  $K$ , multimodal variants of  $K$ , sub-Boolean fragments of modal logics.

In the ordinary modal language, the modal logic  $KD$  is the least normal logic containing the formula  $\Box x \rightarrow \Diamond x$ . It is also the modal logic determined by the class of all frames  $(W, R)$  such that  $R$  is serial on  $W$ . Seeing that  $\Box \perp$ ,  $\Box \top$ ,  $\Diamond \perp$  and  $\Diamond \top$  are, respectively, equivalent in  $KD$  to  $\perp$ ,  $\top$ ,  $\perp$  and  $\top$ , it is a well-known fact that  $KD$ -unification is in *NP*. As for the unification type of  $KD$ , in this paper, following a line of reasoning suggested by (Jerábek, 2013) within the context of the modal logic  $K$ , we demonstrate that  $KD$  is nullary.

## 2. Syntax

Let  $VAR$  be a nonempty countable set of *propositional variables* (with typical members denoted  $x, y$ , etc) and  $PAR$  be a nonempty countable set of *propositional parameters* (with typical members denoted  $p, q$ , etc). In this paper, we will always assume that  $VAR \neq \emptyset$ . The set  $\mathcal{L}$  of all *formulas* (with typical members denoted  $\phi, \psi$ , etc) is inductively defined as follows:

- $\phi, \psi ::= x \mid p \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$ .

We write  $\phi(x_1, \dots, x_n)$  to denote a formula whose variables form a subset of  $\{x_1, \dots, x_n\}$ . The Boolean connectives  $\top, \wedge, \rightarrow$  and  $\leftrightarrow$  are defined by the usual abbreviations. Let  $\Diamond$  be the modal connective defined as follows:

- $\Diamond\phi ::= \neg\Box\neg\phi$ .

For all parameters  $p$ , the modal connective  $[p]$  is defined as follows:

- $[p]\phi ::= \Box(p \rightarrow \phi)$ .

For all parameters  $p$ , the modal connective  $[p]^k$  is inductively defined as follows for each  $k \in \mathbb{N}$ :

- $[p]^0\phi ::= \phi$ ,
- $[p]^{k+1}\phi ::= [p][p]^k\phi$ .

For all parameters  $p$ , the modal connective  $[p]^{<k}$  is inductively defined as follows for each  $k \in \mathbb{N}$ :

- $[p]^{<0}\phi ::= \top$ ,
- $[p]^{<k+1}\phi ::= [p]^{<k}\phi \wedge [p]^k\phi$ .

We adopt the standard rules for omission of the parentheses.

**Example 1.**  $\phi = (x \rightarrow p) \wedge (x \rightarrow [p]x)$  is a readable abbreviation for the less readable formula  $\neg(\neg(\neg x \vee p) \vee \neg(\neg x \vee \Box(\neg p \vee x)))$ .

The *degree* of a formula  $\phi$  (in symbols  $\text{deg}(\phi)$ ) is inductively defined as follows:

- $\text{deg}(x) = 0$ ,
- $\text{deg}(p) = 0$ ,
- $\text{deg}(\perp) = 0$ ,
- $\text{deg}(\neg\phi) = \text{deg}(\phi)$ ,
- $\text{deg}(\phi \vee \psi) = \max\{\text{deg}(\phi), \text{deg}(\psi)\}$ ,
- $\text{deg}(\Box\phi) = \text{deg}(\phi) + 1$ .

A *substitution* is a function  $\sigma$  associating to each variable  $x$  a formula  $\sigma(x)$ . We shall say that a substitution  $\sigma$  is *closed* if for all variables  $x$ ,  $\sigma(x)$  is a variable-free formula. For all formulas  $\phi(x_1, \dots, x_m)$ , let  $\sigma(\phi(x_1, \dots, x_m))$  be  $\phi(\sigma(x_1), \dots, \sigma(x_m))$ . The *composition*  $\sigma \circ \tau$  of the substitutions  $\sigma$  and  $\tau$  associates to each variable  $x$  the formula  $\tau(\sigma(x))$ .

**Example 2.** If  $\phi$  is the formula considered in Example 1 and  $\sigma_p$  is the substitution defined by  $\sigma_p(x) = p$  then  $\sigma_p(\phi) = (p \rightarrow p) \wedge (p \rightarrow [p]p)$ .

**Example 3.** If  $\phi$  is the formula considered in Example 1,  $k \in \mathbb{N}$  and  $\sigma_k$  is the substitution defined by  $\sigma_k(x) = p \wedge [p]^{<k}x \wedge [p]^k\perp$  then  $\sigma_k(\phi) = (p \wedge [p]^{<k}x \wedge [p]^k\perp \rightarrow p) \wedge (p \wedge [p]^{<k}x \wedge [p]^k\perp \rightarrow [p](p \wedge [p]^{<k}x \wedge [p]^k\perp))$ .

### 3. Semantics

A *frame* is a relational structure of the form  $\mathcal{F} = (W, R)$  where  $W$  is a nonempty set of states (with typical members denoted  $s, t$ , etc) and  $R$  is a binary relation on  $W$ . A *model* based on a frame  $\mathcal{F} = (W, R)$  is a relational structure of the form  $\mathcal{M} = (W, R, V)$  where  $V$  is a function associating to each variable  $x$  a set  $V(x)$  of states and to each parameter  $p$  a set  $V(p)$  of states. The relation *formula  $\phi$  is true in model  $\mathcal{M}$  at state  $s$*  (in symbols  $\mathcal{M}, s \models \phi$ ) is inductively defined as follows:

- $\mathcal{M}, s \models x$  iff  $s \in V(x)$ ,
- $\mathcal{M}, s \models p$  iff  $s \in V(p)$ ,
- $\mathcal{M}, s \not\models \perp$ ,
- $\mathcal{M}, s \models \neg\phi$  iff  $\mathcal{M}, s \not\models \phi$ ,
- $\mathcal{M}, s \models \phi \vee \psi$  iff either  $\mathcal{M}, s \models \phi$ , or  $\mathcal{M}, s \models \psi$ ,
- $\mathcal{M}, s \models \Box\phi$  iff for all states  $t \in W$ , if  $sRt$  then  $\mathcal{M}, t \models \phi$ .

Let  $\mathcal{C}$  be a class of frames. We shall say that a formula  $\phi$  is  *$\mathcal{C}$ -valid* (in symbols  $\mathcal{C} \models \phi$ ) if for all frames  $\mathcal{F} = (W, R)$  in  $\mathcal{C}$ , for all models  $\mathcal{M} = (W, R, V)$  based on  $\mathcal{F}$  and for all states  $s \in W$ ,  $\mathcal{M}, s \models \phi$ .

**Example 4.** The following formulas are valid in the class of all frames:

- $[p]p$ ,
- $[p]^{<k}x \wedge [p]^k\perp \rightarrow [p]([p]^{<k}x \wedge [p]^k\perp)$ .

Let  $\mathcal{C}$  be a class of frames. We shall say that a substitution  $\sigma$  is  *$\mathcal{C}$ -equivalent* to a substitution  $\tau$  (in symbols  $\sigma \simeq_{\mathcal{C}} \tau$ ) if for all variables  $x$ ,  $\mathcal{C} \models \sigma(x) \leftrightarrow \tau(x)$ . We shall say that a substitution  $\sigma$  is more  *$\mathcal{C}$ -general* than a substitution  $\tau$  (in symbols  $\sigma \preceq_{\mathcal{C}} \tau$ ) if

there exists a substitution  $v$  such that  $\sigma \circ v \simeq_{\mathcal{C}} \tau$ .

#### 4. Unification problem

Let  $\mathcal{C}$  be a class of frames. We shall say that a formula  $\phi$  is  $\mathcal{C}$ -unifiable if there exists a substitution  $\sigma$  such that  $\mathcal{C} \models \sigma(\phi)$ . In that case,  $\sigma$  is a  $\mathcal{C}$ -unifier of  $\phi$ .

**Example 5.** Let  $\mathcal{C}$  be a class of frames. If  $\phi$  is the formula considered in Example 1 then the substitution  $\sigma_p$  considered in Example 2 is a  $\mathcal{C}$ -unifier of  $\phi$ .

**Example 6.** Let  $\mathcal{C}$  be a class of frames. If  $\phi$  is the formula considered in Example 1 and  $k \in \mathbb{N}$  then the substitution  $\sigma_k$  considered in Example 3 is a  $\mathcal{C}$ -unifier of  $\phi$ .

Given a class  $\mathcal{C}$  of frames, an important question is the following:

**$\mathcal{C}$ -unification:** given a formula  $\phi$ , decide whether  $\phi$  is  $\mathcal{C}$ -unifiable.

Let  $\mathcal{C}^{ser}$  be the class of all serial frames,  $\mathcal{C}^{ref}$  be the class of all reflexive frames and  $\mathcal{C}^{sym}$  be the class of all symmetrical frames.

**Proposition 1.** If  $PAR = \emptyset$  then  $\mathcal{C}^{ser}$ -unification and  $\mathcal{C}^{ref}$ -unification are in NP.

*Proof.* Suppose  $PAR = \emptyset$ .

**$\mathcal{C}^{ser}$ -unification:** Hence, in  $\mathcal{C}^{ser}$ , every variable-free formula is equivalent to  $\perp$  or  $\top$ . This is a well-known fact. It partly follows from the fact that  $\Box\perp$ ,  $\Box\top$ ,  $\Diamond\perp$  and  $\Diamond\top$  are, respectively,  $\mathcal{C}^{ser}$ -equivalent to  $\perp$ ,  $\top$ ,  $\perp$  and  $\top$ . Thus, every closed substitution  $\sigma$  is  $\mathcal{C}^{ser}$ -equivalent to a substitution  $\tau$  such that for each variable  $x$ ,  $\tau(x) = \perp$  or  $\tau(x) = \top$ . Moreover, if a formula  $\phi$  possesses a  $\mathcal{C}^{ser}$ -unifier then  $\phi$  possesses a closed  $\mathcal{C}^{ser}$ -unifier. This follows from the fact that for all  $\mathcal{C}^{ser}$ -unifiers  $\sigma$  of  $\phi$  and for all closed substitutions  $\tau$ ,  $\sigma \circ \tau$  is a closed  $\mathcal{C}^{ser}$ -unifier of  $\phi$ . Consequently, for all formulas  $\phi$ , the following conditions are equivalent:  $\phi$  is  $\mathcal{C}^{ser}$ -unifiable; there exists a  $\mathcal{C}^{ser}$ -unifier  $\sigma$  of  $\phi$  such that for all variables  $x$ ,  $\sigma(x) = \perp$  or  $\sigma(x) = \top$ . Hence, for all formulas  $\phi(x_1, \dots, x_n)$ , to decide whether  $\phi(x_1, \dots, x_n)$  is  $\mathcal{C}^{ser}$ -unifiable, it suffices to nondeterministically guess  $\psi_1, \dots, \psi_n \in \{\perp, \top\}$  and to determine whether  $\phi(\psi_1, \dots, \psi_n)$  is  $\mathcal{C}^{ser}$ -equivalent to  $\perp$  or  $\top$ . Obviously, this can be done in polynomial time.

**$\mathcal{C}^{ref}$ -unification:** Similar to  $\mathcal{C}^{ser}$ -unification. □

The decidability status of  $\mathcal{C}^{ser}$ -unification and  $\mathcal{C}^{ref}$ -unification are unknown when  $PAR \neq \emptyset$ . The decidability status of  $\mathcal{C}^{sym}$ -unification is unknown both when  $PAR \neq \emptyset$  and when  $PAR = \emptyset$ . Let  $\mathcal{C}^{par}$  be the class of all partitions.

**Proposition 2.** If  $PAR = \emptyset$  then  $\mathcal{C}^{par}$ -unification is in NP.

*Proof.* Similar to the proof of Proposition 1. □

$\mathcal{C}^{par}$ -unification remains decidable when  $PAR \neq \emptyset$ . See (Balbiani, & Gencer, 2015) for details. Let  $\mathcal{C}^{det}$  be the class of all deterministic frames.

**Proposition 3.**  $\mathcal{C}^{det}$ -unification is in PSPACE.

*Proof.* See (Balbiani, & Tinchev, 2014). □

Let  $\mathcal{C}^{tra}$  be the class of all transitive frames,  $\mathcal{C}^{rt}$  be the class of all reflexive transitive frames and  $\mathcal{C}^{twf}$  be the class of all transitive well-founded frames.

**Proposition 4.** (1)  $\mathcal{C}^{tra}$ -unification is decidable.

(2)  $\mathcal{C}^{rt}$ -unification is decidable (in NP when  $PAR = \emptyset$ ).

(3)  $\mathcal{C}^{twf}$ -unification is decidable.

*Proof.* See (Ghilardi, 2000). □

As for the decidability status of unification in the class  $\mathcal{C}^{all}$  of all frames, it is unknown.

## 5. Unification type

Let  $\mathcal{C}$  be a class of frames. We shall say that a set  $\Sigma$  of unifiers of a unifiable formula  $\phi$  is *complete* if for all unifiers  $\sigma$  of  $\phi$ , there exists a unifier  $\tau$  of  $\phi$  in  $\Sigma$  such that  $\tau \preceq_{\mathcal{C}} \sigma$ . An important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is “yes”, how large is this set? We shall say that a unifiable formula

- $\phi$  is *nullary* if there exists no minimal complete set of unifiers of  $\phi$ ,
- $\phi$  is *infinitary* if there exists an infinite minimal complete set of unifiers of  $\phi$  but there exists no finite one,
- $\phi$  is *finitary* if there exists a finite minimal complete set of unifiers of  $\phi$  but there exists no with cardinality 1,
- $\phi$  is *unitary* if there exists a minimal complete set of unifiers of  $\phi$  with cardinality 1.

We shall say that

- $\mathcal{C}$  is *nullary* if there exists a nullary formula,
- $\mathcal{C}$  is *infinitary* if every unifiable formula is either infinitary, or finitary, or unitary and there exists a infinitary formula,
- $\mathcal{C}$  is *finitary* if every unifiable formula is either finitary, or unitary and there exists a finitary formula,
- $\mathcal{C}$  is *unitary* if every unifiable formula is unitary.

**Proposition 5.** *If  $PAR \neq \emptyset$  then  $\mathcal{C}^{ser}$  is nullary.*

*Proof.* See Section 6. □

The unification type of  $\mathcal{C}^{ser}$  is unknown when  $PAR = \emptyset$ . The unification type of  $\mathcal{C}^{ref}$  and the unification type of  $\mathcal{C}^{sym}$  are unknown both when  $PAR \neq \emptyset$  and when  $PAR = \emptyset$ .

**Proposition 6.**  *$\mathcal{C}^{par}$  is unitary.*

*Proof.* See (Baader:, & Ghilardi, 2011). □

**Proposition 7.**  *$\mathcal{C}^{det}$  is nullary.*

*Proof.* See (Balbiani, & Tinchev, 2014). □

**Proposition 8.** *(1)  $\mathcal{C}^{tra}$  is finitary.*

*(2)  $\mathcal{C}^{rt}$  is finitary.*

*(3)  $\mathcal{C}^{twf}$  is finitary.*

*Proof.* See (Ghilardi, 2000). □

As for the unification type of  $\mathcal{C}^{all}$ , it is nullary (Jerábek, 2013).

## 6. $KD$ is nullary

In this section, we will always assume that  $PAR \neq \emptyset$ . Let  $\mathcal{C}$  be a class of frames such that for all  $k, l \in \mathbb{N}$ , if  $k < l$  then  $\mathcal{C} \not\models p \wedge [p]^l \perp \rightarrow [p]^k \perp$ . Remark that the class  $\mathcal{C}^{ser}$  of all serial frames is of that kind whereas neither the class  $\mathcal{C}^{ref}$  of all reflexive frames, nor the class  $\mathcal{C}^{sym}$  of all symmetrical frames are of that kind. Let  $\phi = (x \rightarrow p) \wedge (x \rightarrow [p]x)$  be the formula considered in Example 1. Let  $\Sigma = \{\sigma_p\} \cup \{\sigma_k : k \in \mathbb{N}\}$  where  $\sigma_p$  is the substitution defined by  $\sigma_p(x) = p$  and considered in Example 2 and for all  $k \in \mathbb{N}$ ,  $\sigma_k$  is the substitution defined by  $\sigma_k(x) = p \wedge [p]^{<k}x \wedge [p]^k \perp$  and considered in Example 3. By Examples 5 and 6, we know that  $\Sigma$  is a set of unifiers of  $\phi$ .

**Lemma 1.** *Let  $k, l \in \mathbb{N}$ . If  $k \leq l$  then  $\sigma_l \preceq_{\mathcal{C}} \sigma_k$ .*

*Proof.* Suppose  $k \leq l$ . Let  $v$  be the substitution defined by  $v(x) = x \wedge [p]^k \perp$ . The reader may easily verify that  $\mathcal{C} \models v(\sigma_l(x)) \leftrightarrow \sigma_k(x)$ . Hence,  $\sigma_l \preceq_{\mathcal{C}} \sigma_k$ .  $\square$

**Lemma 2.** *Let  $k, l \in \mathbb{N}$ . If  $k < l$  then  $\sigma_k \not\preceq_{\mathcal{C}} \sigma_l$ .*

*Proof.* Suppose  $k < l$  and  $\sigma_k \preceq_{\mathcal{C}} \sigma_l$ . Let  $v$  be a substitution such that  $\sigma_k \circ v \simeq_{\mathcal{C}} \sigma_l$ . Hence,  $\mathcal{C} \models v(\sigma_k(x)) \leftrightarrow \sigma_l(x)$ . Thus,  $\mathcal{C} \models p \wedge [p]^{<l}x \wedge [p]^l \perp \rightarrow [p]^k \perp$ . Consequently,  $\mathcal{C} \models p \wedge [p]^l \perp \rightarrow [p]^k \perp$ : a contradiction.  $\square$

Remark that the proof of Lemma 2 is the only one in this section where we use the fact that for all  $k, l \in \mathbb{N}$ , if  $k < l$  then  $\mathcal{C} \not\models p \wedge [p]^l \perp \rightarrow [p]^k \perp$ . As for Lemma 2, its use will be crucial in the proof of Lemma 7.

**Lemma 3.** *Let  $\sigma$  be a substitution. The following conditions are equivalent:*

- (1)  $\sigma_p \circ \sigma \simeq_{\mathcal{C}} \sigma$ .
- (2)  $\sigma_p \preceq_{\mathcal{C}} \sigma$ .
- (3)  $\mathcal{C} \models \sigma(x) \leftrightarrow p$ .

*Proof.* (1.  $\Rightarrow$  2) : By definition of  $\preceq_{\mathcal{C}}$ .

(2.  $\Rightarrow$  3) : Suppose  $\sigma_p \preceq_{\mathcal{C}} \sigma$ . Let  $v$  be a substitution such that  $\sigma_p \circ v \simeq_{\mathcal{C}} \sigma$ . Hence,  $\mathcal{C} \models v(\sigma_p(x)) \leftrightarrow \sigma(x)$ . Thus,  $\mathcal{C} \models \sigma(x) \leftrightarrow p$ .

(3.  $\Rightarrow$  1) : Suppose  $\mathcal{C} \models \sigma(x) \leftrightarrow p$ . Hence,  $\mathcal{C} \models \sigma(\sigma_p(x)) \leftrightarrow \sigma(x)$ . Thus,  $\sigma_p \circ \sigma \simeq_{\mathcal{C}} \sigma$ .  $\square$

**Lemma 4.** *Let  $k \in \mathbb{N}$ . Let  $\sigma$  be a unifier of  $\phi$ . The following conditions are equivalent:*

- (1)  $\sigma_k \circ \sigma \simeq_{\mathcal{C}} \sigma$ .
- (2)  $\sigma_k \preceq_{\mathcal{C}} \sigma$ .
- (3)  $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$ .

*Proof.* (1.  $\Rightarrow$  2) : By definition of  $\preceq_{\mathcal{C}}$ .

(2.  $\Rightarrow$  3) : Suppose  $\sigma_k \preceq_{\mathcal{C}} \sigma$ . Let  $v$  be a substitution such that  $\sigma_k \circ v \simeq_{\mathcal{C}} \sigma$ . Hence,  $\mathcal{C} \models v(\sigma_k(x)) \leftrightarrow \sigma(x)$ . Thus,  $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$ .

(3.  $\Rightarrow$  1) : Suppose  $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$ . Since  $\sigma$  is a unifier of  $\phi$ , therefore  $\mathcal{C} \models \sigma(x) \rightarrow p$  and  $\mathcal{C} \models \sigma(x) \rightarrow [p]\sigma(x)$ . Hence,  $\mathcal{C} \models \sigma(x) \rightarrow [p]^{<k}\sigma(x)$ . Since  $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$  and  $\mathcal{C} \models \sigma(x) \rightarrow p$ , therefore  $\mathcal{C} \models \sigma(x) \rightarrow \sigma(\sigma_k(x))$ . Now, we consider the following 2 cases.

**Case  $k = 0$ :** Thus,  $\mathcal{C} \models [p]^k \perp \rightarrow \sigma(x)$ .

**Case  $k \geq 1$ :** Consequently,  $\mathcal{C} \models [p]^{<k}\sigma(x) \rightarrow \sigma(x)$ .

In both cases,  $\mathcal{C} \models \sigma(\sigma_k(x)) \leftrightarrow \sigma(x)$ . Since  $\mathcal{C} \models \sigma(x) \rightarrow \sigma(\sigma_k(x))$ , therefore  $\mathcal{C} \models \sigma(\sigma_k(x)) \leftrightarrow \sigma(x)$ . Hence,  $\sigma_k \circ \sigma \simeq_{\mathcal{C}} \sigma$ .  $\square$

**Lemma 5.** *Let  $\sigma$  be a unifier of  $\phi$ . If  $\mathcal{C} = \mathcal{C}^{ser}$  then one of the following conditions holds:*



- (1)  $\sigma_p \preceq_C \sigma$ .
- (2) There exists  $k \in \mathbb{N}$  such that  $\sigma_k \preceq_C \sigma$ .

*Proof.* Suppose  $\mathcal{C} = \mathcal{C}^{ser}$  and none of the above conditions holds. By Lemmas 3 and 4,  $\mathcal{C} \not\models \sigma(x) \leftrightarrow p$  and  $\mathcal{C} \not\models \sigma(x) \rightarrow [p]^{\deg(\sigma(x))} \perp$ . Since  $\sigma$  is a unifier of  $\phi$ , therefore  $\mathcal{C} \models \sigma(x) \rightarrow p$ . Let  $\mathcal{F} = (W, R)$  and  $\mathcal{F}' = (W', R')$  be frames in  $\mathcal{C}$ ,  $\mathcal{M} = (W, R, V)$  and  $\mathcal{M}' = (W', R', V')$  be models based respectively on  $\mathcal{F}$  and  $\mathcal{F}'$  and  $s \in W$  and  $s' \in W'$  be pairwise distinct states such that  $\mathcal{M}, s \not\models p \rightarrow \sigma(x)$  and  $\mathcal{M}', s' \not\models \sigma(x) \rightarrow [p]^{\deg(\sigma(x))} \perp$ . Hence,  $\mathcal{M}, s \models p$ ,  $\mathcal{M}, s \not\models \sigma(x)$ ,  $\mathcal{M}', s' \models \sigma(x)$  and  $\mathcal{M}', s' \not\models [p]^{\deg(\sigma(x))} \perp$ . Let  $t'_0, \dots, t'_{\deg(\sigma(x))} \in W'$  be states such that  $t'_0 = s'$  and for all  $i \in \mathbb{N}$ , if  $i < \deg(\sigma(x))$  then  $t'_i R' t'_{i+1}$  and  $t'_{i+1} \in V'(p)$ . Since  $\mathcal{C} = \mathcal{C}^{ser}$ , therefore without loss of generality, we may assume that  $t'_0, \dots, t'_{\deg(\sigma(x))}$  are pairwise distinct and constitute the shortest  $p$ -path in  $W'$  between  $t'_0$  and  $t'_{\deg(\sigma(x))}$ . Thus, this path is of length  $\deg(\sigma(x))$ . Let  $\mathcal{M}'' = (W'', R'', V'')$  be the model defined as follows:

- $W'' = W \cup W'$ ,
- $R'' = R \cup R' \cup \{(t'_{\deg(\sigma(x))}, s)\}$ ,
- $V'' = V \cup V'$ .

Since  $\mathcal{M}, s \models p$  and  $\mathcal{M}, s \not\models \sigma(x)$ , therefore  $\mathcal{M}'', s \models p$  and  $\mathcal{M}'', s \not\models \sigma(x)$ . Since  $t'_0, \dots, t'_{\deg(\sigma(x))}$  is the shortest  $p$ -path in  $W'$  between  $t'_0$  and  $t'_{\deg(\sigma(x))}$ ,  $\mathcal{M}', s' \models \sigma(x)$ ,  $t'_0 = s'$  and for all  $i \in \mathbb{N}$ , if  $i < \deg(\sigma(x))$  then  $t'_i R' t'_{i+1}$  and  $t'_{i+1} \in V'(p)$ , therefore  $\mathcal{M}'', s' \models \sigma(x)$ . Since  $\sigma$  is a unifier of  $\phi$ , therefore  $\mathcal{C} \models \sigma(x) \rightarrow [p]\sigma(x)$ . Since  $\mathcal{M}'', s \models p$  and  $\mathcal{M}'', s' \models \sigma(x)$ , therefore  $\mathcal{M}'', s \models \sigma(x)$ : a contradiction.  $\square$

**Lemma 6.** *If  $\mathcal{C} = \mathcal{C}^{ser}$  then  $\Sigma$  is a complete set of unifiers of  $\phi$ .*

*Proof.* By Lemma 5.  $\square$

**Lemma 7.** *If  $\mathcal{C} = \mathcal{C}^{ser}$  then there exists no minimal complete set of unifiers of  $\phi$ .*

*Proof.* Suppose  $\mathcal{C} = \mathcal{C}^{ser}$  and let  $\Gamma$  be a minimal complete set of unifiers of  $\phi$ . Remark that by definition of  $\Sigma$ ,  $\sigma_0 \in \Sigma$ . Moreover, in any class of frames,  $\sigma_0(x)$  is equivalent to  $\perp$ . Let  $\gamma \in \Gamma$  be such that  $\gamma \preceq_C \sigma_0$ . Since  $\mathcal{C} = \mathcal{C}^{ser}$ , therefore by Lemma 6, let  $\sigma \in \Sigma$  be such that  $\sigma \preceq_C \gamma$ . Now, we consider the following 2 cases.

**Case  $\sigma = \sigma_p$ :** Since  $\gamma \preceq_C \sigma_0$ , therefore  $\sigma \preceq_C \sigma_0$ . Let  $v$  be a substitution such that  $\sigma \circ v \simeq_C \sigma_0$ . Hence,  $\mathcal{C} \models v(\sigma(x)) \leftrightarrow \sigma_0(x)$ . Thus,  $\mathcal{C} \models \neg p$ : a contradiction.

**Case  $\sigma = \sigma_k$  for some  $k \in \mathbb{N}$ :** Remark that by definition of  $\Sigma$ ,  $\sigma_{k+1} \in \Sigma$ . Let  $\gamma' \in \Gamma$  be such that  $\gamma' \preceq_C \sigma_{k+1}$ . Since  $\sigma \preceq_C \gamma$ , therefore by Lemma 1,  $\gamma' \preceq_C \gamma$ . Since  $\Gamma$  is a minimal complete set of unifiers of  $\phi$ , therefore  $\gamma' = \gamma$ . Since  $\gamma' \preceq_C \sigma_{k+1}$  and  $\sigma \preceq_C \gamma$ , therefore  $\sigma_k \preceq_C \sigma_{k+1}$ . Since  $k < k+1$ , therefore by Lemma 2,  $\sigma_k \not\preceq_C \sigma_{k+1}$ : a contradiction.  $\square$

Remark that the last argument in the proof of Lemma 7 heavily relies on Lemma 2. Therefore, unfortunately, it cannot be repeated when, as in the cases  $\mathcal{C} = \mathcal{C}^{ref}$  and  $\mathcal{C} = \mathcal{C}^{sym}$ , there exists  $k, l \in \mathbb{N}$  such that  $k < l$  and  $\mathcal{C} \models p \wedge [p]^l \perp \rightarrow [p]^k \perp$ . Finally, we obtain the

**Proposition 9.**  *$\mathcal{C}^{ser}$  is nullary.*

*Proof.* By Lemma 7.  $\square$

## 7. Additional comments

In the context of modal logics, classes of frames such as the ones underlying  $K$  and  $KD$  give rise to quite similar sets of valid formulas for what concerns axiomatization and decidability. Putting known results adapted from (Baader, & Ghilardi, 2011; Dzik, 2003; Dzik, 2007; Ghilardi, 2000; Jerábek, 2013) together with new ones enables us to establish basic facts and outline open problems. See Tab. 1. While the study of  $K$  and  $KD$  has now limited mathematical interest for what concerns axiomatization and decidability, considering unification types in modal logics is justified from the following perspectives: methods for deciding the unifiability of formulas can be used to improve the efficiency of automated theorem provers (Babenyshev, Rybakov, Schmidt, & Tishkovsky, 2010); deciding the unifiability of formulas like  $\phi \leftrightarrow \psi$  helps us to understand what is the overlap between the properties  $\phi$  and  $\psi$  correspond to (Baader, & Ghilardi, 2011); in description logics, unification algorithms are used to detect redundancies in knowledge-based systems (Baader, Borgwardt, & Morawska, 2012). One readily observes that, while attacking the above-mentioned problems, little, if anything, from the standard tools in modal logics (canonical models, filtrations, etc) is helpful. In order to successfully solve them, new techniques in modal logics must be developed and much remains to be done. The study of unification types in modal logics has still many secrets to reveal. The most intriguing of them possibly being, both when  $PAR \neq \emptyset$  and when  $PAR = \emptyset$ , the unification types of  $\mathcal{C}^{ref}$  and  $\mathcal{C}^{sym}$ .

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Class	$PAR$	Computability	Type
$\mathcal{C}^{ser}$	$= \emptyset$	in $NP$ (Proposition 1)	?
$\mathcal{C}^{ref}$	$= \emptyset$	in $NP$ (Proposition 1)	?
$\mathcal{C}^{sym}$	$= \emptyset$	?	?
$\mathcal{C}^{par}$	$= \emptyset$	in $NP$ (Proposition 2)	unitary (Baader, & Ghilardi, 2011)
$\mathcal{C}^{det}$	$= \emptyset$	in $PSPACE$ (Balbiani, Tinchev, 2014)	nullary (Balbiani, & Tinchev, 2014)
$\mathcal{C}^{tra}$	$= \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
$\mathcal{C}^{rt}$	$= \emptyset$	in $NP$ (Proposition 2)	finitary (Ghilardi, 2000)
$\mathcal{C}^{twf}$	$= \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
$\mathcal{C}^{all}$	$= \emptyset$	?	nullary (Jerábek, 2013)
$\mathcal{C}^{ser}$	$\neq \emptyset$	?	nullary (Proposition 9)
$\mathcal{C}^{ref}$	$\neq \emptyset$	?	?
$\mathcal{C}^{sym}$	$\neq \emptyset$	?	?
$\mathcal{C}^{par}$	$\neq \emptyset$	decidable (Balbiani, & Gencer, 2015)	unitary
$\mathcal{C}^{det}$	$\neq \emptyset$	in $PSPACE$ (Balbiani, & Tinchev, 2014)	nullary (Balbiani, & Tinchev, 2014)
$\mathcal{C}^{tra}$	$\neq \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
$\mathcal{C}^{rt}$	$\neq \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
$\mathcal{C}^{twf}$	$\neq \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
$\mathcal{C}^{all}$	$\neq \emptyset$	?	nullary (Jerábek, 2013)

Table 1.