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► **To cite this version:**

Laurent Serlet. Explicit Laws for the Records of the Perturbed Random Walk on

Z

. Séminaire de probabilités XLIX, Lecture Notes in Mathematics 2215, pp.495-519, 2018, 10.1007/978-3-319-92420-5_14 . hal-02364930

HAL Id: hal-02364930

<https://hal.science/hal-02364930>

Submitted on 15 Nov 2019

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Explicit laws for the records of the perturbed random walk on \mathbb{Z}

Laurent Serlet

Key words: perturbed random walk, once-reinforced random walk, perturbed Brownian motion, records, invariance principle, recurrence

1 Introduction

We consider a process $(X_n)_{n \geq 0}$ with values in \mathbb{Z} , started at 0 ($X_0 = 0$), which is a nearest neighbor random walk on \mathbb{Z} that is, for every $n \geq 0$, we have $X_{n+1} \in \{X_n - 1, X_n + 1\}$. We denote its maximum and minimum up to time n by $\overline{X}_n = \max\{X_0, X_1, \dots, X_n\}$ and $\underline{X}_n = \min\{X_0, X_1, \dots, X_n\}$. We say that $(X_n)_{n \geq 0}$ is a perturbed random walk (**PRW**) with reinforcement parameters $\beta, \gamma \in (0, +\infty)$ if the transition probability

$$\mathbb{P}(X_{n+1} = X_n + 1 \mid X_0, X_1, \dots, X_n)$$

is equal to

- $1/2$ if $\underline{X}_n < X_n < \overline{X}_n$ or $n = 0$
- $1/(1 + \beta)$ if $X_n = \overline{X}_n$ and $n \geq 1$
- $\gamma/(1 + \gamma)$ if $X_n = \underline{X}_n$ and $n \geq 1$.

When $\beta = \gamma = 1$, we obtain a standard random walk (**SRW**). When $\beta = \gamma$, we obtain a symmetric perturbed random walk (**SPRW**) with parameter $\beta \in (0, +\infty)$. We interpret the case $\beta > 1$ as a self-attractive walk whereas for $\beta \in (0, 1)$ the walk is self-repulsive.

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This process belongs to the broad class of processes with reinforcement which has generated an important amount of literature. Pemantle gives in [Pe07] a nice survey with lots of references. More precisely, our PRW is sometimes called the once-reinforced random walk. This once-reinforced random walk can also be defined in \mathbb{Z}^d for $d > 1$ and some fundamental questions are still open in these dimensions but, in the present paper, we stay in dimension 1 which enables a much easier treatment and in particular, explicit computations of laws.

In [Da96] and [Da99], Davis introduces a diffusive rescaling by setting $X_t^n = \frac{1}{\sqrt{n}}X_{nt}$ (after linear interpolation of X between integer times) and he proves that the process $(X_t^n)_{t \geq 0}$ converges in law to a process $(W_t)_{t \geq 0}$ which is the unique solution of the equation

$$W_t = B_t - (\beta - 1) \sup_{s \leq t} W_s + (\gamma - 1) \inf_{s \leq t} W_s \quad (1)$$

where $(B_t)_{t \geq 0}$ is a Brownian motion.

The solutions of (1) have been studied by several authors under the name ‘‘Brownian motion perturbed at its extrema’’, see for instance [CPY98], [We95], [Da96], [Da99], [PW97], [CDH00] and the references therein.

The present paper uses a different approach since it is based on explicit computations for the random walk and we get results on the limiting continuous objects as by-products. The methods are similar to those in [Se13] where we treated the case of the reflecting random walk perturbed at the maximum. We will of course refer to this paper for several proofs which are identical to that case. However since our approach is based on explicit computations, the non-reflecting case generates different formulas than the reflecting case and the non-reflecting case also adds new questions related to the signs, that we address in the present paper.

Since the PRW behaves as a SRW when it stays away from the extrema, we concentrate on the study of the process when it reaches an extremum, in particular for the first time and in that case we call it a *record time*. More precisely, we set

$$V_k = \overline{X}_k - \underline{X}_k + 1$$

for the number of visited points up to time k i.e. the number of distinct values in the set $\{X_0, X_1, \dots, X_k\}$. Then we define $T_0 = 0$ and for $n \geq 1$, we call

$$T_n = \inf\{k \geq 1; V_k = n + 1\}$$

the time of the n -th record. Then $R_n = X_{T_n}$ is the value of the n -th record and the sign of R_n is denoted by $\chi_n \in \{-1, 1\}$. As we will see the sequence $(\chi_n)_{n \geq 1}$ is a time-inhomogeneous Markov chain for which the transition matrix is easily computed. As a consequence we will derive an invariance principle. We will also note that the record values $(R_n)_{n \geq 0}$ can be reconstructed from

the sequence of signs $(\chi_n)_{n \geq 1}$ and thus derive an invariance principle for the record values.

Then we introduce the rescaled record time process $(\tau_t^n)_{t \geq 0}$ by

$$\tau_t^n = \frac{1}{n^2} T_{[nt]} \quad (2)$$

where $[\cdot]$ denotes the integer part. We want an invariance principle for this process. First we work conditionally on the record signs and then without conditioning. Unfortunately in this latter case, we are unable to obtain a result in the general case and we restrict to symmetric perturbation ($\beta = \gamma$).

This invariance principle has consequences on the process of the number of visited points $(V_k)_{k \geq 1}$. As in the standard case, V_k is of order \sqrt{k} and we obtain in particular the asymptotic law of V_k/\sqrt{k} . Finally we will examine the possibility of “positive recurrence” for the PRW.

The paper is organized as follows. The next section is a precise statement of our main results, ending with two open questions. The following sections are devoted to proofs, beginning with a section of technical preliminary lemmas.

2 Statement of the results

Most of the processes that we consider in this section have their trajectories in the space $d([0, +\infty), \mathbb{R})$ of càdlàg functions that we endow with the usual Skorohod topology. Weak convergence of probability laws on this space is simply called in the sequel “convergence in law”. However special care will be needed in Proposition 2 where we have to restrict to compact intervals of $(0, +\infty)$. In the sequel the notations \cosh , \sinh and \tanh refer to the usual functions of hyperbolic trigonometry.

Let $(X_n)_{n \geq 0}$ denote a PRW with parameters $\beta, \gamma \in (0, +\infty)$. First note that the sequence of record values $(R_n)_{n \geq 0}$ is easily reconstructed from the sequence of record signs $(\chi_n)_{n \geq 1}$ because, for any $n \geq 1$,

$$R_n = \chi_n \sum_{k=1}^n \mathbf{1}_{\{\chi_k = \chi_n\}} \quad (3)$$

and it justifies that we first focus on $(\chi_n)_{n \geq 1}$. We start with an easy fact.

Proposition 1 *The sequence of the signs of records $(\chi_n)_{n \geq 1}$ of the PRW is a time-inhomogeneous Markov chain with transition matrix*

$$Q_n = \begin{pmatrix} \frac{\beta+n}{\gamma+\beta+n} & \frac{\gamma}{\gamma+\beta+n} \\ \frac{\beta}{\gamma+\beta+n} & \frac{\gamma+n}{\gamma+\beta+n} \end{pmatrix}. \quad (4)$$

We notice that the off-diagonal terms of this transition matrix are of order $1/n$ so we speed up time by factor n to get a limit in law.

2.1 Asymptotic results for the signs of records

Proposition 2 *Let the rescaled sequence of the record signs of the PRW be defined by*

$$\forall t > 0, \quad \chi_t^n = \chi_{[nt]}. \quad (5)$$

There is a process $(x_t)_{t \in (0, +\infty)}$ with values in $\{-1, +1\}$ uniquely defined in law such that, for any $a > 0$,

$$\mathbb{P}(x_a = -1) = \frac{\beta}{\beta + \gamma}; \quad \mathbb{P}(x_a = 1) = \frac{\gamma}{\beta + \gamma} \quad (6)$$

and $(x_t)_{t \in (a, +\infty)}$ is a time-inhomogeneous Markov jump process with generator

$$\begin{pmatrix} -\frac{\gamma}{t} & \frac{\gamma}{t} \\ \frac{\beta}{t} & -\frac{\beta}{t} \end{pmatrix} \quad (7)$$

and transition probability matrix

$$T(s, t) = \frac{1}{\beta + \gamma} \begin{pmatrix} \beta + \gamma \left(\frac{s}{t}\right)^{\beta + \gamma} & \gamma \left(1 - \left(\frac{s}{t}\right)^{\beta + \gamma}\right) \\ \beta \left(1 - \left(\frac{s}{t}\right)^{\beta + \gamma}\right) & \gamma + \beta \left(\frac{s}{t}\right)^{\beta + \gamma} \end{pmatrix}. \quad (8)$$

For all $0 < a < b$, the sequence of processes $(\chi_t^n)_{t \in [a, b]}$ converges in law to $(x_t)_{t \in [a, b]}$, as laws on the space $\mathbb{D}([a, b], \{-1, 1\})$ of càdlàg functions from $[a, b]$ to $\{-1, 1\}$ endowed with the Skorohod topology.

Let us remark that setting $\tilde{x}_t = x_{e^t}$ for $t \in \mathbb{R}$ gives a new process $(\tilde{x}_t)_{t \in \mathbb{R}}$ which is a time-homogeneous Markov jump process on $\{-1, 1\}$ or, in other words, an alternating renewal process.

2.2 Consequences on the sequence of records

We derive a corollary which is the continuous-time counterpart of (3).

Corollary 3 *Let $(y_t)_{t > 0}$ be the process defined by*

$$\forall t > 0, \quad y_t = x_t \int_0^t \mathbf{1}_{\{x_u = x_t\}} du$$

where $(x_t)_{t>0}$ is the process introduced in Proposition 2.

Then, the process $(R_{[nt]}/n)_{t>0}$ converges in law to the process $(y_t)_{t>0}$.

In particular R_n/n converges in law to y_1 . One must not be deceived by the formula

$$\frac{R_n}{n} = \begin{cases} -\frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{\chi_k=-1\}} & \text{if } \chi_n = -1 \\ \frac{1}{n} \sum_{k=1}^n \mathbf{1}_{\{\chi_k=1\}} & \text{if } \chi_n = 1 \end{cases} \quad (9)$$

(which is a reformulation of (3)) and the fact that $(\chi_n)_{n \geq 1}$ is a Markov chain converging in law to the probability $\beta/(\beta + \gamma) \delta_{-1} + \gamma/(\beta + \gamma) \delta_{+1}$. In the time-homogeneous case, (9) would imply by the ergodic Theorem that R_n/n accumulates almost surely on the two limit points $-\frac{\beta}{\beta+\gamma}$ and $\frac{\gamma}{\beta+\gamma}$. But here the Markov chain $(\chi_n)_{n \geq 1}$ is time-inhomogeneous and the almost sure behaviour of R_n/n is completely different as one can see in the following proposition which holds whatever the values of $\beta, \gamma \in (0, +\infty)$.

Proposition 4 *Almost surely, the sequence $\{R_n/n; R_n > 0\}$ has $\limsup 1$ and $\liminf 0$ and similarly $\{R_n/n; R_n < 0\}$ has $\limsup 0$ and $\liminf -1$ that is, for all $\varepsilon \in (0, 1)$ and $N \geq 1$ there exist $n_1, n_2, n_3, n_4 \geq N$ such that*

$$\frac{R_{n_1}}{n_1} > 1 - \varepsilon, \quad 0 < \frac{R_{n_2}}{n_2} < \varepsilon, \quad -\varepsilon < \frac{R_{n_3}}{n_3} < 0, \quad \frac{R_{n_4}}{n_4} < -1 + \varepsilon.$$

2.3 Invariance principle for the record times, conditionally on the signs

We pass to the study of $(\tau_t^n)_{t \geq 0}$ the rescaled process of the record times of the PRW as defined by (2).

We have seen in the previous subsection the convergence in law of the rescaled record signs (on compact sets away from 0). By the Skorohod representation Theorem, we could suppose –concerning the properties that involve the law– that this convergence holds almost surely. In that case, we want to show the convergence of the conditional law of (rescaled) record times knowing these (rescaled) record signs. Let us introduce some notation. Let $(x^n(t))_{t>0}$ be a sequence of càdlàg functions taking their values in $\{-1, +1\}$ which converges to a function $(x(t))_{t>0}$ with respect to the Skorohod topology when t varies in any compact sets of $(0, +\infty)$. We set, for $0 \leq s < t$, $\mathcal{D}(x; s, t) = \{r \in (s, t); x(r-) \neq x(r)\}$ for the set of discontinuities of $x(\cdot)$ and similarly $\mathcal{D}(x^n; \cdot, \cdot)$ for $x^n(\cdot)$. We suppose that $\mathcal{D}(x^n; 0, +\infty) \subset \frac{1}{n}\mathbb{N}$ for all n and that $\mathcal{D}(x; s, t)$ is finite for all $0 < s < t < +\infty$.

Proposition 5 *As $n \rightarrow +\infty$, the conditional law of $(\tau_t^n)_{t \geq 0}$ knowing $(\chi_t^n)_{t>0} = (x^n(t))_{t>0}$ converges weakly to the law of a process $(\tau_t^{(x)})_{t \geq 0}$ –defined conditionally on $(x(t))_{t>0}$ – such that it has independent non-negative increments with distribution given, for $0 < s < t$, by the Laplace transform*

$$\mathbb{E} \left[e^{-\frac{\mu^2}{2} (\tau_t^{(x)} - \tau_s^{(x)})} \right] = \left(\prod_{r \in \mathcal{D}(x;s,t)} \frac{\mu r}{\sinh(\mu r)} \right) \times \exp \left[\int_s^t \delta(x(u)) \left(\frac{1}{u} - \mu \coth \mu u \right) du \right] \quad (10)$$

where

$$\delta(y) = \beta \mathbf{1}_{\{y=1\}} + \gamma \mathbf{1}_{\{y=-1\}}.$$

In the case of the SPRW (i.e. $\beta = \gamma$), this formula simplifies into

$$\mathbb{E} \left[e^{-\frac{\mu^2}{2} (\tau_t^{(x)} - \tau_s^{(x)})} \right] = \left(\frac{\sinh(\mu s)}{\mu s} \right)^\beta \left(\prod_{r \in \mathcal{D}(x;s,t)} \frac{\mu r}{\sinh(\mu r)} \right) \left(\frac{\mu t}{\sinh(\mu t)} \right)^\beta. \quad (11)$$

2.4 Invariance principle for the record times of the SPRW

Our goal is to state an invariance principle for $(\tau_t^n)_{t \geq 0}$ without any conditioning. But the same approach as the one leading to Proposition 5 stumbles over a computational difficulty that we will explain later and we are compelled to restrict to the case of symmetric perturbation.

Theorem 6 *Let $(\tau_t^n)_{t \geq 0}$ be the rescaled record process of the SPRW ($\beta = \gamma$).*

Then, as $n \rightarrow +\infty$, the process $(\tau_t^n)_{t \geq 0}$ converges in law to a process $(\tau_t)_{t \geq 0}$ with independent non-negative increments whose law is given, for $0 < s < t$, by the Laplace transform

$$\mathbb{E} \left[e^{-\frac{\mu^2}{2} (\tau_t - \tau_s)} \right] = \left(\frac{\cosh(\frac{\mu}{2} s)}{\cosh(\frac{\mu}{2} t)} \right)^{2\beta}. \quad (12)$$

This process has strictly increasing trajectories and is self-similar :

$$\forall a > 0, \quad (\tau_{at})_{t \geq 0} \stackrel{(d)}{=} (a^2 \tau_t)_{t \geq 0}. \quad (13)$$

For any $t > 0$, the density of τ_t on $(0, +\infty)$ is a (signed) mixture of $1/2$ -stable laws :

$$\phi_{\tau_t} : x \rightarrow \frac{2^{2\beta}}{\sqrt{2\pi}} \sum_{k=0}^{+\infty} \binom{-2\beta}{k} \frac{(\beta+k)t}{x^{3/2}} e^{-\frac{(\beta+k)^2 t^2}{2x}} \quad (14)$$

(using the usual notation (19)). Moreover we have the representation

$$\tau_t = \int_0^t \int_{\mathbb{R}_+} x \mathcal{N}(ds dx)$$

where $\mathcal{N}(ds dx)$ is a Poisson point measure on $\mathbb{R}_+ \times \mathbb{R}_+$ with intensity $f_s(x) ds dx$ where

$$f_s(x) = \frac{2\pi^2}{s^3} \beta \sum_{n=1}^{+\infty} (2n-1)^2 e^{-\frac{(2n-1)^2 \pi^2}{2} \frac{x}{s^2}}.$$

2.5 Criterion of positive recurrence for the PRW

From the value of the transition matrix (4), it is clear that the signs of records cannot be asymptotically constant; indeed the infinite product of the diagonal terms of that transition matrix tends to 0. The PRW, whatever its parameters $\beta, \gamma \in (0, +\infty)$, is thus recurrent in the sense that every level is visited infinitely often, as it is the case for the SRW. But compared to the SRW which is null recurrent, the PRW can also become “positive recurrent” if the reinforcement parameters are high enough that is if the process is sufficiently self-attractive. We define the notion of positive recurrence in the non-Markov setting of PRW as follows but note that this property clearly implies that the return time to any level has finite mean. For $q \geq 1$, we introduce

$$C_q = \inf\{j > q; \chi_j = -\chi_q\}$$

as the index of the first record after record q of opposite sign. We say that the PRW is *positive recurrent* if and only if, for every integer $q \geq 1$,

$$\mathbb{E}(T_{C_q} - T_q) < +\infty.$$

Theorem 7 *The PRW is positive recurrent if and only if $\beta, \gamma \in (2, +\infty)$.*

2.6 Number of visited points for the SPRW

The number of visited points of the PRW is the inverse of the record time process : $V_k = \inf\{n \geq 1; T_n > k\}$. In the case of the SPRW ($\beta = \gamma$) the invariance principle stated in Theorem 6, with limit process $(\tau_t)_{t \geq 0}$, implies an invariance principle for $(V_k)_{k \geq 1}$.

Proposition 8 *Let $(Y_s)_{s \geq 0}$ be the non-decreasing process defined by $Y_s = \inf\{t; \tau_t > s\}$. This process has continuous trajectories and is self-similar : $\forall a > 0, (Y_{as})_{s \geq 0} \stackrel{(d)}{=} (\sqrt{a} Y_s)_{s \geq 0}$. Its marginal laws are*

$$\forall s > 0, \quad Y_s \stackrel{(d)}{=} \sqrt{\frac{1}{\tau_1/\sqrt{s}}} \stackrel{(d)}{=} \sqrt{\frac{s}{\tau_1}} \quad (15)$$

and, for any $s > 0$, the variable Y_s admits the density on \mathbb{R}_+ given by :

$$\phi_{Y_s}(x) = \frac{2s}{x^3} \phi_{\tau_1}\left(\frac{s}{x^2}\right) \quad (16)$$

$$= \frac{2^{2\beta+1}}{\sqrt{2\pi s}} \sum_{k=0}^{+\infty} \binom{-2\beta}{k} (\beta+k) e^{-\frac{(\beta+k)^2 x^2}{2s}}. \quad (17)$$

Moreover, as $k \rightarrow +\infty$, the process $\left(\frac{V_{[ks]}}{\sqrt{k}}\right)_{s \geq 0}$ converges in law to the process $(Y_s)_{s \geq 0}$.

As a corollary of the invariance principle for the PRW proved by Davis, we have the representation

$$Y_s = \max_{u \in [0, s]} W_u - \min_{u \in [0, s]} W_u \quad (18)$$

where (W_u) is the perturbed Brownian motion as introduced in (1).

Also the process $(\tau_t)_{t \geq 0}$ obtained in Theorem 6 is the inverse of $(Y_s)_{s \geq 0}$. As a consequence it can be interpreted in terms of the perturbed Brownian motion via the representation (18) of $(Y_s)_{s \geq 0}$ given above.

2.7 Open questions

Here are two questions we were unable to solve :

- find the law of y_1 ; this will describe how “non-centered” the range can be, asymptotically;
- obtain a generalization of Theorem 6 to the case $\beta \neq \gamma$.

As we will see later, both questions amount to find the value of an infinite matrix product. Alternatively the question about the law of y_1 can also be stated in terms of alternating renewal process.

3 Preliminary lemmas

In the sequel we will use the classical notation of matrix exponential :

$$\exp[M] = \sum_{k=0}^{+\infty} \frac{1}{k!} M^k$$

for any real or complex square matrix M . We also recall the usual notation for generalized binomial coefficients

$$\binom{-\beta}{k} = \prod_{j=1}^k \frac{-\beta - j + 1}{j} = \frac{(-1)^k}{k!} \frac{\Gamma(k + \beta)}{\Gamma(\beta)} \quad (19)$$

where $\Gamma(\cdot)$ is the classical Gamma function. Let $0 < s < t$ be fixed. We introduce a notion of approximate equality of two quantities $a(k, n)$ and $b(k, n)$ up to terms of order $1/n^2$ by

$$a(k, n) \approx b(k, n) \Leftrightarrow \sup_{n \geq 1; [ns] \leq k \leq [nt]} n^2 |a(k, n) - b(k, n)| < +\infty. \quad (20)$$

Denote by $\mathcal{M}_{d \times d}$ the set of real $d \times d$ matrices. If $A(k, n) = (A_{x,y}(k, n))_{1 \leq x, y \leq d}$ and $B(k, n) = (B_{x,y}(k, n))_{1 \leq x, y \leq d}$ belong to $\mathcal{M}_{d \times d}$, we extend the previous notion by setting $A(k, n) \approx B(k, n)$ if and only if $A_{x,y}(k, n) \approx B_{x,y}(k, n)$ for all x, y .

Lemma 9 *Let us suppose that $g(\cdot)$ is a $\mathcal{M}_{d \times d}$ -valued function which is continuous on $[s, t]$ and $f(k, n)$ is a $\mathcal{M}_{d \times d}$ -valued function such that*

$$f(k, n) \approx I + \frac{1}{n} g\left(\frac{k}{n}\right).$$

Then

$$\lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]} f(k, n) = \lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]} \exp\left[\frac{1}{n} g\left(\frac{k}{n}\right)\right] \quad (21)$$

provided the limit on the right-hand side exists. Moreover, when $g(x)g(y) = g(y)g(x)$ for all $x, y \in [s, t]$, then

$$\lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]} f(k, n) = \exp\left[\int_s^t g(x) dx\right]. \quad (22)$$

Proof. As all matrix norms induce the same topology we may choose a matrix norm $\|\cdot\|$ which has the supplementary property that $\|AB\| \leq \|A\| \|B\|$ for all matrices A, B and, as a consequence, $\|\exp[A]\| \leq e^{\|A\|}$. Also, using this property of the norm, the assumption easily implies that

$$f(k, n) = \exp\left[\frac{1}{n} g\left(\frac{k}{n}\right)\right] + \frac{1}{n^2} R(k, n) \text{ with } \sup_{n \geq 1} \sup_{[ns] \leq k \leq [nt]} \|R(k, n)\| < +\infty.$$

Then we have

$$\prod_{k=[ns]}^{[nt]} f(k, n) = \prod_{k=[ns]}^{[nt]} \exp \left[\frac{1}{n} g \left(\frac{k}{n} \right) \right] + \text{Rem}$$

where the remainder term is

$$\text{Rem} = \sum_{j=1}^{[nt]-[ns]+1} \sum_{J: \#(J)=j} \prod_{k=[ns]}^{[nt]} \left(\exp \left[\frac{1}{n} g \left(\frac{k}{n} \right) \right] \mathbf{1}_{\{k \notin J\}} + \frac{1}{n^2} R(k, n) \mathbf{1}_{\{k \in J\}} \right).$$

But the term in the product on the right-hand side has a norm bounded by $e^{H_\infty/n}$ if $k \notin J$ and by R_∞/n^2 if $k \in J$ where H_∞ and R_∞ are two constants. It follows that

$$\begin{aligned} \|\text{Rem}\| &\leq \sum_{j=1}^{[nt]-[ns]+1} \binom{[nt]-[ns]+1}{j} \left(e^{\frac{H_\infty}{n}} \right)^{[nt]-[ns]+1-j} \left(\frac{R_\infty}{n^2} \right)^j \\ &= \left(e^{\frac{H_\infty}{n}} + \frac{R_\infty}{n^2} \right)^{[nt]-[ns]+1} - \left(e^{\frac{H_\infty}{n}} \right)^{[nt]-[ns]+1} \\ &= \left(e^{\frac{H_\infty}{n}} \right)^{[nt]-[ns]+1} \left[\left(1 + \frac{R_\infty e^{-\frac{H_\infty}{n}}}{n^2} \right)^{[nt]-[ns]+1} - 1 \right]. \end{aligned}$$

The first term is bounded and the second one tends to zero ; hence the remainder term converges to zero and we get (21). When the commutation property is satisfied by g , the classical property of the matrix exponential entails

$$\prod_{k=[ns]}^{[nt]} \exp \left[\frac{1}{n} g \left(\frac{k}{n} \right) \right] = \exp \left[\sum_{k=[ns]}^{[nt]} \frac{1}{n} g \left(\frac{k}{n} \right) \right]$$

and (22) follows immediately by a Riemann sum argument.

Lemma 10 *Let $(Y_n)_{n \geq 1}$ be a sequence of nonnegative random variables, ν be a probability on \mathbb{R}_+ and denote its Laplace transform by*

$$L_\nu(\mu) = \int_0^{+\infty} e^{-\frac{\mu^2}{2}x} \nu(dx).$$

Assume that

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\frac{1}{\cosh(\mu/n)} \right)^{Y_n} \right] = L_\nu(\mu)$$

uniformly with respect to μ belonging to a compact neighborhood of any positive value.

Then $\frac{Y_n}{n^2}$ converges in law to ν .

Proof : easy and omitted.

The following elementary lemma is for instance a part of Lemma 12 of [Se13] but we recall it for the convenience of the reader.

Lemma 11 (Time spent in a strip by a SRW) *Let $(X_n)_{n \geq 0}$ be a SRW started at 1 and ξ be the hitting time of $\{0, k\}$. Then*

$$J_{+1}(k, \lambda) = \mathbb{E} \left[(\cosh \lambda)^{-(1+\xi)} \mathbf{1}_{\{X_\xi=0\}} \right] = 1 - \frac{\tanh \lambda}{\tanh(k\lambda)} \quad (23)$$

and

$$J_{-1}(k, \lambda) = \mathbb{E} \left[(\cosh \lambda)^{-(1+\xi)} \mathbf{1}_{\{X_\xi=k\}} \right] = \frac{\tanh \lambda}{\sinh(k\lambda)}. \quad (24)$$

Lemma 12 (Time needed for the PRW to exit the strip of visited points)

Let $k \in \{1, 2, 3, \dots\}$ and $(X_n)_{n \geq 0}$ be the Markov chain on \mathbb{Z} whose transition probabilities $(p(x, y); x, y \in \mathbb{Z})$ are given by

- $p(x, x+1) = p(x, x-1) = 1/2$ if $x \in \{1, \dots, k-1\}$
- $p(0, 1) = \frac{\gamma}{1+\gamma}$, $p(0, -1) = \frac{1}{1+\gamma}$
- $p(k, k+1) = \frac{1}{1+\beta}$, $p(k, k-1) = \frac{\beta}{1+\beta}$,

other transition probabilities being irrelevant for what follows. Let ζ be the hitting time of $\{-1, k+1\}$. We use the notation $\mathbb{E}_a[\cdot] = \mathbb{E}[\cdot | X_0 = a]$ and define,

$$G_k(\lambda, 1, 1) = \mathbb{E}_k \left[(\cosh \lambda)^{-\zeta} \mathbf{1}_{\{X_\zeta=k+1\}} \right] \quad (25)$$

$$G_k(\lambda, -1, -1) = \mathbb{E}_0 \left[(\cosh \lambda)^{-\zeta} \mathbf{1}_{\{X_\zeta=-1\}} \right] \quad (26)$$

$$G_k(\lambda, 1, -1) = \mathbb{E}_k \left[(\cosh \lambda)^{-\zeta} \mathbf{1}_{\{X_\zeta=-1\}} \right] \quad (27)$$

$$G_k(\lambda, -1, 1) = \mathbb{E}_0 \left[(\cosh \lambda)^{-\zeta} \mathbf{1}_{\{X_\zeta=k+1\}} \right]. \quad (28)$$

Then we have

$$G_k(\lambda, 1, 1) = \frac{\cosh \lambda \sinh(k\lambda) + \gamma \sinh \lambda \cosh(k\lambda)}{D_k(\lambda)} \quad (29)$$

$$G_k(\lambda, -1, -1) = \frac{\cosh \lambda \sinh(k\lambda) + \beta \sinh \lambda \cosh(k\lambda)}{D_k(\lambda)} \quad (30)$$

$$G_k(\lambda, 1, -1) = \frac{\beta \sinh \lambda}{D_k(\lambda)} \quad (31)$$

$$G_k(\lambda, -1, 1) = \frac{\gamma \sinh \lambda}{D_k(\lambda)} \quad (32)$$

where

$$D_k(\lambda) = \sinh(k\lambda) \left(1 + (1 + \beta\gamma) \sinh^2 \lambda \right) + \frac{\beta + \gamma}{2} \sinh(2\lambda) \cosh(k\lambda). \quad (33)$$

Proof. Let us start with $G_k(\lambda, -1, -1)$. To simplify notation we set $z = 1/(\cosh \lambda)$. We condition on the value of X_1 to get

$$G_k(\lambda, -1, -1) = \frac{1}{1+\gamma} z + \frac{\gamma}{1+\gamma} \mathbb{E}_1 \left[z^{1+\xi+\zeta \circ \theta_\xi} \mathbf{1}_{\{X_{\xi+\zeta \circ \theta_\xi} = -1\}} \right]$$

where ξ is the duration needed to reach $\{0, k\}$ and $\zeta \circ \theta_\xi$ is the duration needed after that time to hit -1 or $k+1$. We now use the strong Markov property at the stopping time ξ to get

$$\begin{aligned} & \mathbb{E}_1 \left[z^{1+\xi+\zeta \circ \theta_\xi} \mathbf{1}_{\{X_{\xi+\zeta \circ \theta_\xi} = -1\}} \right] \\ &= \mathbb{E}_1 \left[z^{1+\xi} \mathbf{1}_{\{X_\xi=0\}} \mathbb{E}_0 (z^\zeta \mathbf{1}_{\{X_\zeta=-1\}}) \right] \\ & \quad + \mathbb{E}_1 \left[z^{1+\xi} \mathbf{1}_{\{X_\xi=k\}} \mathbb{E}_k (z^\zeta \mathbf{1}_{\{X_\zeta=-1\}}) \right] \\ &= J_{+1}(k, \lambda) G_k(\lambda, -1, -1) + J_{-1}(k, \lambda) G_k(\lambda, 1, -1) \end{aligned}$$

where $J_{-1}(k, \lambda)$ and $J_1(k, \lambda)$ are the functions introduced in Lemma 11. So we get a first equation on the G_k 's as displayed on the first line below and we add three more equations by similar reasoning :

$$\begin{aligned} G_k(\lambda, -1, -1) &= \frac{1}{1+\gamma} z + \frac{\gamma}{1+\gamma} J_{+1}(k, \lambda) G_k(\lambda, -1, -1) \\ & \quad + \frac{\gamma}{1+\gamma} J_{-1}(k, \lambda) G_k(\lambda, 1, -1) \\ G_k(\lambda, 1, 1) &= \frac{1}{1+\beta} z + \frac{\beta}{1+\beta} J_{+1}(k, \lambda) G_k(\lambda, 1, 1) \\ & \quad + \frac{\beta}{1+\beta} J_{-1}(k, \lambda) G_k(\lambda, -1, 1) \\ G_k(\lambda, -1, 1) &= \frac{\gamma}{1+\gamma} J_{+1}(k, \lambda) G_k(\lambda, -1, 1) + \frac{\gamma}{1+\gamma} J_{-1}(k, \lambda) G_k(\lambda, 1, 1) \\ G_k(\lambda, 1, -1) &= \frac{\beta}{1+\beta} J_{+1}(k, \lambda) G_k(\lambda, 1, -1) + \frac{\beta}{1+\beta} J_{-1}(k, \lambda) G_k(\lambda, -1, -1). \end{aligned}$$

From the second and third equations we derive

$$G_k(\lambda, -1, 1) = \frac{\frac{1}{1+\beta} \frac{\gamma}{1+\gamma} z J_{-1}(k, \lambda)}{\left(1 - \frac{\beta}{1+\beta} J_1(k, \lambda)\right) \left(1 - \frac{\gamma}{1+\gamma} J_1(k, \lambda)\right) - \frac{\beta}{1+\beta} \frac{\gamma}{1+\gamma} J_{-1}^2(k, \lambda)}.$$

We replace $J_{-1}(k, \lambda)$ and $J_1(k, \lambda)$ by their explicit values in terms of hyperbolic trigonometric functions. After a few lines of computation, we get (32). Then, by the third equation of the system above we obtain (29). Finally, (31) and (30) can be obtained by the substitution $\gamma \leftrightarrow \beta$.

4 Signs of records of the PRW

We start with the **proof of Proposition 1**. The fact that $(\chi_n)_{n \geq 1}$ is a (time-inhomogeneous) Markov chain is clear. For $n \geq 1$ and $x, y \in \{-1, 1\}$, the transition probabilities are given by

$$p_n(x, y) = \mathbb{P}(\chi_{n+1} = y \mid \chi_n = x) = \lim_{\lambda \rightarrow 0} G_n(\lambda, x, y) \quad (34)$$

where the G_n 's were introduced in (25–28) (for the second equality above apply Lebesgue's dominated convergence Theorem). Using the explicit values given by (29–32), it is straightforward to compute these limits and this completes the proof of (4).

We pass to the **proof of Proposition 2**. The requirements on $(x_t)_{t \in (0, +\infty)}$ impose the finite-dimensional marginal laws thus the uniqueness in law of $(x_t)_{t \in (0, +\infty)}$ is clear. The existence of this law is a consequence of the standard Kolmogorov extension Theorem, the compatibility condition following from the invariance of the probability defined by (6) for the transition matrices $T(s, t)$.

In order to prove the convergence in law of $(\chi_t^n)_{t \in [r, A]}$, for any $0 < r < A$, we first show the tightness. Recalling for instance Corollary 7.4 of [EK86], it suffices, in the present context, to show that, for any $\eta > 0$, we may find $\delta > 0$ such that for all n large enough, the probability that $(\chi_t^n)_{t \in [r, A]}$ has 2 jumps separated by less than δ is lower than η . But this probability is lower than

$$\begin{aligned} & \sum_{k=[nr]}^{[nA]} \sum_{j=1}^{[n\delta]} \mathbb{P}(\chi_{k+1} = -\chi_k, \chi_{k+j+1} = -\chi_{k+j}) \\ & \leq \sum_{k=[nr]}^{[nA]} \sum_{j=1}^{[n\delta]} \frac{\beta \vee \gamma}{\beta + \gamma + k} \frac{\beta \vee \gamma}{\beta + \gamma + k + j}. \end{aligned} \quad (35)$$

Using the usual expansion of the partial sums of the harmonic series, we have

$$\sum_{j=1}^{[n\delta]} \frac{\beta \vee \gamma}{\beta + \gamma + k + j} \leq c \log \left(1 + \frac{n\delta}{k} \right)$$

so that we can bound the expression (35) above by

$$c \sum_{k=[nr]}^{[nA]} \frac{1}{k} \log \left(1 + \frac{n\delta}{k} \right) \leq c \int_{([nr]-1)/n}^{[nA]/n} \frac{1}{x} \log \left(1 + \frac{\delta}{x} \right) dx$$

and this quantity tends to 0 as $\delta \downarrow 0$, uniformly in n large enough. This ends the proof of tightness.

To complete the proof of the Proposition, it suffices to show firstly that, for $r > 0$, the law of χ_r^n converges to the law given by (6) which will follow from

$$\lim_{n \rightarrow +\infty} \prod_{k=1}^{\lfloor nr \rfloor} Q_k = \frac{1}{\beta + \gamma} \begin{pmatrix} \beta & \gamma \\ \beta & \gamma \end{pmatrix} \quad (36)$$

and secondly that the transition kernels also converge that is, for all $t > s > 0$,

$$\lim_{n \rightarrow +\infty} \prod_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor} Q_k = T(s, t). \quad (37)$$

We will only prove (37) since (36) is similar. Note that $Q_k = I + \frac{1}{\beta + \gamma + k} A$ where

$$A = \begin{pmatrix} -\gamma & \gamma \\ \beta & -\beta \end{pmatrix} = \Omega \begin{pmatrix} 0 & 0 \\ 0 & -\beta - \gamma \end{pmatrix} \Omega^{-1}$$

with

$$\Omega = \begin{pmatrix} 1 & \gamma \\ 1 & -\beta \end{pmatrix} \text{ and } \Omega^{-1} = \frac{1}{\beta + \gamma} \begin{pmatrix} \beta & \gamma \\ 1 & -1 \end{pmatrix}.$$

We deduce that

$$\lim_{n \rightarrow +\infty} \prod_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor} Q_k = \Omega \begin{pmatrix} 1 & 0 \\ 0 & L \end{pmatrix} \Omega^{-1} \quad (38)$$

where

$$L = \lim_{n \rightarrow +\infty} \prod_{k=\lfloor ns \rfloor}^{\lfloor nt \rfloor} \left(1 - \frac{\beta + \gamma}{\beta + \gamma + k} \right) \quad (39)$$

but this limit L is easily shown to be $(s/t)^{\beta + \gamma}$, for instance by expressing the product in values of the Gamma function and using the fact that, for $a > 0$, as $x \rightarrow +\infty$,

$$\frac{\Gamma(x + a)}{\Gamma(x)} \sim x^a. \quad (40)$$

Finally we check that the matrix product on the right-hand side of (38) is equal to $T(s, t)$ and it concludes the proof.

5 Record values of the PRW

Let us start with the **proof of Corollary 3**. We rewrite (3) as

$$R_k = \chi_k \left(1 + \int_1^k \mathbf{1}_{\{\chi_{\lfloor u \rfloor} = \chi_k\}} du \right)$$

Changing the variable in the integral and substituting $[nt]$ for k we get

$$\frac{1}{n} R_{[nt]} = \chi_t^n \left(\frac{1}{n} + \int_{1/n}^{[nt]/n} \mathbf{1}_{\{\chi_y^n = \chi_t^n\}} dy \right).$$

For our purpose of convergence in law, by Skorohod representation Theorem and Proposition 2, we can suppose that, almost surely, (χ_t^n) converges to (x_t) in the Skorohod topology over every compact of $(0, +\infty)$. It follows easily that the right-hand side above converges –again with respect to the Skorohod topology over all compacts of $(0, +\infty)$ – to the process (y_t) defined in the statement of the proposition.

We now give a **proof of Proposition 4**. As was noticed by Formula (3), the sequence $(R_n)_{n \geq 1}$ can be reconstructed from the sequence of record signs $(\chi_n)_{n \geq 1}$. The idea is to show that there are long sequences of records with the same sign.

Let q be a large integer. For $k \geq 1$ we introduce the events

$$\begin{aligned} A_k^1 &= \{\chi_{q^{4k}} = 1\}, \\ A_k^2 &= \{\forall j \in [q^{4k}, q^{4k+1}], \chi_j = \chi_{q^{4k}} = 1\}, \\ A_k^3 &= \{\exists! j_0 \in (q^{4k+1}, q^{4k+2}), \chi_{j_0+1} = -\chi_{j_0}\}, \\ A_k^4 &= \{\forall j \in [q^{4k+2}, q^{4k+3}], \chi_j = \chi_{q^{4k+2}} = -1\}, \end{aligned}$$

and $A_k = A_k^1 \cap A_k^2 \cap A_k^3 \cap A_k^4$. The probability of A_k^1 converges to $\gamma/(\beta + \gamma)$ as $k \rightarrow +\infty$. The probability of $A_k^1 \cap A_k^2$ is equal to $\mathbb{P}(A_k^1)$ multiplied by

$$\prod_{j=q^{4k}}^{q^{4k+1}-1} \left(1 - \frac{\beta}{\beta + \gamma + j} \right) = \frac{\Gamma(\beta + \gamma + q^{4k}) \Gamma(\gamma + q^{4k+1})}{\Gamma(\gamma + q^{4k}) \Gamma(\beta + \gamma + q^{4k+1})}.$$

This term converges to $q^{-\beta} > 0$ using (40).

Thus we claim that $\mathbb{P}(A_k^1 \cap A_k^2)$ is bounded from below by a positive constant. Now $\mathbb{P}(A_k^1 \cap A_k^2 \cap A_k^3)$ is equal to $\mathbb{P}(A_k^1 \cap A_k^2)$ multiplied by

$$\sum_{j_0=q^{4k+1}+1}^{q^{4k+2}-1} \prod_{j=j_0}^{j_0-1} \left(1 - \frac{\beta}{\beta + \gamma + j} \right) \frac{\beta}{\beta + \gamma + j_0} \prod_{j=j_0+1}^{q^{4k+2}-1} \left(1 - \frac{\gamma}{\beta + \gamma + j} \right)$$

which is bounded from below by a positive constant, by the same arguments as above. Now $\mathbb{P}(A_k) = \mathbb{P}(A_k^1 \cap A_k^2 \cap A_k^3 \cap A_k^4)$ is equal to $\mathbb{P}(A_k^1 \cap A_k^2 \cap A_k^3)$ multiplied by

$$\prod_{j=q^{4k+2}}^{q^{4k+3}-1} \left(1 - \frac{\gamma}{\beta + \gamma + j} \right)$$

and repeating once more the same arguments we obtain finally that $\mathbb{P}(A_k)$ is bounded from below by a positive constant. We deduce that with positive probability the events A_k holds infinitely often. Note that on A_k , we have

$$\begin{aligned} \frac{R_{q^{4k+1}}}{q^{4k+1}} &\geq \frac{q^{4k+1} - q^{4k}}{q^{4k+1}} = 1 - \frac{1}{q}, \\ \frac{R_{q^{4k+3}}}{q^{4k+3}} &\leq -\frac{q^{4k+3} - q^{4k+2}}{q^{4k+3}} = -\left(1 - \frac{1}{q}\right), \\ 0 &> \frac{R_{j_0+1}}{j_0+1} \geq \frac{-q^{4k}}{q^{4k+1}} = -\frac{1}{q}, \\ 0 &< \frac{R_{j_1}}{j_1} \leq \frac{q^{4k+2}}{q^{4k+3}} = \frac{1}{q}, \end{aligned}$$

where j_1 denotes the time of the first change of sign of χ_n after time q^{4k+3} . From these remarks we deduce that the statements of Proposition 4 hold with positive probability. To conclude with probability 1 and thus complete the proof, we use the following zero-one law.

Proposition 13 (Zero-one law) *Every event in the asymptotic σ -algebra*

$$\mathcal{A}(\chi) = \bigcap_n \sigma(\chi_k; k \geq n)$$

has probability zero or one.

Proof. Let A belong to $\mathcal{A}(\chi)$ and take $B \in \sigma(\chi_k; k \leq m)$ of the form

$$B = \{\chi_1 = x_1, \dots, \chi_m = x_m\}$$

where $x_1, \dots, x_m \in \{-1, 1\}$. Take any $n > m$. Since $A \in \sigma(\chi_k; k \geq n)$ we may write

$$\begin{aligned} \mathbb{P}(A|B) - \mathbb{P}(A) &= \mathbb{P}(A|\chi_m = x_m) - \mathbb{P}(A) \\ &= \mathbb{P}(A|\chi_n = 1) [\mathbb{P}(\chi_n = 1|\chi_m = x_m) - \mathbb{P}(\chi_n = 1)] \\ &\quad + \mathbb{P}(A|\chi_n = -1) [\mathbb{P}(\chi_n = -1|\chi_m = x_m) - \mathbb{P}(\chi_n = -1)]. \end{aligned} \quad (41)$$

By the same computation as the one leading to (36), we obtain

$$\begin{aligned} \lim_{n \rightarrow +\infty} &\left(\frac{\mathbb{P}(\chi_n = -1|\chi_m = -1)}{\mathbb{P}(\chi_n = -1|\chi_m = 1)} \frac{\mathbb{P}(\chi_n = 1|\chi_m = -1)}{\mathbb{P}(\chi_n = 1|\chi_m = 1)} \right) \\ &= \lim_{n \rightarrow +\infty} \prod_{k=m}^{n-1} Q_k = \frac{1}{\beta + \gamma} \begin{pmatrix} \beta & \gamma \\ \beta & \gamma \end{pmatrix} \end{aligned}$$

so that both quantities in square brackets in (41) tend to zero as $n \rightarrow +\infty$. As a consequence $\mathbb{P}(A|B) = \mathbb{P}(A)$ i.e. A is independent of B . Since this holds

for all B of the specified form, in particular for all m , we deduce that $\mathcal{A}(\chi)$ is independent of the σ -algebra generated by all the variables $\chi_i, i \geq 1$. But this σ -algebra contains $\mathcal{A}(\chi)$. Hence $\mathcal{A}(\chi)$ is independent of itself which ends the proof.

6 Record times of the PRW : conditional case

Let us **prove Proposition 5**. We first concentrate on the convergence of finite-dimensional marginal laws. We will give the justification of tightness at the end of section 7.

For $x, y \in \{-1, 1\}$ and $k \geq 1$, the quantity

$$\tilde{G}_k(\lambda, x, y) = \mathbb{E} \left[\left(\frac{1}{\cosh \lambda} \right)^{T_{k+1} - T_k} \mid \chi_k = x, \chi_{k+1} = y \right] \quad (42)$$

is equal to $G_k(\lambda, x, y)/p_k(x, y)$ because of (25–28) and (34). For further reference we gather the explicit values in a matrix :

$$\begin{aligned} & \begin{pmatrix} \tilde{G}_k(\lambda, -1, -1) & \tilde{G}_k(\lambda, -1, 1) \\ \tilde{G}_k(\lambda, 1, -1) & \tilde{G}_k(\lambda, 1, 1) \end{pmatrix} = \frac{k + \beta + \gamma}{D_k(\lambda)} \dots \\ & \dots \times \begin{pmatrix} \frac{\cosh \lambda \sinh k\lambda + \beta \sinh \lambda \cosh k\lambda}{k + \beta} & \sinh \lambda \\ \sinh \lambda & \frac{\cosh \lambda \sinh k\lambda + \gamma \sinh \lambda \cosh k\lambda}{k + \gamma} \end{pmatrix}, \end{aligned} \quad (43)$$

recalling the $D_k(\lambda)$ is given by (33). The increments of the record times $T_{k+1} - T_k$, $k \geq 1$ are independent random variables, even conditionally on the record signs $(\chi_k)_{k \geq 1}$. It follows that

$$\begin{aligned} & \mathbb{E} \left[\left(\frac{1}{\cosh \lambda} \right)^{T_{[nt]} - T_{[ns]}} \mid (\chi_k)_{k \geq 1} = (x^n(k/n))_{k \geq 1} \right] \\ &= \prod_{k=[ns]}^{[nt]-1} \tilde{G}_k \left(\lambda, x^n \left(\frac{k}{n} \right), x^n \left(\frac{k+1}{n} \right) \right) \\ &= \prod_{\substack{[ns] \leq k < [nt] \\ x^n(k/n) \neq x^n((k+1)/n)}} (\sinh \lambda) \frac{k + \beta + \gamma}{D_k(\lambda)} \\ &\times \prod_{\substack{[ns] \leq k < [nt] \\ x^n(k/n) = x^n((k+1)/n)}} \frac{(k + \beta + \gamma) \left(\cosh \lambda \sinh(k\lambda) + \tilde{\delta}(x^n(k/n)) \sinh \lambda \cosh(k\lambda) \right)}{D_k(\lambda) \left(k + \tilde{\delta}(x^n(k/n)) \right)} \end{aligned}$$

where $\tilde{\delta}(x) = \gamma$ if $x = 1$ and $\tilde{\delta}(x) = \beta$ if $x = -1$. We denote the set of discontinuities of the càdlàg function $x^n(\cdot)$ over $[s, t]$ by $\mathcal{D}(x^n; s, t) = \{r \in (s, t); x^n(r-) \neq x^n(r)\}$. In order to get the asymptotic behaviour, we regroup the terms and set $\lambda = \frac{\mu}{n}$:

$$\begin{aligned} & \mathbb{E} \left[\left(\frac{1}{\cosh \frac{\mu}{n}} \right)^{T_{[nt]} - T_{[ns]}} \mid (\chi_k)_{k \geq 1} = \left(x^n \left(\frac{k}{n} \right) \right)_{k \geq 1} \right] \\ &= \prod_{k=[ns]}^{[nt]-1} \frac{(k + \beta + \gamma) \left(\cosh \frac{\mu}{n} \sinh \left(\frac{k\mu}{n} \right) + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \sinh \frac{\mu}{n} \cosh \left(\frac{k\mu}{n} \right) \right)}{D_k \left(\frac{\mu}{n} \right) \left(k + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \right)} \\ & \times \prod_{\frac{k+1}{n} \in \mathcal{D}(x^n; s, t)} \frac{\sinh \left(\frac{k\mu}{n} \right) \left(k + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \right)}{\left(\cosh \frac{\mu}{n} \sinh \left(\frac{k\mu}{n} \right) + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \sinh \frac{\mu}{n} \cosh \left(\frac{k\mu}{n} \right) \right)}. \quad (44) \end{aligned}$$

Let us now perform asymptotic expansions up to the order $1/n^2$ in the sense of (20). We obtain that

$$D_k \left(\frac{\mu}{n} \right) \approx \sinh \left(\frac{k\mu}{n} \right) + (\beta + \gamma) \frac{\mu}{n} \cosh \left(\frac{k\mu}{n} \right).$$

Also we get

$$\begin{aligned} \frac{(k + \beta + \gamma)}{\left(k + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \right)} &\approx 1 + \frac{1}{n} \left(\beta + \gamma - \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \right) \frac{1}{k/n} \\ &\approx 1 + \frac{1}{n} \delta \left(x^n \left(\frac{k}{n} \right) \right) \frac{1}{k/n}, \end{aligned}$$

recalling that $\delta(y) = \beta + \gamma - \tilde{\delta}(y)$. Then we do similarly for the other terms in (44). Now we use Lemma 9 (1-dimensional case) to deduce that

$$\begin{aligned} & \lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]-1} \frac{(k + \beta + \gamma) \left(\cosh \frac{\mu}{n} \sinh \left(\frac{k\mu}{n} \right) + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \sinh \frac{\mu}{n} \cosh \left(\frac{k\mu}{n} \right) \right)}{D_k \left(\frac{\mu}{n} \right) \left(k + \tilde{\delta} \left(x^n \left(\frac{k}{n} \right) \right) \right)} \\ &= \exp \left(\int_s^t \delta(x(u)) \left(\frac{1}{u} - \mu \coth \mu u \right) du \right). \quad (45) \end{aligned}$$

To be precise we apply this Lemma a finite number of time, on each interval where $\delta(x^n(\cdot))$ is constant and at the limit, reunite all the integrals over these intervals into a single one.

The set $\mathcal{D}(x; s, t)$ is finite and the convergence of $(x^n(r))_{s \leq r \leq t}$ toward $(x(r))_{s \leq r \leq t}$ with respect to the Skorohod topology implies that, for any continuous function φ ,

$$\lim_{n \rightarrow +\infty} \prod_{\frac{k+1}{n} \in \mathcal{D}(x^n; s, t)} \varphi(k/n) = \prod_{r \in \mathcal{D}(x; s, t)} \varphi(r).$$

As a consequence we deduce that,

$$\begin{aligned} \lim_{n \rightarrow +\infty} \prod_{k \in \mathcal{D}(x^n; s, t)} \frac{\sinh\left(\frac{k\mu}{n}\right) \left(k + \tilde{\delta}\left(x^n\left(\frac{k}{n}\right)\right)\right)}{\left(\cosh\frac{\mu}{n} \sinh\left(\frac{k\mu}{n}\right) + \tilde{\delta}\left(x^n\left(\frac{k}{n}\right)\right) \sinh\frac{\mu}{n} \cosh\left(\frac{k\mu}{n}\right)\right)} \\ = \prod_{r \in \mathcal{D}(x; s, t)} \frac{\mu r}{\sinh \mu r}. \end{aligned}$$

Moreover by inspecting all the proof we see that the limits above are uniform for μ varying in any compact neighborhood of a fixed positive value. By Lemma 10, the proof of the convergence of finite-dimensional marginal laws is complete.

7 Record times of the PRW : unconditional case

We start with the general case $\beta, \gamma \in (0, +\infty)$ to see how far we can go before being compelled to restrict to $\beta = \gamma$. The main step is to compute the limit in law of a rescaled increment $(T_{[nt]} - T_{[ns]})/n^2$ and, as before, this is done by computing the limit of

$$\mathbb{E} \left[\left(\frac{1}{\cosh \frac{\mu}{n}} \right)^{T_{[nt]} - T_{[ns]}} \right].$$

We set $\lambda = \mu/n$ and $z = 1/\cosh \lambda$, as before. By the repeated use of the Markov property and the definition of $G_k(\cdot, \cdot, \cdot)$, we get, for fixed $x_{[ns]} \in \{-1, 1\}$,

$$\begin{aligned}
& \mathbb{E} \left[z^{T_{[nt]} - T_{[ns]}} \mid \chi_{[ns]} = x_{[ns]} \right] \\
&= \sum_{\substack{x_k \in \{-1, 1\} \\ [ns] < k \leq [nt]}} \mathbb{E} \left[\left(\prod_{k=[ns]}^{[nt]-1} z^{T_{k+1} - T_k} \right) \prod_{k=[ns]+1}^{[nt]} \mathbf{1}_{\{\chi_k = x_k\}} \mid \chi_{[ns]} = x_{[ns]} \right] \\
&= \sum_{\substack{x_k \in \{-1, 1\} \\ [ns] < k \leq [nt]}} \prod_{k=[ns]}^{[nt]-1} G_k(\lambda, x_k, x_{k+1}) \\
&= \sum_{x_{[nt]} \in \{-1, 1\}} \left(\prod_{k=[ns]}^{[nt]-1} G_k(\lambda) \right) (x_{[ns]}, x_{[nt]}) \tag{46}
\end{aligned}$$

where $G_k(\lambda) = (G_k(\lambda, x, y))_{x, y \in \{-1, 1\}}$ is a 2×2 matrix. We recall that

$$G_k(\lambda) = \frac{\cosh \lambda \sinh(k\lambda)}{D_k(\lambda)} \begin{pmatrix} 1 + \beta \tanh \lambda \operatorname{coth}(k\lambda) & \frac{\gamma \tanh \lambda}{\sinh(k\lambda)} \\ \frac{\beta \tanh \lambda}{\sinh(k\lambda)} & 1 + \gamma \tanh \lambda \operatorname{coth}(k\lambda) \end{pmatrix}.$$

Changing λ into μ/n and without conditioning, Formula (46) writes as

$$\begin{aligned}
& \mathbb{E} \left[\left(\frac{1}{\cosh \mu/n} \right)^{T_{[nt]} - T_{[ns]}} \right] \\
&= \sum_{x \in \{-1, 1\}} \mathbb{P}(\chi_{[ns]} = x) \sum_{y \in \{-1, 1\}} \left(\prod_{k=[ns]}^{[nt]-1} G_k(\mu/n) \right) (x, y). \tag{47}
\end{aligned}$$

We want to pass to the limit $n \rightarrow +\infty$. It is easy to see, using again the notation (20), that

$$G_k \left(\frac{\mu}{n} \right) \approx I + \frac{1}{n} \tilde{H} \left(\frac{k}{n} \right)$$

where

$$\tilde{H}(x) = \begin{pmatrix} -\gamma \mu \operatorname{coth}(\mu x) & \gamma \mu \sinh^{-1}(\mu x) \\ \beta \mu \sinh^{-1}(\mu x) & -\beta \mu \operatorname{coth}(\mu x) \end{pmatrix}.$$

Then we would like to apply Lemma 9 and conclude that

$$\lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]-1} G_k(\mu/n) = \lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]-1} \exp \left[\frac{1}{n} \tilde{H} \left(\frac{k}{n} \right) \right]$$

provided the limit on the right-hand side exists. Unfortunately we are not able to prove the existence of the limit in the general case. The problem is that the matrices $\tilde{H}(\cdot)$ do not commute in the general case. We listed this

limit as one of the open problems of Section 2.7. But in the particular case of symmetric perturbation $\beta = \gamma$, the matrices $\tilde{H}(\cdot)$ do commute and we deduce by Lemma 9 that

$$\lim_{n \rightarrow +\infty} \prod_{k=[ns]}^{[nt]} \exp \left[\frac{1}{n} \tilde{H} \left(\frac{k}{n} \right) \right] = \exp \left[\int_s^t \tilde{H}(x) dx \right].$$

Moreover in this case we check that

$$\int_s^t \tilde{H}(x) dx = H(t) - H(s)$$

with

$$H(t) = \begin{pmatrix} -\beta \log \sinh(\mu t) & \beta \log \tanh(\mu t/2) \\ \beta \log \tanh(\mu t/2) & -\beta \log \sinh(\mu t) \end{pmatrix}.$$

Moreover we know that the probabilities $\mathbb{P}(\chi_{[ns]} = x)$, $x \in \{-1, 1\}$ appearing in (47) simply converge in this case to $1/2$. Combining all these facts, the passage to the limit in (47) gives

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\frac{1}{\cosh \frac{\mu}{n}} \right)^{T_{[nt]} - T_{[ns]}} \right] = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \exp [H(t) - H(s)] \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (48)$$

where, of course, the right-hand side should be read as a product of three matrices. But the matrix $H(t) - H(s)$ has the special form

$$H(t) - H(s) = \beta \begin{pmatrix} -a & b \\ b & -a \end{pmatrix}$$

where

$$a = \log \left(\frac{\sinh(\mu t)}{\sinh(\mu s)} \right) \text{ and } b = \log \left(\frac{\tanh \frac{\mu t}{2}}{\tanh \frac{\mu s}{2}} \right).$$

So, computing the exponential is easily done via the diagonalization

$$\begin{pmatrix} -a & b \\ b & a \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -a + b & 0 \\ 0 & -a - b \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and we obtain finally

$$\lim_{n \rightarrow +\infty} \mathbb{E} \left[\left(\frac{1}{\cosh \frac{\mu}{n}} \right)^{T_{[nt]} - T_{[ns]}} \right] = \left(\frac{\cosh(\frac{\mu}{2} s)}{\cosh(\frac{\mu}{2} t)} \right)^{2\beta}$$

which leads to the desired convergence of finite-dimensional marginal laws, via the usual argument.

Now we want to address the problem of tightness of the laws of the processes $((\tau_t^n)_{t \geq 0}, n \geq 1)$. We can use for instance the criterion stated in [Bi68]

Theorem 15.6 which consists, for any $T > 0$, in finding a nondecreasing continuous function F such that, for all $0 \leq t_1 \leq t \leq t_2 \leq T$ and all n large enough,

$$\mathbb{E}[(\tau_t^n - \tau_{t_1}^n)(\tau_{t_2}^n - \tau_t^n)] \leq [F(t_2) - F(t_1)]^2. \quad (49)$$

Note that only the case $t_2 - t_1 \geq 1/n$ has to be considered, otherwise the left-hand side vanishes. Let us recall that trivially

$$\mathbb{E}(\tau_t^n - \tau_s^n) = \frac{1}{n^2} \sum_{k=[ns]}^{[nt]-1} \mathbb{E}(T_{k+1} - T_k). \quad (50)$$

By the Definition (42) we derive easily

$$\mathbb{E}(T_{k+1} - T_k | \chi_k = x, \chi_{k+1} = y) = - \lim_{\lambda \rightarrow 0} \frac{1}{\lambda} \frac{d}{d\lambda} \tilde{G}_k(\lambda, x, y). \quad (51)$$

The expressions of $\tilde{G}_k(\lambda, x, y)$ for $x, y \in \{-1, 1\}$ are explicitly given by (43). So it suffices to differentiate $\tilde{G}_k(\lambda, x, y)$ and replace every hyperbolic trigonometric function by its Taylor expansion around 0 (up to order 3) to get the value of the limit in (51). The computations are a bit tedious and left to the reader but the important fact is that there exists a constant c such that, for all k ,

$$\mathbb{E}(T_{k+1} - T_k | \chi_{k+1} = \chi_k) \leq c k \quad (52)$$

and

$$\mathbb{E}(T_{k+1} - T_k | \chi_{k+1} = -\chi_k) \leq c k^2. \quad (53)$$

Moreover we have seen that, for a certain (other) constant c , $\mathbb{P}(\chi_{k+1} = -\chi_k) \leq \frac{c}{k}$ so that we conclude that $\mathbb{E}(T_{k+1} - T_k) \leq c k$ and as a consequence,

$$\mathbb{E}(\tau_t^n - \tau_s^n) \leq c ([nt] - [ns])/n.$$

Using the independence of the increments, (49) easily follows, with a linear function $F(\cdot)$ and tightness is assured.

Let us come back shortly to the conditional case where we claim that a similar proof of tightness can be constructed on every compact interval of $(0, +\infty)$. Indeed the same argument works on a time interval where the signs of the corresponding records are constant. Any time interval $[\varepsilon, T]$ with $T > \varepsilon > 0$ can be decomposed for every n into a (finite) partition such that on each interval of this partition the signs of the corresponding records, given by $x^n(\cdot)$ are constant. Because of the convergence in Skorohod topology of $x^n(\cdot)$ toward $x(\cdot)$, these partitions converge to the partition ruling the sign of $x(\cdot)$ over $[\varepsilon, T]$. So the relative compactness of the conditional laws over $[\varepsilon, T]$ can be deduced from the tightness guaranteed on each sub-interval of the partition.

The **end of the proof of Theorem 6** is similar to the proof of Theorem 1 in [Se13], except the multiplication by a power of 2 from places to places.

Also we omit the **proof of Proposition 8** which is identical to the proof of Proposition 2 in [Se13].

8 Positive recurrence

We now want to **prove Theorem 7**. By symmetry, it suffices to prove that $\beta > 2$ is equivalent to

$$\mathbb{E} \left(T_{\inf\{j>q; \chi_j=-1\}} - T_q \mid \chi_q = 1 \right) < +\infty.$$

This (conditional) expectation equals

$$\sum_{j=q}^{+\infty} \left\{ \left(\sum_{k=q}^{j-1} \mathbb{E}(T_{k+1} - T_k \mid \chi_{k+1} = \chi_k = 1) \right) + \mathbb{E}(T_{j+1} - T_j \mid \chi_j = 1 = -\chi_{j+1}) \right\} \\ \times \mathbb{P}(\chi_{q+1} = \dots = \chi_j = 1 = -\chi_{j+1} \mid \chi_q = 1). \quad (54)$$

But, by Proposition 1,

$$\mathbb{P}(\chi_{q+1} = \dots = \chi_j = 1 = -\chi_{j+1} \mid \chi_q = 1) \\ = \left(\prod_{k=q}^{j-1} \left(1 - \frac{\beta}{\beta + \gamma + k} \right) \right) \frac{\beta}{\beta + \gamma + j} \quad (55)$$

and it is easy to see that this quantity is equivalent to $C j^{-\beta-1}$ where C is a constant.

We discussed at the end of the previous section the procedure to get the conditional means of $T_{k+1} - T_k$, see (52) and (53). This procedure shows also that, as $k, j \rightarrow +\infty$,

$$\mathbb{E}(T_{k+1} - T_k \mid \chi_{k+1} = \chi_k = 1) \sim \frac{2\beta}{3} k$$

and

$$\mathbb{E}(T_{j+1} - T_j \mid \chi_j = 1 = -\chi_{j+1}) \sim \frac{j^2}{3}.$$

Combining this with the estimate for (55) already obtained, we get that the expression of the (conditional) expectation given by (54) behaves like $\sum_j j^{1-\beta}$ hence is finite for $\beta > 2$ as announced.

References

- [Bi68] Billingsley P. : *Convergence of Probability Measures*. Wiley.
- [CPY98] Carmona P., Petit F., Yor M. : Beta variables as times spent in $[0, +\infty)$ by certain perturbed Brownian motions. *J. London Math. Soc.* 58, 239-256.
- [CDH00] Chaumont L., Doney R.A., Hu Y. : Upper and lower limits of doubly perturbed Brownian motion. *Ann. Inst. H. Poincaré, Prob. Stat.* 36 (2), 219-249.
- [Da96] Davis B. : Weak limits of perturbed random walks and the equation $Y_t = B_t + \alpha \sup\{Y_s : s \leq t\} + \beta \inf\{Y_s : s \leq t\}$ *Ann. Probab.* 24 (4), 2007-2023.
- [Da99] Davis B. : Brownian motion and random walk perturbed at extrema. *Probab. Th. Rel. Fields* 113, 501-518.
- [EK86] Ethier S.N., Kurtz T.G. : *Markov processes; Characterization and convergence*. Wiley.
- [Pe07] Pemantle R. : A survey of random processes with reinforcement. *Probab. Surveys*, (4), 1-79.
- [PW97] Perman M., Werner W. : Perturbed Brownian motions. *Probab. Th. Rel. Fields* 108, 357-383.
- [Se13] Serlet L. : Hitting times for the perturbed reflecting random walk. *Stoch. Proc. Appl.* 123, 110-130.
- [We95] Werner W. : Some remarks on perturbed reflecting Brownian motion. *Séminaire de Probabilités XXIX*. Lecture Notes in Math. 1613. Springer, 37-43.