



HAL
open science

KD is nullary

Philippe Balbiani, Cigdem Gencer

► **To cite this version:**

Philippe Balbiani, Cigdem Gencer. KD is nullary. *Journal of Applied Non-Classical Logics*, 2017, 27 (3-4), pp.196-205. 10.1080/11663081.2018.1428000 . hal-02365665

HAL Id: hal-02365665

<https://hal.science/hal-02365665v1>

Submitted on 15 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Open Archive Toulouse Archive Ouverte

OATAO is an open access repository that collects the work of Toulouse researchers and makes it freely available over the web where possible

This is an author's version published in:

<http://oatao.univ-toulouse.fr/22052>

Official URL

DOI : <https://doi.org/10.1080/11663081.2018.1428000>

To cite this version: Balbiani, Philippe and Gencer, Cigdem *KD is nullary*. (2017) Journal of Applied Non-Classical Logics, 27 (3-4). 196-205. ISSN 1166-3081

Any correspondence concerning this service should be sent to the repository administrator: tech-oatao@listes-diff.inp-toulouse.fr

KD is nullary

Philippe Balbiani¹ and Çiğdem Gencer²

¹Institut de recherche en informatique de Toulouse
CNRS — Université de Toulouse

²Faculty of Engineering and Natural Sciences
Sabancı University

In the ordinary modal language, *KD* is the modal logic determined by the class of all serial frames. In this paper, we demonstrate that *KD* is nullary.

1. Introduction

The unification problem in a logical system L can be defined as follows: given a formula $\phi(x_1, \dots, x_n)$, determine whether there exists formulas ψ_1, \dots, ψ_n such that $\phi(\psi_1, \dots, \psi_n)$ is in L . The research on unification was motivated by a closely related and more general decision problem, namely the admissibility problem for rules of inference: given a rule $\frac{\phi_1(x_1, \dots, x_n), \dots, \phi_m(x_1, \dots, x_n)}{\psi(x_1, \dots, x_n)}$, decide whether for all formulas χ_1, \dots, χ_n , if $\phi_1(\chi_1, \dots, \chi_n), \dots, \phi_m(\chi_1, \dots, \chi_n)$ are in L then $\psi(\chi_1, \dots, \chi_n)$ is in L . The admissibility problem for rules was put forward by (Friedman, 1975) who asked whether there exists a decision procedure for deciding whether a given rule preserves validity in intuitionistic logic.

Friedman's problem was solved by (Rybakov, 1984; Rybakov, 1985) who demonstrated that the admissibility problem in intuitionistic logic and modal logic $S4$ is decidable. See also (Iemhoff, 2001; Rybakov, 2001; Rybakov, Terziler, & Gencer, 1999) for a study of unification and inference rules for modal logics. Later on, (Ghilardi, 1999), proving that intuitionistic logic has a finitary unification type, yielded a new solution of Friedman's problem, seeing that deciding whether a given rule preserves validity in intuitionistic logic is equivalent to checking whether the finitely many maximal unifiers of its premises are unifiers of its conclusion. See also (Ghilardi, & Sacchetti, 2004) for a study of unification and most general unifiers in modal logics. With respect to the complexity issue, (Jerábek, 2007) established the *coNEXPTIME*-completeness of the admissibility problem for several intermediate logics and several $K4$ -extensions, in contrast with the admissibility problem for modal logics contained in $K4$ which is undecidable if one considers a language with the universal modality (Wolter, & Zakharyashev, 2008). See also (Gencer, & de Jongh, 2009) for a study of unifiability in extensions of $K4$.

Is the situation better if the language is restricted? (Cintula, & Metcalfe, 2010) considered the negation-implication fragment of intuitionistic logic and proved that the associated admissibility problem was *PSPACE*-complete. Unification of concept terms has been introduced by (Baader, & Narendran, 2001) as a tool for detecting

redundancies in knowledge bases. In this respect, (Baader, & Küsters, 2001) established the *EXPTIME*-completeness of the unification problem in the description logic \mathcal{FL}_0 whereas (Baader, & Morawska, 2009; Baader, & Morawska, 2010) established the *NP*-completeness of the unification problem in the description logic \mathcal{EL} .

Tense logics and epistemic logics provide formalisms for expressing properties about programs, time, knowledge, etc. Within their context, (Dzik, 2007; Dzik, 2011) has studied the relationships between the unification type of a fusion of modal logics and the unification types of the modal logics composing this fusion. The unification type of non-classical logics such as common knowledge logics and linear temporal logics has also been studied by (Babenyshv, & Rybakov, 2011) and (Rybakov, 2002; Rybakov, 2005; Rybakov, 2008).

Nevertheless, very little is known about the unification problem in some of the most important description and modal logics considered in Computer Science and Artificial Intelligence. For example, the decidability of the unification problem for the following description and modal logics remains open: description logic \mathcal{ALC} , modal logic K , multimodal variants of K , sub-Boolean fragments of modal logics.

In the ordinary modal language, the modal logic KD is the least normal logic containing the formula $\Box x \rightarrow \Diamond x$. It is also the modal logic determined by the class of all frames (W, R) such that R is serial on W . Seeing that $\Box \perp$, $\Box \top$, $\Diamond \perp$ and $\Diamond \top$ are, respectively, equivalent in KD to \perp , \top , \perp and \top , it is a well-known fact that KD -unification is in *NP*. As for the unification type of KD , in this paper, following a line of reasoning suggested by (Jerábek, 2013) within the context of the modal logic K , we demonstrate that KD is nullary.

2. Syntax

Let VAR be a nonempty countable set of *propositional variables* (with typical members denoted x, y , etc) and PAR be a nonempty countable set of *propositional parameters* (with typical members denoted p, q , etc). In this paper, we will always assume that $VAR \neq \emptyset$. The set \mathcal{L} of all *formulas* (with typical members denoted ϕ, ψ , etc) is inductively defined as follows:

- $\phi, \psi ::= x \mid p \mid \perp \mid \neg\phi \mid (\phi \vee \psi) \mid \Box\phi$.

We write $\phi(x_1, \dots, x_n)$ to denote a formula whose variables form a subset of $\{x_1, \dots, x_n\}$. The Boolean connectives $\top, \wedge, \rightarrow$ and \leftrightarrow are defined by the usual abbreviations. Let \Diamond be the modal connective defined as follows:

- $\Diamond\phi ::= \neg\Box\neg\phi$.

For all parameters p , the modal connective $[p]$ is defined as follows:

- $[p]\phi ::= \Box(p \rightarrow \phi)$.

For all parameters p , the modal connective $[p]^k$ is inductively defined as follows for each $k \in \mathbb{N}$:

- $[p]^0\phi ::= \phi$,
- $[p]^{k+1}\phi ::= [p][p]^k\phi$.

For all parameters p , the modal connective $[p]^{<k}$ is inductively defined as follows for each $k \in \mathbb{N}$:

- $[p]^{<0}\phi ::= \top$,
- $[p]^{<k+1}\phi ::= [p]^{<k}\phi \wedge [p]^k\phi$.

We adopt the standard rules for omission of the parentheses.

Example 1. $\phi = (x \rightarrow p) \wedge (x \rightarrow [p]x)$ is a readable abbreviation for the less readable formula $\neg(\neg(\neg x \vee p) \vee \neg(\neg x \vee \Box(\neg p \vee x)))$.

The *degree* of a formula ϕ (in symbols $\text{deg}(\phi)$) is inductively defined as follows:

- $\text{deg}(x) = 0$,
- $\text{deg}(p) = 0$,
- $\text{deg}(\perp) = 0$,
- $\text{deg}(\neg\phi) = \text{deg}(\phi)$,
- $\text{deg}(\phi \vee \psi) = \max\{\text{deg}(\phi), \text{deg}(\psi)\}$,
- $\text{deg}(\Box\phi) = \text{deg}(\phi) + 1$.

A *substitution* is a function σ associating to each variable x a formula $\sigma(x)$. We shall say that a substitution σ is *closed* if for all variables x , $\sigma(x)$ is a variable-free formula. For all formulas $\phi(x_1, \dots, x_m)$, let $\sigma(\phi(x_1, \dots, x_m))$ be $\phi(\sigma(x_1), \dots, \sigma(x_m))$. The *composition* $\sigma \circ \tau$ of the substitutions σ and τ associates to each variable x the formula $\tau(\sigma(x))$.

Example 2. If ϕ is the formula considered in Example 1 and σ_p is the substitution defined by $\sigma_p(x) = p$ then $\sigma_p(\phi) = (p \rightarrow p) \wedge (p \rightarrow [p]p)$.

Example 3. If ϕ is the formula considered in Example 1, $k \in \mathbb{N}$ and σ_k is the substitution defined by $\sigma_k(x) = p \wedge [p]^{<k}x \wedge [p]^k\perp$ then $\sigma_k(\phi) = (p \wedge [p]^{<k}x \wedge [p]^k\perp \rightarrow p) \wedge (p \wedge [p]^{<k}x \wedge [p]^k\perp \rightarrow [p](p \wedge [p]^{<k}x \wedge [p]^k\perp))$.

3. Semantics

A *frame* is a relational structure of the form $\mathcal{F} = (W, R)$ where W is a nonempty set of states (with typical members denoted s, t , etc) and R is a binary relation on W . A *model* based on a frame $\mathcal{F} = (W, R)$ is a relational structure of the form $\mathcal{M} = (W, R, V)$ where V is a function associating to each variable x a set $V(x)$ of states and to each parameter p a set $V(p)$ of states. The relation *formula ϕ is true in model \mathcal{M} at state s* (in symbols $\mathcal{M}, s \models \phi$) is inductively defined as follows:

- $\mathcal{M}, s \models x$ iff $s \in V(x)$,
- $\mathcal{M}, s \models p$ iff $s \in V(p)$,
- $\mathcal{M}, s \not\models \perp$,
- $\mathcal{M}, s \models \neg\phi$ iff $\mathcal{M}, s \not\models \phi$,
- $\mathcal{M}, s \models \phi \vee \psi$ iff either $\mathcal{M}, s \models \phi$, or $\mathcal{M}, s \models \psi$,
- $\mathcal{M}, s \models \Box\phi$ iff for all states $t \in W$, if sRt then $\mathcal{M}, t \models \phi$.

Let \mathcal{C} be a class of frames. We shall say that a formula ϕ is \mathcal{C} -*valid* (in symbols $\mathcal{C} \models \phi$) if for all frames $\mathcal{F} = (W, R)$ in \mathcal{C} , for all models $\mathcal{M} = (W, R, V)$ based on \mathcal{F} and for all states $s \in W$, $\mathcal{M}, s \models \phi$.

Example 4. The following formulas are valid in the class of all frames:

- $[p]p$,
- $[p]^{<k}x \wedge [p]^k\perp \rightarrow [p]([p]^{<k}x \wedge [p]^k\perp)$.

Let \mathcal{C} be a class of frames. We shall say that a substitution σ is \mathcal{C} -*equivalent* to a substitution τ (in symbols $\sigma \simeq_{\mathcal{C}} \tau$) if for all variables x , $\mathcal{C} \models \sigma(x) \leftrightarrow \tau(x)$. We shall say that a substitution σ is more \mathcal{C} -*general* than a substitution τ (in symbols $\sigma \preceq_{\mathcal{C}} \tau$) if

there exists a substitution v such that $\sigma \circ v \simeq_{\mathcal{C}} \tau$.

4. Unification problem

Let \mathcal{C} be a class of frames. We shall say that a formula ϕ is \mathcal{C} -unifiable if there exists a substitution σ such that $\mathcal{C} \models \sigma(\phi)$. In that case, σ is a \mathcal{C} -unifier of ϕ .

Example 5. Let \mathcal{C} be a class of frames. If ϕ is the formula considered in Example 1 then the substitution σ_p considered in Example 2 is a \mathcal{C} -unifier of ϕ .

Example 6. Let \mathcal{C} be a class of frames. If ϕ is the formula considered in Example 1 and $k \in \mathbb{N}$ then the substitution σ_k considered in Example 3 is a \mathcal{C} -unifier of ϕ .

Given a class \mathcal{C} of frames, an important question is the following:

\mathcal{C} -unification: given a formula ϕ , decide whether ϕ is \mathcal{C} -unifiable.

Let \mathcal{C}^{ser} be the class of all serial frames, \mathcal{C}^{ref} be the class of all reflexive frames and \mathcal{C}^{sym} be the class of all symmetrical frames.

Proposition 1. If $PAR = \emptyset$ then \mathcal{C}^{ser} -unification and \mathcal{C}^{ref} -unification are in NP.

Proof. Suppose $PAR = \emptyset$.

\mathcal{C}^{ser} -unification: Hence, in \mathcal{C}^{ser} , every variable-free formula is equivalent to \perp or \top . This is a well-known fact. It partly follows from the fact that $\Box\perp$, $\Box\top$, $\Diamond\perp$ and $\Diamond\top$ are, respectively, \mathcal{C}^{ser} -equivalent to \perp , \top , \perp and \top . Thus, every closed substitution σ is \mathcal{C}^{ser} -equivalent to a substitution τ such that for each variable x , $\tau(x) = \perp$ or $\tau(x) = \top$. Moreover, if a formula ϕ possesses a \mathcal{C}^{ser} -unifier then ϕ possesses a closed \mathcal{C}^{ser} -unifier. This follows from the fact that for all \mathcal{C}^{ser} -unifiers σ of ϕ and for all closed substitutions τ , $\sigma \circ \tau$ is a closed \mathcal{C}^{ser} -unifier of ϕ . Consequently, for all formulas ϕ , the following conditions are equivalent: ϕ is \mathcal{C}^{ser} -unifiable; there exists a \mathcal{C}^{ser} -unifier σ of ϕ such that for all variables x , $\sigma(x) = \perp$ or $\sigma(x) = \top$. Hence, for all formulas $\phi(x_1, \dots, x_n)$, to decide whether $\phi(x_1, \dots, x_n)$ is \mathcal{C}^{ser} -unifiable, it suffices to nondeterministically guess $\psi_1, \dots, \psi_n \in \{\perp, \top\}$ and to determine whether $\phi(\psi_1, \dots, \psi_n)$ is \mathcal{C}^{ser} -equivalent to \perp or \top . Obviously, this can be done in polynomial time.

\mathcal{C}^{ref} -unification: Similar to \mathcal{C}^{ser} -unification. □

The decidability status of \mathcal{C}^{ser} -unification and \mathcal{C}^{ref} -unification are unknown when $PAR \neq \emptyset$. The decidability status of \mathcal{C}^{sym} -unification is unknown both when $PAR \neq \emptyset$ and when $PAR = \emptyset$. Let \mathcal{C}^{par} be the class of all partitions.

Proposition 2. If $PAR = \emptyset$ then \mathcal{C}^{par} -unification is in NP.

Proof. Similar to the proof of Proposition 1. □

\mathcal{C}^{par} -unification remains decidable when $PAR \neq \emptyset$. See (Balbiani, & Gencer, 2015) for details. Let \mathcal{C}^{det} be the class of all deterministic frames.

Proposition 3. \mathcal{C}^{det} -unification is in PSPACE.

Proof. See (Balbiani, & Tinchev, 2014). □

Let \mathcal{C}^{tra} be the class of all transitive frames, \mathcal{C}^{rt} be the class of all reflexive transitive frames and \mathcal{C}^{twf} be the class of all transitive well-founded frames.

Proposition 4. (1) \mathcal{C}^{tra} -unification is decidable.

(2) \mathcal{C}^{rt} -unification is decidable (in NP when $PAR = \emptyset$).

(3) \mathcal{C}^{twf} -unification is decidable.

Proof. See (Ghilardi, 2000). □

As for the decidability status of unification in the class \mathcal{C}^{all} of all frames, it is unknown.

5. Unification type

Let \mathcal{C} be a class of frames. We shall say that a set Σ of unifiers of a unifiable formula ϕ is *complete* if for all unifiers σ of ϕ , there exists a unifier τ of ϕ in Σ such that $\tau \preceq_{\mathcal{C}} \sigma$. An important question is the following: when a formula is unifiable, has it a minimal complete set of unifiers? When the answer is “yes”, how large is this set? We shall say that a unifiable formula

- ϕ is *nullary* if there exists no minimal complete set of unifiers of ϕ ,
- ϕ is *infinitary* if there exists an infinite minimal complete set of unifiers of ϕ but there exists no finite one,
- ϕ is *finitary* if there exists a finite minimal complete set of unifiers of ϕ but there exists no with cardinality 1,
- ϕ is *unitary* if there exists a minimal complete set of unifiers of ϕ with cardinality 1.

We shall say that

- \mathcal{C} is *nullary* if there exists a nullary formula,
- \mathcal{C} is *infinitary* if every unifiable formula is either infinitary, or finitary, or unitary and there exists a infinitary formula,
- \mathcal{C} is *finitary* if every unifiable formula is either finitary, or unitary and there exists a finitary formula,
- \mathcal{C} is *unitary* if every unifiable formula is unitary.

Proposition 5. *If $PAR \neq \emptyset$ then \mathcal{C}^{ser} is nullary.*

Proof. See Section 6. □

The unification type of \mathcal{C}^{ser} is unknown when $PAR = \emptyset$. The unification type of \mathcal{C}^{ref} and the unification type of \mathcal{C}^{sym} are unknown both when $PAR \neq \emptyset$ and when $PAR = \emptyset$.

Proposition 6. *\mathcal{C}^{par} is unitary.*

Proof. See (Baader:, & Ghilardi, 2011). □

Proposition 7. *\mathcal{C}^{det} is nullary.*

Proof. See (Balbiani, & Tinchev, 2014). □

Proposition 8. *(1) \mathcal{C}^{tra} is finitary.*

(2) \mathcal{C}^{rt} is finitary.

(3) \mathcal{C}^{twf} is finitary.

Proof. See (Ghilardi, 2000). □

As for the unification type of \mathcal{C}^{all} , it is nullary (Jerábek, 2013).

6. KD is nullary

In this section, we will always assume that $PAR \neq \emptyset$. Let \mathcal{C} be a class of frames such that for all $k, l \in \mathbb{N}$, if $k < l$ then $\mathcal{C} \not\models p \wedge [p]^l \perp \rightarrow [p]^k \perp$. Remark that the class \mathcal{C}^{ser} of all serial frames is of that kind whereas neither the class \mathcal{C}^{ref} of all reflexive frames, nor the class \mathcal{C}^{sym} of all symmetrical frames are of that kind. Let $\phi = (x \rightarrow p) \wedge (x \rightarrow [p]x)$ be the formula considered in Example 1. Let $\Sigma = \{\sigma_p\} \cup \{\sigma_k : k \in \mathbb{N}\}$ where σ_p is the substitution defined by $\sigma_p(x) = p$ and considered in Example 2 and for all $k \in \mathbb{N}$, σ_k is the substitution defined by $\sigma_k(x) = p \wedge [p]^{<k}x \wedge [p]^k \perp$ and considered in Example 3. By Examples 5 and 6, we know that Σ is a set of unifiers of ϕ .

Lemma 1. *Let $k, l \in \mathbb{N}$. If $k \leq l$ then $\sigma_l \preceq_{\mathcal{C}} \sigma_k$.*

Proof. Suppose $k \leq l$. Let v be the substitution defined by $v(x) = x \wedge [p]^k \perp$. The reader may easily verify that $\mathcal{C} \models v(\sigma_l(x)) \leftrightarrow \sigma_k(x)$. Hence, $\sigma_l \preceq_{\mathcal{C}} \sigma_k$. \square

Lemma 2. *Let $k, l \in \mathbb{N}$. If $k < l$ then $\sigma_k \not\preceq_{\mathcal{C}} \sigma_l$.*

Proof. Suppose $k < l$ and $\sigma_k \preceq_{\mathcal{C}} \sigma_l$. Let v be a substitution such that $\sigma_k \circ v \simeq_{\mathcal{C}} \sigma_l$. Hence, $\mathcal{C} \models v(\sigma_k(x)) \leftrightarrow \sigma_l(x)$. Thus, $\mathcal{C} \models p \wedge [p]^{<l}x \wedge [p]^l \perp \rightarrow [p]^k \perp$. Consequently, $\mathcal{C} \models p \wedge [p]^l \perp \rightarrow [p]^k \perp$: a contradiction. \square

Remark that the proof of Lemma 2 is the only one in this section where we use the fact that for all $k, l \in \mathbb{N}$, if $k < l$ then $\mathcal{C} \not\models p \wedge [p]^l \perp \rightarrow [p]^k \perp$. As for Lemma 2, its use will be crucial in the proof of Lemma 7.

Lemma 3. *Let σ be a substitution. The following conditions are equivalent:*

- (1) $\sigma_p \circ \sigma \simeq_{\mathcal{C}} \sigma$.
- (2) $\sigma_p \preceq_{\mathcal{C}} \sigma$.
- (3) $\mathcal{C} \models \sigma(x) \leftrightarrow p$.

Proof. (1. \Rightarrow 2) : By definition of $\preceq_{\mathcal{C}}$.

(2. \Rightarrow 3) : Suppose $\sigma_p \preceq_{\mathcal{C}} \sigma$. Let v be a substitution such that $\sigma_p \circ v \simeq_{\mathcal{C}} \sigma$. Hence, $\mathcal{C} \models v(\sigma_p(x)) \leftrightarrow \sigma(x)$. Thus, $\mathcal{C} \models \sigma(x) \leftrightarrow p$.

(3. \Rightarrow 1) : Suppose $\mathcal{C} \models \sigma(x) \leftrightarrow p$. Hence, $\mathcal{C} \models \sigma(\sigma_p(x)) \leftrightarrow \sigma(x)$. Thus, $\sigma_p \circ \sigma \simeq_{\mathcal{C}} \sigma$. \square

Lemma 4. *Let $k \in \mathbb{N}$. Let σ be a unifier of ϕ . The following conditions are equivalent:*

- (1) $\sigma_k \circ \sigma \simeq_{\mathcal{C}} \sigma$.
- (2) $\sigma_k \preceq_{\mathcal{C}} \sigma$.
- (3) $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$.

Proof. (1. \Rightarrow 2) : By definition of $\preceq_{\mathcal{C}}$.

(2. \Rightarrow 3) : Suppose $\sigma_k \preceq_{\mathcal{C}} \sigma$. Let v be a substitution such that $\sigma_k \circ v \simeq_{\mathcal{C}} \sigma$. Hence, $\mathcal{C} \models v(\sigma_k(x)) \leftrightarrow \sigma(x)$. Thus, $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$.

(3. \Rightarrow 1) : Suppose $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$. Since σ is a unifier of ϕ , therefore $\mathcal{C} \models \sigma(x) \rightarrow p$ and $\mathcal{C} \models \sigma(x) \rightarrow [p]\sigma(x)$. Hence, $\mathcal{C} \models \sigma(x) \rightarrow [p]^{<k}\sigma(x)$. Since $\mathcal{C} \models \sigma(x) \rightarrow [p]^k \perp$ and $\mathcal{C} \models \sigma(x) \rightarrow p$, therefore $\mathcal{C} \models \sigma(x) \rightarrow \sigma(\sigma_k(x))$. Now, we consider the following 2 cases.

Case $k = 0$: Thus, $\mathcal{C} \models [p]^k \perp \rightarrow \sigma(x)$.

Case $k \geq 1$: Consequently, $\mathcal{C} \models [p]^{<k}\sigma(x) \rightarrow \sigma(x)$.

In both cases, $\mathcal{C} \models \sigma(\sigma_k(x)) \leftrightarrow \sigma(x)$. Since $\mathcal{C} \models \sigma(x) \rightarrow \sigma(\sigma_k(x))$, therefore $\mathcal{C} \models \sigma(\sigma_k(x)) \leftrightarrow \sigma(x)$. Hence, $\sigma_k \circ \sigma \simeq_{\mathcal{C}} \sigma$. \square

Lemma 5. *Let σ be a unifier of ϕ . If $\mathcal{C} = \mathcal{C}^{ser}$ then one of the following conditions holds:*

- (1) $\sigma_p \preceq_C \sigma$.
- (2) There exists $k \in \mathbb{N}$ such that $\sigma_k \preceq_C \sigma$.

Proof. Suppose $\mathcal{C} = \mathcal{C}^{ser}$ and none of the above conditions holds. By Lemmas 3 and 4, $\mathcal{C} \not\models \sigma(x) \leftrightarrow p$ and $\mathcal{C} \not\models \sigma(x) \rightarrow [p]^{\deg(\sigma(x))} \perp$. Since σ is a unifier of ϕ , therefore $\mathcal{C} \models \sigma(x) \rightarrow p$. Let $\mathcal{F} = (W, R)$ and $\mathcal{F}' = (W', R')$ be frames in \mathcal{C} , $\mathcal{M} = (W, R, V)$ and $\mathcal{M}' = (W', R', V')$ be models based respectively on \mathcal{F} and \mathcal{F}' and $s \in W$ and $s' \in W'$ be pairwise distinct states such that $\mathcal{M}, s \not\models p \rightarrow \sigma(x)$ and $\mathcal{M}', s' \not\models \sigma(x) \rightarrow [p]^{\deg(\sigma(x))} \perp$. Hence, $\mathcal{M}, s \models p$, $\mathcal{M}, s \not\models \sigma(x)$, $\mathcal{M}', s' \models \sigma(x)$ and $\mathcal{M}', s' \not\models [p]^{\deg(\sigma(x))} \perp$. Let $t'_0, \dots, t'_{\deg(\sigma(x))} \in W'$ be states such that $t'_0 = s'$ and for all $i \in \mathbb{N}$, if $i < \deg(\sigma(x))$ then $t'_i R' t'_{i+1}$ and $t'_{i+1} \in V'(p)$. Since $\mathcal{C} = \mathcal{C}^{ser}$, therefore without loss of generality, we may assume that $t'_0, \dots, t'_{\deg(\sigma(x))}$ are pairwise distinct and constitute the shortest p -path in W' between t'_0 and $t'_{\deg(\sigma(x))}$. Thus, this path is of length $\deg(\sigma(x))$. Let $\mathcal{M}'' = (W'', R'', V'')$ be the model defined as follows:

- $W'' = W \cup W'$,
- $R'' = R \cup R' \cup \{(t'_{\deg(\sigma(x))}, s)\}$,
- $V'' = V \cup V'$.

Since $\mathcal{M}, s \models p$ and $\mathcal{M}, s \not\models \sigma(x)$, therefore $\mathcal{M}'', s \models p$ and $\mathcal{M}'', s \not\models \sigma(x)$. Since $t'_0, \dots, t'_{\deg(\sigma(x))}$ is the shortest p -path in W' between t'_0 and $t'_{\deg(\sigma(x))}$, $\mathcal{M}', s' \models \sigma(x)$, $t'_0 = s'$ and for all $i \in \mathbb{N}$, if $i < \deg(\sigma(x))$ then $t'_i R' t'_{i+1}$ and $t'_{i+1} \in V'(p)$, therefore $\mathcal{M}'', s' \models \sigma(x)$. Since σ is a unifier of ϕ , therefore $\mathcal{C} \models \sigma(x) \rightarrow [p]\sigma(x)$. Since $\mathcal{M}'', s \models p$ and $\mathcal{M}'', s' \models \sigma(x)$, therefore $\mathcal{M}'', s \models \sigma(x)$: a contradiction. \square

Lemma 6. *If $\mathcal{C} = \mathcal{C}^{ser}$ then Σ is a complete set of unifiers of ϕ .*

Proof. By Lemma 5. \square

Lemma 7. *If $\mathcal{C} = \mathcal{C}^{ser}$ then there exists no minimal complete set of unifiers of ϕ .*

Proof. Suppose $\mathcal{C} = \mathcal{C}^{ser}$ and let Γ be a minimal complete set of unifiers of ϕ . Remark that by definition of Σ , $\sigma_0 \in \Sigma$. Moreover, in any class of frames, $\sigma_0(x)$ is equivalent to \perp . Let $\gamma \in \Gamma$ be such that $\gamma \preceq_C \sigma_0$. Since $\mathcal{C} = \mathcal{C}^{ser}$, therefore by Lemma 6, let $\sigma \in \Sigma$ be such that $\sigma \preceq_C \gamma$. Now, we consider the following 2 cases.

Case $\sigma = \sigma_p$: Since $\gamma \preceq_C \sigma_0$, therefore $\sigma \preceq_C \sigma_0$. Let v be a substitution such that $\sigma \circ v \simeq_C \sigma_0$. Hence, $\mathcal{C} \models v(\sigma(x)) \leftrightarrow \sigma_0(x)$. Thus, $\mathcal{C} \models \neg p$: a contradiction.

Case $\sigma = \sigma_k$ for some $k \in \mathbb{N}$: Remark that by definition of Σ , $\sigma_{k+1} \in \Sigma$. Let $\gamma' \in \Gamma$ be such that $\gamma' \preceq_C \sigma_{k+1}$. Since $\sigma \preceq_C \gamma$, therefore by Lemma 1, $\gamma' \preceq_C \gamma$. Since Γ is a minimal complete set of unifiers of ϕ , therefore $\gamma' = \gamma$. Since $\gamma' \preceq_C \sigma_{k+1}$ and $\sigma \preceq_C \gamma$, therefore $\sigma_k \preceq_C \sigma_{k+1}$. Since $k < k+1$, therefore by Lemma 2, $\sigma_k \not\preceq_C \sigma_{k+1}$: a contradiction. \square

Remark that the last argument in the proof of Lemma 7 heavily relies on Lemma 2. Therefore, unfortunately, it cannot be repeated when, as in the cases $\mathcal{C} = \mathcal{C}^{ref}$ and $\mathcal{C} = \mathcal{C}^{sym}$, there exists $k, l \in \mathbb{N}$ such that $k < l$ and $\mathcal{C} \models p \wedge [p]^l \perp \rightarrow [p]^k \perp$. Finally, we obtain the

Proposition 9. *\mathcal{C}^{ser} is nullary.*

Proof. By Lemma 7. \square

7. Additional comments

In the context of modal logics, classes of frames such as the ones underlying K and KD give rise to quite similar sets of valid formulas for what concerns axiomatization and decidability. Putting known results adapted from (Baader, & Ghilardi, 2011; Dzik, 2003; Dzik, 2007; Ghilardi, 2000; Jerábek, 2013) together with new ones enables us to establish basic facts and outline open problems. See Tab. 1. While the study of K and KD has now limited mathematical interest for what concerns axiomatization and decidability, considering unification types in modal logics is justified from the following perspectives: methods for deciding the unifiability of formulas can be used to improve the efficiency of automated theorem provers (Babenyshev, Rybakov, Schmidt, & Tishkovsky, 2010); deciding the unifiability of formulas like $\phi \leftrightarrow \psi$ helps us to understand what is the overlap between the properties ϕ and ψ correspond to (Baader, & Ghilardi, 2011); in description logics, unification algorithms are used to detect redundancies in knowledge-based systems (Baader, Borgwardt, & Morawska, 2012). One readily observes that, while attacking the above-mentioned problems, little, if anything, from the standard tools in modal logics (canonical models, filtrations, etc) is helpful. In order to successfully solve them, new techniques in modal logics must be developed and much remains to be done. The study of unification types in modal logics has still many secrets to reveal. The most intriguing of them possibly being, both when $PAR \neq \emptyset$ and when $PAR = \emptyset$, the unification types of \mathcal{C}^{ref} and \mathcal{C}^{sym} .

Acknowledgement

This paper has been written on the occasion of a 3-months visit of Çiğdem Gencer during the Fall 2015 in Toulouse that was financially supported by the *Paul Sabatier University* (“Professeurs invités 2015”) and the *Institut de recherche en informatique de Toulouse* (“Actions spécifiques 2015”).

Special acknowledgement

Special acknowledgement is heartily granted to Luis Fariñas del Cerro. His research on modal logic as a general tool for the formalization of human reasoning has exerted a profound influence on our research. To our knowledge, this paper does not relate in any way to Luis’ own papers. We hope it can constitute the subject of a joint work in the near future.

References

- Baader, F., Borgwardt, S., & Morawska, B. (2012). Extending unification in \mathcal{EL} towards general TBoxes. In Brewka, G. *et al.* (editors): *Principles of Knowledge Representation and Reasoning*. AAAI Press (2012) 568–572.
- Baader, F., & Ghilardi, S. (2011). Unification in modal and description logics. *Logic Journal of the IGPL* **19** (2011) 705–730.
- Baader, F., & Küsters, R. (2001). Unification in a description logic with transitive closure of roles. In Nieuwebhuis, R., Voronkov, A. (editors): *Logic for Programming and Automated Reasoning*. Springer (2001) 217–232.
- Baader, F., & Morawska, B. (2009). Unification in the description logic \mathcal{EL} . In Treinen, R. (editor): *Rewriting Techniques and Applications*. Springer (2009) 350–364.

- Baader, F., & Morawska, B. (2010). SAT encoding of unification in \mathcal{EL} . In Fermüller, C., Voronkov, A. (editors): *Logic for Programming, Artificial Intelligence, and Reasoning*. Springer (2010) 97–111.
- Baader, F., & Narendran, P. (2001). Unification of concept terms in description logics. *Journal of Symbolic Computation* **31** (2001) 277–305.
- Babenyshv, S., & Rybakov, V. (2011). Unification in linear temporal logic *LTL*. *Annals of Pure and Applied Logic* **162** (2011) 991–1000.
- Babenyshv, S., Rybakov, V., Schmidt, R., & Tishkovsky, D. (2010). A tableau method for checking rule admissibility in *S4*. *Electronic Notes in Theoretical Computer Science* **262** (2010) 17–32.
- Balbiani, P., & Gencer, Ç. (to appear). Unification in epistemic logics. *Journal of Applied Non-Classical Logics* (to appear).
- Balbiani, P., & Tinchev, T. (2016). Unifiability and unification types in the normal modal logic *Alt₁*. In: *Advances in Modal Logic*. College Publications (2016) 117–134.
- Cintula, P., & Metcalfe, G. (2010). Admissible rules in the implication-negation fragment of intuitionistic logic. *Annals of Pure and Applied Logic* **162** (2010) 162–171.
- Dzik, W. (2003). Unitary unification of *S5* modal logics and its extensions. *Bulletin of the Section of Logic* **32** (2003) 19–26.
- Dzik, W. (2007). Unification Types in Logic. Wydawnictwo Uniwersytetu Śląskiego (2007).
- Dzik, W. (2011). Remarks on projective unifiers. *Bulletin of the Section of Logic* **40** (2011) 37–46.
- Friedman, H. (1975). One hundred and two problems in mathematical logic. *Journal of Symbolic Logic* **40** (1975) 113–129.
- Gencer, Ç., & de Jongh, D. (2009). Unifiability in extensions of *K4*. *Logic Journal of the IGPL* **17** (2009) 159–172.
- Ghilardi, S. (1999). Unification in intuitionistic logic. *Journal of Symbolic Logic* **64** (1999) 859–880.
- Ghilardi, S. (2000). Best solving modal equations. *Annals of Pure and Applied Logic* **102** (2000) 183–198.
- Ghilardi, S., & Sacchetti, L. (2004). Filtering unification and most general unifiers in modal logic. *Journal of Symbolic Logic* **69** (2004) 879–906.
- Iemhoff, R. (2001). On the admissible rules of intuitionistic propositional logic. *Journal of Symbolic Computation* **66** (2001) 281–294.
- Jerábek, E. (2007). Complexity of admissible rules. *Archive for Mathematical Logic* **46** (2007) 73–92.
- Jerábek, E. (2015). Blending margins: the modal logic *K* has nullary unification type. *Journal of Logic and Computation* **25** (2015) 1231–1240.
- Rybakov, V. (1984). A criterion for admissibility of rules in the model system *S4* and the intuitionistic logic. *Algebra and Logic* **23** (1984) 369–384.
- Rybakov, V. (1985). Bases of admissible rules of the logics *S4* and *Int*. *Algebra and Logic* **24** (1985) 55–68.
- Rybakov, V. (1997). Admissibility of Logical Inference Rules. Elsevier Science (1997).
- Rybakov, V. (2001). Construction of an explicit basis for rules admissible in modal system *S4*. *Mathematical Logic Quarterly* **47** (2001) 441–446.
- Rybakov, V. (2002). Unification in common knowledge logics. Bulletin of the Section of Logic **31** (2002) 207–215.
- Rybakov, V. (2005). Logical consecutions in discrete linear temporal logic. *Journal of Symbolic Logic* **70** (2005) 1137–1149.
- Rybakov, V. (2008). Multi-modal and temporal logics with universal formula — reduction of admissibility to validity and unification. *Journal of Logic and Computation*

Rybakov, V., Terziler, M., & Gencer, Ç. (1999). An essay on unification and inference rules for modal logics. *Bulletin of the Section of Logic* **28** (1999) 145–157.

Wolter, F., & Zakharyashev, M. (2008) Undecidability of the unification and admissibility problems for modal and description logics. *ACM Transactions on Computational Logic* **9** (2008) 25:1–25:20.

Class	PAR	Computability	Type
\mathcal{C}^{ser}	$= \emptyset$	in NP (Proposition 1)	?
\mathcal{C}^{ref}	$= \emptyset$	in NP (Proposition 1)	?
\mathcal{C}^{sym}	$= \emptyset$?	?
\mathcal{C}^{par}	$= \emptyset$	in NP (Proposition 2)	unitary (Baader, & Ghilardi, 2011)
\mathcal{C}^{det}	$= \emptyset$	in $PSPACE$ (Balbiani, Tinchev, 2014)	nullary (Balbiani, & Tinchev, 2014)
\mathcal{C}^{tra}	$= \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
\mathcal{C}^{rt}	$= \emptyset$	in NP (Proposition 2)	finitary (Ghilardi, 2000)
\mathcal{C}^{twf}	$= \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
\mathcal{C}^{all}	$= \emptyset$?	nullary (Jerábek, 2013)
\mathcal{C}^{ser}	$\neq \emptyset$?	nullary (Proposition 9)
\mathcal{C}^{ref}	$\neq \emptyset$?	?
\mathcal{C}^{sym}	$\neq \emptyset$?	?
\mathcal{C}^{par}	$\neq \emptyset$	decidable (Balbiani, & Gencer, 2015)	unitary
\mathcal{C}^{det}	$\neq \emptyset$	in $PSPACE$ (Balbiani, & Tinchev, 2014)	nullary (Balbiani, & Tinchev, 2014)
\mathcal{C}^{tra}	$\neq \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
\mathcal{C}^{rt}	$\neq \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
\mathcal{C}^{twf}	$\neq \emptyset$	decidable (Rybakov, 1997)	finitary (Ghilardi, 2000)
\mathcal{C}^{all}	$\neq \emptyset$?	nullary (Jerábek, 2013)

Table 1.