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Basic Postulates for Inconsistency Measures

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Abstract. Postulates for inconsistency measures are examined, the set of postulates proposed by Hunter and Konieczny being the starting point. The focus is on two postulates that were questioned by various authors. Studying the first suggests a systematic transformation to guard postulates against a certain kind of counter-examples. The second postulate under investigation here is devoted to independence, for which a general version is proposed that avoids the pitfalls mentioned in the literature. Combining these two additions with some postulates previously introduced by the same author, a set of basic postulates alternative to the core set given by Hunter and Konieczny arises.

1 Inconsistency Measures

There are plenty of reasons for belief bases to be inconsistent. Unfortunately, inconsistency is a nuisance on a number of counts (it makes deductive reasoning to collapse, it allows decision-making to simultaneously enforce two mutually exclusive options, ...). In short, inconsistency in belief bases is bad. How bad? This is the question that inconsistency measures have been taking seriously. Informally speaking, an inconsistency measure tells *to what extent* a belief base is inconsistent. Indeed, there seems to be degrees. Consider e.g. the statement “*This item is robust and affordable*”. One way to contradict it is by means of the statement “*If it’s robust then it is not affordable*”. Another way is by means of the statement “*It is neither robust nor affordable*”. The latter expresses that both claims (i.e., “*the item is robust*” and “*the item is affordable*”) in the initial statement are false but the former only objects that either “*the item is robust*” is false or “*the item is affordable*” is false. Accordingly, the belief base

$$K_1 = \left\{ \begin{array}{l} \textit{This item is robust and affordable} \\ \textit{If it’s robust then it is not affordable} \end{array} \right\}$$

can be viewed as *less inconsistent than* the belief base

$$K_2 = \left\{ \begin{array}{l} \textit{This item is robust and affordable} \\ \textit{It is neither robust nor affordable} \end{array} \right\}.$$

Formally, for I denoting an inconsistency measure,

$$I(K_1) < I(K_2).$$

A host of inconsistency measures exist, e.g., [1, 8, 9, 12, 14, 15, 20, 22, 23]. An example of a well-known inconsistency measure is [10], an approach based on counting contradicted atoms (a belief base consists of propositional formulas). Using $\mathcal{A}|_K$ to denote the set of atoms occurring in K , it is defined as

$$I_{3Mod}(K) \stackrel{\text{def}}{=} \frac{\min_{M \in 3Mod(K)} |\{a \in \mathcal{A}|_K : M(a) = B\}|}{|\mathcal{A}|_K|}$$

where $3Mod$ is a system of $\{T, F, B\}$ -valuations (with T and B being the designated truth-values) in which T stands for *True* and F stands for *False* while B stands for *True and False*. The truth tables are:

\neg		\wedge	F	T	B	\rightarrow	F	T	B
F	T	F	F	F	F	F	T	T	T
T	F	T	F	T	B	T	F	T	B
B	B	B	F	B	B	B	F	T	T

Let the atom a represent the statement “*the item is affordable*” and the atom r represent the statement “*the item is robust*”. Then, the belief bases K_1 and K_2 above can be written formally as

$$\begin{aligned} K_1 &= \{r \wedge a, r \rightarrow \neg a\} \\ K_2 &= \{r \wedge a, \neg r \wedge \neg a\} \end{aligned}$$

The $\{T, F, B\}$ -models of K_1 are

$$3Mod(K_1) = \left\{ \begin{array}{l} a \mapsto B, \quad a \mapsto B, \quad a \mapsto T \\ r \mapsto T, \quad r \mapsto B, \quad r \mapsto B \end{array} \right\}$$

of which the first and the third are such that only one atom is assigned the truth-value B (whereas in the second model, a and r are assigned the truth-value B). Then, the minimum number of atoms that are assigned B in models of K_1 is 1 (in symbols, $\min_{M \in 3Mod(K_1)} |\{a \in \mathcal{A}|_{K_1} : M(a) = B\}| = 1$). There are exactly two atoms occurring in K_1 hence $|\mathcal{A}|_K| = 2$. Therefore,

$$I_{3Mod}(K_1) = \frac{1}{2}.$$

There is a single $\{T, F, B\}$ -model of K_2 , namely

$$3Mod(K_2) = \left\{ \begin{array}{l} a \mapsto B \\ r \mapsto B \end{array} \right\}$$

hence the minimum number of atoms assigned B in models of K_2 is 2. Thus,

$$I_{3Mod}(K_2) = \frac{2}{2} = 1.$$

Summing up, $I_{3Mod}(K_1) < I_{3Mod}(K_2)$ which means that, according to the I_{3Mod} inconsistency measure, K_1 is less inconsistent than K_2 .

An approach based on counting formulas underlying contradictions is in [5]. Let $MI(K)$ denote the set of Minimal Unsatisfiable Subsets of K (in symbols, $MI(K) = \{K' \subseteq K : K' \vdash \perp \text{ and } K' \setminus \{\varphi\} \not\vdash \perp \text{ for all } \varphi \in K'\}$). Next, define

$$I_P(K) \stackrel{\text{def}}{=} \left| \bigcup MI(K) \right|.$$

Back to the above illustration, the only MUS of $\{r \wedge a, r \rightarrow \neg a\}$ is itself and the only MUS of $\{r \wedge a, \neg r \wedge \neg a\}$ is also itself. That is, $MI(K_1) = \{K_1\}$ and $MI(K_2) = \{K_2\}$. Despite contrary intuition (in fact, MUSes are not fine-grained enough to discriminate between contents of formulas), it follows that

$$I_P(K_1) = 2 = I_P(K_2).$$

Intuitively, an inconsistency measure I is supposed to indicate how much inconsistency a knowledge base K carries (where a more inconsistent belief base is ascribed a larger value). Of course, not every function I can do! A list of requirements over I is needed. To this end, postulates can ensure I to make sense for the purpose of inconsistency measuring.

This note is an investigation into such requirements as postulates for inconsistency measures have indeed been proposed on the following grounds:

- The context is classical logic \vdash over a language \mathcal{L} .
- Belief bases are finite sets of formulas of \mathcal{L} .
- I maps all finite sets of formulas of \mathcal{L} to values in $\mathbb{R}^+ \cup \{\infty\}$.

2 Postulates

Hunter and Konieczny [7] proposed a few postulates for inconsistency measures. The core set (of the Hunter-Konieczny postulates) is:

- $I(K) = 0$ iff $K \not\vdash \perp$ *(Consistency Null)*
 - $I(K \cup K') \geq I(K)$ *(Monotony)*
 - If $\alpha \vdash \beta$ and $\alpha \not\vdash \perp$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ *(Dominance)*
 - If α is free for K then $I(K \cup \{\alpha\}) = I(K)$ *(Free Formula Independence)*
- where a formula φ is free for X iff $Y \cup \{\varphi\} \vdash \perp$ for no consistent subset Y of X*

The last two postulates have been questioned on various grounds (see e.g., [13]). This note first gives a general principle that, among others, backs (Dominance) in a slightly amended form, and, second, proposes an independence postulate more sound than (Free Formula Independence) and also stronger than another well-known substitute [18] called (Safe Formula Independence).

In the course of this study, we emphasize the importance of the postulate for the elimination of tautologies (in a sense, the weakest independence postulate) and the role of the postulate expressing that the amount of inconsistency does not increase when a conjunction is replaced by one of its conjuncts.

In other words, we suggest a starting point for a body of postulates alternative to the core set provided by Hunter and Konieczny in their pioneering article.

2.1 Formalities

All the postulates to be discussed refer to propositional logic \vdash with a language \mathcal{L} based on a set of propositional variables denoted $Atoms(\mathcal{L})$ as well as the propositional constants \perp and \top . The symbols we use for the connectives are \neg (negation), \wedge (conjunction), \vee (disjunction). Turning to meta-level notation, \equiv denotes logical equivalence, i.e., $p \equiv q$ means $p \vdash q$ and $q \vdash p$. In addition, $\alpha, \beta, \gamma, \dots$ denote formulas of \mathcal{L} while K, K', \dots are called belief bases and denote finite sets of formulas of \mathcal{L} . Lastly, $\mathcal{K}_{\mathcal{L}}$ is comprised of all belief bases over \mathcal{L} .

Formally, inconsistency measures are maps $I : \mathcal{K}_{\mathcal{L}} \rightarrow \mathbb{R}^+ \cup \{\infty\}$. Intuitively, $I(K)$ indicates how inconsistent a belief base $K \in \mathcal{K}_{\mathcal{L}}$ is: A more inconsistent K is ascribed a larger value by I (with 0 being the least).

A number of inconsistency measures actually have $[0, 1]$ as their codomain. Equivalently, they can be viewed as satisfying the following postulate from [7]

$$- 0 \leq I(K) \leq 1 \quad (Normalization)$$

In any case, postulates are meant to capture some aspects of rationality for inconsistency measures.

3 Restriction on Membership

As already mentioned, (Dominance) has been argued against on various grounds. Here is a specific illustration.

Proposition 1. *Assuming (Monotony), (Dominance) is equivalent with:*

$$- \text{For } \alpha \in K, \text{ if } \alpha \not\vdash \perp \text{ and } \alpha \vdash \beta \text{ then } I(K \cup \{\beta\}) = I(K) \quad (A_1)$$

Proof. Let α and β be such that $\alpha \not\vdash \perp$ and $\alpha \vdash \beta$. Assume (A_1) . Trivially, $\alpha \in K \cup \{\alpha\}$ so that (A_1) applies to give $I(K \cup \{\alpha\} \cup \{\beta\}) = I(K \cup \{\alpha\})$. Due to (Monotony), $I(K \cup \{\alpha\} \cup \{\beta\}) \geq I(K \cup \{\beta\})$. By transitivity, it follows that $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$. Conversely, assume (Dominance). Consider K such that $\alpha \in K$. As a consequence of (Dominance), $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$. Accordingly, $I(K) \geq I(K \cup \{\beta\})$ since $I(K \cup \{\alpha\}) = I(K)$ in view of $\alpha \in K$. The converse, i.e., $I(K \cup \{\beta\}) \geq I(K)$ holds by (Monotony).

Proposition 1 really pinpoints the fact that (Dominance) may get I to go astray when α is in K , should β be involved in a MUS of $K \cup \{\beta\}$. Indeed, (A_1) then expresses that $I(K \cup \{\beta\}) = I(K)$ which may happen to be counterintuitive as the set of MUSes of K and $K \cup \{\beta\}$ need not be the same.

Example 1 (Adapted from [13]). Consider $K = \{p, \neg q, p \wedge q\}$. Take $\alpha = p \wedge q$ and $\beta = \neg p \vee q$. So, $\alpha \not\vdash \perp$ and $\alpha \vdash \beta$. By Proposition 1, (Dominance) would require $I(K \cup \{\beta\}) = I(K)$, i.e.,

$$I \left(\left(\left\{ \begin{array}{l} p, \\ \neg q, \\ p \wedge q, \\ \neg p \vee q \end{array} \right\} \right) \right) = I \left(\left(\left\{ \begin{array}{l} p, \\ \neg q, \\ p \wedge q \end{array} \right\} \right) \right)$$

although the MUSes of K consist of $\{\{\neg q, p \wedge q\}\}$ and the MUSes of $K \cup \{\beta\}$ consist of $\{\{\neg q, p \wedge q\}, \{p, \neg q, \neg p \vee q\}\}$.

This suggests a general principle, denoted **-principle* in the sequel, as follows. Would-be postulates of the form

- if ... then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ *(Postulate)*

should be turned into the form

- for $\alpha \notin K$, if ... then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ *(Postulate*)*

The expected proviso $\beta \notin K$ is omitted because $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ is ensured by (Monotony) for $\beta \in K$.

Fact 1 *For all inconsistency measure I , if I satisfies (Postulate) then I also satisfies (Postulate*).*

The *-principle gives rise to a slightly restricted version of (Dominance), i.e.,

- For $\alpha \notin K$, if $\alpha \vdash \beta$ and $\alpha \not\vdash \perp$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ *(Dominance*)*

Proposition 1 no longer holds if (Dominance) is replaced by (Dominance*). Moreover, (Dominance*) does not impose $I(K \cup \{\beta\}) = I(K)$ in Example 1. However, Example 2 introduced by Mu, Liu, Jin and Bell in [13] to show that the I_{MI} inconsistency measure (it simply counts the number of MUSes, i.e., $I_{MI}(K) = |\{M \subseteq K : M \text{ is a MUS of } K\}|$) fails (Dominance) also shows that I_{MI} fails (Dominance*).

Example 2 [13]. Let $K = \{p, p \wedge r, \neg q\}$. Let $\alpha = p \wedge r \wedge (\neg p \vee q)$ and $\beta = \neg p \vee q$. $K \cup \{\alpha\}$ has a single MUS $\{\neg q, p \wedge r \wedge (\neg p \vee q)\}$ and $K \cup \{\beta\}$ has two MUSes, which are $\{p, \neg q, \neg p \vee q\}$ and $\{p \wedge r, \neg q, \neg p \vee q\}$, hence both (Dominance) and (Dominance*) fail here because $I_{MI}(K \cup \{\alpha\}) = 1 < 2 = I_{MI}(K \cup \{\beta\})$.

A most worthwhile application of the *-principle is with a postulate introduced in [3], to the effect that inconsistency does not increase when a conjunction is replaced by any one of its conjuncts. In symbols, $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha \wedge \beta\})$. This actually conveys the very same idea than (Monotony): Extra information (whether in the form of an extra formula or in the form of an extra conjunct) cannot make the amount of inconsistency to decrease. Keeping the non-starred name although applying the *-principle, the postulate writes

- If $\alpha \wedge \beta \notin K$ then $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha \wedge \beta\})$ *(Conjunction Dominance)*

The need for the proviso $\alpha \wedge \beta \notin K$ in (Conjunction Dominance) can be illustrated by means of Example 3.

Example 3. Let $K = \{p \wedge \neg p \wedge q\}$. Take α to be $p \wedge \neg p$ and take β to be q . If it were not for the proviso $\alpha \wedge \beta \notin K$, (Conjunction Dominance) would give $I(\{p \wedge \neg p \wedge q, p \wedge \neg p\}) \leq I(\{p \wedge \neg p \wedge q\})$ —thereby precluding the intuitive possibility $I(\{p \wedge \neg p \wedge q, p \wedge \neg p\}) > I(\{p \wedge \neg p \wedge q\})$.

The *-principle extends in a natural way to would-be postulates of the form *if ... then $I(K \cup \{\alpha\}) = I(K \cup \{\beta\})$* , to be turned into *for $\alpha \notin K$ and $\beta \notin K$, if ... then $I(K \cup \{\alpha\}) = I(K \cup \{\beta\})$* .

4 Independence

A most welcome consequence of (Free Formula Independence) is

- If $\alpha \equiv \top$ then $I(K \cup \{\alpha\}) = I(K)$ *(Tautology Independence)*

Unless $\beta \not\vdash \perp$ (i.e., the case that would let (Dominance) to apply), there is unfortunately no guarantee that the following holds:

- for $\alpha \equiv \top$, if $\alpha \wedge \beta \notin K$ and $\beta \notin K$ then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$
(\top -conjunct Independence)

Addressing the concern that (Free Formula Independence) applies in some undesirable cases, Thimm [17] proposed a postulate called (Weak Independence) which was examined by Hunter and Konieczny [7] in the following form

- If α is safe for K then $I(K \cup \{\alpha\}) = I(K)$ *(Safe Formula Independence)*
where a formula φ is safe for X iff $\varphi \not\vdash \perp$ and $Atoms(\varphi) \cap Atoms(X) = \emptyset$

Unfortunately, (Safe Formula Independence) is weaker than expected. First, it may fail to apply to a formula $\alpha \wedge (p \vee \neg p)$ although applying to α . Second, (Safe Formula Independence) does not entail (Tautology Independence).

Anyway, postulates about independence attempt to capture the idea that if a formula can be satisfied with no impact on the truth value of critical items then such a formula is not to count for measuring inconsistency. We now propose a general version. In order to express syntactically the fact that a subformula has a truth value determining the truth of the entire formula, we follow the \mathcal{NP} -form approach [16]. However, we focus on atoms and we can thus simplify Schütte’s inductive definition, instead resorting to a special class of substitutions:

Definition 1. $\sigma : Atoms(\mathcal{L}) \rightarrow Atoms(\mathcal{L}) \cup \{\top, \perp\}$ is a Boolean substitution iff for all $a \in Atoms(\mathcal{L})$, either $\sigma(a) = a$ or $\sigma(a) = \top$ or $\sigma(a) = \perp$.

A generalization of the notion of a safe formula is now in order, according to the intuitions just expressed.

Definition 2. A formula φ is safely consistent for X if there exists a Boolean substitution σ such that $\sigma\varphi$ is a tautology and $\sigma(a) = a$ for all $a \in Atoms(X)$.

A new independence postulate can then be formulated—with a due proviso from the *-principle—as follows.

- If $\alpha \wedge \beta \notin K$, $\beta \notin K$, and α is safely consistent for $K \cup \{\beta\}$ then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$ *(Conjunct Independence)*

Clearly, if φ is safely consistent for K then φ is free for K . This means that (Free Formula Independence) is at least as strong as (Conjunct Independence). Example 4 shows that the converse is untrue.

Example 4. Consider $K = \{p \wedge r, q \wedge \neg r\}$. Let α be $(\neg p \vee \neg q) \wedge \top$. Clearly, α is free for K hence (Free Formula Independence) applies although $\neg p \vee \neg q$ causes a contradiction with two conjunctively consistent parts of formulas of K (see [3]). In other words, $I(\{p \wedge r, q \wedge \neg r, (\neg p \vee \neg q) \wedge \top\}) = I(\{p \wedge r, q \wedge \neg r\})$ is required by (Free Formula Independence) but not by (Conjunct Independence).

The next results show that (Conjunct Independence) entails several other independence postulates, possibly with the help of (Tautology Independence).

Proposition 2. *(Conjunct Independence) entails (\top -conjunct Independence).*

Proof. For α tautologous, σ can be taken to be identity. It is then possible to apply (Conjunct Independence) which gives $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$.

Proposition 3. *Assuming (Tautology Independence), (Conjunct Independence) entails*

- if α is safely consistent for K then $I(K \cup \{\alpha\}) = I(K)$

Proof. If $\alpha \in K$ then $I(K \cup \{\alpha\}) = I(K)$. Therefore, assume $\alpha \notin K$. Since K is finite, there exists n such that $\alpha \wedge \bigwedge_{i=1}^n \top \notin K$ and $\bigwedge_{i=1}^n \top \notin K$. Clearly, (Tautology Independence) gives $I(K) = I(K \cup \{\bigwedge_{i=1}^n \top\})$. Since α is safely consistent for K , it is safely consistent for $K \cup \{\bigwedge_{i=1}^n \top\}$. Due to $\bigwedge_{i=1}^n \top \notin K$ and $\alpha \wedge \bigwedge_{i=1}^n \top \notin K$, (Conjunct Independence) can thus be applied so that $I(K \cup \{\bigwedge_{i=1}^n \top\}) = I(K \cup \{\alpha \wedge \bigwedge_{i=1}^n \top\})$ is obtained. In view of $\alpha \wedge \bigwedge_{i=1}^n \top \notin K$ and $\alpha \notin K$, it happens that $I(K \cup \{\alpha \wedge \bigwedge_{i=1}^n \top\}) = I(K \cup \{\alpha\})$ by means of (\top -conjunct Independence) which is available according to Proposition 2. Summing up, $I(K) = I(K \cup \{\alpha\})$.

Proposition 4. *Assuming (Tautology Independence), (Conjunct Independence) entails (Safe Formula Independence).*

Proof. Let α be safe for K , that is, $\alpha \not\vdash \perp$ and $Atoms(K) \cap Atoms(\alpha) = \emptyset$. From completeness of \vdash , there exists a valuation v over $Atoms(\alpha)$ satisfying $v(\alpha) = true$. Thus, α has the truth value *true* if each atom a in $Atoms(\alpha)$ has the truth value $v(a)$. Define σ as $\sigma(a) = a$ if $a \in \mathcal{L} \setminus Atoms(\alpha)$ whereas $\sigma(a) = \top$ for $a \in Atoms(\alpha)$ s. t. $v(a) = true$ and $\sigma(a) = \perp$ for $a \in Atoms(\alpha)$ s. t. $v(a) = false$. Accordingly, for all atomic formulas a in α , $v(a) = v(\sigma a)$. By induction, $v(\alpha) = v(\sigma \alpha)$. Thus, $\sigma \alpha$ is true under v . However, all atomic formulas in $\sigma \alpha$ are \top and \perp hence the truth value of $\sigma \alpha$ is independent of v . Since $\sigma \alpha$ is true under v , this means that $\sigma \alpha$ is a tautology. In view of $\sigma(a) = a$ for all $a \in \mathcal{L} \setminus Atoms(\alpha)$, it is clear that $\sigma(a) = a$ for all $a \in Atoms(K)$ because $Atoms(K) \cap Atoms(\alpha) = \emptyset$. Therefore, α is safely consistent for K . Finally, $I(K \cup \{\alpha\}) = I(K)$ by Proposition 3.

Example 5 shows that the converse of Proposition 4 is untrue.

Example 5. Consider $K = \{p \wedge q \wedge \neg q\}$ and let α be $\neg q \vee [r \leftrightarrow (s \wedge (q \vee \neg q))]$ (for readability, α is abbreviated using the symbol \leftrightarrow with its usual meaning). Since α is not safe for K , (Safe Formula Independence) fails to apply. However, replacing r and s by \perp in α results in a tautology, i.e., (Conjunct Independence) and (Tautology Independence) give $I(K \cup \{\alpha\}) = I(K)$ (see Proposition 3).

5 Paradigm

Example 6. A paradigmatic case is

$$K_0 = \left\{ \begin{array}{l} p \wedge \neg p \\ q \wedge \neg r \wedge s \end{array} \right\} \quad K_1 = \left\{ \begin{array}{l} p \wedge \neg p \\ p \wedge \neg p \wedge s \end{array} \right\} \quad K_2 = \left\{ \begin{array}{l} p \wedge \neg p \\ q \wedge \neg q \wedge s \end{array} \right\}$$

where it is expected that $I(K_0) \leq I(K_1)$ and $I(K_0) \leq I(K_2)$ and $I(K_2) \not\leq I(K_1)$.

The codomain of I is totally ordered hence $I(K_2) \not\leq I(K_1)$ is in fact equivalent with $I(K_1) \leq I(K_2)$, and what is expected in Example 6 is in fact

$$I(K_0) \leq I(K_1) \leq I(K_2).$$

Intuition also suggests

$$I(K_0) < I(K_2).$$

Hence, either $I(K_0) < I(K_1)$ or $I(K_1) < I(K_2)$ (or both) must hold. That is,

- the number of occurrences of the *same* atomic contradiction makes a difference,
- or the number of *distinct* atomic contradictions makes a difference.

Now, $I(K_0) \leq I(K_1)$ (and $I(K_0) \leq I(K_2)$ in a similar way) can be shown using (Monotony) and (Safe Formula Independence) as follows. Since $q \wedge \neg r \wedge s$ is safe for $\{p \wedge \neg p\}$, (Safe Formula Independence) can be applied to give

$$I\left(\left\{ \begin{array}{l} p \wedge \neg p \\ q \wedge \neg r \wedge s \end{array} \right\}\right) = I(\{p \wedge \neg p\})$$

while (Monotony) of course gives

$$I(\{p \wedge \neg p\}) \leq I\left(\left\{ \begin{array}{l} p \wedge \neg p \\ p \wedge \neg p \wedge s \end{array} \right\}\right)$$

and $I(K_0) \leq I(K_1)$ ensues by transitivity, i.e.,

$$I\left(\left\{ \begin{array}{l} p \wedge \neg p \\ q \wedge \neg r \wedge s \end{array} \right\}\right) \leq I\left(\left\{ \begin{array}{l} p \wedge \neg p \\ p \wedge \neg p \wedge s \end{array} \right\}\right)$$

However, (Safe Formula Independence) is not general enough to deal with a variant where an intuitively safe formula actually fails to be safe because it has a conjunct which is a tautology over an atom in K ; e.g., for our running example:

$$K'_0 = \left\{ \begin{array}{l} p \wedge \neg p \\ q \wedge \neg r \wedge s \wedge (p \vee \neg p) \end{array} \right\} \quad K_1 = \left\{ \begin{array}{l} p \wedge \neg p \\ p \wedge \neg p \wedge s \end{array} \right\} \quad K_2 = \left\{ \begin{array}{l} p \wedge \neg p \\ q \wedge \neg q \wedge s \end{array} \right\}$$

The combination “(Safe Formula Independence) + (Monotony)” can be generalized by substituting (Conjunct Independence) and (Tautology Independence) for (Safe Formula Independence). To generalize further, supplement (Monotony) with (Conjunction Dominance).

In the example, $I(K'_0) = I(\{p \wedge \neg p\})$ is given by (Conjunct Independence) together with (Tautology Independence) whereas $I(\{p \wedge \neg p\}) \leq I(K_1)$ comes from (Monotony). That is, $I(K'_0) \leq I(K_1)$ can be shown using these postulates.

6 Exhibiting Conjunctions

The reader might be concerned that some important postulates above explicitly mention a distinctive connective, namely conjunction. Such a concern is well-taken because there are a host of forms, $\neg(\neg\alpha \vee \neg\beta)$ for instance, under which conjunctions may hide. Display logic [2] tells us what to do: if a subformula α is in essence a conjunct of the whole formula then it is possible to rewrite the formula in such a way that α *actually* occurs as a conjunct. Fortunately, there is no need here to resort to the full-fledged system (display logic is meant to cover a large family of logics) because we are only interested in classical logic.

Thus, define α' to be a *prenormal form* of α if α' results from α by applying (possibly repeatedly) one or more of the principles: commutativity, associativity and distribution for \wedge and \vee , De Morgan laws, double negation equivalence.

- If β is a prenormal form of α then $I(K \cup \{\alpha\}) = I(K \cup \{\beta\})$ (*Rewriting*)

There is another way the form of a belief base should be irrelevant with respect to inconsistency measuring, and that is, the propositional symbols chosen to express formulas in a belief base must be interchangeable as follows.

- If σ and σ' are substitutions s.t. $\sigma K = K'$ and $\sigma' K' = K$ then $I(K) = I(K')$ (*Variant Equality*)

It seems indeed rational to hold that $\{p, q \wedge \neg p\}$ for example conveys exactly the same amount of inconsistency than $\{r, s \wedge \neg r\}$ and the like.

7 Towards an Alternative Set of Basic Postulates

Summing up, an alternative set of basic postulates would be:

- $I(K) = 0$ iff $K \not\vdash \perp$ (*Consistency Null*)
- $I(K \cup K') \geq I(K)$ (*Monotony*)
- For $\alpha \notin K$, if $\alpha \vdash \beta$ and $\alpha \not\vdash \perp$ then $I(K \cup \{\alpha\}) \geq I(K \cup \{\beta\})$ (*Dominance**)
- If $\alpha \equiv \top$ then $I(K \cup \{\alpha\}) = I(K)$ (*Tautology Independence*)
- If $\alpha \wedge \beta \notin K$, $\beta \notin K$, and α is safely consistent for $K \cup \{\beta\}$ then $I(K \cup \{\alpha \wedge \beta\}) = I(K \cup \{\beta\})$ (*Conjunct Independence*)
- If β is a prenormal form of α then $I(K \cup \{\alpha\}) = I(K \cup \{\beta\})$ (*Rewriting*)
- If σ and σ' are substitutions s.t. $\sigma K = K'$ and $\sigma' K' = K$ then $I(K) = I(K')$ (*Variant Equality*)

In a very insightful comparison, Thimm [21] checks many inconsistency measures against various postulates. An interesting conclusion is that no postulate except (Inconsistency Null) is satisfied by all those inconsistency measures. How well do our basic postulates would fare? In fact, all those inconsistency measures satisfy (Variant Equality) and (Rewriting). The other way around, an inconsistency measure such as I_{MIC} (defined in [6] by $I_{MIC}(K) = \Sigma_{M \in MI(K)} 1/|M|$ with $MI(K)$ denoting the set of MUSes of K) satisfies all our basic postulates.

Despite the argument that (Monotony) is only one way to express that extra information cannot make the amount of inconsistency to decrease while another way is by means of the postulate

- If $\alpha \wedge \beta \notin K$ then $I(K \cup \{\alpha\}) \leq I(K \cup \{\alpha \wedge \beta\})$ (*Conjunction Dominance*)

the latter is not in the set. The reason is that (Conjunction Dominance) rules out inconsistency measures based on Minimal Unsatisfiable Subsets. As an example, considering $K = \{p, \neg q, \neg r\}$ when taking α to be $s \wedge (\neg p \vee (q \wedge r))$ and taking β to be $\neg p$ gives $I_{MIC}(K \cup \{\alpha\}) = 1/3 + 1/3 > 1/2 = I_{MIC}(K \cup \{\alpha \wedge \beta\})$. That is, (Conjunction Dominance) is failed by I_{MIC} .

With the advent of a number of new inconsistency measures (see e.g. [9] as well as the special issue [11] in the Journal of Approximate Reasoning) or related approaches [4, 19], further investigations are afoot.

8 Conclusion

We have examined basic postulates for inconsistency measures, unearthing a simple transformation to secure would-be postulates against a family of counter-examples and investigating independence postulates. We have also considered a case study in the form of three belief bases close to each other, illustrating the impact of the linearity of the codomain of an inconsistency measure (so that an intuition about $I(K) \not\leq I(K')$ actually means $I(K) \geq I(K')$) leading to the fact that on some occasions it *must* either be that (1) the number of occurrences of the same atomic contradiction makes a difference, or be that (2) the number of distinct atomic contradictions makes a difference.

In the end, we propose a set of basic postulates that can be viewed as an alternative to the core set introduced by Hunter and Konieczny. Among these are two postulates expressing that changing the form of a belief base through e.g. exchanging the order of disjuncts in a disjunction or using fresh propositional symbols should be of no consequence when it comes to inconsistency measuring. We have also discussed a postulate along the lines of the rather uncontroversial (Monotony) postulate, to the effect that extending a belief base cannot make the amount of inconsistency to decrease, regardless of whether extra information comes in the form of an extra formula or of an extra conjunct. A major avenue for future work is exhaustiveness, both through providing further postulates and through offering criteria to fix bounds to the introduction of basic postulates for inconsistency measures.

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