Efficiency and volatility of spot and futures agricultural markets: Impact of trade frequencies

David Batista Soares, Alain Bretto, Joël Priolon

Efficiency and volatility of spot and futures agricultural markets: Impact of trade frequencies

David Batista Soares*[1,2], Alain Bretto[1], and Joël Priolon[2]

[1]Normandie Université, Unicaen, Computer Science Department, CNRS-GREYC UMR-6072, Campus II, Bd Maréchal Juin BP 5186, 14032 Caen cedex, France
[2]Economie Publique, AgroParisTech, INRA, Université Paris-Saclay, 78850, Thiverval-Grignon, France

February 16, 2019

Abstract

In efficient markets, asset prices are equal to their fundamentals. This classical view is considered valid for agricultural commodities' spot and futures markets. However, fragmentation of orders impacts price dynamics, leading to modification in spot and futures' trade frequency, relative trade frequency, and quantities exchanged. To highlight public policies on the impacts of fragmentation of orders, it is necessary to improve the understanding of its theoretical consequences.

Based on a sequential trading framework, our main result showed that unbiased prices and a minimal volatility of fundamental basis are achieved not with optimal trade frequencies but with an optimal relative trade frequency.

Keywords– commodities; spot and futures prices; market efficiency; volatility; stock and fragmentation

1 Introduction

The physical market (also called spot market) and the financial market (referred to as the futures market where futures contracts are exchanged), are linked for a single standardized agricultural raw material (also called a commodity; cf. [1], [17]). There exists retro-action between the prices in these two markets. The basis, defined as the spread between these two prices, is a source of risk, and its volatility is a major stake for farmers and processors: it represents a major component of their production and selling decisions (cf. [13]). However, price in the futures market evolves faster than that in the spot market (cf. [15], [18]). Fragmentation of orders, defined as the division of one specific order into several suborders, occurs in both markets. It entails a decrease in quantities exchanged per transaction and an increase in the number of transactions in a market (cf. [14], [15], [19]). Consequently, to buy the same quantity, agents pass more orders and trade frequency increases. Although such fragmentation of orders is used in both spot and futures markets, transaction costs in these two markets are not the same (cf. [18]). Fragmentation of orders is neither equal or proportional in these two markets: it impacts both relative trade frequency (RTF) and quantities exchanged per transaction in these markets. Several empirical studies analyzed the impact of order fragmentation on market liquidity and price volatility (cf. [3], [6], [11]), but only for equity markets. However, equity markets and agricultural commodity markets differ in many ways (cf. [2] for a review). The crucial difference between the current study and this work is that we show how fragmentation of orders in the spot market modifies the fundamental futures' commodity price and fundamental volatility, in particular regarding its impact on the released information and its price impact. The link between financial information and market prices has been popularized by Eugene Fama (cf. [8], [7]):

*Corresponding author: `david.batista-soares@unicaen.fr`

> "I take the market efficiency hypothesis to be the simple statement that security prices fully reflect all available information." – (cf. [7], p1575)

In fact, the market efficiency hypothesis is a double one : first the market prices are supposed to be fundamentally determined by information. Second, the process that turns information into prices is supposed to be as efficient as it should be. The market efficiency research program belongs mainly to empirical economics. The main conclusion of Fama is that the program that has been followed by so many financial economists has never lead to reject the hypothesis. In this work, we use the central part of Fama's concept that is market prices result from the processing of financial information. We can sum it up in a general manner: $p_t = g(I_t)$ where p_t is the price at period t, I_t is the set of available information and g is an economic function that turns information into prices (see below subsection 3.5). The main difference with Fama's tradition is that we use the concept of market efficiency for a research that is, in a first stage, a theoretical framework; that framework is dedicated to the study of the formation of two series of market prices that are perfectly connected at the level of the underlying fundamental values but which are not in fact perfectly correlated. Spot and futures prices normally evolve in a like manner but they are not thoroughly linked at least because their dynamics are not perfectly synchronized.

This study aims to examine the extent to which the difference in trade frequencies between the two markets and the evolution of that difference influence market efficiency as well as to examine the fundamental volatility of the basis, the spread between the two prices. Finally, we consider whether there is an optimal level of trade frequencies that satisfies these two objectives. Our main result has a major implication. We show that the use of limit order book pricing with fragmentation of orders in the futures market coupled with fragmentation of orders in the spot market is not optimal, except if order fragmentation in the futures market leads to infinite trade frequency in that market. Otherwise, we argue that synchronization of trade frequency allows unbiased prices and minimal fundamental basis volatility. We extrapolate from this result that there is a dilemma between market liquidity and the two objectives.

The article is organized as follows. Section 2 presents all parameters and variables whereas section 3 presents the definitions and hypothesis of the model. Section 4 derives the influence of trade's frequencies and section 5 discusses in detail the impacts of fragmentation on whether only the spot market, only the futures market, or proportional on both markets. In section 6, we study the existence of optimal trade's frequencies. Section 7 concludes and discusses the limitations of the model.

2 Parameters and variables

2.1 The parameters

The following set of parameters is used for the futures and spot markets:

Futures market	t_r	ω_f	e	a	Q_{op}	T_r
Spot market	t_r	ω_s	e			

- t_r Unit of real time considered, such as a second or minute.
- ω_f Historical trade frequency in the futures market per unit of real time: $\left(\frac{\text{Number of transactions in the futures market}}{t_r}\right)$
- ω_s Historical trade frequency in the spot market per unit of real time: $\left(\frac{\text{Number of transactions in the spot market}}{t_r}\right)$
- e Trend of the spot market information delivered in monetary value per unit of real time.
- a Advantages of possessing a unit of the storable commodity in monetary value, per unit of real time.
- Q_{op} Optimal stock level of the commodity for agents.
- T_r Maturity of the futures contract, expressed in a unit of real time.

2.2 The variables

The following set of variables is used for the futures and spot markets:

Futures market	t	T	$I_{]t_1,t_2]}$		Q_t	CY_t
Spot market	t		$I_{]t_1,t_2]}$	q	Q_t	

Futures market	$\widetilde{f_{t+1}}$	f_t	FV_{f_t}
Spot market	$\widetilde{s_{t+1}}$	s_t	FV_{s_t}

t	Time step of the model, which represents a transaction in the futures market: $t = t_r \times \omega_f$ and $t \in \mathbb{N}$.
T	Maturity expressed in the time scale t: $T = T_r \times \omega_f$ and $T \in \mathbb{N}$.
$I_{]t_1,t_2]}$	Spot market information delivered in monetary value between t_1 and t_2, t_1 excluded.
q	Quantities traded in the spot market at each transaction on the spot market.
Q_t	Available stock of the commodity at time t.
CY_t	Convenience yield (advantage in detaining one unit of stock to face uncertainty) in monetary value at time t until maturity T.
$\widetilde{s_t}, \widetilde{f_t}, \widetilde{Q_t}$	Agents' expectations of the spot price, futures price, and commodity available stock, respectively, at time t.
s_t, f_t	Observed spot price and futures price, respectively, at time t.
FV_{s_t}, FV_{f_t}	Fundamental value of the spot price and futures price, respectively, at time t.

3 Definitions and hypotheses

3.1 Time scale and frequencies

We define $\mathbb{F}_s$ and $\mathbb{F}_f$ as the sets of all possible frequencies on the spot and futures markets. We have $0 \notin (\mathbb{F}_s \cup \mathbb{F}_f)$, which means that the two markets exist. Therefore, $(\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f$. Furthermore, we assume that $\omega_s \leq \omega_f$. Futures are more frequently traded than the commodity itself since transaction costs are inferior (cf. [18]). This explains why we choose the transaction on the futures as the time step (the smallest one). We assume that transactions on the futures markets are equally spaced out, according to the futures market trade frequency (FTF),ω_f. The real time between two transactions on the futures market is constant. The effective FTF during the period is known, and the effective spot market trade frequency (STF), is unknown. The historical STF, ω_s, only gives the probability of having a transaction in the spot market per unit of real time.

At each transaction in the futures market, there is an independent probability $\frac{\omega_s}{\omega_f}$ of having a transaction in the spot market. This probability is equal to the historical RTF.

Let $\lessdot$ define a relation wherein t' represents the latest period when there was a synchronized transaction in the spot and futures markets until t so that the following is true:
$t' \lessdot t$ if $t' \leq t$, and there is no element $y' \neq t'$ nor $y' \neq t$ such that $t' \leq y' \leq t$. Thus, $(t-1)'$ is the latest period when there was a synchronized transaction in the spot market and futures market until $t-1$, and $(t-1)' \lessdot t-1$. However, we do not have $(t-1)' \lessdot t'-1$. Indeed, t' and $(t-1)'$ can both be equal to 0 such that $0 \leq -1$, which is impossible.

3.2 Spot market information

We assume that the monetary impact of incoming spot market information delivered at time t, denoted $I_{]t-1,t]}$, follows any probabilistic law $\mathcal{L}$ whose mean is $E\left(I_{]t-1,t]}\right) = \frac{e}{\omega_f}\left(t-(t-1)\right) = \frac{e}{\omega_f}$. This mean is a strictly decreasing function of ω_f. As spot market information arrives between each transaction in the futures market, we can naturally assume that if transactions are more frequent in the futures market, spot market information between two transactions in the futures market has, on average, a lower price impact.

So, $I_{]t',t]}$ represents the spot market information delivered in monetary value since the last transaction on the spot market (t').

3.3 Quantities exchanged and influence of stock dynamics

We now define q, which indicates the quantities traded per transaction in the spot market as a function such that

$$\begin{array}{rccl} q: & \mathbb{F}_s & \to & \mathbb{R}^{+,*} \\ & \omega_s & \mapsto & q(\omega_s) = \omega_s^{-\epsilon}, \epsilon \in \mathbb{Q}^+ \end{array} \tag{1}$$

From the above function, quantities traded per transaction in the spot market are decreasing when the spot market trade frequency increases. Note that parameter ϵ corresponds to the absolute value of the elasticity of quantities traded per transactions in the spot market to the STF.

The available stock of the commodity at time t is defined using a discrete random variable (D.R.V).

$$Q_0 > 0 \text{ and } Q_t := \left\{ \begin{array}{lll} Q_{t-1} & if & t' \neq t \\ Q_{t-1} - q(\omega_s) & if & t' = t \end{array} \right\} = Q_{t-1} - q(\omega_s) \mathbb{1}_{\{t\}}(t') \ \ \forall \ t \in [1, T] \ \ \in \mathbb{N} \tag{2}$$

We assume that Q_0 is such that for any ω_s in $\mathbb{F}_s$ and any t in $[0; T-1]$, $Q_t - q(\omega_s) \geq q(\omega_s)$. We naturally assume that the available commodity is only purchased in the spot market during the period considered (at maturity, commodity is purchased and consumed according to the commitments taken in the futures market). We assume that production or harvest of the commodity is less frequent than the consumption. Therefore, we do not insert it since it would only have a symmetric effect to the stock consumption on the expectations and fundamental value. However, we can do it easily. Consequently, the available stock evolves when and only when there is a transaction in the spot market. It is important to underline that this assumption requires us to not consider that commodity is purchased only to be sold later. For the sake of simplicity, there is no speculation on the spot market. Once again, we could integrate a speculation Bernoulli D.R.V.

We assume that the storage cost per unit of *real* time is the same during the period considered, and, consequently, they are linear with the time and will be perfectly expected. For the sake of simplicity, we do not model them, but it can be done easily.

Despite positive stocks, the futures' price can be inferior to the spot price because of the necessity for corporates to maintain their stocks to face uncertainty (cf. [10]). This advantage (or disadvantage) to detain a unit in stock at transaction t in monetary value until maturity, also called convenience yield and denoted by CY_t, is given by the following equation:

$$CY_t := \frac{a}{\omega_f}(T-t)(Q_{op} - Q_t) \tag{3}$$

We assume that Q_{op} is exogenous to the model. This is the same for all agents. When the stock is above this level, there is no advantage to detain more units, and the convenience yield is negative. When the stock is under this level, there is an advantage to detain the units of stock. This advantage also depends on the number of transactions on the futures market remaining until maturity weighted by the advantage per unit of real time between a transaction on the futures market ($\frac{a}{\omega_f}(T-t) = a(T_r - t_r)$).

3.4 Agents' expectations

We assume that all information is freely available and that, at time t, all agents know all prices and market characteristics. Therefore, $\Phi_t = \{s_t, f_t, Q_t, I_{]t',t]}, CY_t\}$ is the common knowledge at time t. Furthermore, we assume that agents form rational expectations in the sense that, on average, their expectations reflect fundamental prices. They compute the average expected price evolution.

The expectation of the spot price is computed by

$$E(\widetilde{s_{t+1}}|\Phi_t) = s_{t'} + I_{]t',t]} + \frac{e}{\omega_f} \tag{4}$$

Recall that t' represents the latest period when there was a synchronized transaction in the spot market and futures market until t. Agents add to the current spot price (at period t), all spot market information that should have been included in the spot price $I_{]t',t]}$, and the expected incoming spot market information for its average value $E\left(I_{]t,t+1]}\right) = \frac{e}{\omega_f}$.

Expectations of the stock dynamics are given by

$$E\left(\widetilde{Q_{t+1}}|\Phi_t\right) = Q_t - q\left(\omega_s\right)\frac{\omega_s}{\omega_f} \tag{5}$$

The expected stock evolution is given by quantities exchanged at each transaction on the spot market (that we suppose consumed) weighted by the probability of having a transaction on the spot market at the next transaction on the futures market. The convenience yield is expected such that

$$E\left(\widetilde{CY_{t+1}}|\Phi_t\right) = \frac{a}{\omega_f}\left(T-(t+1)\right)\left(Q_{op} - E\left(\widetilde{Q_{t+1}}|\Phi_t\right)\right) \tag{6}$$

To prevent an arbitrage operation (defined as an operation that guarantees a positive profit without risk of loss; cf. [16]), the basis must include the current level of stock since the lower stocks are, the more agents have an interest in detaining it depending on the level of their optimal stock. Between each transaction in the futures market, the advantage in detaining stocks decreases by $\frac{a}{\omega_f}$ for the same level of stock than at t. However, there is a probability of $\frac{\omega_s}{\omega_f}$ of having a stock movement, and the average stock evolution is not null. Thus, expected stock movement affects the expected advantage in detaining stocks until maturity.

Finally, expectations of the futures price for the next period are

$$E\left(\widetilde{f_{t+1}}|\Phi_t\right) := E\left(\widetilde{s_{t+1}} - \widetilde{CY_{t+1}}|\Phi_t\right) \tag{7}$$

Equation 7 translates the fact that agents know what is the no arbitrage condition.

3.5 Price dynamics and fundamental values

Prices evolve according to the agents' expectations if and only if a transaction occurs. Agents expect the next prices but do not necessarily pass an order at these prices. Thus,

$$\begin{array}{rcl} s_{t+1} & := & \left\{\begin{array}{lll} s_{t'} & if & (t+1)' \neq t+1 \\ E\left(\widetilde{s_{t+1}}|\Phi_t\right) & if & (t+1)' = t+1 \end{array}\right\} = s_{t'} + \mathbb{1}_{\{t+1\}}\left((t+1)'\right)\left[I_{]t',t]} + \frac{e}{\omega_f}\right] \\ f_{t+1} & := & E\left(\widetilde{f_{t+1}}|\Phi_t\right) \end{array} \tag{8}$$

The fundamental value of a price corresponds to the instantaneous integration of all available information on the price (cf. [9]). Assuming that $FV_{s_0} = s_0$ and $FV_{f_0} = f_0$, the fundamental values are naturally defined by

$$\begin{array}{rcl} FV_{s_{t+1}} & := & s_0 + I_{]0,t+1]} \\ FV_{f_{t+1}} & := & FV_{s_{t+1}} - CY_{t+1} \end{array} \tag{9}$$

The futures' fundamental value is computed using a classical view, by its no arbitrage value (cf. [4], [5], [12]).

3.6 Market efficiency and fundamental volatility

We study price bias and we refer to it as market efficiency, denoted by B. We are aware that an unbiased price does not ensure that at each time t, the price fully reflects the information on the market, but unbiased price is a *sine qua non* condition. We give the following function for the market efficiency:

$$\begin{array}{rcl} B: \mathbb{R}^+ & \to & \mathbb{R}^+ \\ p_t & \mapsto & B(p_t) = \left|E\left(p_t - FV_{p_t}\right)\right|,\ p_t \in \{s_t, f_t\} \text{ where } |\ .\ | \text{ denote the absolute value} \end{array} \tag{10}$$

Then, we consider the fundamental volatility criterion, denoted by V. It is computed using a standard variance such that at time t

$$\begin{array}{llll} V: & \mathbb{R}^{+} & \rightarrow & \mathbb{R}^{+} \\ & FV_{p_t} & \mapsto & V(FV_{p_t}) = Var\left(FV_{p_t}\right), \ p_t \in \{s_t, f_t\} \end{array} \tag{11}$$

4 Impact of trade frequencies

In this section, we study the impact of trade frequencies on market efficiency and fundamental basis volatility. Market efficiency regarding the fragmentation of orders has already been studied empirically in previous literature (cf. [3], [11], [14], [15]). We provide a theoretical explanation of its impacts. Finally, we show within this framework that fragmentation of orders also impacts fundamental basis volatility.

4.1 Impact of trade frequencies on the market efficiency

Theorem 1. Let SM and FM respectively be a spot market and a futures market under our hypothesis. For $e \neq 0$ and $\omega_s \neq \omega_f$, SM is not an efficient market, and the spot price bias is equal to

$$B\left(s_t\right) = \frac{|-e|}{\omega_f} \sum_{k=1}^{t} \left(1 - \frac{\omega_s}{\omega_f}\right)^k > 0 \tag{12}$$

Proof. See appendix A □

If $e \neq 0$ and $\omega_s \neq \omega_f$, trade frequencies have an impact on spot market efficiency. They generate a bias on the spot price dynamic that results in an adjustment delay in a number of transactions. However, spot market information is released between transactions. The adjustment delay in monetary value is strictly positive.

Remark 1. For $e = 0$, $B\left(s_t\right) = 0 \ \forall \ t \in [1, T]$.
When incoming information has no impact on average on the spot price, an adjustment delay in the number of transactions does not generate a bias in the market. As the released spot market information has on average no impact on monetary value, the adjustment delay in monetary value corresponding to the bias is null.

For $\omega_s = \omega_f$, $t' = t \ \forall \ t \in [1, T]$, we have $E\left(t'\right) = t$; hence, $B\left(s_t\right) = 0 \ \forall \ t \in [1, T]$.
When transactions are synchronized in both the spot and futures markets, there is no adjustment delay in number of transactions on the futures market. Thus, the rational expectations assumption ensures an unbiased spot price.

In reality, the FTF is higher than the STF. Furthermore, commodity prices have a seasonal tendency such that $e \neq 0$ (cf. [2]). Then, the adjustment delay of the spot market can be important.

Corollary 1. Let FM be a futures market under our hypothesis. FM is an unbiased market, and thus it can be an efficient market.

By construction, agents form rational expectations, and information arrives between two transactions in the futures market. They perfectly expect, on average, the fundamental prices. As the futures price always evolves according to the expectations, this leads to an unbiased futures price.

Remark 2. The futures market efficiency is independent from trade frequencies. However, the STF has an indirect impact on stock dynamics and its expectations (cf. equations 2 and 5). One can infer that this indirect impact is, on average, equal on both futures' price and fundamental value dynamics.

4.2 Impact of trade frequencies on the fundamental basis volatility

Theorem 2. Let SM and FM respectively be a spot market and a futures market under our hypothesis. For $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, we have the following properties:

(i) The volatility of the available stock at time t, denoted by $Var(Q_t)$, is impacted by both STF and FTF.

$$Var(Q_t) = tq(\omega_s)^2 \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} \tag{13}$$

(ii) The fundamental basis volatility at time t, denoted by $Var(FV_{f_t} - FV_{s_t})$, is impacted by both STF and FTF.

$$Var(FV_{f_t} - FV_{s_t}) = \left(\frac{a}{\omega_f}\right)^2 ((T_r\omega_f) - t)^2 q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f} \tag{14}$$

Proof. See appendix B □

If $a \neq 0$, $\omega_s \neq \omega_f$, and $t < T$, trade frequencies have an impact on the fundamental basis volatility. They generate uncertainty on the dynamic of the commodity stock. Furthermore, the STF has an impact on quantities exchanged in the spotmarket according to the absolute value of the elasticity ϵ. A representation of the sensitivity of the fundamental basis volatility to trade frequencies according to the elasticity of quantities traded at each transaction on the spot market to the STF is given in figures 5, 6 and 7.

Remark 3. For $a = 0$, $Var(FV_{f_t} - FV_{s_t}) = 0 \; \forall \; t \in [1, T]$. If there is no advantage in detaining stock until maturity, as we do not integrate storage costs, both fundamental basis volatility and fundamental basis are null.

For $\omega_s = \omega_f$, $Var(FV_{f_t} - FV_{s_t}) = 0 \; \forall \; t \in [1, T]$. There are only synchronized transactions in both markets. Consequently, there is no uncertainty; There is a transaction of $q(\omega_s)$ unit(s) of the commodity at each transaction in the futures market.

For $t = T$, $Var(FV_{f_t} - FV_{s_t}) = 0$. As a consequence of the no arbitrage condition, if there is no more time until maturity, the basis must be null and so its volatility.

In reality, the FTF is more important than the STF. Transaction costs on the spot market are significant, and they prevent the STF from increasing. Then, quantities exchanged at each transaction on the spot market are significant. Furthermore, the advantage of detaining stock is a major component of the basis, such that $a \neq 0$. Then, basis fundamental volatility is higher with these market characteristics.

5 Impact of a fragmentation of orders

From theorems 1 and 2, we proved that under the assumptions made, trade frequencies influence the fundamental basis volatility and the spot price bias. We study in this section how this impact evolves when these frequencies increase. We present some simulations and the associated mechanisms. We study the impact of the fragmentation of orders on the spot market and futures market separately as well as the impact of a proportional fragmentation of orders on the two markets. First, the spot price bias (see below subsection 5.1) and, second, the fundamental basis volatility (see below subsection 5.2).

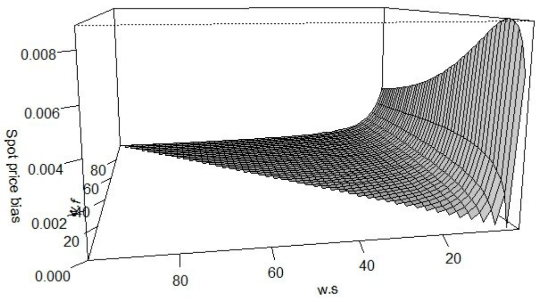

Figure 1: Representation of the spot price bias $B(s_t)$. Parameters used are $e = 0.01$ and $t = 40$.

5.1 Impact of a fragmentation of orders on the spot price bias

Theorem 3. For $e \neq 0$, $a \neq 0$, and $\omega_s \neq \omega_f$, we have the following properties:

(i) If the STF increases, the spot price bias strictly decreases.

(ii) If the FTF increases

- For $\frac{\omega_s}{\omega_f} \in \left]0; 1 - \left(\frac{1}{t+1}\right)^{\frac{1}{t}}\right[\subset \left]0; \frac{1}{2}\right[$, the spot price bias strictly decreases;
- For $\frac{\omega_s}{\omega_f} \in \left]1 - \left(\frac{1}{t+1}\right)^{\frac{1}{t}}; 1\right[\supset \left]\frac{1}{2}; 1\right[$, the spot price bias strictly increases.

(iii) If both STF and FTF increase such that the RTF, equal to $\frac{\omega_s}{\omega_f}$, is unchanged, the spot price bias strictly decreases.

Proof. See appendix C □

An increase in the STF (i.e., fragmentation of orders on the spot market) decreases the average adjustment delay in number of transactions on the futures market (cf. Figure 1; the bias is strictly decreasing when ω_s increases). As the average spot market information impact between two transactions in the futures market is unchanged, the spot price bias decreases in monetary value. This explains the first assertion of theorem 3.

An increase in the FTF (i.e., fragmentation of orders on the futures market) increases the average adjustment delay in number of transactions in the futures market. However, the average spot market information impact between two transactions in the futures market decreases. The first effect overcomes the second one when the RTF is initially sufficiently low. The impact of the FTF on the RTF and thus on the probability of having a transaction in the spot market at each transaction on the futures market is not linear. Figure 1 graphically represents the upper level of the plan, wherein for a given (little) STF, there is an FTF above that the bias decreases. This explains the second assertion of theorem 3.

Remark 4. The above condition depends on the number of transactions in the futures market considered, denoted by t. To have fragmentation in the futures market that reduces the spot price bias, the more we consider a high t, the lower the RTF must be (i.e., $\omega_f >> \omega_s$). With a high initial FTF, an increase in the FTF has a small effect on the RTF. Figures 2, 3 and 4 illustrate the effect of t considered. Looking at the upper part of the plan, the higher t is when $\omega_f >> \omega_s$, the less the spot price bias decreases.

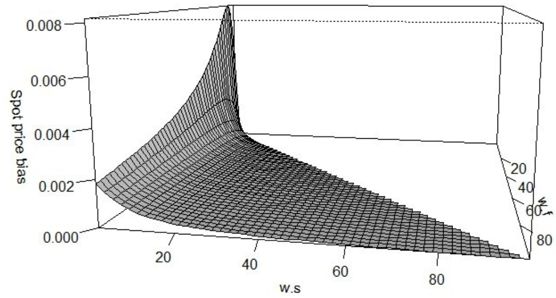

Figure 2: Representation of the spot price bias $B(s_t)$. Parameters used are $e = 0,01$ and $t = 20$.

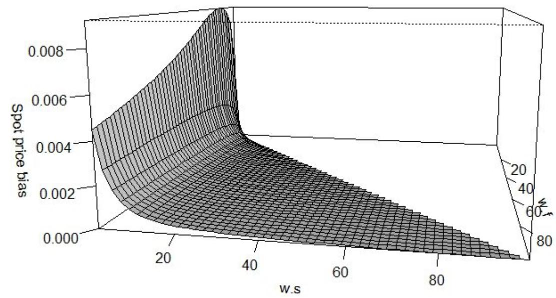

Figure 3: Representation of the spot price bias $B(s_t)$. Parameters used are $e = 0,01$ and $t = 60$.

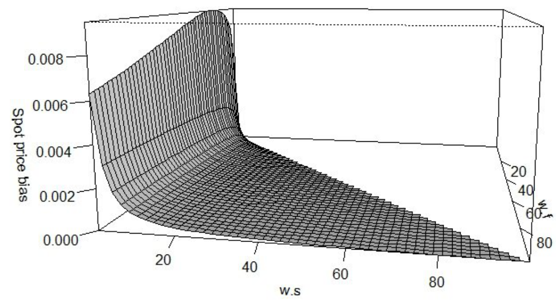

Figure 4: Representation of the spot price bias $B(s_t)$. Parameters used are $e = 0,01$ and $t = 100$.

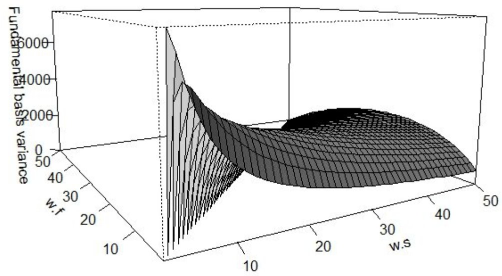

Figure 5: Representation of the fundamental basis volatility $Var\left(FV_{f_t} - FV_{s_t}\right)$. Parameters used are $a = 0.01$, $t = 50$, $T_r = 2500$, and $\epsilon = 0.2$.

A proportional increase in both STF and FTF such that $\frac{\omega_s}{\omega_f}$ is unchanged does not modify the probability of synchronized transaction's occurrence. However, between two transactions in the futures market, the average monetary impact of the information on the futures market strictly decreases. Thus, the spot price bias strictly decreases. This result explains the third assertion of theorem 3. Figure 1 illustrates it. When following a parallel of the first bisector, the RTF is unchanged and the spot price bias decreases.

5.2 Impact of a fragmentation of orders on the fundamental basis volatility

Theorem 4. For $e \neq 0$, $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, we have the following properties:

(i) If the STF increases,

- For $\frac{\omega_s}{\omega_f} \in \left]0, \frac{1}{2}\right[$, the fundamental basis volatility strictly decreases if the following condition is verified:

$$-\epsilon < -\frac{\omega_f - 2\omega_s}{2\left(\omega_f - \omega_s\right)}, \text{ where } -\epsilon \text{ is the elasticity of } q\left(\omega_s\right) \text{ to } \omega_s \text{ (see equation 1)} \tag{15}$$

- For $\frac{\omega_s}{\omega_f} \in \left]\frac{1}{2}, 1\right[$, the fundamental basis volatility strictly decreases.

(ii) If the FTF increases,

- For $\frac{\omega_s}{\omega_f} \in \left]0, \frac{1}{2}\right[$, the fundamental basis volatility strictly decreases if the following condition is verified:

$$-\frac{\omega_f - 2\omega_s}{2\left(\omega_f - \omega_s\right)} < -\frac{t}{T_r\omega_f - t} \tag{16}$$

- For $\frac{\omega_s}{\omega_f} \in \left[\frac{1}{2}, 1\right[$, the fundamental basis volatility increases.

(iii) If both STF and FTF increase such that the RTF, equal to $\frac{\omega_s}{\omega_f}$, is unchanged, the fundamental basis volatility strictly decreases if the following condition is verified:

$$-\epsilon < -\frac{t}{T_r\omega_f - t} \tag{17}$$

Proof. See appendix D □

Remark 5. Let f be a function such that

$$\begin{array}{llll} f: & \mathbb{F}_s \backslash \{\omega_f\} \times \mathbb{F}_f & \rightarrow & \mathbb{R} \\ & (\omega_s, \omega_f) & \mapsto & f(\omega_s, \omega_f) = -\frac{\omega_f - 2\omega_s}{2(\omega_f - \omega_s)} \end{array}$$

It is easy to prove that f is a C1 class function.
For $\frac{\omega_s}{\omega_f} \in \left]0, \frac{1}{2}\right[$, $\frac{\partial f(\omega_s,\omega_f)}{\partial \omega_s} = -\frac{-2(2(\omega_f-\omega_s))-(\omega_f-2\omega_s)(-2)}{(2(\omega_f-\omega_s))^2} = \frac{\omega_f}{2(\omega_f-\omega_s)^2} > 0$. The higher ω_s is, the higher the elasticity of spot quantities traded to ω_s can be to verify equation 27.
For $\frac{\omega_s}{\omega_f} \in \left]0, \frac{1}{2}\right[$, $\frac{\partial f(\omega_s,\omega_f)}{\partial \omega_f} = -\frac{12(\omega_f-\omega_s)-(\omega_f-2\omega_s)2}{4(\omega_f-\omega_s)^2} = -\frac{\omega_s}{2(\omega_f-\omega_s)^2} < 0$. The higher ω_f is, the lower ϵ has to be to verify equation 27.

Remark 6. Let g be a function such that

$$\begin{array}{llll} g: & \mathbb{F}_f \times [1; (T_r \times \omega_f) - 1] & \rightarrow & \mathbb{R} \\ & (\omega_f, t) & \mapsto & g(\omega_f, t) = -\frac{t}{(T_r\omega_f - t)} \end{array} \tag{18}$$

It is easy to prove that g is a C1 class function.
For $\frac{\omega_s}{\omega_f} \in \left]0, \frac{1}{2}\right[$, $\frac{\partial g(\omega_f,t)}{\partial \omega_f} = \frac{-t(-T_r)}{(T_r\omega_f-t)^2} > 0$. Referring to remark 5, the left term of the condition given by equation 16 is decreasing when ω_f increases. The higher ω_f is, the less restrictive is the condition given by equation 16.
For $\frac{\omega_s}{\omega_f} \in \left]0, \frac{1}{2}\right[$, $\frac{\partial g(\omega_f,t)}{\partial t} = -\frac{1(T_r\omega_f-t)-t(-1)}{(T_r\omega_f-t)^2} = -\frac{T_r\omega_f}{(T_r\omega_f-t)^2} < 0$. The higher is the number of transactions on the futures market t, the more restrictive is the condition of equation 16.

An increase in the STF (i.e., fragmentation of orders on the spot market) reduces the volatility of the fundamental basis by two mechanisms if the RTF is superior to $\frac{1}{2}$ (cf. Figure 5; the fundamental basis volatility curve decreases with an increasing STF when $\frac{\omega_s}{\omega_f} > \frac{1}{2}$). First, the volatility of the occurrence of transactions in the spot market decreases. Second, quantities traded per transaction in the spot market are inferior, smoothing the stock dynamic. If the RTF is inferior to $\frac{1}{2}$, the first mechanism does not hold but the second still holds. The volatility of the occurrence of transactions on the spot market increases. The "smoothing per trade exchanged quantities" effect must overcome the "increasing occurrence of transaction's volatility" effect. To smooth the stock dynamic enough, the elasticity must be under a threshold (sufficiently high in absolute value) such that quantities exchanged per trade sufficiently decrease. This condition is given by equation 15 and illustrated in Figures 6 and 7. For $\epsilon = 0,8$ (cf. Figure 7), the elasticity is high in absolute value, and the volatility is decreasing with an increasing ω_s. On the contrary, for $\epsilon = 0,2$ (cf. figure 5), the volatility is increasing when the STF is originally low. These results explain the first assertion of theorem 4. A study of the condition is presented in Remark 5, and highlights the following remark.

Remark 7. This condition depends on the initial levels of trade frequencies. The higher the STF initially is, the lower are the quantities exchanged per transaction in the spot market. Thus, the higher the elasticity can be (the lower in absolute value) such that quantities traded will decrease enough, and overcome the increasing volatility in the stock dynamic. Near the first bisector, the fundamental basis volatility decreases with an increasing STF for all values of ϵ (cf. Figures 5, 6, and 7).
Based on the same reasoning, the higher the FTF is, the more restrictive is the threshold. The increasing part of the curve for a given STF is more important when the FTF increases. To overcome this increasing effect, the elasticity must be sufficiently high in absolute value to ensure a sufficient decrease in quantities exchanged per trade on the spot market. The analytic condition is given in equation 16. In Figures 5, 6, and 7, the plan twists according to the value of ϵ, representing this threshold. We observe that the higher ϵ is, the less the plan is twisted starting from low values of FTF.

An increase in the FTF (i.e., fragmentation of orders in the futures market) strictly increases the volatility of the fundamental basis by two mechanisms if the RTF is superior to $\frac{1}{2}$ (cf.

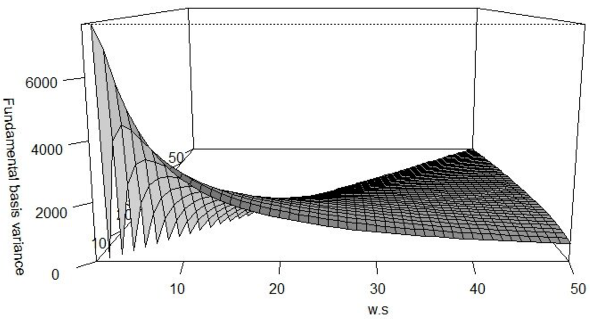

Figure 6: Representation of the fundamental basis volatility $Var\left(FV_{f_t} - FV_{s_t}\right)$. Parameters used are $a = 0.01$, $t = 50$, $T_r = 2500$ and $\epsilon = 0,4$.

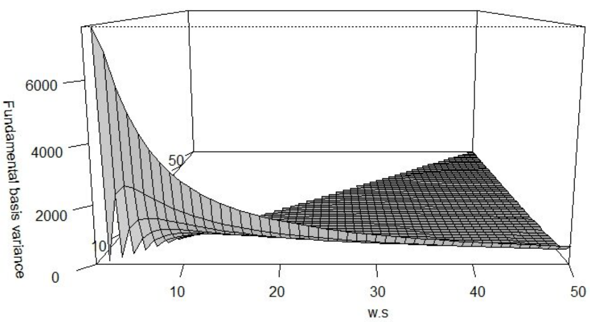

Figure 7: Representation of the fundamental basis volatility $Var\left(FV_{f_t} - FV_{s_t}\right)$. Parameters used are $a = 0.01$, $t = 50$, $T_r = 2500$ and $\epsilon = 0,8$.

Figure 5; the fundamental basis volatility plan increases with an increasing ω_f when $\frac{\omega_s}{\omega_f} > \frac{1}{2}$). The volatility of the occurrence of synchronized transactions increases. Furthermore, a transaction in the spot market long before the maturity has more impact on the convenience yield than if it happens at maturity, as a consequence of the no arbitrage value and the time remaining until maturity. Here, we have horizon t, expressed by a number of transactions in the futures market, to study the fundamental basis volatility. An increase in the FTF reduces the real time at which the first t transactions are made. Each transaction takes place earlier, and there is more real time remaining until maturity. An increase in the FTF gives more weight to the first t possible transactions in the spot market. It consequently increases the fundamental basis volatility. However, when the RTF is inferior to $\frac{1}{2}$, an increasing FTF decreases the volatility of synchronized transaction's occurrence. This effect overcomes the real-time scale effect if the condition given by equation 16 is verified. Figures 8, 9 and 10 illustrate the latest condition. These results explain the second assertion of theorem 4.

A proportional increase in the STF and the FTF such that $\frac{\omega_s}{\omega_f}$ is unchanged does not modify the occurrence of synchronized transaction's volatility. However, for a given *real time*, there are more transactions in the futures' market. When considering a given number of transactions in the futures market t, it reduces the "real time" considered. Therefore, each potential evolution of the stock has more impact on the convenience yield, since the real time remaining between the first t transactions and maturity is superior. The increase of the STF reduces quantities traded per transaction in the spot market such that it reduces the stock dynamic volatility at an unchanged time horizon. To ensure that this effect overcomes the higher importance of stock evolution until maturity, the elasticity of quantities traded per transaction on the spot market to the STF must be under the threshold given by equation 17 (i.e., sufficiently high in absolute value). This result explain the third assertion of theorem 4. A study of this condition is presented in Remark 6.

Remark 8. Equation 17 underlies the importance of the choice in the number of transactions considered to study the effect of a proportional increase in the STF and the FTF.
This threshold firstly depends on t. The nearer to maturity t is, the more restrictive is the condition given by equation 17. The impact of the modification of the real-time scale is higher if the initial real time considered is important. Consequently, the longest the initial horizon is, the higher the quantity effect has to be.
Second, this threshold depends on the FTF. The modification of the "time scale effect" relies on the initial level of the FTF. Hence, a study of the condition shows that the higher the FTF is, the less restrictive is the condition given by equation 17. If the t transactions considered occur with a high FTF, only a little real time is considered. Consequently, an increase in the FTF reduces the real time considered but not significantly, and the quantity effect is not strong.

Remark 9. The condition given by equation 17 is independent from the RTF. Since the increases in the STF and the FTF are proportional such that the RTF is unchanged, there is no effect on the volatility of occurrence of synchronized transactions. Then, the only condition to ensure a reducing fundamental basis volatility relies on the elasticity of quantities traded at each transaction on the spot market to the STF and the initial levels of both STF and FTF.

The latest result has strong implications. In subsection 5.1, we demonstrated that a proportional fragmentation reduces spot market bias. One could think that proportional fragmentation can reduce fundamental basis volatility. However, this intuition is unverified if the decreasing quantities exchanged per transaction in the spot market are insufficient. This strong result shows the extent to which the spot market and its structure influences the futures market and can reduce the benefit of proportional fragmentation in both markets.

The elasticity of quantities traded per transaction in the spot market to the STF is constant. Otherwise, we could face a condition (given by equation 17) depending on the current level of the

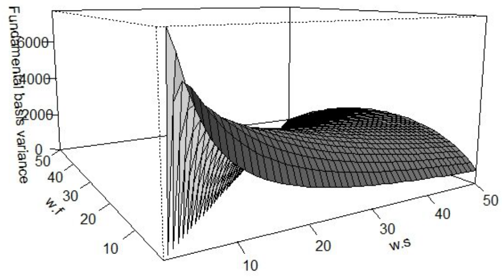

Figure 8: Representation of the fundamental basis volatility $Var\left(FV_{f_t} - FV_{s_t}\right)$. Parameters used are $a = 0.01$, $\epsilon = 0.2$, $T_r = 250$, and $t = 25$.

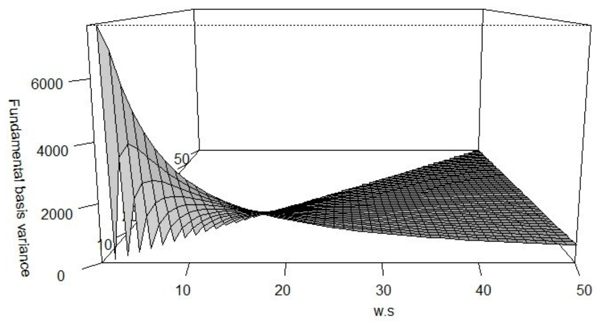

Figure 9: Representation of the fundamental basis volatility $Var\left(FV_{f_t} - FV_{s_t}\right)$. Parameters used are $a = 0.01$, $\epsilon = 0.2$, $T_r = 250$, and $t = 100$.

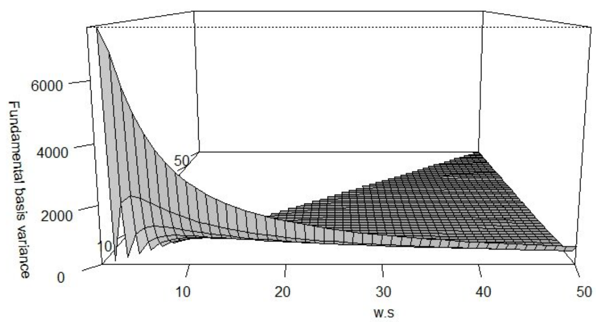

Figure 10: Representation of the fundamental basis volatility $Var\left(FV_{f_t} - FV_{s_t}\right)$. Parameters used are $a = 0.01$, $\epsilon = 0.2$, $T_r = 250$, and $t = 200$.

STF. Based on the same reasoning, one could assume an elasticity of quantities traded per transaction in the spot market depending on the FTF. Spot and futures markets are linked, and we can imagine that when there are more trades in the futures market, producers and buyers of the commodity adjust their orders' frequency and thus quantities traded. A further extension of this work could be to study the evolution of these results according to the form of the q function.

6 Existence of an optimal STF and FTF

A policymaker's objectives in commodity markets are to minimize the fundamental basis volatility and increase market efficiency in both spot and futures markets. As the futures price is unbiased (cf. corollary 1), the policymaker focuses on spot market efficiency. We consider here that trade frequencies are the policymaker's tools, and thus the policymaker can implement a fixed pricing and transaction frequency. This section has major issues. At some point, it can question the commodity futures pricing by limit order book if the optimal trade frequencies are inferior to the actual FTF where a limit order book pricing operates.

Definition 1. We define a utility function U in the following way:

$$\begin{array}{llllll} U: & \mathbb{F}_s \times \mathbb{F}_f & \xrightarrow{U_1} & \mathbb{R}^+ \times \mathbb{R}^+ & \xrightarrow{U_2} X & \mathbb{R}^- \\ & (\omega_s, \omega_f) & \longmapsto & (B(s_t), V(FV_{f_t} - FV_{s_t})) & \longmapsto & U(B(s_t), V(FV_{f_t} - FV_{s_t})) \end{array} \tag{19}$$

Hence, we have $U(\omega_s, \omega_f) = U_2 \circ U_1(\omega_s, \omega_f)$, verifying the following two properties:

(i) $U_2(0,0) = 0$;

(ii) $\frac{\partial U_2\left(B(s_t), V\left(FV_{f_t} - FV_{s_t}\right)\right)}{\partial B(s_t)} < 0$, and $\frac{\partial U_2\left(B(s_t), V\left(FV_{f_t} - FV_{s_t}\right)\right)}{\partial V\left(FV_{f_t} - FV_{s_t}\right)} < 0$.

We assume that a policymaker, to maximize the market efficiency and fundamental basis volatility, uses the utility function U as defined above. Properties (i) and (ii) above correspond respectively to the following natural insights:

(i) If the spot price bias and fundamental basis volatility are both null, we assume that U_2 takes its maximal value on $\mathbb{R}^-$: 0;

(ii) If either the spot price bias or fundamental basis volatility increases, the policymaker's utility decreases.

The policymaker consequently has the following maximization program:

$$\begin{cases} \max_{\omega_s, \omega_f} U(\omega_s, \omega_f) = U\left(\frac{|-e|}{\omega_f} \sum_{k=1}^{t} \left(1 - \frac{\omega_s}{\omega_f}\right)^k, \left(\frac{a}{\omega_f}\right)^2 (T_r \omega_f - t)^2 q(\omega_s)^2 t \left(1 - \frac{\omega_s}{\omega_f}\right) \frac{\omega_s}{\omega_f}\right) \\ \text{s.t. } \omega_f \geq \omega_s \end{cases} \tag{20}$$

Theorem 5. For $e \neq 0$, $a \neq 0$, $t < T$, and $\mathbb{F}_s \cap \mathbb{F}_f \neq \emptyset$, a policymaker whose utility function is U has an optimal solution and an optimal solution by limit:

(i) A policymaker does not have to fix the trade frequencies of both spot and futures markets but has to synchronize them. This implies an RTF equal to 1 (the subset of optimal solutions is $O = \left\{(\omega_s, \omega_f) \in (\mathbb{F}_s \cap \mathbb{F}_f)^2 : \frac{\omega_s}{\omega_f} = 1\right\}$);

(ii) For any value of ω_s except $\omega_s \to +\infty$, a policymaker can allow infinite fragmentation in the futures contract. This implies an RTF converging to 0 (the subset of optimal solutions by limit is
$O_{bl} = \left\{(\omega_s; \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0, \omega_f \mapsto +\infty\right\}$).

Proof. See appendix E □

This result implies that independently of the absolute value of both STF and FTF, the synchronization of the trade frequencies ensures minimal fundamental basis volatility and minimal spot price bias.

Remark 10. If $\omega_s = \omega_f$, there is no uncertainty on the stock dynamic and no unexpected elements on the basis (cf. equation 5; $E\left(\widetilde{s_{t+1}}|\Phi_t\right) = s_{t+1} \; \forall \; t$). With synchronized transactions and rational expectations, the basis is always equal to its fundamental value.

This result implies that the links between a spot market and a futures market of a commodity responding to our framework prevent the STF and the FTF from being disconnected in order to achieve market efficiency and minimal fundamental basis volatility. Only more fragmentation on the futures market (i.e., an increase in the FTF) can lead to an increasing spot price bias and increasing fundamental basis volatility (cf. section 5), except if the fragmentation is relatively infinite. There are no optimal values for the STF and FTF, but there are co-dependent conditions.

Theorem 6. For $e \neq 0$, $a \neq 0$, and $t < T$, a policymaker whose utility function is U has optimal solutions by limit relying on the following market characteristic:

(i) For $-\epsilon > -\frac{1}{2}$, there is no conditional optimal solution different from unconditional optimal solutions to the policymaker's utility maximization program given in theorem 5;

(ii) For $-\epsilon < -\frac{1}{2}$, the policymaker's utility is maximized by limit for all values in the following subset: $O_{bl}^{\epsilon > \frac{1}{2}} = \left\{(\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 1,\, \omega_s \mapsto +\infty,\, \omega_f \mapsto +\infty\right\}$;

(iii) For $-\epsilon = -\frac{1}{2}$, there is no conditional optimal solution different from unconditional optimal solutions of the policymaker's utility maximization program given in theorem 5.

Proof. See appendix F □

There is also a subset of optimal values by limit for the policymaker that is conditional to market characteristics (value of ϵ). The analytical proof of this result is presented in theorem 6. If $-\epsilon \geq -\frac{1}{2}$, there is no conditional optimal solution different from the unconditional optimal solutions of the policymaker's utility maximization program. However, if $-\epsilon < -\frac{1}{2}$, the optimal solution by limit for the policymaker can be to implement the centralization of orders by brokers and reduce transaction costs as much as possible in the spot market such that $\frac{\omega_s}{\omega_f} \to 1$ and $\omega_f \to +\infty$ independently of the FTF (the RTF is converging to 1). However, despite the possibility of having high elasticity, implementing a market structure such that $\omega_s \to +\infty$ is impossible since there are structural transaction costs. We can qualify this conditional subset of optimal solutions as a limit case.

7 Conclusion

This work has major implications. It shows that commodity futures pricing by limit order book is not an efficient policy regarding the two objectives when fragmentation in the futures market is not infinite. Although previous empirical studies argued that fragmentation allows for better market efficiency, their finding does not necessarily contradict our results (depending on the initial RTF). Our results recommend the implementation of a fixing price operating at the same frequency as the trade of the commodity in the associated spot market. Indeed, a relatively infinite fragmentation in futures market, the only other optimal solution independent from the market characteristics, seems compromised. This can also be understood as an arbitrage between liquidity in the futures market and achievement of the two objectives; We showed in section 5 that proportional increase in both frequencies could lead to increasing fundamental basis volatility according to the value of ϵ. A way to improve liquidity and market efficiency of the spot market could be a regulation of the spot market structure to facilitate the transactions in this market, in order to allow an increase in the absolute value of ϵ. To conclude, the simultaneous determination of spot and futures prices is optimal and independent from the market characteristics. We are aware that this model presents some limits. Some extensions of this work could be made depending on the form of the q function as well as in the implementation of several commodities (or other markets in general). Finally, we could also integrate the possibility of speculation in the spot market, such that trade in the spot market would not necessarily imply commodity consumption.

References

[1] Michail Anthropelos, Michael Kupper, and Antonis Papapantoleon. An Equilibrium Model for Spot and Forward Prices of Commodities. *Mathematics of Operations Research*, 43(1):152–180, February 2018.

[2] Janis Back and Marcel Prokopczuk. Commodity price dynamics and derivative valuation: A review. *International Journal of Theoretical and Applied Finance*, 16(06):1350032, September 2013.

[3] Paul Bennett and Li Wei. Market structure, fragmentation, and market quality. *Journal of Financial Markets*, 9(1):49–78, February 2006.

[4] Fischer Black. The pricing of commodity contracts. *Journal of Financial Economics*, 3(1):167–179, January 1976.

[5] Fischer Black and Myron Scholes. The Pricing of Options and Corporate Liabilities. *Journal of Political Economy*, 81(3):637–654, 1973.

[6] Beatrice Boehmer and Ekkehart Boehmer. Trading your neighbor's ETFs: Competition or fragmentation? *Journal of Banking & Finance*, 27(9):1667–1703, September 2003.

[7] Eugene F. Fama. Efficient capital markets: II. *The journal of finance*, 46(5):1575–1617, 1991.

[8] Eugene F. Fama, Lawrence Fisher, Michael C. Jensen, and Richard Rol. Efficient capital markets: a review of theory and empirical work. *Journal of Finance*, 25(2):383–417, 1970.

[9] Eugene F. Fama and Merton H. Miller. *The theory of finance.* Dryden Press, Hindsdale (Ill.), Etats-Unis d'Amérique, 1972.

[10] Nicholas Kaldor. Speculation and Economic Stability. *The Review of Economic Studies*, 7(1):1–27, 1939.

[11] Amy Kwan, Ronald Masulis, and Thomas H. McInish. Trading rules, competition for order flow and market fragmentation. *Journal of Financial Economics*, 115(2):330–348, February 2015.

[12] Robert C. Merton. Theory of Rational Option Pricing. *The Bell Journal of Economics and Management Science*, 4(1):141, 1973.

[13] Giancarlo Moschini and David A Hennessy. Uncertainty, risk aversion, and risk management for agricultural producers. *Handbook of agricultural economics*, 1:87–153, 2001.

[14] Maureen O'Hara and Mao Ye. Is market fragmentation harming market quality? *Journal of Financial Economics*, 100(3):459–474, June 2011.

[15] Maureen O'Hara. High frequency market microstructure. *Journal of Financial Economics*, 116(2):257–270, May 2015.

[16] Patrice Poncet, Roland Portait, and Serge Hayat. *Mathématiques financieres: Evaluation des actifs et analyse du risque.* Dalloz, 1996.

[17] Jerome L. Stein. The simultaneous determination of spot and futures prices. *The American Economic Review*, pages 1012–1025, 1961.

[18] Holbrook Working. Futures trading and hedging. In *The Economics of Futures Trading*, pages 68–82. Springer, 1976.

[19] S. C. P. Yam and W. Zhou. Optimal Liquidation of Child Limit Orders. *Mathematics of Operations Research*, 42(2):517–545, May 2017.

A Proof of theorem 1

Proof. Using equation 8, we have

$$\begin{array}{rcl} s_{(1)'} &=& s_0 + \mathbb{1}_{\{1\}}(1')\left[I_{]0',0]} + \frac{e}{\omega_f}\right] \\ s_{(2)'} &=& s_1 + \mathbb{1}_{\{2\}}(2')\left[I_{]1',1]} + \frac{e}{\omega_f}\right] \\ s_{(2)'} &=& s_0 + \mathbb{1}_{\{1\}}(1')\left[I_{]0',0]} + \frac{e}{\omega_f}\right] + \mathbb{1}_{\{2\}}(2')\left[I_{]1',1]} + \frac{e}{\omega_f}\right] \end{array}$$

Hence, we express $s_{(t-1)'}$ according to s_0 as follows:

$$s_{(t-1)'} = s_0 + \sum_{k=1}^{(t-1)'} \mathbb{1}_{\{k\}}(k')\left[I_{](k-1)',k-1]} + \frac{e}{\omega_f}\right] \tag{21}$$

We express the dynamic of s_t according to s_0 using equations 8 and 21:

$$s_t = s_0 + \sum_{k=1}^{t} \mathbb{1}_{\{k\}}(k')\left[I_{](k-1)',k-1]} + \frac{e}{\omega_f}\right]$$

Then, its average value is

$$\begin{array}{rcl} E(s_t) &=& E\left(s_0 + \sum_{k=1}^{t} \mathbb{1}_{\{k\}}(k')\left[I_{](k-1)',k-1]} + \frac{e}{\omega_f}\right]\right) = \\ &=& s_0 + \sum_{k=1}^{t} E\left(\mathbb{1}_{\{k\}}(k')\left[I_{](k-1)',k-1]} + \frac{e}{\omega_f}\right]\right) \\ &=& s_0 + \sum_{k=1}^{t} E\left(\mathbb{1}_{\{k\}}(k')\right) E\left(I_{](k-1)',k-1]} + \frac{e}{\omega_f}\right) \\ &=& s_0 + \frac{\omega_s}{\omega_f}\frac{e}{\omega_f}\sum_{k=1}^{t}\left(k - 1 - E\left((k-1)'\right) + 1\right) \\ E(s_t) &=& s_0 + \frac{\omega_s}{\omega_f}\frac{e}{\omega_f}\sum_{k=1}^{t}\left(k - E\left((k-1)'\right)\right) \end{array}$$

We compute the average spot fundamental value at time t:

$$E(FV_{s_t}) = E\left(s_0 + I_{]0,t]}\right) = s_0 + \frac{e}{\omega_f}t$$

We have the following expression of the bias:

$$\begin{array}{rcl} B(s_t) &=& \left|s_0 + \frac{\omega_s}{\omega_f}\frac{e}{\omega_f}\left[\sum_{k=1}^{t}\left(k - E\left[(k-1)'\right]\right)\right] - s_0 - \frac{e}{\omega_f}t\right| \\ B(s_t) &=& \left|\frac{e}{\omega_f}\frac{\omega_s}{\omega_f}\sum_{k=1}^{t}\left(k - \frac{\omega_f}{\omega_s} - E\left[(k-1)'\right]\right)\right| \end{array} \tag{22}$$

Focusing on the t' variable, we have $Pr(t'=k) = \frac{\omega_s}{\omega_f}\left(1-\frac{\omega_s}{\omega_f}\right)^{t-k}$ $\forall\, k \in [1;t]$ and $Pr(t'=0) = \left(1-\frac{\omega_s}{\omega_f}\right)^{t}$. Hence, we compute

$$\begin{array}{rcl} E(t') &=& \sum_{k=0}^{t} kPr(t'=k) = \sum_{k=1}^{t} k\frac{\omega_s}{\omega_f}\left(1-\frac{\omega_s}{\omega_f}\right)^{t-k} \\ E(t') &=& -\frac{\omega_s}{\omega_f}\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}\sum_{k=0}^{t} -k\left(1-\frac{\omega_s}{\omega_f}\right)^{-k-1} \end{array}$$

Let $z=\left(1-\frac{\omega_s}{\omega_f}\right)$, $0\le z<1$. For a fixed t, we have

$$\begin{aligned}\sum_{k=0}^{t}-kz^{-k-1} &= \sum_{k=0}^{t}\frac{d\left[\left(z^{-1}\right)^k\right]}{dz}=\frac{d\left[\sum_{k=0}^{t}\left(z^{-1}\right)^k\right]}{dz}=\frac{d\left[\sum_{k=0}^{t}\left(z^{-1}\right)^k\right]}{dz}\\ &= \frac{d\left[\left(z^{-1}\right)^0\frac{1-\left(z^{-1}\right)^{t+1}}{1-z^{-1}}\right]}{dz}=\frac{(t+1)z^{-t}-tz^{-(t+1)}-1}{(z-1)^2}\end{aligned}$$

Then, we have

$$\begin{aligned}E\left(t'\right) &= -\frac{\omega_s}{\omega_f}\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}\frac{(t+1)\left(1-\frac{\omega_s}{\omega_f}\right)^{-t}-t\left(1-\frac{\omega_s}{\omega_f}\right)^{-(t+1)}-1}{\left(\frac{\omega_s}{\omega_f}\right)^2}\\ &= \frac{-(t+1)\left(1-\frac{\omega_s}{\omega_f}\right)+t+\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}}{\left(\frac{\omega_s}{\omega_f}\right)}=\frac{-t-1+t\frac{\omega_s}{\omega_f}+\frac{\omega_s}{\omega_f}+t+\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}}{\left(\frac{\omega_s}{\omega_f}\right)}\\ E\left(t'\right) &= t+\frac{-1+\frac{\omega_s}{\omega_f}+\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}}{\left(\frac{\omega_s}{\omega_f}\right)}=t-\left(1-\frac{\omega_s}{\omega_f}\right)\frac{1-\left(1-\frac{\omega_s}{\omega_f}\right)^t}{1-\left(1-\frac{\omega_s}{\omega_f}\right)}\end{aligned}$$

This can be rewritten as follows:

$$E\left(t'\right)=t-\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k \tag{23}$$

Remark 11. The D.R.V t' can be decomposed into two parts. First, t is the maximal value of t' (since $t'\lessdot t$). The second part is stochastic and can be assimilated to a truncated geometric law with t experiences – maximum – starting from t in backwardation and ending to 0. Hence, its mean is equal to its maximal value minus the average delay for having a synchronized transaction looking backward in number of transactions on the futures market.

Replacing the value of equation 23 in 22, we obtain the following expression of the bias:

$$\begin{aligned}B\left(s_t\right) &= \left|\frac{e}{\omega_f}\frac{\omega_s}{\omega_f}\sum_{k=1}^{t}\left(k-\frac{\omega_f}{\omega_s}-\left[k-1-\sum_{j=1}^{k-1}\left(1-\frac{\omega_s}{\omega_f}\right)^j\right]\right)\right|\\ &= \left|\frac{e}{\omega_f}\frac{\omega_s}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_f}{\omega_s}+\sum_{j=1}^{k-1}\left(1-\frac{\omega_s}{\omega_f}\right)^j\right)\right|\\ &= \left|\frac{-e}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}-\frac{\omega_s}{\omega_f}\sum_{j=1}^{k-1}\left(1-\frac{\omega_s}{\omega_f}\right)^j\right)\right|\\ &= \left|\frac{-e}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}-\frac{\omega_s}{\omega_f}\left(1-\frac{\omega_s}{\omega_f}\right)\frac{1-\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}}{1-\left(1-\frac{\omega_s}{\omega_f}\right)}\right)\right|\\ &= \left|\frac{-e}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}-\left(1-\frac{\omega_s}{\omega_f}\right)\left[1-\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\right]\right)\right|\\ B\left(s_t\right) &= \frac{|-e|}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k\end{aligned}$$

For $e\neq 0$ and $\omega_s\neq\omega_f$, $B\left(s_t\right)>0$. Hence, the spot price is biased, and SM cannot be an efficient market. $\square$

B Proof of theorem 2

Proof. First, we compute $Var\left(Q_t\right)$ and show that it depends on parameters ω_s and ω_f.

$$\begin{aligned}
Var\left(Q_t\right) &= Var\left(Q_0 - q\left(\omega_s\right)\sum_{k=1}^{t}\mathbb{1}_{\{t\}}\left(t'\right)\right)\\
&= q\left(\omega_s\right)^2 Var\left(\sum_{k=1}^{t}\mathbb{1}_{\{t\}}\left(t'\right)\right) = q\left(\omega_s\right)^2\sum_{k=1}^{t} Var\left(\mathbb{1}_{\{t\}}\left(t'\right)\right)\\
Var\left(Q_t\right) &= q\left(\omega_s\right)^2\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)\frac{\omega_s}{\omega_f} = q\left(\omega_s\right)^2 t\left(1-\frac{\omega_s}{\omega_f}\right)\frac{\omega_s}{\omega_f}
\end{aligned}$$

The expression of $Var\left(Q_t\right)$ depends on ω_s and ω_f, which proves the first assertion. Then, we compute the fundamental variance of the spread between the futures price and spot price. We show that it also depends on parameters ω_s and ω_f.

$$\begin{aligned}
V\left(FV_{f_t} - FV_{s_t}\right) &= Var\left(CY_t\right) = Var\left(\frac{a}{\omega_f}\left(T-t\right)\left(Q_{op}-Q_t\right)\right)\\
V\left(FV_{f_t} - FV_{s_t}\right) &= \left(\frac{a}{\omega_f}\right)^2\left(\left(T_r\omega_f\right)-t\right)^2 Var\left(Q_{op}-Q_t\right) = \left(\frac{a}{\omega_f}\right)^2\left(\left(T_r\omega_f\right)-t\right)^2 Var\left(Q_t\right)
\end{aligned}$$

Knowing the value of $Var\left(Q_t\right)$ from the first assertion's proof above, we obtain the expression of equation 14.

For $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, $V\left(FV_{f_t} - FV_{s_t}\right) > 0$.
The expression of $V\left(FV_{f_t} - FV_{s_t}\right)$ depends on ω_s and ω_f, which proves the second assertion. □

C Proof of theorem 3

Proof. To prove the first assertion, we calculate the derivation of the spot price bias according to ω_s, and we show that it is negative. Using equation A we have

$$\begin{aligned}
\frac{\partial B(s_t)}{\partial\omega_s} &= \frac{\partial\left[\frac{|-e|}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k\right]}{\partial\omega_s}\\
\frac{\partial B(s_t)}{\partial\omega_s} &= \frac{|-e|}{\omega_f}\left[\sum_{k=1}^{t} k\left(\frac{-1}{\omega_f}\right)\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\right] < 0
\end{aligned}$$

Hence, first assertion is proved.

To prove the second assertion, we calculate the derivation of the spot price bias according to ω_f, and we study its sign. Using equation A we have

$$\begin{aligned}
\frac{\partial B(s_t)}{\partial\omega_f} &= \frac{\partial\left[\frac{|-e|}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k\right]}{\partial\omega_f}\\
&= |-e|\,\frac{\partial\left[\frac{1}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k\right]}{\partial\omega_f}\\
&= |-e|\left[-\frac{1}{\omega_f{}^2}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k + \frac{1}{\omega_f}\sum_{k=1}^{t} k\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\frac{\omega_s}{\omega_f{}^2}\right]\\
&= \frac{|-e|}{\omega_f{}^2}\left[-\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^k + \sum_{k=1}^{t} k\left(\frac{\omega_s}{\omega_f}\right)\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\right]\\
&= \frac{|-e|}{\omega_f{}^2}\left[-\left(1-\frac{\omega_s}{\omega_f}\right)\frac{1-\left(1-\frac{\omega_s}{\omega_f}\right)^t}{1-\left(1-\frac{\omega_s}{\omega_f}\right)} + \right.\\
&\quad \left.\frac{\omega_s}{\omega_f}\left[\frac{\left(1-(t+1)\left(1-\frac{\omega_s}{\omega_f}\right)^t\right)\left(1-\left(1-\frac{\omega_s}{\omega_f}\right)\right)+\left(\left(1-\frac{\omega_s}{\omega_f}\right)-\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}\right)}{\left(1-\left(1-\frac{\omega_s}{\omega_f}\right)\right)^2}\right]\right]\\
&= \frac{|-e|}{\omega_f\omega_s}\left[-\left(\left(1-\frac{\omega_s}{\omega_f}\right)-\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}\right)\right.\\
&\quad \left.+\left(1-(t+1)\left(1-\frac{\omega_s}{\omega_f}\right)^t\right)\left(\frac{\omega_s}{\omega_f}\right)+\left(\left(1-\frac{\omega_s}{\omega_f}\right)-\left(1-\frac{\omega_s}{\omega_f}\right)^{t+1}\right)\right]\\
\frac{\partial B(s_t)}{\partial\omega_f} &= \frac{|-e|}{\omega_f{}^2}\left[1-(t+1)\left(1-\frac{\omega_s}{\omega_f}\right)^t\right]
\end{aligned} \tag{24}$$

Equation 24 is strictly negative if and only if

$$\begin{array}{lll} & 1-(t+1)\left(1-\frac{\omega_s}{\omega_f}\right)^t & < & 0 \\ \Leftrightarrow & 1 & < & (t+1)\left(1-\frac{\omega_s}{\omega_f}\right)^t \\ \Leftrightarrow & \frac{\omega_s}{\omega_f} & < & 1-\left(\frac{1}{t+1}\right)^{\frac{1}{t}} \end{array}$$

Let h be a function such that

$$\begin{array}{llll} h: & [1;T_r\times\omega_f]\subset\mathbb{N} & \rightarrow & \mathbb{R} \\ & t & \mapsto & h(t)=1-\left(\frac{1}{t+1}\right)^{\frac{1}{t}} \end{array} \tag{25}$$

It is easy to prove that h is a C1 class function. Then, $\frac{dh(t)}{dt}=-\left(-\frac{1}{t^2}\right)\left(\frac{1}{t+1}\right)^{\frac{1}{t}-1}\frac{-1\times 1}{(t+1)^2}=-\frac{1}{t^2(t+1)^2}\left(\frac{1}{t+1}\right)^{\frac{1-t}{t}}<0$. The maximal value of h is $h(1)=\frac{1}{2}$. The minimal value of h is then $\lim\limits_{t\rightarrow T_r\times\omega_f\rightarrow\infty}h(t)=1-0^0=0$. This proves the second assertion.

To prove the last assertion, we compute the total derivation of spot price bias and study its sign. A proportional increase of the frequencies implies that the RTF is unchanged such that $\frac{\omega_s+d\omega_s}{\omega_f+d\omega_f}=\frac{\omega_s}{\omega_f}\Leftrightarrow\omega_f(\omega_s+d\omega_s)=\omega_s(\omega_f+d\omega_f)\Leftrightarrow d\omega_f=\frac{\omega_f d\omega_s}{\omega_s}$. This gives

$$\begin{array}{lll} d(B(s_t)) & = & \frac{\partial(B(s_t))}{\partial\omega_s}d\omega_s+\frac{\partial(B(s_t))}{\partial\omega_f}d\omega_f \\ & = & \frac{|-e|}{\omega_f}\left[\sum_{k=1}^t k\left(\frac{-1}{\omega_f}\right)\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\right]d\omega_s \\ & & +\frac{|-e|}{{\omega_f}^2}\left[-\sum_{k=1}^t\left(1-\frac{\omega_s}{\omega_f}\right)^k+\sum_{k=1}^t k\left(\frac{\omega_s}{\omega_f}\right)\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\right]\frac{\omega_f}{\omega_s}d\omega_s \\ & = & \frac{|-e|}{{\omega_f}^2}d\omega_s\left[-\sum_{k=1}^t k\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}-\frac{\omega_f}{\omega_s}\sum_{k=1}^t\left(1-\frac{\omega_s}{\omega_f}\right)^k+\sum_{k=1}^t k\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1}\right] \\ d(B(s_t)) & = & -\frac{|-e|}{\omega_s\omega_f}d\omega_s\sum_{k=1}^t\left(1-\frac{\omega_s}{\omega_f}\right)^k<0 \end{array}$$

This proves the last assertion. □

D Proof of theorem 4

Proof. To prove the first assertion, we calculate the derivation of the fundamental basis volatility according to ω_s, and we study its sign. Using equation 14 we have

$$\begin{array}{lll} \frac{\partial V\left(FV_{f_t}-FV_{s_t}\right)}{\partial\omega_s} & = & \frac{\partial\left[\left(\frac{a}{\omega_f}\right)^2\left((T_r\omega_f)-t\right)^2 q(\omega_s)^2 t\left(1-\frac{\omega_s}{\omega_f}\right)\frac{\omega_s}{\omega_f}\right]}{\partial\omega_s} \\ & = & \left(\frac{a}{\omega_f}\right)^2\left((T_r\omega_f)-t\right)^2 t\frac{1}{\omega_f}\frac{\partial\left[q(\omega_s)^2\omega_s-\frac{q(\omega_s)^2{\omega_s}^2}{\omega_f}\right]}{\partial\omega_s} \\ & = & \left(\frac{a}{\omega_f}\right)^2\left((T_r\omega_f)-t\right)^2 t\frac{1}{\omega_f}\left[q(\omega_s)^2+2\omega_s q(\omega_s)\frac{dq(\omega_s)}{d\omega_s}\right. \\ & & \left.-\frac{1}{\omega_f}\left(2\omega_s q(\omega_s)^2+2{\omega_s}^2 q(\omega_s)\frac{dq(\omega_s)}{q\omega_s}\right)\right] \\ \frac{\partial V\left(FV_{f_t}-FV_{s_t}\right)}{\partial\omega_s} & = & \left(\frac{a}{\omega_f}\right)^2\left((T_r\omega_f)-t\right)^2 t_r q(\omega_s)\times \\ & & \left[q(\omega_s)\left[1-2\frac{\omega_s}{\omega_f}\right]+2\omega_s\frac{dq(\omega_s)}{d\omega_s}\left[1-\frac{\omega_s}{\omega_f}\right]\right] \end{array} \tag{26}$$

Equation 26 is strictly negative if and only if

$$\begin{array}{llll} & q(\omega_s)\left[1-2\frac{\omega_s}{\omega_f}\right]+2\omega_s\frac{dq(\omega_s)}{d\omega_s}\left[1-\frac{\omega_s}{\omega_f}\right] & < & 0 \\ \Leftrightarrow & q(\omega_s)\left[\frac{\omega_f-2\omega_s}{\omega_f}\right] & < & -2\omega_s\frac{dq(\omega_s)}{d\omega_s}\left[\frac{\omega_f-\omega_s}{\omega_f}\right] \\ \Leftrightarrow & \left[\frac{\omega_f-2\omega_s}{\omega_f}\right]\left[\frac{\omega_f}{2\left(\omega_f-\omega_s\right)}\right] & < & -\frac{\omega_s\frac{dq(\omega_s)}{d\omega_s}}{q(\omega_s)} \\ \Leftrightarrow & -\epsilon & < & -\frac{\omega_f-2\omega_s}{2\left(\omega_f-\omega_s\right)} \end{array} \tag{27}$$

This condition is always verified when $\frac{\omega_f-2\omega_s}{2(\omega_f-\omega_s)}<0 \Leftrightarrow \frac{\omega_s}{\omega_f}>\frac{1}{2}$. Otherwise, the relation expressed in equation 27 must be verified. This proves the first assertion.

To prove the second assertion, we derive the fundamental basis volatility regarding ω_f and study its sign.

$$\begin{aligned}
\frac{\partial V\left(FV_{f_t}-FV_{s_t}\right)}{\partial \omega_f} &= \frac{\partial\left[\left(\frac{a}{\omega_f}\right)^2\left((T_r\omega_f)-t\right)^2 q(\omega_s)^2 t\left(1-\frac{\omega_s}{\omega_f}\right)\frac{\omega_s}{\omega_f}\right]}{\partial \omega_f}\\
&= a^2 q\left(\omega_s\right)^2 t\omega_s \frac{\partial\left[\frac{1}{\omega_f{}^2}\left((T_r\omega_f)-t\right)^2\left(1-\frac{\omega_s}{\omega_f}\right)\frac{1}{\omega_f}\right]}{\partial \omega_f}\\
&= a^2 q\left(\omega_s\right)^2 t\omega_s \frac{\partial\left[\left((T_r\omega_f)-t\right)^2\left(\frac{\omega_f-\omega_s}{\omega_f{}^4}\right)\right]}{\partial \omega_f}\\
&= a^2 q\left(\omega_s\right)^2 t\omega_s \Big[2T_r\left((T_r\omega_f)-t\right)\left(\frac{\omega_f-\omega_s}{\omega_f{}^4}\right)\\
&\quad +\left((T_r\omega_f)-t\right)^2\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^5}\right)\Big]\\
\frac{\partial V\left(FV_{f_t}-FV_{s_t}\right)}{\partial \omega_f} &= a^2 q\left(\omega_s\right)^2 t\omega_s\left((T_r\omega_f)-t\right)\Big[2T_r\left(\frac{\omega_f-\omega_s}{\omega_f{}^4}\right)\\
&\quad +\left((T_r\omega_f)-t\right)\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^5}\right)\Big]
\end{aligned} \tag{28}$$

This term is strictly negative if and only if

$$\begin{aligned}
& 2T_r\left(\frac{\omega_f-\omega_s}{\omega_f{}^4}\right)+\left((T_r\omega_f)-t\right)\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^5}\right) && < 0\\
\Leftrightarrow\ & 2T_r\left(\frac{\omega_f-\omega_s}{\omega_f{}^4}\right) && < -\left((T_r\omega_f)-t\right)\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^5}\right)\\
\Leftrightarrow\ & \frac{T_r\omega_f}{(T_r\omega_f)-t} && < -\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^4}\right)\left(\frac{\omega_f{}^4}{2(\omega_f-\omega_s)}\right)\\
\Leftrightarrow\ & \frac{T_r\omega_f}{T_r\omega_f-t} && < 1+\frac{\omega_f-2\omega_s}{2(\omega_f-\omega_s)}\\
\Leftrightarrow\ & -\frac{t}{T_r\omega_f-t} && > -\frac{\omega_f-2\omega_s}{2(\omega_f-\omega_s)}
\end{aligned} \tag{29}$$

For $t\in[0;T_r\times\omega_f-1]\subset\mathbb{N}$, $\frac{t}{T_r\omega_f-t}\geq 0$. Then, this condition is never verified for

$$\frac{\omega_f-2\omega_s}{2\left(\omega_f-\omega_s\right)}<0 \Leftrightarrow \frac{\omega_s}{\omega_f}>\frac{1}{2} \tag{30}$$

For $a\neq 0$, $t<T$, and $\omega_s\neq\omega_f$, $\frac{\partial V\left(FV_{f_t}-FV_{s_t}\right)}{\partial \omega_f}>0$ when $\frac{\omega_s}{\omega_f}>\frac{1}{2}$. Otherwise, the relation expressed in equation 29 must be verified. This proves the second assertion.

To prove the third assertion, we compute the total derivation of the fundamental basis volatility considering a proportional increase of the two frequencies.

Using equations 26, 28, and the variation of ω_f explained with the variation of ω_s computed above, we get:

$$\begin{aligned}
dV\left(FV_{f_t}-FV_{s_t}\right) &= \frac{\partial Var\left(FV_{f_t}-FV_{s_t}\right)}{\partial\omega_s}d\omega_s+\frac{\partial Var\left(FV_{f_t}-FV_{s_t}\right)}{\partial\omega_f}d\omega_f\\
&= \left(\frac{a}{\omega_f}\right)^2(T-t)^2 t\frac{1}{\omega_f}q\left(\omega_s\right)\times\\
&\quad \left[q\left(\omega_s\right)\left[1-2\frac{\omega_s}{\omega_f}\right]+2\omega_s\frac{dq(\omega_s)}{d\omega_s}\left[1-\frac{\omega_s}{\omega_f}\right]\right]d\omega_s\\
&\quad +a^2 q\left(\omega_s\right)^2 t\omega_s\left(T-t\right)\times\\
&\quad \left[2T_r\left(\frac{\omega_f-\omega_s}{\omega_f{}^4}\right)+(T-t)\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^5}\right)\right]\frac{\omega_f}{\omega_s}d\omega_s\\
&= a^2\left(T_r-t_r\right)^2 t\frac{1}{\omega_f}q\left(\omega_s\right)^2\left[1-2\frac{\omega_s}{\omega_f}+2\frac{\omega_s}{q(\omega_s)}\frac{dq(\omega_s)}{d\omega_s}\left[1-\frac{\omega_s}{\omega_f}\right]\right]d\omega_s\\
&\quad +a^2 q\left(\omega_s\right)^2 t\omega_f\left(T_r-t_r\right)^2\frac{\omega_f}{\omega_f{}^3}\\
&\quad \left[2\frac{T_r}{(T_r-t_r)}\left(\frac{\omega_f-\omega_s}{\omega_f}\right)+\frac{(T-t)}{(T_r-t_r)}\left(\frac{4\omega_s-3\omega_f}{\omega_f{}^2}\right)\right]d\omega_s\\
&= a^2\left(T_r-t_r\right)^2\frac{t}{\omega_f}q\left(\omega_s\right)^2 d\omega_s\Big[1-2\frac{\omega_s}{\omega_f}+2\frac{\omega_s}{q(\omega_s)}\frac{dq(\omega_s)}{d\omega_s}\left(1-\frac{\omega_s}{\omega_f}\right)\\
&\quad +2\frac{T_r}{(T_r-t_r)}\left(1-\frac{\omega_s}{\omega_f}\right)+4\frac{\omega_s}{\omega_f}-3\Big]\\
dV\left(FV_{f_t}-FV_{s_t}\right) &= a^2\left(T_r-t_r\right)^2 t_r q\left(\omega_s\right)^2 2\left(1-\frac{\omega_s}{\omega_f}\right)d\omega_s\left[\frac{\omega_s}{q(\omega_s)}\frac{dq(\omega_s)}{d\omega_s}+\frac{T_r}{(T_r-t_r)}-1\right]
\end{aligned} \tag{31}$$

Equation 31 is strictly negative if and only if

$$
\begin{array}{lll}
 & -1+\frac{\omega_s}{q(\omega_s)}\frac{dq(\omega_s)}{d\omega_s}+\frac{T_r}{(T_r-t_r)} & < 0 \\
\Leftrightarrow & -\epsilon & < 1-\frac{T_r}{(T_r-t_r)} \\
\Leftrightarrow & -\epsilon & < -\frac{t}{(T_r\omega_f-t)}
\end{array}
$$

This proves the third assertion. □

E Proof of Theorem 5

Proof. The maximal value of the utility function is 0. According to properties (i) and (ii) of U, the maximal value of U is reached if and only if $B(s_t) = V(FV_{f_t} - FV_{s_t}) = 0$ (or $B(s_t) \to 0$ and $V(FV_{f_t} - FV_{s_t}) \to 0$). From theorems 1 and 2 as well as remarks 1 and 3, the only frequencies allowing $B(s_t) = V(FV_{f_t} - FV_{s_t}) = 0$ are given by the subset $O = \left\{(\omega_s, \omega_f) \in (\mathbb{F}_s \cap \mathbb{F}_f)^2 : \frac{\omega_s}{\omega_f} = 1\right\}$. This proves the first assertion.

For $e \neq 0$, $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, we have $B(s_t) > 0$ and $V(FV_{f_t} - FV_{s_t}) > 0$. We look for optimal solutions by limit for the two arguments. This gives the following conditions for the second argument:

$$
\begin{array}{llll}
 & \left(\frac{a}{\omega_f}\right)^2 (T_r\omega_f - t)^2 q(\omega_s)^2 t\frac{\omega_s}{\omega_f}\left(1-\frac{\omega_s}{\omega_f}\right) & = & 0 \\
\Leftrightarrow & q(\omega_s)^2 \frac{\omega_s}{{\omega_f}^3}(T_r\omega_f - t)^2 & = & 0 \\
\Leftrightarrow & (\omega_s^{-\epsilon})^2 \frac{\omega_s}{{\omega_f}^3}(T_r\omega_f - t)^2 & = & 0 \\
\Leftrightarrow & \frac{{\omega_s}^{\frac{1}{2}-\epsilon}}{{\omega_f}^{\frac{3}{2}}}(T_r\omega_f - t) & = & 0
\end{array}
\tag{32}
$$

As $\lim_{\substack{\frac{\omega_s}{\omega_f} \to 0 \\ \omega_f \to +\infty}} \frac{{\omega_s}^{\frac{1}{2}-\epsilon}}{{\omega_f}^{\frac{3}{2}}}(T_r\omega_f - t) = 0$, a subset of optimal solutions by limit can exist. This subset is $O_{bl} = \left\{(\omega_s;\omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0,\ \omega_f \mapsto +\infty\right\}$, and it is independent of the value of ϵ.

Focusing on the first argument gives the following:

$$
\begin{array}{llll}
 & \frac{|-e|}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1} & = & 0 \\
\Leftrightarrow & \frac{1}{\omega_f}\sum_{k=1}^{t}\left(1-\frac{\omega_s}{\omega_f}\right)^{k-1} & = & 0 \\
\Leftrightarrow & \frac{1}{\omega_f}\frac{\left(1-\frac{\omega_s}{\omega_f}\right)}{\left(1-\frac{\omega_s}{\omega_f}\right)}\frac{1-\left(1-\frac{\omega_s}{\omega_f}\right)^{t-1+1}}{1-\left(1-\frac{\omega_s}{\omega_f}\right)} & = & 0 \\
\Leftrightarrow & \frac{1}{\omega_s}\left(1-\left(1-\frac{\omega_s}{\omega_f}\right)^t\right) & = & 0
\end{array}
\tag{33}
$$

We study the limit of the left term of equation 33 within the subset O_{bl}. It gives:

$$
\begin{array}{l}
\lim_{\substack{\frac{\omega_s}{\omega_f} \to 0 \\ \omega_f \to +\infty}} \frac{1}{\omega_s} = \frac{1}{\omega_s} \\
\lim_{\substack{\frac{\omega_s}{\omega_f} \to 0 \\ \omega_f \to +\infty}} \frac{\omega_s}{\omega_f} = 0 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \to 0 \\ \omega_f \to +\infty}} 1-\left(1-\frac{\omega_s}{\omega_f}\right)^t = 0 \\
\text{By product } \lim_{\substack{\frac{\omega_s}{\omega_f} \to 0 \\ \omega_f \to +\infty}} \frac{1}{\omega_s}\left(1-\left(1-\frac{\omega_s}{\omega_f}\right)^t\right) = 0
\end{array}
$$

Hence, O_{bl} is also a subset of optimal solutions by limit to the first argument of U. It is consequently the subset of all optimal solutions by limit independent from market characteristics (ϵ value). This proves the second assertion of the theorem. $\square$

F Proof of theorem 6

Proof. We use the proof of theorem 5 to identify the other optimal and quasi-optimal trade frequencies.

For $e \neq 0$, $a \neq 0$, $t < T$, and $\omega_s \neq \omega_f$, we have $B(s_t) > 0$ and $V(FV_{f_t} - FV_{s_t}) > 0$.

Equation 32 gives the conditions for the second argument to be equal to 0.

If $\frac{1}{2} - \epsilon > 0 \Leftrightarrow -\epsilon > -\frac{1}{2}$, no ω_s exists that satisfies equation 32. However, the subset $O_{bl}^{\epsilon<\frac{1}{2}} = \left\{(\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0^+ \text{ with } \omega_s \mapsto 0^+\right\}$ gives optimal solutions by limit for the second argument of the utility function U. This subset, conditional to the values of ϵ, is such that

$$O_{bl}^{\epsilon<\frac{1}{2}} \subset O_{bl}$$

Hence, there are no other optimal solutions by limit for the second argument in the case of $-\epsilon > -\frac{1}{2}$. This proves the first assertion.

If $\frac{1}{2} - \epsilon < 0 \Leftrightarrow -\epsilon < -\frac{1}{2}$, there exists no ω_s satisfying equation 32. However, $O_{bl}^{\epsilon>\frac{1}{2}} = \left\{(\omega_s, \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 1,\ \omega_s \mapsto +\infty,\ \omega_f \mapsto +\infty\right\}$ gives optimal solutions by limit for the second argument of the utility function U.

Equation 33 gives the condition for the first argument of the utility function U.

We study the limit of the left term of equation 33 within the subset $O_{bl}^{\epsilon>\frac{1}{2}}$. We get:

$$\lim_{\substack{\frac{\omega_s}{\omega_f} \to 1 \\ \omega_f \to +\infty}} \frac{1}{\omega_s} = 0$$

$$\lim_{\substack{\frac{\omega_s}{\omega_f} \to 1 \\ \omega_f \to +\infty}} \frac{\omega_s}{\omega_f} = 1 \Rightarrow \lim_{\substack{\frac{\omega_s}{\omega_f} \to 1 \\ \omega_f \to +\infty}} 1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t = 1$$

$$\text{By product} \lim_{\substack{\frac{\omega_s}{\omega_f} \to 1 \\ \omega_f \to +\infty}} \frac{1}{\omega_s}\left(1 - \left(1 - \frac{\omega_s}{\omega_f}\right)^t\right) = 0$$

$O_{bl}^{\epsilon>\frac{1}{2}}$ allows to verify equation 33. Hence, $O_{bl}^{\epsilon>\frac{1}{2}}$ is a subset of optimal solutions by limit which maximizes U. This subset is such that

$$O_{bl}^{\epsilon>\frac{1}{2}} \cap O_{bl} = \emptyset$$

This proves the second assertion.

If $\frac{1}{2} - \epsilon = 0 \Leftrightarrow -\epsilon = -\frac{1}{2}$, there is no ω_s satisfying equation 32. However, the subset $O_{bl}^{\epsilon=\frac{1}{2}} = \left\{(\omega_s; \omega_f) \in \mathbb{F}_s \times \mathbb{F}_f : \frac{\omega_s}{\omega_f} \mapsto 0,\ \omega_f \mapsto +\infty\right\}$ gives optimal solutions by limit for the second argument of the utility function U. However, this subset is such that

$$O_{bl}^{\epsilon=\frac{1}{2}} = O_{bl}$$

Hence, there are no other optimal solutions by limit in the case of $-\epsilon = -\frac{1}{2}$. This proves the third assertion. $\square$