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ELEMENTARY UNIFICATION IN MODAL LOGIC *KD45*

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Abstract

KD45 is the least modal logic containing the formulas $\Box x \rightarrow \Diamond x$, $\Box x \rightarrow \Box \Box x$ and $\Diamond x \rightarrow \Box \Diamond x$. It is determined by the class of all serial, transitive and Euclidean frames. The elementary unifiability problem in *KD45* is to determine, given a formula $\varphi(x_1, \dots, x_n)$, whether there exists formulas ψ_1, \dots, ψ_n such that $\varphi(\psi_1, \dots, \psi_n)$ is in *KD45*. It is well-known that the elementary unifiability problem in *KD45* is *NP*-complete. In our paper, we show that every *KD45*-unifiable formula has a projective unifier. As a corollary, we conclude that *KD45* has unitary type for elementary unification.

Keywords: Modal logic *KD45*. Elementary unification. Most general unifier. Projective formula. Unification type.

1 Introduction

Modal logics like *S5* or *KD45* are essential to the design of logical systems that capture elements of reasoning about knowledge [13, 21]. There exists variants of these logics with one or several agents, with or without common knowledge, etc. As in any modal logic, the questions addressed in their setting usually concern their axiomatizability and their decidability. Another desirable question which one should address whenever possible concerns the unifiability of formulas. A formula $\varphi(x_1, \dots, x_n)$ is unifiable in a modal logic *L* iff there exists formulas ψ_1, \dots, ψ_n such that $\varphi(\psi_1, \dots, \psi_n)$ is in *L*. See [1, 11, 14, 15] for details.

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Results about the unifiability problem have been already obtained in many modal logics. Rybakov [22, 23] demonstrated that the unifiability problem in transitive modal logics like $K4$ and G is decidable. Wolter and Zakharyashev [24] showed that the unifiability problem is undecidable for any modal logic between K and $K4$ extended with the universal modality. The notion of projectivity has been introduced by Ghilardi [15] to determine the unification type, finitary, of transitive modal logics like $K4$ and G . The unification type, nullary, of modal logics like K , KD and KT has been established in [6, 18].

Within the context of description logics, checking subsumption of concepts is not sufficient and new inference capabilities are required. One of them, the unifiability of concept terms, has been introduced by Baader and Narendran [4] for \mathcal{FL}_0 . Baader and Küsters [2] established the *EXPTIME*-completeness of the unifiability problem in \mathcal{FL}_{reg} whereas Baader and Morawska [3] established the *NPTIME*-completeness of the unifiability problem in \mathcal{EL} . Much remains to be done, seeing that the computability of the unifiability problem and the unification types are unknown in multifarious modal logics and description logics.

$KD45$ is the least modal logic containing the formulas $\Box x \rightarrow \Diamond x$, $\Box x \rightarrow \Box \Box x$ and $\Diamond x \rightarrow \Box \Diamond x$. It is determined by the class of all serial, transitive and Euclidean frames. The elementary unifiability problem in $KD45$ is to determine, given a parameter-free formula $\varphi(x_1, \dots, x_n)$, whether there exists parameter-free formulas ψ_1, \dots, ψ_n such that $\varphi(\psi_1, \dots, \psi_n)$ is in $KD45$. It is well-known that the elementary unifiability problem in $KD45$ is *NP*-complete. Moreover, as proved by Ghilardi and Sacchetti [16], the unifiability problem in $KD45$ is directed and, consequently, $KD45$ has either unitary type, or nullary type. See also [7, 19]. The directedness of $KD45$ is a consequence of the characterization by Ghilardi and Sacchetti of the normal extensions of $K4$ with a directed unifiability problem. This characterization uses advanced notions from algebraic and relational semantics of normal modal logics.

In our paper, we directly show that every $KD45$ -unifiable parameter-free formula has a projective unifier. As an immediate corollary, we conclude that $KD45$ has unitary type for elementary unification. Section 2 defines the syntax and the semantics of $KD45$. In Section 3, definitions about the elementary unifiability problem in $KD45$ are given. Sections 4–6 introduce and study arrows, setarrows and tips which will be our main tools for proving our results. In Section 7, definitions about acceptable agreements as a simplified version of bounded morphisms are given. Section 8 introduces and studies types which are sets of tips. In Sections 9–11, intermediate results about types needed to show that every $KD45$ -unifiable parameter-free formula has a projective unifier are proved.

2 Syntax and semantics

Let VAR be a countable set of *variables* (with typical members denoted x, y , etc). Let (x_1, x_2, \dots) be an enumeration of VAR without repetitions. The set FOR of all *formulas* (with typical members denoted φ, ψ , etc) is inductively defined as follows:

- $\varphi ::= x \mid \perp \mid \neg\varphi \mid (\varphi \vee \psi) \mid \Box\varphi$.

We write $\varphi(x_1, \dots, x_n)$ to denote a formula whose variables form a subset of $\{x_1, \dots, x_n\}$. The result of the replacement in $\varphi(x_1, \dots, x_n)$ of variables x_1, \dots, x_n in their places with formulas ψ_1, \dots, ψ_n will be denoted by $\varphi(\psi_1, \dots, \psi_n)$. We define the other Boolean constructs as usual. We will follow the standard rules for omission of the parentheses. Let φ be a formula. We will write $\Diamond\varphi$ for $\neg\Box\neg\varphi$. We will respectively write φ^\perp and φ^\top for $\neg\varphi$ and φ . Let Γ be a finite set of formulas. Considering that $\bigvee \emptyset = \perp$ and $\bigwedge \emptyset = \top$, we will write $\nabla\Gamma$ for the conjunction of the following formulas:

- $\Box\bigvee\{\varphi : \varphi \text{ is a formula in } \Gamma\}$,
- $\bigwedge\{\Diamond\varphi : \varphi \text{ is a formula in } \Gamma\}$.

A *model* is a function $V : VAR \rightarrow 2^{\mathbb{N}}$ associating to each variable x a set $V(x)$ of non-negative integers. We inductively define the *truth* of a formula φ in model V at nonnegative integer s , in symbols $V, s \models \varphi$, as follows:

- $V, s \models x$ iff $s \in V(x)$,
- $V, s \not\models \perp$,
- $V, s \models \neg\varphi$ iff $V, s \not\models \varphi$,
- $V, s \models \varphi \vee \psi$ iff either $V, s \models \varphi$, or $V, s \models \psi$,
- $V, s \models \Box\varphi$ iff for all positive integers t , $V, t \models \varphi$.

As a result, $V, s \models \Diamond\varphi$ iff there exists a positive integer t such that $V, t \models \varphi$. Moreover, $V, s \models \varphi^\perp$ iff $V, s \not\models \varphi$ and $V, s \models \varphi^\top$ iff $V, s \models \varphi$. In other respect, $V, s \models \nabla\Gamma$ iff

- for all positive integers t , there exists $\varphi \in \Gamma$ such that $V, t \models \varphi$,
- for all $\varphi \in \Gamma$, there exists a positive integer t such that $V, t \models \varphi$.

We shall say that a model V is *uniform* iff for all variables x , either $V(x) = \emptyset$, or $V(x) = \mathbb{N}$. We shall say that a formula φ is *satisfiable* iff there exists a model V such that $V, 0 \models \varphi$. We shall say that a formula φ is *valid*, in symbols $\models \varphi$, iff for all models V , $V, 0 \models \varphi$. The following result is well-known and can be proved by using the canonical model construction, the technique of the generated subframe and the bounded morphism lemma [10].

Proposition 1. *For all formulas φ , $\models \varphi$ iff $\varphi \in KD45$.*

Proof. Left to the reader. □

3 Unification

A *substitution* is a function $\sigma : VAR \rightarrow FOR$ associating to each variable a formula. We shall say that a substitution σ is *closed* iff for all variables x , $\sigma(x)$ is a variable-free formula. For all formulas $\varphi(x_1, \dots, x_n)$, let $\sigma(\varphi(x_1, \dots, x_n))$ be $\varphi(\sigma(x_1), \dots, \sigma(x_n))$. The *composition* $\sigma \circ \tau$ of the substitutions σ and τ is the substitution associating to each variable x the formula $\tau(\sigma(x))$. We shall say that a substitution σ is *equivalent* to a substitution τ , in symbols $\sigma \simeq \tau$, iff $\models \sigma(x) \leftrightarrow \tau(x)$ for all variables x . We shall say that a substitution σ is more *general* than a substitution τ , in symbols $\sigma \preceq \tau$, iff there exists a substitution υ such that $\sigma \circ \upsilon \simeq \tau$. Note that the notation $\tau \preceq \sigma$ is also used in many papers. We shall say that a formula φ is *unifiable* iff there exists a substitution σ such that $\models \sigma(\varphi)$. In that case, σ is a *unifier* of φ . We shall say that a unifiable formula φ is *projective* iff there exists a unifier σ of φ such that $\models \varphi \wedge \Box\varphi \rightarrow (\sigma(x) \leftrightarrow x)$ for all variables x . The following results are well-known [1].

Proposition 2. *Let φ be a formula. If φ is unifiable then φ possesses a closed unifier.*

Proof. Since the set of all valid formulas is closed with respect to the rule of uniform substitution, therefore if σ is a unifier of φ then for all closed substitutions τ , $\sigma \circ \tau$ is a closed unifier of φ . □

Proposition 3. *Let $\varphi(x_1, \dots, x_n)$ be a \Box -free formula. The following conditions are equivalent:*

1. $\varphi(x_1, \dots, x_n)$, considered as a Boolean formula, is satisfiable.
2. $\varphi(x_1, \dots, x_n)$, considered as a modal formula, is unifiable.

Proof. Suppose $\varphi(x_1, \dots, x_n)$, considered as a Boolean formula, is satisfiable. Hence, there exists (ψ_1, \dots, ψ_n) in $\{\perp, \top\}^n$ such that $\varphi(\psi_1, \dots, \psi_n)$ is classically equivalent to \top . Thus, $\varphi(\psi_1, \dots, \psi_n)$ is *KD45*-equivalent to \top . Consequently, $\varphi(x_1, \dots, x_n)$, considered as a modal formula, is unifiable.

Reciprocally, suppose $\varphi(x_1, \dots, x_n)$, considered as a modal formula, is unifiable. Let σ be a unifier of $\varphi(x_1, \dots, x_n)$. Let V be a model. Since σ is a unifier of $\varphi(x_1, \dots, x_n)$, therefore $V, 0 \models \varphi(\sigma(x_1), \dots, \sigma(x_n))$. Let (ψ_1, \dots, ψ_n) in $\{\perp, \top\}^n$ be such that for all $i \in \{1, \dots, n\}$, if $V, 0 \not\models \sigma(x_i)$ then $\psi_i = \perp$ else $\psi_i = \top$. Since $V, 0 \models \varphi(\sigma(x_1), \dots, \sigma(x_n))$, therefore $\varphi(\psi_1, \dots, \psi_n)$ is classically equivalent to \top . Hence, $\varphi(x_1, \dots, x_n)$, considered as a Boolean formula, is satisfiable. □

Proposition 4. *The elementary unifiability problem in $KD45$ is NP-complete.*

Proof. Remark that every variable-free formula is either $KD45$ -equivalent to \perp , or $KD45$ -equivalent to \top . Hence, by Proposition 2, in order to determine if a given formula $\varphi(x_1, \dots, x_n)$ is unifiable, it suffices to nondeterministically choose (ψ_1, \dots, ψ_n) in $\{\perp, \top\}^n$ such that $\models \varphi(\psi_1, \dots, \psi_n)$. Since the validity of a given variable-free formula can be checked in polynomial time, therefore the elementary unifiability problem in $KD45$ is in NP. As for the NP-hardness of the elementary unifiability problem in $KD45$, it follows from Proposition 3. \square

Proposition 5. *Let φ be a unifiable formula. If φ is projective then φ possesses a most general unifier.*

Proof. Suppose φ is projective. Let σ be a unifier of φ such that $\models \varphi \wedge \Box\varphi \rightarrow (\sigma(x) \leftrightarrow x)$ for all variables x . Let τ be a unifier of φ and x be a variable. Hence, $\models \tau(\varphi)$ and $\models \varphi \wedge \Box\varphi \rightarrow (\sigma(x) \leftrightarrow x)$. Thus, $\models \tau(\varphi) \wedge \Box\tau(\varphi)$. Since $\models \varphi \wedge \Box\varphi \rightarrow (\sigma(x) \leftrightarrow x)$, therefore $\models \tau(\varphi) \wedge \Box\tau(\varphi) \rightarrow ((\sigma \circ \tau)(x) \leftrightarrow \tau(x))$. Since $\models \tau(\varphi) \wedge \Box\tau(\varphi)$, therefore $\models (\sigma \circ \tau)(x) \leftrightarrow \tau(x)$. Since x is an arbitrary variable, therefore $\sigma \circ \tau \simeq \tau$. Consequently, $\sigma \preceq \tau$. \square

From now on, let us fix $n \in \mathbb{N}$.

Formulas of the form $\varphi(x_1, \dots, x_n)$ will be called n -formulas. From now on, they will be denoted $\varphi(\vec{x})$.

4 Arrows

We define $A_n = \{\perp, \top\}^n$. Elements of A_n are n -tuples of bits. They will be called n -arrows. They will be denoted α, β , etc. Remark that $Card(A_n) = 2^n$. For all n -arrows $\alpha = (\alpha_1, \dots, \alpha_n)$, we will write $\tilde{\alpha}(\vec{x})$ for the associated n -formula

- $\tilde{\alpha}(\vec{x}) = x_1^{\alpha_1} \wedge \dots \wedge x_n^{\alpha_n}$.

The following result says that the n -formula associated to an n -arrow is always satisfiable.

Lemma 6. *Let α be an n -arrow. There exists a model V such that $V, 0 \models \tilde{\alpha}(\vec{x})$.*

Proof. Left to the reader. \square

Remark that for all n -arrows $\alpha = (\alpha_1, \dots, \alpha_n)$ and for all n -tuples $\vec{\psi}$ of formulas, $\tilde{\alpha}(\vec{\psi}) = \psi_1^{\alpha_1} \wedge \dots \wedge \psi_n^{\alpha_n}$. As a result,

Lemma 7. Let $\vec{\psi}$ be an n -tuple of formulas, V be a model and s be a nonnegative integer. For all n -arrows α, β , if $V, s \models \tilde{\alpha}(\vec{\psi})$ and $V, s \models \tilde{\beta}(\vec{\psi})$ then $\alpha = \beta$.

Proof. Let α, β be n -arrows. Suppose $V, s \models \tilde{\alpha}(\vec{\psi})$, $V, s \models \tilde{\beta}(\vec{\psi})$ and $\alpha \neq \beta$. Let $i \in \{1, \dots, n\}$ be such that either $\alpha_i = \perp$ and $\beta_i = \top$, or $\alpha_i = \top$ and $\beta_i = \perp$. Without loss of generality, assume $\alpha_i = \perp$ and $\beta_i = \top$. Since $V, s \models \tilde{\alpha}(\vec{\psi})$ and $V, s \models \tilde{\beta}(\vec{\psi})$, therefore $V, s \models \neg\psi_i$ and $V, s \models \psi_i$: a contradiction. \square

For all n -tuples $\vec{\psi}$ of formulas, for all models V and for all nonnegative integers s , let $\alpha[\vec{\psi}, V, s]$ be the n -arrow such that for all $i \in \{1, \dots, n\}$,

- if $V, s \not\models \psi_i$ then $\alpha_i[\vec{\psi}, V, s] = \perp$ else $\alpha_i[\vec{\psi}, V, s] = \top$.

As a result, $\tilde{\alpha}[\vec{\psi}, V, s](\vec{x}) = x_1^{\alpha_1[\vec{\psi}, V, s]} \wedge \dots \wedge x_n^{\alpha_n[\vec{\psi}, V, s]}$ and

Lemma 8. Let $\vec{\psi}$ be an n -tuple of formulas, V be a model and s be a nonnegative integer. $V, s \models \tilde{\alpha}[\vec{\psi}, V, s](\vec{\psi})$.

Proof. By definition of $\alpha[\vec{\psi}, V, s]$. \square

Moreover,

Lemma 9. Let $\vec{\psi}$ be an n -tuple of formulas, V be a model and s be a nonnegative integer. $\alpha[\vec{\psi}, V, s]$ is the unique n -arrow α such that $V, s \models \tilde{\alpha}(\vec{\psi})$.

Proof. By Lemmas 7 and 8. \square

The following result will be useful when we study the most general unifiers of unifiable n -formulas.

Lemma 10. Let V be a model. For all n -arrows α , there exists a model V' such that $V', 0 \models \tilde{\alpha}(\vec{x})$ and for all variables x and for all positive integers s , $s \in V'(x)$ iff $s \in V(x)$.

Proof. Left to the reader. \square

5 Setarrows

Let $S_n = 2^{A_n} \setminus \{\emptyset\}$. Elements of S_n are nonempty sets of n -arrows. They will be called n -setarrows. They will be denoted a, b , etc. Remark that $\text{Card}(S_n) = 2^{2^n} - 1$. For all n -setarrows $a = \{\alpha^0, \dots, \alpha^k\}$, we will write $\tilde{a}(\vec{x})$ the associated n -formula

- $\tilde{a}(\vec{x}) = \nabla\{\tilde{\alpha}^0(\vec{x}), \dots, \tilde{\alpha}^k(\vec{x})\}$.

The following result says that the n -formula associated to an n -setarrow is always satisfiable.

Lemma 11. *Let a be an n -setarrow. There exists a model V such that $V, 0 \models \tilde{a}(\vec{x})$.*

Proof. Left to the reader. □

Remark that for all n -setarrows $a = \{\alpha^0, \dots, \alpha^k\}$ and for all n -tuples $\vec{\psi}$ of formulas, $\tilde{a}(\vec{\psi}) = \nabla\{\tilde{\alpha}^0(\vec{\psi}), \dots, \tilde{\alpha}^k(\vec{\psi})\}$. As a result,

Lemma 12. *Let $\vec{\psi}$ be an n -tuple of formulas and V be a model. For all n -setarrows a, b , if $V, 0 \models \tilde{a}(\vec{\psi})$ and $V, 0 \models \tilde{b}(\vec{\psi})$ then $a = b$.*

Proof. Let a, b be n -setarrows. Suppose $V, 0 \models \tilde{a}(\vec{\psi})$, $V, 0 \models \tilde{b}(\vec{\psi})$ and $a \neq b$. Let α be an n -arrow such that either $\alpha \in a$ and $\alpha \notin b$, or $\alpha \notin a$ and $\alpha \in b$. Without loss of generality, assume $\alpha \in a$ and $\alpha \notin b$. Since $V, 0 \models \tilde{a}(\vec{\psi})$, therefore there exists a positive integer s such that $V, s \models \tilde{\alpha}(\vec{\psi})$. Since $V, 0 \models \tilde{b}(\vec{\psi})$ and $\alpha \notin b$, therefore by Lemma 7, for all positive integers s , $V, s \not\models \tilde{\alpha}(\vec{\psi})$: a contradiction. □

For all n -tuples $\vec{\psi}$ of formulas and for all models V , let $a[\vec{\psi}, V]$ be the n -setarrow

- $a[\vec{\psi}, V] = \{\alpha[\vec{\psi}, V, s] : s \text{ is a positive integer}\}$.

As a result, $\tilde{a}[\vec{\psi}, V](\vec{x}) = \nabla\{x_1^{\alpha_1[\vec{\psi}, V, s]} \wedge \dots \wedge x_n^{\alpha_n[\vec{\psi}, V, s]} : s \text{ is a positive integer}\}$ and

Lemma 13. *Let $\vec{\psi}$ be an n -tuple of formulas and V be a model. $V, 0 \models \tilde{a}[\vec{\psi}, V](\vec{\psi})$.*

Proof. By definition of $a[\vec{\psi}, V]$. □

Moreover,

Lemma 14. *Let $\vec{\psi}$ be an n -tuple of formulas and V be a model. $a[\vec{\psi}, V]$ is the unique n -setarrow a such that $V, 0 \models \tilde{a}(\vec{\psi})$.*

Proof. By Lemmas 12 and 13. □

The following result will be useful when we study the most general unifiers of unifiable n -formulas. It can be proved by induction on $\varphi(\vec{x})$.

Lemma 15. *Let $\vec{\psi}$ be an n -tuple of formulas. Let V, V' be models such that $a[\vec{\psi}, V] = a[\vec{\psi}, V']$. Let $\varphi(\vec{x})$ be an n -formula. For all nonnegative integers s, s' , if $\alpha[\vec{\psi}, V, s] = \alpha[\vec{\psi}, V', s']$ then $V, s \models \varphi(\vec{\psi})$ iff $V', s' \models \varphi(\vec{\psi})$.*

Proof. Left to the reader. □

6 Tips

Let $\mathcal{P}_n = A_n \times S_n$. Elements of \mathcal{P}_n are couples consisting of an n -arrow component and an n -setarrow component. They will be called n -tips. They will be denoted p, q , etc. Remark that $Card(\mathcal{P}_n) = 2^n \times (2^{2^n} - 1)$. For all n -tips $p = (\alpha, a)$, we will write $\tilde{p}(\vec{x})$ the associated n -formula

- $\tilde{p}(\vec{x}) = \tilde{\alpha}(\vec{x}) \wedge \tilde{a}(\vec{x})$.

The following result says that the n -formula associated to an n -tip is always satisfiable.

Lemma 16. *Let p be an n -tip. There exists a model V such that $V, 0 \models \tilde{p}(\vec{x})$.*

Proof. Left to the reader. □

Remark that for all n -tips $p = (\alpha, a)$ and for all n -tuples $\vec{\psi}$ of formulas, $\tilde{p}(\vec{\psi}) = \tilde{\alpha}(\vec{\psi}) \wedge \tilde{a}(\vec{\psi})$. As a result,

Lemma 17. *Let $\vec{\psi}$ be an n -tuple of formulas and V be a model. For all n -tips p, q , if $V, 0 \models \tilde{p}(\vec{\psi})$ and $V, 0 \models \tilde{q}(\vec{\psi})$ then $p = q$.*

Proof. By Lemmas 7 and 12. □

For all n -tuples $\vec{\psi}$ of formulas and for all models V , let $p[\vec{\psi}, V]$ be the n -tip

- $p[\vec{\psi}, V] = (\alpha[\vec{\psi}, V, 0], a[\vec{\psi}, V])$.

As a result, $\tilde{p}[\vec{\psi}, V](\vec{x}) = x_1^{\alpha_1[\vec{\psi}, V, 0]} \wedge \dots \wedge x_n^{\alpha_n[\vec{\psi}, V, 0]} \wedge \nabla \{x_1^{\alpha_1[\vec{\psi}, V, s]} \wedge \dots \wedge x_n^{\alpha_n[\vec{\psi}, V, s]} : s \text{ is a positive integer}\}$ and

Lemma 18. *Let $\vec{\psi}$ be an n -tuple of formulas and V be a model. $V, 0 \models \tilde{p}[\vec{\psi}, V](\vec{\psi})$.*

Proof. By definition of $p[\vec{\psi}, V]$. □

Moreover,

Lemma 19. *Let $\vec{\psi}$ be an n -tuple of formulas and V be a model. $p[\vec{\psi}, V]$ is the unique n -tip p such that $V, 0 \models \tilde{p}(\vec{\psi})$.*

Proof. By Lemmas 17 and 18. □

7 Acceptable agreements

In this section, we give definitions of acceptable agreements as a simplified version of bounded morphisms. We shall say that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ associating to each nonnegative integer a nonnegative integer is *acceptable* iff for all positive integers s , $f(s)$ is a positive integer and $f^{-1}(s)$ contains a positive integer. We shall say that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ associating to each nonnegative integer a nonnegative integer is an *n-agreement* between models V and V' iff for all $i \in \{1, \dots, n\}$ and for all nonnegative integers s , $s \in V(x_i)$ iff $f(s) \in V'(x_i)$.

Lemma 20. *Let f be an acceptable n-agreement between models V and V' . Let $\varphi(\vec{x})$ be an n-formula. For all nonnegative integers s , $V, s \models \varphi(\vec{x})$ iff $V', f(s) \models \varphi(\vec{x})$.*

Proof. By induction on $\varphi(\vec{x})$. □

We shall say that a function $f : \mathbb{N} \rightarrow \mathbb{N}$ associating to each nonnegative integer a nonnegative integer is an ω -agreement between models V and V' iff for all variables x and for all nonnegative integers s , $s \in V(x)$ iff $f(s) \in V'(x)$.

Lemma 21. *Let f be an acceptable ω -agreement between models V and V' . Let φ be a formula. For all nonnegative integers s , $V, s \models \varphi$ iff $V', f(s) \models \varphi$.*

Proof. By induction on φ . □

8 Types

Let $\mathcal{T}_n = 2^{A_n \times S_n}$. Elements of \mathcal{T}_n are sets of n -tips. They will be called *n-types*. They will be denoted T, U , etc. Remark that $\text{Card}(\mathcal{T}_n) = 2^{2^n \times (2^{2^n} - 1)}$. We shall say that an n -type T is *complete* for an n -setarrow a iff for all n -arrows α , $(\alpha, a) \in T$. We shall say that an n -type T is *empty* for an n -setarrow a iff for all n -arrows α , if $\alpha \in a$ then $(\alpha, a) \notin T$. We shall say that an n -type T is *full* for an n -setarrow a iff for all n -arrows α , if $\alpha \in a$ then $(\alpha, a) \in T$. We shall say that an n -type T is *saturated* iff for all n -arrows α, β and for all n -setarrows a , if $(\alpha, a) \in T$ and $\beta \in a$ then $(\beta, a) \in T$. The following result will be of crucial importance in the remaining sections of our paper.

Proposition 22. *Let T be a saturated n -type. For all n -setarrows a , exactly one of the following conditions holds: (i) T is complete for a ; (ii) T is not complete for a and T is empty for a ; (iii) T is not complete for a and T is full for a .*

Proof. Left to the reader. □

We shall say that an n -type T is *closed* iff for all n -setarrows a , there exists an n -arrow γ such that if T is not complete for a then either T is empty for a and $(\gamma, \{\gamma\}) \in T$, or T is full for a and $(\gamma, a) \in T$. We shall say that an n -type is *perfect* iff it is saturated and closed.

9 From tuples of formulas to perfect types

Let $\vec{\psi}$ be an n -tuple of formulas. Let $T[\vec{\psi}]$ be the n -type

- $T[\vec{\psi}] = \{p[\vec{\psi}, V] : V \text{ is a model}\}$.

The aim of this section is to demonstrate that $T[\vec{\psi}]$ is perfect.

Lemma 23. $T[\vec{\psi}]$ is saturated.

Proof. Let β, γ be n -arrows and b be an n -setarrow such that $(\beta, b) \in T[\vec{\psi}]$ and $\gamma \in b$. Let V be a model such that $\beta = \alpha[\vec{\psi}, V, 0]$ and $b = a[\vec{\psi}, V]$. Recall that $\gamma \in b$. Let s be a positive integer such that $\gamma = \alpha[\vec{\psi}, V, s]$. Let V' be the model such that for all variables x , if $s \notin V(x)$ then $V'(x) = V(x) \setminus \{0\}$ else $V'(x) = V(x) \cup \{0\}$. Let f be the acceptable function such that $f(0) = s$ and for all positive integers t , $f(t) = t$. The reader may easily verify that f is an ω -agreement between V' and V . Since $\gamma = \alpha[\vec{\psi}, V, s]$ and $f(0) = s$, therefore by Lemma 21, $\gamma = \alpha[\vec{\psi}, V', 0]$. Moreover, since $b = a[\vec{\psi}, V]$, therefore by Lemma 21, $b = a[\vec{\psi}, V']$. Hence, $(\gamma, b) \in T[\vec{\psi}]$. Since β, γ are arbitrary n -arrows and b is an arbitrary n -setarrow such that $(\beta, b) \in T[\vec{\psi}]$ and $\gamma \in b$, therefore $T[\vec{\psi}]$ is saturated. \square

Lemma 24. There exists an n -arrow γ such that $(\gamma, \{\gamma\}) \in T[\vec{\psi}]$.

Proof. Let V be a uniform model. The reader may easily verify that $a[\vec{\psi}, V] = \{\alpha[\vec{\psi}, V, 0]\}$. Hence, $(\alpha[\vec{\psi}, V, 0], \{\alpha[\vec{\psi}, V, 0]\}) \in T[\vec{\psi}]$. \square

Lemma 25. $T[\vec{\psi}]$ is closed.

Proof. By Lemma 23, $T[\vec{\psi}]$ is saturated. Hence, by Proposition 22, for all n -setarrows a , exactly one of the following conditions holds: (i) $T[\vec{\psi}]$ is complete for a ; (ii) $T[\vec{\psi}]$ is not complete for a and $T[\vec{\psi}]$ is empty for a ; (iii) $T[\vec{\psi}]$ is not complete for a and $T[\vec{\psi}]$ is full for a . By Lemma 24, let γ be an n -arrow such that $(\gamma, \{\gamma\}) \in T[\vec{\psi}]$. For all n -setarrows a , let $\gamma^{T,a}$ be an arbitrary n -arrow if condition (i) holds, the n -arrow γ if condition (ii) holds and an arbitrary n -arrow in a if condition (iii) holds. The reader may easily verify that for all n -setarrows a , if $T[\vec{\psi}]$ is not complete for a then either $T[\vec{\psi}]$ is empty for a and $(\gamma^{T,a}, \{\gamma^{T,a}\}) \in T[\vec{\psi}]$, or $T[\vec{\psi}]$ is full for a and $(\gamma^{T,a}, a) \in T[\vec{\psi}]$. \square

From all this, it follows that

Proposition 26. $T[\vec{\psi}]$ is perfect.

Proof. By Lemmas 23 and 25. \square

10 From perfect types to tuples of formulas

Let T be a perfect n -type. Hence, T is saturated and closed. Thus, by Proposition 22, for all n -setarrows a , exactly one of the following conditions holds: (i) T is complete for a ; (ii) T is not complete for a and T is empty for a ; (iii) T is not complete for a and T is full for a . Since T is closed, therefore for all n -setarrows a , let $\gamma^{T,a}$ be an n -arrow such that if T is not complete for a then either T is empty for a and $(\gamma^{T,a}, \{\gamma^{T,a}\}) \in T$, or T is full for a and $(\gamma^{T,a}, a) \in T$. For all n -tips $p = (\alpha, a)$, let $\delta^{T,p}$ be the n -arrow such that if $p \notin T$ then $\delta^{T,p} = \gamma^{T,a}$ else $\delta^{T,p} = \alpha$. Let $\vec{\psi}[T](\vec{x})$ be the n -tuple of n -formulas such that for all $i \in \{1, \dots, n\}$,

- $\psi_i[T](\vec{x}) = \bigvee \{\tilde{p}(\vec{x}) \wedge \delta_i^{T,p} : p \text{ is an } n\text{-tip}\}$.

The aim of this section is to demonstrate that $T = T[\vec{\psi}[T](\vec{x})]$.

Lemma 27. *Let p be an n -tip. If $p \in T$ then $\models \tilde{p}(\vec{x}) \rightarrow \tilde{p}(\vec{\psi}[T](\vec{x}))$.*

Proof. Suppose $p \in T$. Let β be the n -arrow component of p and b be the n -setarrow component of p . Let V be a model such that $V, 0 \models \tilde{p}(\vec{x})$. Let $i \in \{1, \dots, n\}$. Since $V, 0 \models \tilde{p}(\vec{x})$, therefore by Lemma 17, $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$. Since $p \in T$, therefore $\delta_i^{T,p} = \beta_i$. Since $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$, therefore $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \beta_i$. Since $V, 0 \models \tilde{p}(\vec{x})$, therefore $V, 0 \models \tilde{\beta}(\vec{x})$. Since $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \beta_i$, therefore $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow x_i$. Let s be a positive integer. Since $V, 0 \models \tilde{p}(\vec{x})$, therefore $V, s \models \tilde{b}(\vec{x})$. Recall that s is a positive integer. Let α be an n -arrow such that $\alpha \in b$ and $V, s \models \tilde{\alpha}(\vec{x})$. Let q be the n -tip with n -arrow component α and n -setarrow component b . Since $V, s \models \tilde{b}(\vec{x})$ and $V, s \models \tilde{\alpha}(\vec{x})$, therefore $V, s \models \tilde{q}(\vec{x})$. Hence, by Lemma 17, $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,q}$. Since T is saturated, $p \in T$ and $\alpha \in b$, therefore $q \in T$. Thus, $\delta_i^{T,q} = \alpha_i$. Since $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,q}$, therefore $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \alpha_i$. Since $V, s \models \tilde{\alpha}(\vec{x})$, therefore $V, s \models \psi_i[T](\vec{x}) \leftrightarrow x_i$. Since s is an arbitrary positive integer, therefore $V, 0 \models \square(\psi_i[T](\vec{x}) \leftrightarrow x_i)$. Since $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow x_i$, therefore $V, 0 \models (\psi_i[T](\vec{x}) \leftrightarrow x_i) \wedge \square(\psi_i[T](\vec{x}) \leftrightarrow x_i)$. Since i is arbitrary in $\{1, \dots, n\}$, therefore $V, 0 \models (\psi_1[T](\vec{x}) \leftrightarrow x_1) \wedge \dots \wedge (\psi_n[T](\vec{x}) \leftrightarrow x_n) \wedge \square((\psi_1[T](\vec{x}) \leftrightarrow x_1) \wedge \dots \wedge (\psi_n[T](\vec{x}) \leftrightarrow x_n))$. Since $V, 0 \models \tilde{p}(\vec{x})$, therefore $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$. Since V is an arbitrary model such that $V, 0 \models \tilde{p}(\vec{x})$, therefore $\models \tilde{p}(\vec{x}) \rightarrow \tilde{p}(\vec{\psi}[T](\vec{x}))$. \square

Lemma 28. $T \subseteq T[\vec{\psi}[T](\vec{x})]$.

Proof. Let p be an n -tip such that $p \in T$. By Lemma 16, let V be a model such that $V, 0 \models \tilde{p}(\vec{x})$. Since $p \in T$, therefore by Lemma 27, $\models \tilde{p}(\vec{x}) \rightarrow \tilde{p}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{p}(\vec{x})$, therefore $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$. Hence, by Lemma 19, $p = p[\vec{\psi}[T](\vec{x}), V]$. Thus, $p \in T[\vec{\psi}[T](\vec{x})]$. Since p is an arbitrary n -tip such that $p \in T$, therefore $T \subseteq T[\vec{\psi}[T](\vec{x})]$. \square

Lemma 29. *Let p be an n -tip with n -setarrow component b . If $p \notin T$ then $\models \tilde{p}(\vec{x}) \rightarrow \widetilde{\gamma^{T,b}}(\vec{\psi}[T](\vec{x}))$.*

Proof. Suppose $p \notin T$. Let V be a model such that $V, 0 \models \tilde{p}(\vec{x})$. Let $i \in \{1, \dots, n\}$. Since $V, 0 \models \tilde{p}(\vec{x})$, therefore by Lemma 17, $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$. Since $p \notin T$, therefore $\delta_i^{T,p} = \gamma_i^{T,b}$. Since $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$, therefore $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \gamma_i^{T,b}$. Since i is arbitrary in $\{1, \dots, n\}$, therefore $V, 0 \models \widetilde{\gamma^{T,b}}(\vec{\psi}[T](\vec{x}))$. Since V is an arbitrary model such that $V, 0 \models \tilde{p}(\vec{x})$, therefore $\models \tilde{p}(\vec{x}) \rightarrow \widetilde{\gamma^{T,b}}(\vec{\psi}[T](\vec{x}))$. \square

Lemma 30. *Let b be an n -setarrow. If T is empty for b then $\models \tilde{b}(\vec{x}) \rightarrow \widetilde{\{\gamma^{T,b}\}}(\vec{\psi}[T](\vec{x}))$.*

Proof. Suppose T is empty for b . Let V be a model such that $V, 0 \models \tilde{b}(\vec{x})$. Let $i \in \{1, \dots, n\}$. Let s be a positive integer. Since $V, 0 \models \tilde{b}(\vec{x})$, therefore $V, s \models \tilde{b}(\vec{x})$. Recall that s is a positive integer. Let α be an n -arrow such that $\alpha \in b$ and $V, s \models \tilde{\alpha}(\vec{x})$. Let p be the n -tip with n -arrow component α and n -setarrow component b . Since $V, s \models \tilde{b}(\vec{x})$ and $V, s \models \tilde{\alpha}(\vec{x})$, therefore $V, s \models \tilde{p}(\vec{x})$. Hence, by Lemma 17, $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$. Since T is empty for b and $\alpha \in b$, therefore $p \notin T$. Thus, $\delta_i^{T,p} = \gamma_i^{T,b}$. Since $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$, therefore $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \gamma_i^{T,b}$. Since s is an arbitrary positive integer, therefore $V, 0 \models \square(\psi_i[T](\vec{x}) \leftrightarrow \gamma_i^{T,b})$. Since i is arbitrary in $\{1, \dots, n\}$, therefore $V, 0 \models \square((\psi_1[T](\vec{x}) \leftrightarrow \gamma_1^{T,b}) \wedge \dots \wedge (\psi_n[T](\vec{x}) \leftrightarrow \gamma_n^{T,b}))$. Consequently, $V, 0 \models \widetilde{\{\gamma^{T,b}\}}(\vec{\psi}[T](\vec{x}))$. Since V is an arbitrary model such that $V, 0 \models \tilde{b}(\vec{x})$, therefore $\models \tilde{b}(\vec{x}) \rightarrow \widetilde{\{\gamma^{T,b}\}}(\vec{\psi}[T](\vec{x}))$. \square

Lemma 31. *Let b be an n -setarrow. If T is full for b then $\models \tilde{b}(\vec{x}) \rightarrow \tilde{b}(\vec{\psi}[T](\vec{x}))$.*

Proof. Suppose T is full for b . Let V be a model such that $V, 0 \models \tilde{b}(\vec{x})$. Let $i \in \{1, \dots, n\}$. Let s be a positive integer. Since $V, 0 \models \tilde{b}(\vec{x})$, therefore $V, s \models \tilde{b}(\vec{x})$. Recall that s is a positive integer. Let α be an n -arrow such that $\alpha \in b$ and $V, s \models \tilde{\alpha}(\vec{x})$. Let p be the n -tip with n -arrow component α and n -setarrow component b . Since $V, s \models \tilde{b}(\vec{x})$ and $V, s \models \tilde{\alpha}(\vec{x})$, therefore $V, s \models \tilde{p}(\vec{x})$. Hence, by Lemma 17, $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$. Since T is full for b and $\alpha \in b$, therefore $p \in T$. Thus, $\delta_i^{T,p} = \alpha_i$. Since $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,p}$, therefore $V, s \models \psi_i[T](\vec{x}) \leftrightarrow \alpha_i$. Since $V, s \models \tilde{\alpha}(\vec{x})$, therefore $V, s \models \psi_i[T](\vec{x}) \leftrightarrow x_i$. Since s is an arbitrary positive integer, therefore $V, 0 \models \square(\psi_i[T](\vec{x}) \leftrightarrow x_i)$. Since i is arbitrary in $\{1, \dots, n\}$, therefore $V, 0 \models \square((\psi_1[T](\vec{x}) \leftrightarrow x_1) \wedge \dots \wedge (\psi_n[T](\vec{x}) \leftrightarrow x_n))$. Since $V, 0 \models \tilde{b}(\vec{x})$, therefore $V, 0 \models \tilde{b}(\vec{\psi}[T](\vec{x}))$. Since V is an arbitrary model such that $V, 0 \models \tilde{b}(\vec{x})$, therefore $\models \tilde{b}(\vec{x}) \rightarrow \tilde{b}(\vec{\psi}[T](\vec{x}))$. \square

Lemma 32. $T[\vec{\psi}[T](\vec{x})] \subseteq T$.

Proof. Let p be an n -tip such that $p \in T[\vec{\psi}[T](\vec{x})]$. Let V be a model such that $p = p[\vec{\psi}[T](\vec{x}), V]$. Hence, by Lemma 18, $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$. Let $q = p[\vec{x}, V]$. Thus, by Lemma 18, $V, 0 \models \tilde{q}(\vec{x})$. Case “ $q \in T$ ”: Hence, by Lemma 27, $\models \tilde{q}(\vec{x}) \rightarrow \tilde{q}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{q}(\vec{x})$, therefore $V, 0 \models \tilde{q}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$, therefore by Lemma 17, $p = q$. Since $q \in T$, therefore $p \in T$. Case “ $q \notin T$ ”: Let a be the n -setarrow component of q . Since $q \notin T$, therefore by Lemma 29, $\models \tilde{q}(\vec{x}) \rightarrow \widetilde{\gamma^{T,a}}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{q}(\vec{x})$, therefore $V, 0 \models \widetilde{\gamma^{T,a}}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$, therefore by Lemma 7, $\gamma^{T,a}$ is the n -arrow component of p . Since $q \notin T$, therefore T is not complete for a . Since T is saturated, therefore by Proposition 22, either T is empty for a , or T is full for a . In the former case, $(\gamma^{T,a}, \{\gamma^{T,a}\}) \in T$. Moreover, by Lemma 30, $\models \tilde{a}(\vec{x}) \rightarrow \widetilde{\{\gamma^{T,a}\}}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{q}(\vec{x})$, therefore $V, 0 \models \widetilde{\{\gamma^{T,a}\}}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$, therefore by Lemma 12, $\{\gamma^{T,a}\}$ is the n -setarrow component of p . Since $\gamma^{T,a}$ is the n -arrow component of p and $(\gamma^{T,a}, \{\gamma^{T,a}\}) \in T$, therefore $p \in T$. In the latter case, $(\gamma^{T,a}, a) \in T$. Moreover, by Lemma 31, $\models \tilde{a}(\vec{x}) \rightarrow \tilde{a}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{q}(\vec{x})$, therefore $V, 0 \models \tilde{a}(\vec{\psi}[T](\vec{x}))$. Since $V, 0 \models \tilde{p}(\vec{\psi}[T](\vec{x}))$, therefore by Lemma 12, a is the n -setarrow component of p . Since $\gamma^{T,a}$ is the n -arrow component of p and $(\gamma^{T,a}, a) \in T$, therefore $p \in T$. Since p is an arbitrary n -tip such that $p \in T[\vec{\psi}[T](\vec{x})]$, therefore $T[\vec{\psi}[T](\vec{x})] \subseteq T$. \square

From all this, it follows that

Proposition 33. $T = T[\vec{\psi}[T](\vec{x})]$.

Proof. By Lemmas 28 and 32. \square

11 About most general unifiers

Let $\varphi(\vec{x})$ be an n -formula. Let T be the n -type

- $T = \{p : \models \tilde{p}(\vec{x}) \rightarrow \varphi(\vec{x}) \wedge \square \varphi(\vec{x})\}$.

Lemma 34. T is saturated.

Proof. Let α, β be n -arrows and a be an n -setarrow such that $(\alpha, a) \in T$ and $\beta \in a$. Hence, $\models \tilde{\alpha}(\vec{x}) \wedge \tilde{a}(\vec{x}) \rightarrow \varphi(\vec{x}) \wedge \square \varphi(\vec{x})$. Let V be a model such that $V, 0 \models \tilde{\beta}(\vec{x})$ and $V, 0 \models \tilde{a}(\vec{x})$. By Lemma 10, let V' be a model such that $V', 0 \models \tilde{\alpha}(\vec{x})$ and for all variables x and for all positive integers s , $s \in V'(x)$ iff $s \in V(x)$. Since $V, 0 \models \tilde{a}(\vec{x})$, therefore $V', 0 \models \tilde{a}(\vec{x})$. Since $\models \tilde{\alpha}(\vec{x}) \wedge \tilde{a}(\vec{x}) \rightarrow \varphi(\vec{x}) \wedge \square \varphi(\vec{x})$ and $V', 0 \models \tilde{\alpha}(\vec{x})$, therefore $V', 0 \models \varphi(\vec{x}) \wedge \square \varphi(\vec{x})$. Moreover, recall that $\beta \in a$. Let s_β be a positive integer such that $V', s_\beta \models \tilde{\beta}(\vec{x})$. Since $V', 0 \models \varphi(\vec{x}) \wedge \square \varphi(\vec{x})$, therefore $V', s_\beta \models \varphi(\vec{x}) \wedge \square \varphi(\vec{x})$. Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be the

function associating to each nonnegative integer a nonnegative integer such that $f(0) = s_\beta$ and for all positive integers s , $f(s) = s$. The reader may easily verify that f is an acceptable n -agreement between V and V' . Since $V', s_\beta \models \varphi(\vec{x}) \wedge \Box\varphi(\vec{x})$ and $f(0) = s_\beta$, therefore by Lemma 20, $V, 0 \models \varphi(\vec{x}) \wedge \Box\varphi(\vec{x})$. Since V is an arbitrary model such that $V, 0 \models \widetilde{\beta}(\vec{x})$ and $V, 0 \models \widetilde{a}(\vec{x})$, therefore $\models \widetilde{\beta}(\vec{x}) \wedge \widetilde{a}(\vec{x}) \rightarrow \varphi(\vec{x}) \wedge \Box\varphi(\vec{x})$. Thus, $(\beta, a) \in T$. \square

Lemma 35. *Let $\vec{\chi}$ be an n -tuple of variable-free formulas. There exists an n -arrow γ such that $\models \widetilde{\gamma}(\vec{\chi})$.*

Proof. Left to the reader. \square

From now on, let us assume $\varphi(\vec{x})$ is unifiable.

The aim of this section is to demonstrate that $\varphi(\vec{x})$ is projective.

Lemma 36. *There exists an n -arrow γ such that $(\gamma, \{\gamma\}) \in T$.*

Proof. Since $\varphi(\vec{x})$ is unifiable, therefore by Proposition 2, let σ be a closed substitution such that $\models \sigma(\varphi(\vec{x}))$. Let $\vec{\chi}$ be the n -tuple of variable-free formulas such that for all $i \in \{1, \dots, n\}$, $\chi_i = \sigma(x_i)$. Since $\models \sigma(\varphi(\vec{x}))$, therefore $\models \varphi(\vec{\chi})$. Hence, $\models \varphi(\vec{\chi}) \wedge \Box\varphi(\vec{\chi})$. Since $\vec{\chi}$ is an n -tuple of variable-free formulas, therefore by Lemma 35, let γ be an n -arrow such that $\models \widetilde{\gamma}(\vec{\chi})$. Thus, $\models \widetilde{\gamma}(\vec{\chi}) \wedge \Box\widetilde{\gamma}(\vec{\chi})$. Let V be a model such that $V, 0 \models \widetilde{\gamma}(\vec{\chi})$ and $V, 0 \models \widetilde{\{\gamma\}}(\vec{\chi})$. Since $\models \widetilde{\gamma}(\vec{\chi}) \wedge \Box\widetilde{\gamma}(\vec{\chi})$, therefore $V, 0 \models \widetilde{\gamma}(\vec{\chi})$ and $V, 0 \models \Box\widetilde{\gamma}(\vec{\chi})$. Let $i \in \{1, \dots, n\}$. Since $V, 0 \models \widetilde{\gamma}(\vec{\chi})$ and $V, 0 \models \widetilde{\gamma}(\vec{\chi})$, therefore $V, 0 \models \chi_i \leftrightarrow x_i$. Let s be a positive integer. Since $V, 0 \models \widetilde{\{\gamma\}}(\vec{\chi})$ and $V, 0 \models \Box\widetilde{\gamma}(\vec{\chi})$, therefore $V, s \models \widetilde{\gamma}(\vec{\chi})$ and $V, s \models \widetilde{\gamma}(\vec{\chi})$. Hence, $V, s \models \chi_i \leftrightarrow x_i$. Since s is an arbitrary positive integer, therefore $V, 0 \models \Box(\chi_i \leftrightarrow x_i)$. Since $V, 0 \models \chi_i \leftrightarrow x_i$, therefore $V, 0 \models (\chi_i \leftrightarrow x_i) \wedge \Box(\chi_i \leftrightarrow x_i)$. Since i is arbitrary in $\{1, \dots, n\}$, therefore $V, 0 \models (\chi_1 \leftrightarrow x_1) \wedge \dots \wedge (\chi_n \leftrightarrow x_n) \wedge \Box((\chi_1 \leftrightarrow x_1) \wedge \dots \wedge (\chi_n \leftrightarrow x_n))$. Since $\models \varphi(\vec{\chi}) \wedge \Box\varphi(\vec{\chi})$, therefore $V, 0 \models \varphi(\vec{\chi}) \wedge \Box\varphi(\vec{\chi})$. Since $V, 0 \models (\chi_1 \leftrightarrow x_1) \wedge \dots \wedge (\chi_n \leftrightarrow x_n) \wedge \Box((\chi_1 \leftrightarrow x_1) \wedge \dots \wedge (\chi_n \leftrightarrow x_n))$, therefore $V, 0 \models \varphi(\vec{x}) \wedge \Box\varphi(\vec{x})$. Since V is an arbitrary model such that $V, 0 \models \widetilde{\gamma}(\vec{\chi})$ and $V, 0 \models \widetilde{\{\gamma\}}(\vec{\chi})$, therefore $\models \widetilde{\gamma}(\vec{\chi}) \wedge \widetilde{\{\gamma\}}(\vec{\chi}) \rightarrow \varphi(\vec{x}) \wedge \Box\varphi(\vec{x})$. Thus, $(\gamma, \{\gamma\}) \in T$. \square

Lemma 37. *T is closed.*

Proof. By Lemma 34, T is saturated. Hence, by Proposition 22, for all n -setarrows a , exactly one of the following conditions holds: (i) T is complete for a ; (ii) T is not complete for a and T is empty for a ; (iii) T is not complete for a and T is full for a . By Lemma 36, let γ be an n -arrow such that $(\gamma, \{\gamma\}) \in T$. For all n -setarrows a , let $\gamma^{T,a}$ be an arbitrary n -arrow if condition (i) holds, the n -arrow γ if condition (ii) holds and an arbitrary n -arrow in a if condition (iii) holds. The reader may easily verify that for all n -setarrows a , if T is

not complete for a then either T is empty for a and $(\gamma^{T,a}, \{\gamma^{T,a}\}) \in T$, or T is full for a and $(\gamma^{T,a}, a) \in T$. \square

Lemma 38. T is perfect.

Proof. By Lemmas 34 and 37. \square

By Lemma 37, T is closed. Hence, for all n -setarrows a , let $\gamma^{T,a}$ be an n -arrow such that if T is not complete for a then either T is empty for a and $(\gamma^{T,a}, \{\gamma^{T,a}\}) \in T$, or T is full for a and $(\gamma^{T,a}, a) \in T$. For all n -tips $p = (\alpha, a)$, let $\delta^{T,p}$ be the n -arrow such that if $p \notin T$ then $\delta^{T,p} = \gamma^{T,a}$ else $\delta^{T,p} = \alpha$. Let $\vec{\psi}[T](\vec{x})$ be the n -tuple of n -formulas such that for all $i \in \{1, \dots, n\}$, $\psi_i[T](\vec{x}) = \bigvee \{\tilde{p}(\vec{x}) \wedge \delta_i^{T,p} : p \text{ is an } n\text{-tip}\}$. Thus, by Proposition 33 and Lemma 38, $T = T[\vec{\psi}[T](\vec{x})]$.

Lemma 39. $\models \varphi(\vec{\psi}[T](\vec{x}))$.

Proof. Let V be a model. Since $T = T[\vec{\psi}[T](\vec{x})]$, therefore $p[\vec{\psi}[T](\vec{x}), V] \in T$. Hence, $\models \tilde{p}[\vec{\psi}[T](\vec{x}), V](\vec{x}) \rightarrow \varphi(\vec{x}) \wedge \Box \varphi(\vec{x})$. Thus, $\models \tilde{p}[\vec{\psi}[T](\vec{x}), V](\vec{\psi}[T](\vec{x})) \rightarrow \varphi(\vec{\psi}[T](\vec{x})) \wedge \Box \varphi(\vec{\psi}[T](\vec{x}))$. By Lemma 18 it holds $V, 0 \models \tilde{p}[\vec{\psi}[T](\vec{x}), V](\vec{\psi}[T](\vec{x}))$. Since it holds $\models \tilde{p}[\vec{\psi}[T](\vec{x}), V](\vec{\psi}[T](\vec{x})) \rightarrow \varphi(\vec{\psi}[T](\vec{x})) \wedge \Box \varphi(\vec{\psi}[T](\vec{x}))$, therefore $V, 0 \models \varphi(\vec{\psi}[T](\vec{x})) \wedge \Box \varphi(\vec{\psi}[T](\vec{x}))$. Consequently, $V, 0 \models \varphi(\vec{\psi}[T](\vec{x}))$. Since V is an arbitrary model, therefore $\models \varphi(\vec{\psi}[T](\vec{x}))$. \square

Lemma 40. For all $i \in \{1, \dots, n\}$, $\models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x}) \rightarrow (\psi_i[T](\vec{x}) \leftrightarrow x_i)$.

Proof. Let $i \in \{1, \dots, n\}$. Let V be a model such that $V, 0 \models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x})$. Let $q = p[\vec{x}, V]$. By Lemma 18, $V, 0 \models \tilde{q}(\vec{x})$. Case “ $q \in T$ ”: Hence, by Lemma 17, $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,q}$. Let α be the n -arrow component of q . Since $q \in T$, therefore $\delta_i^{T,q} = \alpha_i$. Since $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \delta_i^{T,q}$, therefore $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \alpha_i$. Since $V, 0 \models \tilde{q}(\vec{x})$, therefore $V, 0 \models \alpha_i \leftrightarrow x_i$. Since $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow \alpha_i$, therefore $V, 0 \models \psi_i[T](\vec{x}) \leftrightarrow x_i$. Case “ $q \notin T$ ”: Hence, $\not\models \tilde{q}(\vec{x}) \rightarrow \varphi(\vec{x}) \wedge \Box \varphi(\vec{x})$. Let V' be a model such that $V', 0 \models \tilde{q}(\vec{x})$ and $V', 0 \not\models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x})$. Since $V, 0 \models \tilde{q}(\vec{x})$, therefore by Lemma 15, $V, 0 \not\models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x})$: a contradiction. Since V is an arbitrary model such that $V, 0 \models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x})$, therefore $\models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x}) \rightarrow (\psi_i[T](\vec{x}) \leftrightarrow x_i)$. \square

From all this, it follows that

Proposition 41. $\varphi(\vec{x})$ is projective.

Proof. Let σ be the substitution such that for all positive integers i , if $i \in \{1, \dots, n\}$ then $\sigma(x_i) = \psi_i[T](\vec{x})$ else $\sigma(x_i) = x_i$. By lemma 39, the reader may easily verify that σ is a unifier of $\varphi(\vec{x})$. Moreover, by Lemma 40, $\models \varphi(\vec{x}) \wedge \Box \varphi(\vec{x}) \rightarrow (\sigma(x) \leftrightarrow x)$ for all variables x . \square

As a corollary, we conclude that

Corollary 42. *KD45 has unitary type for elementary unification, i.e. every unifiable formula possesses a most general unifier.*

Proof. By Propositions 5 and 41. □

12 Conclusion

Much remains to be done. For example, one may consider the unifiability problem when the language is extended by a countable set of parameters (with typical members denoted p, q , etc). In this case, the unifiability problem is said to be non-elementary. It consists to determine, given a formula $\varphi(p_1, \dots, p_m, x_1, \dots, x_n)$, whether there exists formulas ψ_1, \dots, ψ_n such that $\models \varphi(p_1, \dots, p_m, \psi_1, \dots, \psi_n)$. Another example, one may also consider the unifiability problem, the elementary one or the non-elementary one, this time in modal logic $K45$ or in modal logic $K5$. More generally, the unifiability problem, the elementary one or the non-elementary one, in modal logics extending $K5$ is of interest, knowing that these modal logics are *coNP*-complete [17]. Other *coNP*-complete modal logics are all proper extensions of $S5 \times S5$ [8, 9] and all finitely axiomatizable tense logics of linear time flows [20]. Thus, one may consider whether our method is applicable to the unifiability problem in these modal logics. A similar question can be asked as well with respect to the linear temporal logic considered by Babenyshev and Rybakov [5]. In other respect, what becomes of the unifiability problem, the elementary one or the non-elementary one, when the language is extended by the universal modality or the difference modality? Finally, considering the tight relationships between unifiability of formulas and admissibility of inference rules as explained in [1, 12, 15], one may ask whether all normal modal logics extending $K5$ are almost structurally complete, i.e. one may ask whether all admissible non-derivable inference rules are passive in these logics.

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