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► **To cite this version:**

Colette Anné, Olaf Post. NORM CONVERGENCE OF THE RESOLVENT FOR WILD PERTURBATIONS (presentation). Analysis and Geometry on Graphs and Manifolds, Jul 2017, Potsdam, Germany. hal-02364006

HAL Id: hal-02364006

<https://hal.science/hal-02364006>

Submitted on 14 Nov 2019

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NORM CONVERGENCE OF THE RESOLVENT FOR WILD PERTURBATIONS

COLETTE ANNÉ AND OLAF POST

ABSTRACT. We present here recent progress in the convergence of the resolvent of Laplace operators under wild perturbations. In particular, we show convergence in *norm* in a generalised sense. We focus here on the excision of many small balls in a complete Riemannian manifold with bounded geometry.

1. INTRODUCTION

Rauch-Taylor’s contribution on wild perturbations. What kind of convergence can we expect for the Laplace operator under wild perturbations such as removing many small holes or adding many thin handles? Such questions received already quite a lot of answers, following the seminal work of Rauch and Taylor [RT75]. We present here results from [AP18] and focus on convergence of the resolvents in *operator norm*. As the underlying spaces vary with the convergence parameter, we apply an abstract convergence result of the second author [P12] expressed in terms of quadratic forms acting in different Hilbert spaces, see Section 2.

The expression “wild perturbation” goes back to [RT75]. Let us first recall the original result of Rauch and Taylor concerning the excision of small obstacles and convergence of the corresponding Dirichlet Laplacians. A typical result of their paper is as follows: Let $\Omega \subset \mathbb{R}^m$ be an open and bounded set having some mild regularity, namely

$$H_0^1(\Omega) = \{u \in H^1(\mathbb{R}^m), \text{supp } u \subset \overline{\Omega}\}.$$

Let K be a compact subset of Ω . We assume that $\Omega_n \rightarrow \Omega \setminus K$ as $n \rightarrow \infty$ *metrically*, i.e., every compact subset of $\Omega \setminus K$ is eventually in Ω_n , and every compact subset outside $\overline{\Omega \setminus K}$ is eventually outside $\overline{\Omega_n}$.

Let Δ_Ω and Δ_{Ω_n} be the (non-negative) Laplacians on Ω and Ω_n with Dirichlet boundary condition, respectively. Moreover, let $J_n f = \mathbb{1}_{\Omega \cap \Omega_n} f$ be the restriction of $f \in L_2(\Omega)$ onto $\Omega \cap \Omega_n$ extended by 0 on $\Omega_n \setminus \Omega$. Then $J_n^* u = \mathbb{1}_{\Omega \cap \Omega_n} u$ extended by 0 on $\Omega \setminus \Omega_n$.

1.1. Theorem ([RT75, Thm. 2.3]). *If K has capacity zero then for any real-valued continuous and bounded function Φ and any $f \in L_2(\Omega)$ we have $J_n^* \Phi(\Delta_{\Omega_n}) J_n f \rightarrow \Phi(\Delta_\Omega) f$ in $L_2(\mathbb{R}^m)$ as $n \rightarrow \infty$.*

For a characterisation of a set to have capacity zero, we refer e.g. to [RT75, Lem. 2.1]. An example of a set of capacity zero is a finite set of points or more generally a subset of co-dimension 2.

We can think of this result as a (generalised) *strong* resolvent convergence (choose $\Phi(\lambda) = (1 + \lambda)^{-1}$, recall our convention $\Delta \geq 0$). Strong resolvent convergence implies the convergence of the discrete spectrum, as the limit spectrum cannot suddenly expand; but it can shrink suddenly in the limit (see the discussion after Thm. VIII.24 in [RS80]). This is probably the main disadvantage of *strong* resolvent convergence compared to

Date: February 4, 2019. Notes from the conference “Analysis and Geometry on Graphs and Manifolds”, Potsdam, July, 31 – August, 4 2017.

norm resolvent convergence from a spectral viewpoint. Note that the sudden shrinkage leads to so-called *spectral pollution*, i.e., spectral values in the approximation, which do not converge to a spectral value in the limit problem. The opposite effect is called *spectral exactness*, and holds for norm resolvent convergence in general (see [Bög17] and references therein for details).

Wild perturbations and norm resolvent convergence. One of our main question in this article is as follows:

Question. Can we show stronger convergence results for wild perturbations such as *norm* resolvent convergence and results which work also (without much modifications) for *unbounded* domains or manifolds?

As wild perturbation we focus here on the excision of many small balls as obstacle from a (not necessarily compact) Riemannian manifold of bounded geometry and the Dirichlet Laplacian on the manifold without the obstacles. Further results are shrinking *Neumann* obstacles (see [AP18]). Note that our perturbation result also works quite well when neither the perturbed space X_ε nor the limit space X_0 is subset of the other. This is e.g. the case when adding many thin handles to a manifold; we treat this question in a forthcoming publication.

Domain perturbations and convergence results. Domain perturbation and (spectral) convergence results have a long history. We are not trying to give an exhaustive list of references here, but just highlight a few points:

Weidmann [W84] proved the continuous dependency of eigenvalues and eigenfunctions of elliptic differential operators and he also developed a general (strong resolvent) convergence theory for sequences of operators acting in different Hilbert spaces (which can be embedded in a larger common Hilbert space).

The asymptotic behaviour of Neumann eigenvalues was studied for a single hole for bounded domains or compact manifolds in [Oz83, Hem06, LdC12] and the Dirichlet eigenvalues in [CF78, Cou95] where we find precise estimates; it applies also to the ε -neighbourhood of compact subset, see also [CF88] for the calculation of the first correction term.

Daners [Dan03] considers the *norm* convergence of resolvents of Dirichlet Laplacians for perturbations of Euclidean *bounded* domains (or at least those with compact resolvent), the norm convergence follows from the strong one under the assumption of compactness of the limit resolvent, see also [Dan08] for a survey and the references therein. Our approach is more general as we do not assume a priori that the perturbed and unperturbed domains are embedded in a common space as in [Dan03, Dan08]. Moreover, we obtain explicit error estimates in terms of δ_ε . For an older survey about strong resolvent convergence and perturbations of Euclidean domains, we refer to [Hen94].

Finally, the work of Rauch and Taylor [RT75] inspired with their *crushed ice problem* the study of *homogenisation*. There is a critical density of balls removed under which the Dirichlet Laplacian converges to the original Laplacian with a shift in energy. Below that critical parameter, the limit is the original Laplacian itself, above, there are regions that “become solid” in the sense that the limit Laplacian fulfils Dirichlet conditions there. The homogenisation problem is usually also treated showing *strong* resolvent convergence, see [BN98, Bal88] using Γ -convergence, see [DM93]. More recent works can be found in [Kh09] or [Kh13] and references therein. For a similar approach as in this paper using the above mentioned generalised *norm* resolvent convergence in the homogenisation case, we refer to [KhP18] and the references cited therein. For an approach using the already

shown strong resolvent convergence to upgrade to norm resolvent convergence (similarly as in [Dan03, Dan08], but even for general unbounded domains) we refer to [DCR17].

The notion of Γ - or Mosco convergence is another way of defining a convergence of quadratic forms acting in different Hilbert spaces, see e.g. [KS03, Sec. 2]: note that this convergence is more or less equivalent with some sort of generalisation of *strong* resolvent convergence, hence our results are stronger.

2. A GENERALISED NORM RESOLVENT CONVERGENCE

To achieve this goal we apply a rather general result of the second author [P06] (see also the monograph [P12]):

For each $\varepsilon \geq 0$, let \mathcal{H}_ε be a separable Hilbert space together with a closed quadratic form \mathfrak{q}_ε and domain $\mathcal{H}_\varepsilon^1$. We denote by $\Delta_\varepsilon \geq 0$ the corresponding self-adjoint operator. We define the generalised Sobolev spaces $\mathcal{H}_\varepsilon^k$ as $\mathcal{D}(\Delta_\varepsilon^{k/2})$ together with the norms $\|u\|_k = \|(\Delta_\varepsilon + 1)^{k/2} f\|$, and choose the completion of \mathcal{H}_ε with respect to the norm $\|\cdot\|_k$ if $k < 0$. Then all spaces $(\mathcal{H}_\varepsilon^k, \|\cdot\|_k)$ are complete. Note that

$$\|u\|_1^2 = \|u\|_{\mathcal{H}_\varepsilon}^2 + \mathfrak{q}_\varepsilon(u).$$

We suppose there are transplation or identification operators at the level of the Hilbert spaces and also at the level of the quadratic forms (we suppress here and in the following the dependency of ε in the notation):

$$\begin{aligned} J: \mathcal{H}_0 &\rightarrow \mathcal{H}_\varepsilon & J_1: \mathcal{H}_0^1 &\rightarrow \mathcal{H}_\varepsilon^1 \\ J': \mathcal{H}_\varepsilon &\rightarrow \mathcal{H}_0 & J'_1: \mathcal{H}_\varepsilon^1 &\rightarrow \mathcal{H}_0^1. \end{aligned}$$

We assume that these operators are bounded and need some compatibility, also called δ_ε -quasi-unitary equivalence of \mathfrak{q}_ε and \mathfrak{q}_0 , if $\delta_\varepsilon > 0$ and if

$$|\langle J'u, f \rangle - \langle u, Jf \rangle| \leq \delta_\varepsilon \|f\|_1 \|u\|_1, \quad (1a)$$

$$\|f - J'Jf\| \leq \delta_\varepsilon \|f\|_1 \quad \text{and} \quad \|u - JJ'u\| \leq \delta_\varepsilon \|u\|_1, \quad (1b)$$

$$\|(J_1 - J)f\| \leq \delta_\varepsilon \|f\|_1 \quad \text{and} \quad \|(J'_1 - J')u\| \leq \delta_\varepsilon \|u\|_1, \quad (1c)$$

$$|\mathfrak{q}_\varepsilon(J_1f, u) - \mathfrak{q}_0(f, J'_1u)| \leq \delta_\varepsilon \|f\|_2 \|u\|_1 \quad (1d)$$

for all f and u in the respective spaces. We have adopted the definition of quasi-unitary equivalence already to the situation here where the quadratic forms are estimated with respect to the *form* norm $\|\cdot\|_1$ on the perturbed space $\mathcal{H}_\varepsilon^1$ and the *graph* norm $\|\cdot\|_2$ on the unperturbed space $\mathcal{H}_0^2 = \text{dom } \Delta_0$.

We have the following notion of *generalised norm resolvent convergence*:

2.1. Theorem ([P12, Prp. 4.4.15]). *If the quadratic forms \mathfrak{q}_ε and \mathfrak{q}_0 are δ_ε -quasi-unitary equivalent, then the resolvents $R_\varepsilon := (\Delta_\varepsilon + 1)^{-1}$ of the operators Δ_ε associated with \mathfrak{q}_ε satisfy*

$$\|R_\varepsilon J - JR_0\| \leq 4\delta_\varepsilon.$$

Moreover, if $\delta_\varepsilon \rightarrow 0$, then we also have the convergence of (suitable) functions of the operators in norm, of the spectrum, and of the eigenfunctions also in energy norm.

3. REMOVING MANY SMALL BALLS: THE FADING CASE

Let (X, g) be a complete connected Riemannian manifold of dimension m with natural energy form defined by $\mathfrak{q}(f) = \int_X |df|^2 \, d\text{vol}_g$ for $f \in C_0^\infty(X)$. This form is closable (because the manifold is complete) and defines a non-negative self-adjoint operator $\Delta =$

$\Delta_{(X,g)}$ (see e.g. [RS80, Thm. VIII.15] for details), given in local coordinates (y_1, \dots, y_m) by

$$\Delta(f) = - \sum_{1 \leq i, j \leq m} \frac{1}{\rho} \partial_{x_i} (\rho g^{ij} \partial_{x_j} f),$$

where (g^{ij}) is the inverse matrix of the metric tensor (g_{ij}) , and where the Riemannian measure $d \text{vol}_g$ is locally given by $\rho dy_1 \otimes \dots \otimes dy_m$ with $g_{ij} = g(\partial_{y_i}, \partial_{y_j})$.

As an example of application of the above generalised norm resolvent convergence, let us look at the problem of removing many small balls: Assume that (X, g) is a complete Riemannian manifold of dimension $m \geq 2$. Consider the following perturbation:

For any $\varepsilon > 0$, let $(x_j)_{j \in \mathcal{J}_\varepsilon}$ be a family of points in X such that $d(x_j, x_k) \geq 2\eta_\varepsilon$ for some $\eta_\varepsilon \gg \varepsilon$ (typically, we will choose $(\eta_\varepsilon = \varepsilon^\alpha)$ for some $0 < \alpha < 1$). Note that we do not assume any relation between points x_j for $j \in \mathcal{J}_\varepsilon$ for *different* values of ε .

We set

$$X_\varepsilon = X \setminus B_\varepsilon \quad \text{with} \quad B_\varepsilon = \bigcup_{j \in \mathcal{J}_\varepsilon} B(x_j, \varepsilon). \quad (2)$$

In this situation, let

$$\begin{aligned} \mathcal{H}_0 &= L_2(X, g), & \mathcal{H}_\varepsilon &= L_2(X_\varepsilon, g) \\ \mathcal{H}_0^1 &= H^1(X, g), & \mathcal{H}_\varepsilon^1 &= H_0^1(X_\varepsilon, g) \end{aligned}$$

with the transplantation operators

$$\begin{aligned} J: \mathcal{H}_0 &\rightarrow \mathcal{H}_\varepsilon & J_1: \mathcal{H}_0^1 &\rightarrow \mathcal{H}_\varepsilon^1, & Jf &= f \upharpoonright_{X_\varepsilon}, & J_1 f &= \chi_\varepsilon f \\ J': \mathcal{H}_\varepsilon &\rightarrow \mathcal{H}_0 & J'_1: \mathcal{H}_\varepsilon^1 &\rightarrow \mathcal{H}_0^1 & J'u &= \bar{u}, & J'_1 u &= \bar{u}, \end{aligned}$$

where \bar{u} is the extension of $u: X_\varepsilon \rightarrow \mathbb{C}$ onto X by 0, and where χ_ε is a cut-off function on X given by $\chi_\varepsilon(x) = \tilde{\chi}(d(x, x_j))$ if $d(x, x_j) \in [0, \varepsilon^+)$, $\chi_\varepsilon(x) = 0$ if $d(x, x_j) \in [0, \varepsilon]$ and $\chi_\varepsilon(x) = 1$ otherwise. Here,

$$\varepsilon \ll \varepsilon^+ \ll \eta_\varepsilon$$

and $\tilde{\chi}_\varepsilon$ is given by $\tilde{\chi}_\varepsilon(r) = 0$ if $r \in [0, \varepsilon]$ and

$$\tilde{\chi}_\varepsilon(r) = \begin{cases} \frac{1/r^{m-2} - 1/\varepsilon^{m-2}}{1/(\varepsilon^+)^{(m-2)} - 1/\varepsilon^{m-2}}, & \text{for } m \geq 3 \\ \frac{\log(r/\varepsilon)}{\log(\varepsilon^+/\varepsilon)}, & \text{for } m = 2. \end{cases}$$

for $r \in (\varepsilon, \varepsilon^+)$. Note that $\tilde{\chi}_\varepsilon$ and χ_ε are both Lipschitz continuous, hence $\chi_\varepsilon f$ is in $H_0^1(X_\varepsilon)$ if $f \in H^1(X)$. In particular, J_1 is well-defined.

The most difficult part to check in the assumptions of the above Theorem 2.1 is a control of the assumption (1d) on the forms (with $k = 2$). For this (and also the other assumptions (1a)–(1c)) we need the additional assumption of *bounded geometry* on the manifold:

3.1. Definition. The manifold (X, g) has *bounded geometry* if there exist $i_0 > 0$ and k_0 such that the injectivity radius and the Ricci curvature of X satisfy

$$\forall x \in X: \text{Inj}(x) \geq i_0, \quad \text{Ric}(x) \geq k_0 g.$$

We assume throughout this article that (X, g) has bounded geometry. We know, for instance by the book of Hebey [Heb99], that these hypotheses assure the existence of a uniform harmonic radius r_0 , i.e., a radius independent of the point such that inside the ball of this radius there exist *harmonic coordinates*. These coordinate assure a uniform

control of the metric with respect to the Euclidean one: there exists $K > 0$ such that for all $x_0 \in X$ there are harmonic coordinates (y_1, \dots, y_m) in $B(x_0, r_0)$ such that

$$\forall x \in B(x_0, r_0): K^{-1}\delta_{ij} \leq g_x(\partial_{y_i}, \partial_{y_j}) \leq K\delta_{ij} \quad (3)$$

(see e.g. [Heb99, Thm. 1.2]). These coordinates assure also that $C_0^\infty(X)$ is dense in $H^2(X, g) = \mathcal{H}_0^2$. As a consequence the Laplacian defined on $C_0^\infty(X)$ is essentially self-adjoint (see e.g. [Heb99, Prop. 3.3]¹).

Let us now describe the first result in this context. By ‘‘fading’’ we mean that there are not enough balls close to each other, so that one has no effect, i.e., the limit operator is the original Laplacian on X :

3.2. Theorem (many small balls fading). *Let X be a complete Riemannian manifold with bounded geometry and X_ε as in (2). Moreover, let the centres of balls be separated by $2\eta_\varepsilon$ with*

$$\eta_\varepsilon = \begin{cases} \varepsilon^\alpha, & 0 < \alpha < \frac{m-2}{m} \text{ if } m \geq 3 \text{ and} \\ |\log \varepsilon|^{-\alpha}, & 0 < \alpha < \frac{1}{2} \text{ if } m = 2. \end{cases}$$

Then the Laplacian with Dirichlet boundary conditions on X_ε converges in generalised norm resolvent sense to the Laplacian on X .

Remark. The critical power $\varepsilon^{(m-2)/m}$ is related to the *capacity* of the obstacle (the balls of radius ε) being at distance of order $\varepsilon^{(m-2)/2}$ away from other balls: This case needs more assumptions about the spacing of the points x_j ; details about generalised norm resolvent convergence in this situation and capacity can be found in [KhP18]. In particular, the *capacity* determines about the limit behaviour of the crushed ice problem.

Proof. Let us sketch the proof of (1d) for $k = 2$: For all $f \in \mathcal{H}_0^2$ and $u \in \mathcal{H}_\varepsilon^1$ we have

$$\begin{aligned} |\mathfrak{q}_0(f, J^1 u) - \mathfrak{q}_\varepsilon(J^1 f, u)| &= |\langle df - d(\chi_\varepsilon f), du \rangle_{L_2(T^*B_{\varepsilon^+}, g)}| \\ &\leq |\langle (1 - \chi_\varepsilon)df, du \rangle_{L_2(T^*B_{\varepsilon^+}, g)}| + |\langle f d\chi_\varepsilon, du \rangle_{L_2(T^*B_{\varepsilon^+}, g)}| \\ &\leq (\|df\|_{L_2(T^*B_{\varepsilon^+}, g)} + \|f d\chi_\varepsilon\|_{L_2(T^*B_{\varepsilon^+}, g)}) \|du\|_{L_2(T^*B_{\varepsilon^+}, g)} \end{aligned}$$

To control $\|df\|_{L_2(T^*B_{\varepsilon^+}, g)}^2$ which is a sum of integrals on balls, we use the assumption of bounded geometry and (3). Hence, it suffices to control the estimate on Euclidean balls, namely

$$\forall \phi \in H^1(B_{\eta_\varepsilon}, \text{eucl}): \|\phi\|_{L_2(B_{\varepsilon^+}, \text{eucl})} \leq \tau_m \left(\frac{\varepsilon^+}{\eta_\varepsilon} \right) \|\phi\|_{H^1(B_{\eta_\varepsilon}, \text{eucl})} \quad (4)$$

where $\tau_m(r) = O(r)$ for $m \geq 3$ and $\tau_2(r) = O(r |\log r|^{1/2})$. This control, pulled back onto the balls of the manifold can be applied as well to $\phi = |df|$.

Now, for the second term, we conclude from the Hölder inequality that

$$\|f d\chi_\varepsilon\|_{L_2(T^*B_{\varepsilon^+}(x), g)}^2 \leq \|f\|_{L_{2p}(B_{\varepsilon^+}(x), g)}^2 \|d\chi_\varepsilon\|_{L_{2q}(T^*B_{\varepsilon^+}(x), g)}^2$$

for any $p \in (1, \infty)$ with $1/p + 1/q = 1$ and any $j \in \mathcal{J}_\varepsilon$.

In order to control $\|f\|_{L_{2p}(B_{\varepsilon^+}(x), g)}$ we use a Sobolev embedding $H^2(B_2(0)) \hookrightarrow L_{2p}(B_1(0))$ for p small enough, rescaling gives a bad estimate in terms of on ε , but this can be compensated by a rather good estimate of $\|d\chi_\varepsilon\|_{L_{2q}(T^*B_{\varepsilon^+}(x), g)}$ if p is not too small. We assert that, for each dimension m , there exists good p_m, q_m which do the job, and consequently there exists $\delta_\varepsilon = o(1)$ such that

$$\|f d\chi_\varepsilon\|_{L_2(T^*B_{\varepsilon^+}, g)} \leq \delta_\varepsilon \|f\|_{\mathcal{H}_0^2}.$$

¹The identification $H^2(X, g) = \mathcal{H}_0^2$ needs the Bochner-Weitzenböck formula.

The details can be found in [AP18]. Note that $J' = J^*$, and that the remaining (non-trivial) assumptions of (1b)–(1c) also follow by (4) and the bounded geometry assumption. \square

4. REMOVING MANY SMALL BALLS: THE SOLIDIFYING CASE

We obtain also results for the *solidifying* situation (named after [RT75]): here, the density of the removed balls is so high that it solidifies in the limit to an obstacle Ω_0 . Again, (X, g) is a complete Riemannian manifold of dimension $m \geq 2$ and $X_\varepsilon = X \setminus B_\varepsilon$ with $B_\varepsilon = \bigcup_{j \in \mathcal{J}_\varepsilon} B(x_j, \varepsilon)$ for a set of points $(x_j)_{j \in \mathcal{J}_\varepsilon}$. We suppose now that there exists an open subset Ω_0 of X with regular boundary (see Definition 4.2) and that $B_\varepsilon \subset \Omega_0$. Moreover, we assume that there exists $N \in \mathbb{N}$, $\eta_\varepsilon \gg \varepsilon$ and $\alpha_\varepsilon > 0$ such that

$$\Omega_{\alpha_\varepsilon} = \{x \in X; d(x, \Omega_0) < \alpha_\varepsilon\} \subset B_{\eta_\varepsilon} \quad (5a)$$

$$\forall x \in X \forall \varepsilon > 0: \#\{j \in \mathcal{J}_\varepsilon; x \in B(x_j, \eta_\varepsilon)\} \leq N, \quad (5b)$$

where $\#M$ is the cardinality of the set M . The first assertion assures that the family $(x_j)_{j \in \mathcal{J}_\varepsilon}$ is dense enough: at the scale η_ε it covers all Ω_0 and a bit more; it also implies that $\alpha_\varepsilon/\eta_\varepsilon$ is small or at least bounded. The second assertion assures that this cover is not too redundant. In particular, it follows from $B_\varepsilon \subset \Omega_0 \subset B_{\eta_\varepsilon}$ that $X_\varepsilon \xrightarrow{\varepsilon \rightarrow 0} X \setminus \Omega_0 = X_0$.

We also need control of the first eigenvalue λ_ε of the Laplacian on $B_{\mathbb{R}^m}(0, \eta_\varepsilon) \setminus B_{\mathbb{R}^m}(0, \varepsilon)$ with Neumann boundary condition at $r = \eta_\varepsilon$ and Dirichlet boundary condition at $r = \varepsilon$. It is calculated in [RT75] that $\lambda_\varepsilon \geq C\varepsilon^{(m-2)}/\eta_\varepsilon^m$ (respectively $\lambda_\varepsilon \geq C/(\eta_\varepsilon^2 |\log \varepsilon|)$ for $m = 2$), where C depends only on the dimension m ; and this estimate carries over to balls on the manifold (by our assumption of bounded geometry).

4.1. Theorem (many small balls solidifying). *In the situation just described, assume that $\lim_{\varepsilon \rightarrow 0} \alpha_\varepsilon \lambda_\varepsilon = +\infty$, then the Laplacian Δ_ε with Dirichlet boundary conditions on $X_\varepsilon = X \setminus B_\varepsilon$ converges in generalised norm resolvent sense to the Laplacian Δ_0 with Dirichlet boundary conditions on $X \setminus \Omega_0$.*

We check again the conditions of quasi-unitary equivalence in (1a)–(1d). We define here

$$\begin{aligned} J: \mathcal{H}_0 &:= L_2(X_0, g) \longrightarrow \mathcal{H}_\varepsilon := L_2(X_\varepsilon, g), & f &\mapsto \bar{f}, \\ J_1: \mathcal{H}_0^1 &:= H_0^1(X_0, g) \longrightarrow \mathcal{H}_\varepsilon^1 := H_0^1(X_\varepsilon, g), & f &\mapsto \bar{f}, \\ J': \mathcal{H}_\varepsilon &:= L_2(X_\varepsilon, g) \longrightarrow \mathcal{H}_0 = L_2(X_0, g), & u &\mapsto u|_{X_0}, \\ J^{1'}: \mathcal{H}_\varepsilon^1 &:= H_0^1(X_\varepsilon, g) \longrightarrow \mathcal{H}_0^1 = H_0^1(X_0, g), & u &\mapsto \chi_\varepsilon u, \end{aligned}$$

where \bar{f} is the extension of f by 0 onto X_ε , as $X_0 \subset X_\varepsilon$ and χ_ε is now a cut-off function depending on the distance to Ω_0 . In particular, we need some control of the boundary of Ω_0 :

4.2. Definition. We say that the open set $\Omega \subset X$ has a *regular boundary* $Y = \bar{\Omega} \setminus \Omega$ if Y is a smooth sub-manifold of X which admits a *uniform tubular neighbourhood*, i.e., we assume that Y admits a global normal unitary vector field \vec{N} (so that Y is orientable) and that there exists $r_0 > 0$ such that

$$\exp_\nu: Y \times [0, r_0) \rightarrow X, \quad (y, t) \mapsto \exp_y(t\vec{N}(y)) \quad (6)$$

is a diffeomorphism.

4.3. Remark. This regularity assumption (together with the bounded geometry) implies that the principal curvatures of the hypersurface Y are bounded by a constant depending on $1/r_0$ and k_0 , see [HK78, Cor. 3.3.2]. But it is stronger: we need also that Y does not admit arbitrarily close points which are far away with respect to the inner distance.

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