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Stochastic modeling for the financial markets. Part 2. Dynamical Programming

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MINISTRY OF EDUCATION AND SCIENCE
OF THE RUSSIAN FEDERATION
NATIONAL RESEARCH TOMSK STATE UNIVERSITY
FACULTY OF MECHANICS AND MATHEMATICS

**STOCHASTIC MODELLING
FOR THE FINANCIAL MARKETS
PART 2. DYNAMICAL PROGRAMMING**

Lectures notes
for the courses "Methods of optimization" and "Stochastic
modelling" taken by most Mathematics students and Economics
students (directions of training 01.03.01 – Mathematics and
38.04.01 – Economics)

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The main goal of these lectures notes is to study the stochastic dynamic programming method in discrete and continuous time proposed by Bellman. We apply the analytical methods studied in mathematical analysis and theory of probability to the important optimization problems such as the optimal consumption and investment in discrete and continuous time. The notes are intended for students of the Mathematics and Economics Faculties.

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1 Optimal consumption in the discrete time

We consider the optimal consumption problem on the interval $[0, N]$, where $N < \infty$ is the horizon, i.e. we consider a consumer who chooses the amount of its consumption $c_k \geq 0$ at the time moment $0 \leq k \leq N$. Describe the dynamics of his wealth. Let $x_0 = x > 0$ be the initial wealth. So, at the time $k = 0$ we obtain the constraint on the consumption:

$$0 \leq c_0 \leq x.$$

Then he places the rest $x - c_0$ in a risk-free asset with the interest rate $r \geq 0$. So, the next day his wealth is equal to

$$x_1 = R(x - c_0) \quad \text{and} \quad R = 1 + r.$$

At the time $k = 1$ the consumer can consume $0 \leq c_1 \leq x_1$ and its capital at the time $k = 2$ is given by the following equation

$$x_2 = R(x_1 - c_1).$$

Similarly, when $k \geq 1$ the capital x_k is written as

$$x_k = R(x_{k-1} - c_{k-1}), \quad x_0 = x, \tag{1.1}$$

where the consumption vector $c = (c_0, \dots, c_{N-1})'$ holds the following constraints

$$0 \leq c_k \leq x_k, \quad 0 \leq k \leq N - 1. \quad (1.2)$$

Recall that the prime $'$ means transpose.

Let $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a utility function and $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ be a heritage function. It is assumed that the functions U and h are increasing and continuous. In this case the objective function is

$$J_0(x, c_0, \dots, c_{N-1}) = \sum_{k=0}^{N-1} U(c_k) + h(x_N). \quad (1.3)$$

The problem is to find a sequence $c^* = (c_0^*, \dots, c_{N-1}^*)'$ that maximizes the objective function (1.3) under the constraints (1.1) – (1.2), i.e. to solve the following extremal problem

$$J_0^*(x) = \max_{(c_0, \dots, c_{N-1})' \in \mathcal{C}_0(x)} J_0(x, c_0, \dots, c_{N-1}), \quad (1.4)$$

where $\mathcal{C}_0(x) \subset \mathbb{R}^N$ is the set of the vectors $c = (c_0, \dots, c_{N-1})'$ which satisfy the conditions (1.1) – (1.2).

To resolve this problem we will use the Bellman dynamic programming method [1]. Now we describe this approach. First define

the objective functions on intervals $[n, N]$ for $0 \leq n \leq N - 1$. We have

$$J_n(x, c_n, \dots, c_{N-1}) = \sum_{k=n}^{N-1} U(c_k) + h(x_N). \quad (1.5)$$

Here the consumptions c_n, \dots, c_{N-1} satisfy the following conditions

$$0 \leq c_k \leq x_k, \quad n \leq k \leq N - 1, \quad (1.6)$$

where $x_n = x$ and for $k > n$ we get

$$x_k = R(x_{k-1} - c_{k-1}). \quad (1.7)$$

Denote by $\mathcal{C}_n(x)$ the set in \mathbb{R}^{N-n} of the vectors $(c_n, \dots, c_{N-1})'$ which hold the constraints (1.6) – (1.7). For $0 \leq n \leq N - 1$ we set

$$J_n^*(x) = \max_{(c_n, \dots, c_{N-1})' \in \mathcal{C}_n(x)} J_n(x, c_n, \dots, c_{N-1}). \quad (1.8)$$

For $n = N$ put

$$J_N^*(x) = h(x). \quad (1.9)$$

To resolve the problems (1.8) we use the following deterministic dynamic programming principle.

Theorem 1.1. (Deterministic dynamic programming principle)

The functions $(J_n^(x))_{0 \leq n \leq N-1}$ satisfy the following Bellman equa-*

tions:

$$J_n^*(x) = \max_{0 \leq c \leq x} \left(U(c) + J_{n+1}^*(\phi(x, c)) \right), \quad (1.10)$$

where $\phi(x, c) = R(x - c)$.

Proof. First, note that all the sets $\mathcal{C}_n(x)$ are bounded in \mathbb{R}^{N-n} . Therefore all the functions $(J_n^*(x))_{0 \leq n \leq N-1}$ are fined, i.e. the problem (1.8) has a solution for any $0 \leq n \leq N - 1$. Moreover, by the definition, we obtain that

$$\begin{aligned} J_n^*(x) &= \max_{(c_n, \dots, c_{N-1})' \in \mathcal{C}_n(x)} \sum_{k=n}^{N-1} U(c_k) + h(x_N) \\ &= \max_{0 \leq c_n \leq x} \left(\max_{(c_{n+1}, \dots, c_{N-1})' \in Q(x, c_n)} \left(U(c_n) + \sum_{k=n+1}^{N-1} U(c_k) + h(x_N) \right) \right) \\ &= \max_{0 \leq c \leq x} \left(U(c) + \max_{(c_{n+1}, \dots, c_{N-1})' \in Q(x, c)} J_{n+1}(\phi(x, c), c_{n+1}, \dots, c_{N-1}) \right), \end{aligned}$$

where $Q(x, c) = \mathcal{C}_{n+1}(\phi(x, c)) \subset \mathbb{R}^{N-n-1}$. This directly implies the equation (1.10). \square

To find the optimal objective functions $(J^*(n, x))_{0 \leq n \leq N-1}$ we use the back recursion. It means that we start from $n = N - 1$. We have

$$J_{N-1}^*(x) = \max_{0 \leq c \leq x} (U(c) + h(\phi(x, c))). \quad (1.11)$$

We denote by $s_{N-1}(x)$ the solution of this problem. Similarly, by

the recurrence we obtain the solution $s_n(x)$ of the problem (1.10) for $0 \leq n \leq N - 1$. Therefore, finally we obtain $s_0(x)$, where $x > 0$ is initial wealth. This means that $c_0^* = \mathbf{s}_0(x)$. Then, to find the consumption for every $n \geq 1$ we have to calculate the optimal wealths $(x_n^*)_{1 \leq n \leq N}$. We start with $n = 1$. We obtain

$$x_1^* = R(x - c_0^*).$$

This implies that $c_1^* = \mathbf{s}_1(x_1^*)$. This consumption gives us the wealth $n = 2$, i.e.

$$x_2^* = R(x_1^* - c_1^*)$$

and $c_2^* = \mathbf{s}_2(x_2^*)$ etc. So, by the recurrence

$$x_n^* = R(x_{n-1}^* - c_{n-1}^*) \quad \text{and} \quad c_n^* = \mathbf{s}_n(x_n^*).$$

Finally, the solution of (1.4) is given by the vector

$$c^* = (c_0^*, c_1^*, \dots, c_{N-1}^*)'.$$

Let us consider now the following example.

Example. Let $U(x) = h(x) = x^\gamma$ and $0 < \gamma < 1$. We start to resolve the Bellman equations (1.10) with $n = N - 1$, i.e. the

equation (1.11). In this case

$$J_{N-1}^*(x) = \max_{0 \leq c \leq x} (c^\gamma + (x - c)^\gamma R^\gamma) .$$

For $0 < \gamma < 1$ the solution is

$$\mathbf{s}_{N-1}(x) = \alpha_{N-1}x, \quad \alpha_{N-1} = \frac{1}{1 + R^q} \quad \text{and} \quad q = \frac{\gamma}{1 - \gamma} .$$

This means that

$$J_{N-1}^*(x) = \beta_{N-1} x^\gamma \quad \text{and} \quad \beta_{N-1} = \alpha_{N-1}^\gamma + (1 - \alpha_{N-1})^\gamma R^\gamma . \tag{1.12}$$

We will resolve the Bellman equations (1.10) by induction. We assume that for some $1 \leq n \leq N - 1$ the solution has the following form:

$$\mathbf{s}_n(x) = \alpha_n x \quad \text{and} \quad J_n^*(x) = \beta_n x^\gamma \tag{1.13}$$

for some positive constants α_n and β_n . To show that these formulas are true for all $0 \leq n \leq N - 1$ it suffices to verify this for $n - 1$. Using (1.13) in (1.10) one obtains that

$$\begin{aligned} J_{n-1}^*(x) &= \max_{0 \leq c \leq x} (c^\gamma + J^*(n, R(x - c))) \\ &= \max_{0 \leq c \leq x} (c^\gamma + \beta_n (x - c)^\gamma R^\gamma) . \end{aligned}$$

We find that $\mathbf{s}_{n-1}(x) = \alpha_{n-1}x$ and

$$J_{n-1}^*(x) = \mathbf{s}_{n-1}(x)^\gamma + \beta_n R^\gamma (x - \mathbf{s}_{n-1}(x))^\gamma = \beta_{n-1} x^\gamma,$$

where

$$\alpha_{n-1} = \frac{1}{1 + R^q \beta_n^{1+q}} \quad \text{and} \quad \beta_{n-1} = \alpha_{n-1}^\gamma + \beta_n R^\gamma (1 - \alpha_{n-1})^\gamma. \quad (1.14)$$

This implies that (1.13) is true for all $0 \leq n \leq N-1$. So, to find the optimal consumptions it remains to calculate the wealth $(x_n^*)_{0 \leq n \leq N}$. We start by $n = 0$. Our initial wealth is $x > 0$. Therefore,

$$c_0^* = \mathbf{s}_0(x) = \alpha_0 x.$$

Then, for $n = 1$ we get that

$$x_1^* = R(x - c_0^*) = R(1 - \alpha_0)x \quad \text{and} \quad c_1^* = \mathbf{s}_1(x_1^*) = \alpha_1 x_1^*.$$

By the same way for any $n \geq 1$

$$x_n^* = R(x_{n-1} - c_{n-1}^*) = R(1 - \alpha_{n-1})x_{n-1}^* \quad \text{and} \quad c_n^* = \mathbf{s}_n(x_n^*) = \alpha_n x_n^*.$$

This implies that for $1 \leq n \leq N$

$$x_n^* = xR^n \prod_{j=1}^n (1 - \alpha_{j-1}) \quad \text{and} \quad c_n^* = x\alpha_n R^n \prod_{j=1}^n (1 - \alpha_{j-1}).$$

Using here the recurrence equations (1.14) with the values β_{N-1} given in (1.12) we can calculate all the coefficients $(\alpha_j)_{0 \leq j \leq N-2}$.

Exercises

1. Let the utility function $U(x) = \ln x$ and the heritage function $h(x) = \ln x$.
 - (a) Find optimal consumption for $N = 1$, $N = 2$ and any $N \geq 3$.
 - (b) For which the interest rates $r \geq 0$ optimal terminal wealth will be greater than initial wealth x for the cases $N = 1$, $N = 2$ and any $N \geq 3$?
2. Let $U(x) = h(x) = 1 - e^{-x}$. Find optimal consumptions for $N = 1$. Find optimal consumptions for $N = 2$ and any $N \geq 3$ in the case $\mathbf{r} = 0$.
3. Find the optimal consumptions for the following cases:
 - (a) $U(x) = x$ and $h(x) = 0$; $N = 1$, $N = 2$ and $N = 3$.

(b) $U(x) = 0$ and $h(x) = 1 - e^{-x}$; $N = 1$, $N = 2$ and $N = 3$.

2 Optimal consumption and investment in discrete time models

Let (Ω, \mathbf{P}) be a finite probability space with $\Omega = \{\omega_1, \dots, \omega_k\}$. In this space we consider a (B, S) - financial market with two assets on the time interval $[0, N]$:

$$\begin{cases} (B) & B_n = (1+r)B_{n-1}, \\ (S) & S_n = (1+\rho_n)S_{n-1}, \quad 1 \leq n \leq N, \end{cases} \quad (2.1)$$

where $B_0 > 0$ and $S_0 > 0$ are initial values. Assume that interest rates of the risk asset $(\rho_n)_{1 \leq n \leq N}$ is sequence of the independent identically distributed (i.i.d.) random variables with values $\{a_1, \dots, a_m\}$, where $a_1 < \dots < a_m$ and $\mathbf{P}(\rho_n = a_i) = p_i > 0$. To avoid trivial strategies we will assume that

$$-1 < a_1 < r < a_m. \quad (2.2)$$

We consider an investor with an initial capital $x_0 = x > 0$. On the date $n = 0$ we can consume a quantity $0 \leq c_0 \leq x$. Moreover,

it distributes the remaining $x - c_0$ between the assets (B) and (S) as follows. The part $\theta_0(x - c)$ is placed in the asset (S) and $(1 - \theta_0)(x - c)$ in (B). This means that on the date $n = 1$ (the next day) the wealth x_1 is equal to

$$x_1 = R(1 - \theta_0)(x_0 - c_0) + (1 + \rho_1)\theta_0(x_0 - c_0) = \tau_1(\theta_0)(x_0 - c_0),$$

where $\tau_1(\theta) = R + \theta(\rho_1 - r)$ and $R = 1 + r$.

Let now x_n the wealth on the date n . We denote by $0 \leq c_n \leq x_n$ the consumption on the date n and by θ_n the part of the wealth invested in risk asset (S) on the same date. Similarly, we obtain

$$x_{n+1} = \tau_{n+1}(\theta_n)(x_n - c_n), \quad (2.3)$$

where $\tau_n(\theta) = R + \theta(\rho_n - r)$ and $R = 1 + r$.

We need to introduce now an information structure, i.e. we need to describe all possible random variables which can be used for financial strategies $(v_n)_{0 \leq n \leq N}$, where $v_n = (c_n, \theta_n)$. To this end we set $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_n = \sigma\{\rho_1, \dots, \rho_n\}$ for $1 \leq n \leq N$. The sequence $(\mathcal{F}_n)_{0 \leq n \leq N}$ is called *filtration*. Assume that the random variables v_n are measured with respect to \mathcal{F}_n for any $0 \leq n \leq N - 1$. In this case we say that the sequence of random variables

$(v_n)_{0 \leq n \leq N-1}$ is *adapted*. Moreover, the variables $(x_n)_{1 \leq n \leq N}$ should be positive. It means that for any $0 \leq k \leq N-1$

$$\theta_k \in \Theta \quad \text{and} \quad 0 \leq c_k \leq x_k, \quad (2.4)$$

where

$$\Theta = \{\theta \in \mathbb{R} : \inf_{k \geq 1} \tau_k(\theta) \geq 0 \quad \text{a.s.}\}$$

The variables $(x_k)_{1 \leq k \leq N}$ are defined by (2.3) with $x_0 = x$. We denote by $\mathcal{V}_0(x)$ the set of all adapted sequences $v = (v_n)_{0 \leq n \leq N-1}$ which satisfy the constraints (2.4). Generally, we denote by $\mathcal{V}_n(x)$ the set of all adapted sequences $(v_k)_{n \leq k \leq N-1}$ which satisfy the constraints (2.4) for $n \leq k \leq N-1$ with $x_n = x$. We define now the cost function on the time interval $[n, N]$ as

$$J_n(v_n, \dots, v_{N-1}) = \sum_{k=n}^{N-1} U(c_k) + h(x_N). \quad (2.5)$$

Our goal is to find a financial strategy $v^* = (v_k^*)_{0 \leq k \leq N-1}$ in $\mathcal{V}_0(x)$ which maximizes the conditional expectation of the function

$$J_0(x, v_0, \dots, v_{N-1}),$$

i.e.

$$J_0^*(x) = \max_{(v_0, \dots, v_{N-1})' \in \mathcal{V}_0(x)} \mathbf{E} \left(J_0(v_0, \dots, v_{N-1}) | x_0 = x \right). \quad (2.6)$$

For this problem we will use the Bellman stochastic dynamical programming principle. According to this principle we need to find optimal strategies on the intervals $[n, N]$ for $0 \leq n \leq N - 1$, i.e.

$$J_n^*(x) = \max_{(v_n, \dots, v_{N-1})' \in \mathcal{V}_n(x)} \mathbf{E} \left(J_n(v_n, \dots, v_{N-1}) | x_n = x \right). \quad (2.7)$$

Theorem 2.1. (The stochastic dynamic programming principle)
The optimal cost functions $(J_n^(x))_{0 \leq n \leq N-1}$ satisfy the following Bellman equations :*

$$\begin{cases} J_n^*(x) &= \max_{0 \leq c \leq x, \theta \in \Theta} (U(c) + \bar{J}_{n+1}(x, c, \theta)), \\ J_N^*(x) &= h(x), \end{cases} \quad (2.8)$$

where

$$\bar{J}_{n+1}(x, c, \theta) = \mathbf{E} \left(J_{n+1}^*(x_{n+1}) | x_n = x, v_n = (c, \theta) \right).$$

Example. Let us resolve the Bellman equations (2.8) for $U(x) =$

$h(x) = x^\gamma$ with $0 < \gamma < 1$. First, note that

$$\bar{J}_{n+1}(c, \theta) = \mathbf{E} J_{n+1}^*(\tau_1(\theta)(x - c)),$$

where $\tau_1(\theta) = R + \theta(\rho_1 - r)$. Thus, in this case the equation (2.8) has the following form

$$J_n^*(x) = \max_{0 \leq c \leq x} \max_{\theta \in \Theta_n} (U(c) + \mathbf{E} J_{n+1}^*(\tau_1(\theta)(x - c))).$$

Moreover, it is easy to see that

$$\Theta = [\theta_{min}, \theta_{max}], \quad (2.9)$$

where

$$\theta_{min} = -\frac{R}{a_m - r} \quad \text{and} \quad \theta_{max} = \frac{R}{r - a_1}.$$

Therefore, we obtain that for $0 \leq n \leq N - 1$

$$J_n^*(x) = \max_{0 \leq c \leq x} \max_{\theta_{min} \leq \theta \leq \theta_{max}} \left(U(c) + \mathbf{E} J_{n+1}^*(\tau_1(\theta)(x - c)) \right) \quad (2.10)$$

and $J_N^*(x) = h(x) = x^\gamma$.

We start to study these equations from $n = N - 1$. We have

$$\begin{aligned} J_{N-1}^*(x) &= \max_{0 \leq c \leq x} \max_{\theta_{min} \leq \theta \leq \theta_{max}} (c^\gamma + (x - c)^\gamma \mathbf{L}(\theta)) \\ &= \max_{0 \leq c \leq x} (c^\gamma + (x - c)^\gamma \nu). \end{aligned} \quad (2.11)$$

where

$$\nu = \max_{\theta_{min} \leq \theta \leq \theta_{max}} \mathbf{L}(\theta) \quad \text{and} \quad \mathbf{L}(\theta) = \mathbf{E}(\tau_1(\theta))^\gamma.$$

To study this problem one needs to solve the following equation

$$f(\theta) = \mathbf{E} \frac{\rho_1 - r}{(R + \theta(\rho_1 - r))^{1-\gamma}} = \sum_{j=1}^m \frac{a_j - r}{(R + \theta(a_j - r))^{1-\gamma}} p_j = 0. \quad (2.12)$$

One can check directly, that the solution of the problem (2.11) is given by $(\mathbf{s}_{N-1}(x), \kappa_{N-1}(x))$ with

$$\mathbf{s}_{N-1}(x) = \frac{1}{1 + \mu} x \quad \text{and} \quad \kappa_{N-1}(x) = \kappa^*,$$

where $\mu = \nu^q$, $q = 1/(1 - \gamma)$ and κ^* is the unique root of the equation (2.12) on the interval $(\theta_{min}, \theta_{max})$. We set

$$\mathbf{s}_{N-1}(x) = \alpha_{N-1} x, \quad \text{where} \quad \alpha_{N-1} = \frac{1}{1 + \mu}. \quad (2.13)$$

Then, we obtain that

$$J_{N-1}^*(x) = \beta_{N-1} x^\gamma, \quad \beta_{N-1} = \alpha_{N-1}^\gamma + \nu(1 - \alpha_{N-1})^\gamma. \quad (2.14)$$

Now, by induction we assume that for $0 \leq n \leq N - 1$

$$\mathbf{s}_n(x) = \alpha_n x, \quad \kappa_n(x) = \kappa^* \quad \text{and} \quad J_n^*(x) = \beta_n x^\gamma. \quad (2.15)$$

Using these conditions in (2.10), we find

$$J_{n-1}^*(x) = \max_{0 \leq c \leq x} \max_{\theta_{min} \leq \theta \leq \theta_{max}} (c^\gamma + \beta_n (x - c)^\gamma \mathbf{L}(\theta)).$$

In the same way as for the case $n = N - 1$ we get

$$\mathbf{s}_{n-1}(x) = \alpha_{n-1} x, \quad \kappa_{n-1}(x) = \kappa^* \quad \text{and} \quad J_{n-1}^*(x) = \beta_{n-1} x^\gamma, \quad (2.16)$$

where

$$\alpha_{n-1} = (1 + \beta_n^\gamma \mu)^{-1} \quad \text{and} \quad \beta_{n-1} = \alpha_n^\gamma + \beta_n \nu (1 - \alpha_n)^\gamma. \quad (2.17)$$

Through these equations we obtain the solutions of the Bellman equations (2.10) for all $0 \leq n \leq N - 1$.

Now we have to calculate the optimal strategies $(v_0^*, \dots, v_{N-1}^*)$, where $v_j = (c_j^*, \theta_j^*)$. Note that according to (2.16) for all $0 \leq n \leq$

$N - 1$ we have $\theta_n^* = \kappa^*$. Then, taking into account that $x_0 = x$, we obtain $c_0^* = \mathbf{s}(0, x) = \alpha_0 x$. So, for $n = 1$ we get

$$x_1^* = \tau_1(\kappa^*)(x - c_0^*) = \tau_1(\kappa^*)(1 - \alpha_0)x \quad \text{and} \quad c_1^* = \mathbf{s}(1, x_1^*) = \alpha_1 x_1^*.$$

Similarly, for $n > 1$: $x_n^* = \tau_n(\kappa^*)(1 - \alpha_{n-1})x_{n-1}^*$ and $c_n^* = \mathbf{s}(n, x_n^*) = \alpha_n x_n^*$. This implies that

$$x_n^* = x \prod_{k=1}^n \tau_k(\kappa^*)(1 - \alpha_{k-1}) \quad \text{and} \quad c_n^* = \alpha_n x \prod_{k=1}^n \tau_k(\kappa^*)(1 - \alpha_{k-1}).$$

Using now the recurrence equalities (2.17) we can calculate the optimal consumption for any $n \geq 1$.

Exercises

1. Find the optimal strategies for $U(x) = h(x) = \ln x$, $N = 1$, $N = 2$ and for any $N \geq 3$.
2. Find the optimal strategy for $N = 1$, $U(x) = x^\gamma$ and $h(x) = x$ with $0 < \gamma < 1$.
3. Find the optimal strategy for $N = 1$, $U(x) = x$ and $h(x) = x^\gamma$ with $0 < \gamma < 1$.

3 Optimal consumption in continuous time

In this chapter we will consider a financial market in continuous time consisting of a riskless asset

$$\dot{B}(t) = \mathbf{r} B(t) \quad B_0 = 1, \quad 0 \leq t \leq T. \quad (3.1)$$

Here \mathbf{r} is an interest rate. Assume that this function is continuous. Next, let $(c_t)_{0 \leq t \leq T}$ be a positive consumption rate on the time interval $[0, T]$. We assume that this is a nonnegative measurable integrable function, i.e.

$$\int_0^T c_t dt < \infty. \quad (3.2)$$

The integral $\int_0^t c_s ds$ represents the amount consumed on the interval $[0, t]$.

Let X_t be a wealth at the moment t . To obtain an equation for the wealth we divide the interval $[0, t]$ by the sequence $(t_k)_{0 \leq k \leq n}$ defined as

$$t_k = \frac{kt}{n}.$$

So, for $k \geq 1$ by analogy with the discrete time case is obtained

$$X_{t_k} = \left(1 + r_{t_{k-1}} \Delta + d_k\right) \left(X_{t_{k-1}} - \int_{t_{k-1}}^{t_k} c_t dt\right), \quad (3.3)$$

where

$$\Delta = t_k - t_{k-1} = \frac{t}{n} \quad \text{and} \quad d_k = \mathbf{r} \int_{t_{k-1}}^{t_k} \left(B_t - B_{t_{k-1}}\right) dt.$$

It is clear that

$$\lim_{n \rightarrow \infty} \frac{1}{\Delta} \max_{1 \leq k \leq n} |d_k| = 0.$$

One can rewrite the equation (3.3) as

$$X_{t_k} - X_{t_{k-1}} = r_{t_{k-1}} X_{t_{k-1}} \Delta - \int_{t_{k-1}}^{t_k} c_t dt + h_k \Delta$$

with

$$h_k = \frac{d_k}{\Delta} \left(X_{t_{k-1}} - \int_{t_{k-1}}^{t_k} c_t dt\right) - r_{t_{k-1}} \int_{t_{k-1}}^{t_k} c_t dt.$$

It is easy to deduce that

$$\lim_{n \rightarrow \infty} \frac{1}{\Delta} \max_{1 \leq k \leq n} |h_k| = 0.$$

So, for any $n \geq 1$,

$$X_t = x + \sum_{k=1}^n r_{t_{k-1}} X_{t_{k-1}} \Delta - \int_0^t c_u \, du + \sum_{k=1}^n h_k.$$

Then, passing to the limit when $n \rightarrow \infty$, we obtain that for any $t > 0$

$$X_t = x + \int_0^t r_u X_u \, du - \int_0^t c_u \, du,$$

where $x > 0$ is an initial endowment. This means that

$$\dot{X}_t = \mathbf{r} X_t - c_t, \quad X_0 = x. \quad (3.4)$$

Now we define the cost function as

$$J(x, c) := \int_0^T U(c_t) \, dt + h(X_T). \quad (3.5)$$

Our goal is to resolve the following optimization problem

$$J^*(x) = \max_c J(x, c), \quad (3.6)$$

where $c = (c_t)_{0 \leq t \leq T}$. For this problem we will use the Bellman dynamic programming method. To this end we set

$$J(t, x, c) := \int_t^T U(c_v) \, dv + h(X_T)$$

and

$$J^*(t, x) = \max_c J(t, x, c) = J(t, x, c^*).$$

Then we introduce the Hamiltonien function as

$$H(t, x, z) := \sup_{\vartheta \geq 0} H_0(t, x, z, \vartheta) \quad (3.7)$$

with

$$H_0(t, x, z, \vartheta) := z(\mathbf{r}x - \vartheta) + U(\vartheta).$$

To find the solution for the problem (3.6), one needs to study the following Hamilton-Jacobi-Bellman equation

$$\begin{cases} z_t(t, x) + H(t, x, z_x(t, x)) = 0, & t \in [0, T], \\ z(T, x) = h(x), & x > 0. \end{cases} \quad (3.8)$$

Here z_t is the partial derivative of z with respect to t . We use the same notation for all partial derivatives.

We assume that

C₁) *There exists a solution $z(t, x)$ for the equation (3.8).*

C₂) *There exists a measurable function $\check{\vartheta} : [0, T] \times (0, \infty) \rightarrow \mathbb{R}_+$*

such that

$$H(t, x, z_x(t, x)) = H_0(t, x, z_x(t, x), \check{\vartheta}(t, x))$$

for all $0 \leq t \leq T$ and $x > 0$.

C₃) There exists a solution of the following equation

$$dX_t^* = (\mathbf{r} x_t^* - \check{\vartheta}(t, X_t^*)) dt, \quad X_0^* = x. \quad (3.9)$$

The solution of the problem (3.6) is given in the following theorem.

Theorem 3.1. *Assume that the conditions **C₁**)–**C₃**) hold. Then for all $0 \leq t \leq T$ and for all $x > 0$*

$$z(t, x) = J^*(t, x) = J^*(t, x, c^*),$$

where the optimal consumption $c^* = (c_t^* = \vartheta(t, X_t^*))_{0 \leq t \leq T}$ is defined in **C₃**).

Proof. Let $c = (c_t)_{0 \leq t \leq T}$ be a nonnegative integrable function. We have

$$z(T, X_T) - z(t, X_t) = \int_t^T \left(z_t(v, X_v) + z_x(v, X_v) \dot{X}_v \right) dv.$$

Using the equation (3.4) and the conditions $X_t = x$ and $z(T, x) = h(x)$, we obtain that

$$z(t, x) = J(t, x, c) - \int_t^T (z_t(s, X_s) + H_1(s, X_s, c_s)) ds, \quad (3.10)$$

where $H_1(s, x, \vartheta) = H_0(t, x, z_x(t, x), \vartheta)$. This means that $z(t, x) \geq J(t, x, c)$ for all $c = (c_t)_{0 \leq t \leq T}$. Therefore, we get $z(t, x) \geq J^*(t, x)$. Moreover, replacing c in (3.9) by c^* we obtain $z(t, x) = J(t, x, c^*)$. Hence Theorem 3.1. \square

Exercises

1. Write the Hamiltoniens for the following cases:

(a) $U(x) = x^\gamma$ with $0 < \gamma < 1$.

(b) $U(x) = \ln x$.

(c) $U(x) = 1 - e^{-x}$.

2. Find the optimal consumptions for the following problems :

(a) $U(x) = h(x) = x^\gamma$ with $0 < \gamma < 1$.

(b) $U(x) = x^\gamma$ with $0 < \gamma < 1$ and $h(x) = 0$.

(c) $U(x) = h(x) = \ln x$.

(d) $U(x) = \ln x$ and $h(x) = 0$.

(e) $r = 0$, $U(x) = 1 - e^{-x}$ and $h(x) = x/e$.

4 Optimal consumption and investment in continuous time models

4.1 Financial market model

In this section we consider the financial Black-Scholes market consisting of a risk-free asset and a risk asset on the time interval $[0, T]$. Their prices $(B(t))_{0 \leq t \leq T}$ and $(S(t))_{0 \leq t \leq T}$ for $i = 1, \dots, d$ satisfy the following equations

$$\begin{cases} dB(t) = \mathbf{r} B(t)dt, & B(0) = 1; \\ dS(t) = S(t) (\mu dt + \sigma dw_t), & S(0) > 0. \end{cases} \quad (4.1)$$

Here $(w_t)_{0 \leq t \leq T}$ is the Wiener process and the coefficients \mathbf{r} , μ and $\sigma > 0$ are fixed parameters. We recall, that \mathbf{r} is the interest rate, μ is the increasing coefficient and σ is volatility. Moreover, $(\mathcal{F}_t^w)_{0 \leq t \leq T}$ is the filtration generated by the Wiener process $(w_t)_{0 \leq t \leq T}$, i.e. $\mathcal{F}_t^w = \sigma\{w_u, 0 \leq u \leq t\}$. Let now $\phi_t \in \mathbb{R}$ and $\varphi_t \in \mathbb{R}$ be the amounts placed at the time moment t in the riskless and the risk assets respectively. In this case *financial strategy* is defined as the random process $(\phi_t, \varphi_t)_{0 \leq t \leq T}$ with the values in \mathbb{R}^2 progressively

measurable with respect to the filtration $(\mathcal{F}_t^w)_{0 \leq t \leq T}$. The process

$$X_t = \phi_t B(t) + \varphi_t S(t), \quad 0 \leq t \leq T,$$

is called *wealth process (or the value of the portfolio)* at the time moment t .

Moreover let for any $0 \leq t \leq T$ the function c_t be a consumption rate on the time interval $[0, t]$, i.e. it's a non-negative process adapted to the filtration $(\mathcal{F}_t^w)_{0 \leq t \leq T}$ and almost sure integrable on the interval $[0, T]$, i.e.

$$\int_0^T c_t dt < \infty \quad \text{a.s.}$$

The integral $\int_0^t c_s ds$ is the consumption process during the time interval $[0, t]$. A financial strategy $(\phi_t, \varphi_t)_{0 \leq t \leq T}$ with consumption $(c_t)_{0 \leq t \leq T}$ is called *self-financing strategy* if the wealth process satisfies the following stochastic equation

$$X_t = x + \int_0^t \phi_u dB(u) + \int_0^t \varphi_u dS(u) - \int_0^t c_u du, \quad (4.2)$$

where $x > 0$ is the initial endowment. We will work with the relative

quantities in relation to the wealth process, i.e. we set

$$\pi_t := \frac{\varphi_t S(t)}{X_t} \quad \text{and} \quad v_t := \frac{c_t}{X_t}.$$

Taking this into account, we can rewrite the equation for X_t as

$$dX_t = (\mathbf{r} + \sigma\pi_t\theta - v_t)X_t dt + \sigma\pi_t X_t dw_t, \quad X_0 = x, \quad (4.3)$$

where $\theta = (\mu - \mathbf{r})/\sigma$.

Let us describe now the set of *control processes* $\nu = (\nu_t)_{0 \leq t \leq T}$ with $\nu_t = (\pi_t, v_t)$.

Definition 4.1. *The control process $\nu = (\nu_t)_{0 \leq t \leq T} = (\pi_t, v_t)_{0 \leq t \leq T}$ with the values in $\mathbb{R} \times \mathbb{R}_+$ is called admissible if it is progressively measurable with respect to the filtration $(\mathcal{F}_t^w)_{0 \leq t \leq T}$ and such that*

$$\|\pi\|_T^2 = \int_0^T \pi_t^2 dt < \infty, \quad \int_0^T v_t dt < \infty \quad a.s. \quad (4.4)$$

and the equation (4.3) has an unique strong positive solution on the time interval $[0, T]$. We denote by \mathcal{V} the set of all admissible control processes.

To emphasize that a wealth process corresponds to a control process ν we will write X^ν . Now, for the initial wealth $x > 0$ and for a

process of control $(\nu_t)_{t \geq 0}$ in \mathcal{V} , we introduce the *objective function* as

$$J(x, \nu) := \mathbf{E}_x \left(\int_0^T U(v_t X_t^\nu) dt + h(X_T^\nu) \right), \quad (4.5)$$

where \mathbf{E}_x is the conditional expectation for $X_0^\nu = x$, $U : \mathbb{R}_+ \rightarrow \mathbb{R}$ and $h : \mathbb{R}_+ \rightarrow \mathbb{R}$ are utility functions.

The goal of investment and consumption is to maximize the objective function (4.5), i.e. one needs to find

$$\max_{\nu \in \mathcal{V}} J(x, \nu). \quad (4.6)$$

To solve this problem, we will use the Bellman dynamic stochastic programming method which is based on the verification theorems tool for stochastic differential equations. A special version of such theorem is given in the next section.

4.2 Verification theorem

In this section we prove the verification theorem from [4] which will apply to the problem (4.6). We consider the problem of stochas-

tic control for the scalar Itô process on the time interval $[0, T]$

$$\begin{cases} dX_t^\nu &= a(t, X_t^\nu, \nu_t) dt + (b(t, X_t^\nu, \nu_t))' dw_t; \\ X_0^\nu &= x > 0. \end{cases} \quad (4.7)$$

We assume that the control process $\nu = (\nu_t)_{0 \leq t \leq T}$ takes its values in the set $\mathcal{K} \subseteq \mathbb{R}^d$ and that the coefficients

$$a : [0, T] \times \mathbb{R}_+^* \times \mathcal{K} \rightarrow \mathbb{R} \quad \text{and} \quad b : [0, T] \times \mathbb{R}_+^* \times \mathcal{K} \rightarrow \mathbb{R}^d$$

are nonrandom functions such that for any $t \in [0, T]$ the functions $a(t, \cdot, \cdot)$ and $b(t, \cdot, \cdot)$ are continuous on $\mathbb{R}_+^* \times \mathcal{K}$ and for any $\kappa \in \mathcal{K}$ the equation (4.7) with $\nu \equiv \kappa$ has an unique positive strong solution on the interval $[0, T]$. We denote by $\mathbb{R}_+^* = \mathbb{R}_+ \setminus \{0\}$. Moreover, let

$$U : [0, T] \times \mathbb{R}_+^* \times \mathcal{K} \rightarrow \mathbb{R} \quad \text{and} \quad h : \mathbb{R}_+^* \rightarrow \mathbb{R}$$

be the utility functions. It is assumed that U and h are continuous.

In this section we will modify the Definition 4.2 of admissibility for the equation (4.7) as follows.

Definition 4.2. *The control process $\nu = (\nu_t)_{0 \leq t \leq T}$ is admissible if it is progressively measurable with respect to the filtration $(\mathcal{F}_t^w)_{0 \leq t \leq T}$ and the equation (4.7) has a unique strong positive so-*

lution $(X_t^\nu)_{0 \leq t \leq T}$ such that

$$\int_0^T (|a(t, X_t^\nu, \nu_t)| + |b(t, X_t^\nu, \nu_t)|^2) dt < \infty \quad a.s. \quad (4.8)$$

and

$$\mathbf{E} \left[\int_0^T (U(t, X_t^\nu, \nu_t))_- dt + (h(X_T^\nu))_- \right] < \infty, \quad (4.9)$$

where $(a)_- = -\min(0, a)$.

We denote by \mathcal{V} the set of all admissible control processes.

Now let's define the objective functions, by setting

$$J(t, x, \nu) := \mathbf{E}_{t,x} \left[\int_t^T U(s, X_s^\nu, \nu_s) ds + h(X_T^\nu) \right], \quad 0 \leq t \leq T, \quad (4.10)$$

where $\mathbf{E}_{t,x}$ is the conditional expectation for $X_t^\nu = x$.

Our goal is to solve the following optimization problems

$$J^*(t, x) := \sup_{\nu \in \mathcal{V}} J(t, x, \nu), \quad 0 \leq t \leq T. \quad (4.11)$$

To this end, let's introduce the Hamiltonian defined by

$$H(t, x, z_1, z_2) := \sup_{\vartheta \in \mathcal{K}} H_0(t, x, z_1, z_2, \vartheta) \quad (4.12)$$

and

$$H_0(t, x, z_1, z_2, \vartheta) := a(t, x, \vartheta) z_1 + \frac{1}{2} |b(t, x, \vartheta)|^2 z_2 + U(t, x, \vartheta).$$

To find a solution to the problem (4.11), we must study the Hamilton-Jacobi-Bellman (HJB) equation

$$\begin{cases} z_t(t, x) + H(t, x, z_x(t, x), z_{xx}(t, x)) = 0, & t \in [0, T], \\ z(T, x) = h(x), & x > 0. \end{cases} \quad (4.13)$$

Here z_t denotes the partial derivative of z over t . We will use the same notations for all partial derivatives. Assume that

H₁) *There exists a classical solution of the problem (4.13), i.e. there exists $[0, T] \times (0, \infty) \rightarrow \mathbb{R}$ function z which is continuously differentiable over the variable t and two times continuously differentiable over x and satisfies the equation (4.13).*

H₂) *There exists a $[0, T] \times (0, \infty) \rightarrow \mathcal{K}$ measurable function $\check{\vartheta}$ such that for any $t \in [0, T]$ and for any $x > 0$*

$$H(t, x, z_x(t, x), z_{xx}(t, x)) = H_0(t, x, z_x(t, x), z_{xx}(t, x), \check{\vartheta}(t, x)).$$

H₃) *There is an unique strong positive solution for the following stochastic differential equation*

$$dX_t^* = a^*(t, X_t^*) dt + b^*(t, X_t^*) dw_t, \quad X_0^* = x, \quad (4.14)$$

where $a^*(t, x) = a(t, x, \check{\nu}(t, x))$ and $b^*(t, x) = b(t, x, \check{\nu}(t, x))$. Moreover, the optimal control process $\nu^* = (\nu_t^*)_{0 \leq t \leq T}$ with $\nu_t^* = \check{\nu}(t, X_t^*)$ belongs to the set \mathcal{V} and for any $0 \leq t \leq T$ and $x > 0$

$$\mathbf{E}_{t,x} \sup_{t \leq s \leq T} |z(s, X_s^*)| < \infty. \quad (4.15)$$

Now we show that the process ν^* solves the problems (4.11).

Theorem 4.1. *Assume that $\mathcal{V} \neq \emptyset$ and the conditions **H₁)** – **H₃)** hold. Moreover, assume that*

$$z_* = \inf_{0 \leq t \leq T} \inf_{x > 0} z(t, x) > -\infty. \quad (4.16)$$

Then for any $0 \leq t \leq T$ and for any $x > 0$

$$z(t, x) = J^*(t, x) = J^*(t, x, \nu^*),$$

where the strategy ν^* is defined in the condition **H₃)**.

Proof. Let now $\nu \in \mathcal{V}$ and X^ν be the wealth process with $X_t^\nu = x$ for a some fixed $0 \leq t < T$. Next, let us define the following stopping times

$$\tau_n = \inf \left\{ s \geq t : \int_t^s |b(u, X_u^\nu, \nu_u)|^2 z_x^2(u, X_u^\nu) du \geq n \right\} \wedge T.$$

Note that the condition (4.8) implies that $\tau_n \rightarrow T$ as $n \rightarrow \infty$ a.s. Taking into account that the function $z(\cdot, \cdot)$ and the process $(X_t^\nu)_{0 \leq t \leq T}$ are continuous, we get

$$\lim_{n \rightarrow \infty} z(\tau_n, X_{\tau_n}^\nu) = z(T, X_T^\nu) = h(X_T^\nu) \quad \text{a.s.} \quad (4.17)$$

Then, according to Ito's formula, we have

$$\begin{aligned} z(t, x) &= \int_t^{\tau_n} U(s, X_s^\nu, \nu_s) ds + z(\tau_n, X_{\tau_n}^\nu) \\ &\quad - \int_t^{\tau_n} (z_t(s, X_s^\nu) + H_1(s, X_s^\nu, \nu_s)) ds - M_n, \end{aligned} \quad (4.18)$$

where $M_n = \int_t^{\tau_n} b(u, X_u^\nu, \nu_u) z_x(u, X_u^\nu) dw_u$ and

$$H_1(s, x, \vartheta) = H_0(t, x, z_x(t, x), z_{xx}(t, x), \vartheta).$$

The definition of the stopping time τ_n implies that $\mathbf{E}_{t,x} M_n = 0$.

Therefore, we obtain the following inequality

$$z(t, x) \geq \mathbf{E}_{t,x} \int_t^{\tau_n} U_s \, ds + \mathbf{E}_{t,x} z_n, \quad (4.19)$$

where $U_s = U(s, X_s^\nu, \nu_s)$ and $z_n = z(\tau_n, X_{\tau_n}^\nu)$. Consider the first term in this inequality. We have

$$\int_t^{\tau_n} U_s \, ds = \int_t^{\tau_n} (U_s)_+ \, ds - \int_t^{\tau_n} (U_s)_- \, ds,$$

where $(a)_+ = \max(a, 0)$. The condition (4.8) and the monotone convergence theorem imply that

$$\lim_{n \rightarrow \infty} \mathbf{E}_{t,x} \int_t^{\tau_n} U_s \, ds = \mathbf{E}_{t,x} \int_t^T U_s \, ds.$$

Moreover, by the condition (4.16), the sequence $((z_n)_-)_{n \geq 1}$ is bounded, therefore, uniformly integrable. So, through Fatou's lemma we obtain that

$$\liminf_{n \rightarrow \infty} \mathbf{E}_{t,x} z_n \geq \mathbf{E}_{t,x} \lim_{n \rightarrow \infty} z(\tau_n, X_{\tau_n}^\nu) = \mathbf{E}_{t,x} h(X_T^\nu).$$

Therefore, going to the limit as $n \rightarrow \infty$ in (4.19) we get $z(t, x) \geq J^*(t, x)$. In the same way, replacing ν in (4.18) by ν^* we obtain the

following equality

$$z(t, x) = \mathbf{E}_{t,x} \int_t^{\tau_n} U(s, X_s^*, \nu_s^*) ds + \mathbf{E}_{t,x} z(\tau_n, X_{\tau_n}^*).$$

Then in view of the condition (4.15) and the dominated convergence theorem we get

$$\lim_{n \rightarrow \infty} \mathbf{E}_{t,x} z(\tau_n, X_{\tau_n}^*) = \mathbf{E}_{t,x} \lim_{n \rightarrow \infty} z(\tau_n, X_{\tau_n}^*) = \mathbf{E}_{t,x} h(X_T^*).$$

This means that

$$\begin{aligned} z(t, x) &= \lim_{n \rightarrow \infty} \mathbf{E}_{t,x} \int_t^{\tau_n} U(s, X_s^*, \nu_s^*) ds + \lim_{n \rightarrow \infty} \mathbf{E}_{t,x} z(\tau_n, X_{\tau_n}^*) \\ &= \mathbf{E}_{t,x} \left(\int_t^T U(s, X_s^*, \nu_s^*) ds + h(X_T^*) \right) \\ &= J(t, x, \nu^*). \end{aligned}$$

So, using the inequality $z(t, x) \geq J^*(t, x)$ we obtain $z(t, x) = J^*(t, x)$.

Hence Theorem 4.1. \square

In the next section we will see how to apply the verification theorem to the problem (4.6).

4.3 Optimal investment and consumption for the power utility functions

In this section we consider the problem (4.6) in the case where $U(x) = x^{\gamma_1}$ and $h(x) = x^{\gamma_2}$ with $0 < \gamma_1, \gamma_2 < 1$. Let us first define the following functions

$$A_1(t) = \gamma_1^{q_1} \int_t^T e^{\int_t^s \beta_1(u) du} ds \quad \text{and} \quad A_2(t) = \gamma_2^{q_2} e^{\int_t^T \beta_2(u) du}, \quad (4.20)$$

where $q_i = (1 - \gamma_i)^{-1}$ and

$$\beta_i(t) = (q_i - 1) \left(\mathbf{r} + \frac{q_i}{2} |\theta|^2 \right).$$

Then for any $0 \leq t \leq T$ and $x > 0$, we define the function $g(t, x) > 0$ as

$$A_1(t) g^{-q_1}(t, x) + A_2(t) g^{-q_2}(t, x) = x. \quad (4.21)$$

In addition, we set

$$p(t, x) = q_1 A_1(t) g^{-q_1}(t, x) + q_2 A_2(t) g^{-q_2}(t, x).$$

Now we define the functions which will be used to check the condition \mathbf{H}_2), i.e. we set

$$\tilde{\pi}(t, x) = \frac{p(t, x)}{\sigma x} \theta \quad \text{and} \quad \tilde{v}(t, x) = (\gamma_1)^{q_1} \frac{g^{-q_1}(t, x)}{x}. \quad (4.22)$$

The solution in this case is given in the following theorem.

Theorem 4.2. *The optimal value of $J(x, \nu)$ is given by*

$$\max_{\nu \in \mathcal{V}} J(x, \nu) = J(x, \nu^*) = \frac{A_1(0)}{\gamma_1} g^{1-q_1}(0, x) + \frac{A_2(0)}{\gamma_2} g^{1-q_2}(0, x),$$

where optimal control $\nu_t^* = (\pi_t^*, v_t^*)$ has the following form $\pi_t^* = \tilde{\pi}(t, X_t^*)$ and $v_t^* = \tilde{v}(t, X_t^*)$. The optimal wealth process $(X_t^*)_{0 \leq t \leq T}$ satisfies the following stochastic equation

$$dX_t^* = a^*(t, X_t^*) dt + b^*(t, X_t^*) dw_t, \quad (4.23)$$

where $X_0^* = x$, $b^*(t, x) = p(t, x) \theta$ and

$$a^*(t, x) = \mathbf{r} x + p(t, x) |\theta|^2 - \gamma_1^{q_1} g^{-q_1}(t, x).$$

Proof. To prove this theorem we will use the theorem 4.1 for the stochastic differential equation (4.3). First we fixe $\vartheta = (y, v)$ with $y \in \mathbb{R}$ and $v \in [0, \infty)$. For such ϑ the coefficients in the model (4.7)

are defined as

$$a(t, x, \vartheta) = x(\mathbf{r} + \sigma y \theta - v) \quad \text{and} \quad b(t, x, \vartheta) = \sigma x y.$$

In addition, the utility functions are defined as

$$U(t, x, \vartheta) = v^{\gamma_1} x^{\gamma_1} \quad \text{and} \quad h(x) = x^{\gamma_2}.$$

To check the conditions $\mathbf{H}_1) - \mathbf{H}_3)$ calculate the Hamiltonian (4.12) in our case. We have

$$H(t, x, z_1, z_2) = \sup_{\vartheta \in \mathbb{R}^d \times [0, \infty)} H_0(t, x, z_1, z_2, \vartheta),$$

where

$$H_0(t, x, z_1, z_2, \vartheta) = (\mathbf{r} + \sigma y \theta) x z_1 + \frac{1}{2} x^2 \sigma^2 y^2 z_2 + v^{\gamma_1} x^{\gamma_1} - x v z_1.$$

Therefore for $z_2 \leq 0$ we obtain

$$H(t, x, z_1, z_2) = H_0(t, x, z_1, z_2, \vartheta_0) = \mathbf{r} x z_1 + \frac{z_1^2 \theta^2}{2|z_2|} + \frac{1}{q_1} \left(\frac{z_1}{\gamma_1} \right)^{\frac{\gamma_1}{\gamma_1 - 1}},$$

where $\vartheta_0 = \vartheta_0(t, x, z_1, z_2) = (y_0(t, x, z_1, z_2), v_0(t, x, z_1, z_2)) = (y_0, v_0)$

and

$$y_0 = \frac{z_1}{\sigma x |z_2|} \theta \quad \text{and} \quad v_0 = \left(\frac{\gamma_1}{z_1} \right)^{q_1} x^{-1}. \quad (4.24)$$

Now we have to solve the problem (4.13) which has for our case the following form

$$\begin{cases} z_t(t, x) + \mathbf{r} x z_x(t, x) + \frac{z_x^2(t, x) \theta^2}{2|z_{xx}(t, x)|} + \frac{1}{q_1} \left(\frac{z_x(t, x)}{\gamma_1} \right)^{\frac{\gamma_1}{\gamma_1-1}} = 0, \\ z(T, x) = x^{\gamma_2}. \end{cases}$$

We check directly that the solution for this problem (see [4]) for $0 \leq t < T$ and $x > 0$ is

$$z(t, x) = \frac{A_1(t)}{\gamma_1} g^{1-q_1}(t, x) + \frac{A_2(t)}{\gamma_2} g^{1-q_2}(t, x), \quad (4.25)$$

which fulfills the condition (4.16) for $z_* = 0$. Therefore, we have

$$H(t, x, z_x(t, x), z_{xx}(t, x)) = H_0(t, x, z_x(t, x), z_{xx}(t, x), \check{v}(t, x)),$$

where the function $\check{v}(t, x) = (\check{\pi}(t, x), \check{v}(t, x))$ is defined in (4.22). In [4] it is shown that the conditions $\mathbf{H}_1) - \mathbf{H}_3)$ hold for this case. So, Theorem 4.1 implies Theorem 4.2. \square

Now we consider the problem (4.6) for $U(x) = h(x) = x^\gamma$ with

$0 < \gamma < 1$. We set

$$\tilde{g}_\gamma(t) = \exp\left(\gamma \mathbf{r} + \frac{\gamma \theta^2}{2(1-\gamma)} t\right).$$

Corollary 4.1. *Let $U(x) = h(x) = x^\gamma$ and $0 < \gamma < 1$. Then the optimal value of $J(x, \nu)$ in (4.6) is*

$$J^*(x) = \max_{\nu \in \mathcal{V}} J(x, \nu) = J(x, \nu^*) = x^\gamma \left(\|\tilde{g}_\gamma\|_{q,T}^q + \tilde{g}_\gamma^q(T) \right)^{1/q},$$

where the optimal control process $\nu_t^* = (\pi_t^*, v_t^*)$ for $0 \leq t \leq T$ has the following form

$$\pi_t^* = \frac{\theta}{\sigma(1-\gamma)} \quad \text{and} \quad v_t^* = \frac{\tilde{g}_\gamma^q(t)}{\tilde{g}_\gamma^q(T) + \int_t^T \tilde{g}_\gamma^q(s) ds}. \quad (4.26)$$

The optimal wealth process $(X_t^*)_{0 \leq t \leq T}$ satisfies the following stochastic differential equation

$$dX_t^* = \left(\mathbf{r} - v_t^* + \frac{\theta^2}{1-\gamma} \right) X_t^* dt + \frac{\theta}{1-\gamma} X_t^* dw_t \quad (4.27)$$

and $X_0^* = x$.

Remark 4.1. *It should be noted that the strategy (4.26) – (4.27) is the well known Merton strategy obtained in [6].*

Exercises

1. Write the Hamilton function for the following cases:

(b) $U(x) = \ln x$.

(c) $U(x) = 1 - e^{-x}$.

2. Find the optimal consumptions for the following problems:

(b) $U(x) = x^\gamma$ with $0 < \gamma < 1$ and $h(x) = 0$.

(c) $U(x) = h(x) = \ln x$.

(d) $U(x) = \ln x$ and $h(x) = 0$.

Conclusion

In this manual we considered the main optimization portfolio problems for the financial markets modeled by the stochastic differential and stochastic differential equations. We considered the optimal consumption, the optimal investment and consumption problems in discrete time studied in [2], then we considered the same problems in continuous time using the approach developed in [3]. For all these problems we used the stochastic calculus developed in [5, 7]. For all these problems we used the dynamical programming method proposed by Bellman for the optimization problem for the utilities functions of the integral forms [1]. For the Bellman principle in continuous time we studied the verification theorem method developed in [3, 4, 8, 9].

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