Robust optimal identification experiment design for multisine excitation
Xavier Bombois, Federico Morelli, H. Hjalmarsson, Laurent Bako, Kévin Colin

To cite this version:
Xavier Bombois, Federico Morelli, H. Hjalmarsson, Laurent Bako, Kévin Colin. Robust optimal identification experiment design for multisine excitation. Automatica, 2021, 10.1016/j.automatica.2020.109431. hal-02363067

HAL Id: hal-02363067
https://hal.science/hal-02363067
Submitted on 14 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Robust optimal identification experiment design for multisine excitation

X. Bombois a,b, F. Morelli a, H. Hjalmarsson c, L. Bako a, K. Colin a

a Laboratoire Ampère, Ecole Centrale de Lyon, Université de Lyon, 36 avenue Guy de Collongue, Ecully, France
b Centre National de la Recherche Scientifique (CNRS), France
c Automatic control, KTH, Stockholm, Sweden

Abstract

In least costly experiment design, the optimal spectrum of an identification experiment is determined in such a way that the cost of the experiment is minimized under some accuracy constraint on the identified parameter vector. Like all optimal experiment design problems, this optimization problem depends on the unknown true system, which is generally replaced by an initial estimate. One important consequence of this is that we can underestimate the actual cost of the experiment and that the accuracy of the identified model can be lower than desired. Here, based on an a-priori uncertainty set for the true system, we propose a convex optimization approach that allows to prevent these issues from happening. We do this when the to-be-determined spectrum is the one of a multisine signal.

Key words: Optimal experiment design, System identification

1 Introduction

We consider in this paper the problem of optimally designing the spectrum \( \Phi_u \) of the excitation signal \( u \) of an open-loop identification experiment. By optimal spectrum, we here mean the spectrum yielding the smallest experiment cost while guaranteeing that the accuracy of the identified parameter vector of the plant transfer function is larger than a given threshold. We thus consider the least costly experiment design framework [5], but the approach can easily be adapted to other (dual) frameworks [10,17,13]. The experiment cost \( J \) can be defined as a linear combination of the power of the excitation signal \( u \) and of the power of the part of the output signal induced by \( u \). The experiment cost will therefore be a function of the spectrum \( \Phi_u \), but also of the unknown true parameter vector \( \theta_0 \) (we therefore denote the cost as \( J(\theta_0, \Phi_u) \)). Likewise, the accuracy constraint will also depend on \( \theta_0 \) and on \( \Phi_u \) since the classical accuracy constraints are of the type \( P^{-1}(\theta_0, \Phi_u) \geq \text{R}_{\text{adm}} \) where \( P(\theta_0, \Phi_u) \) is the covariance matrix of the to-be-identified parameter vector (which depends on \( \theta_0 \) and \( \Phi_u \)) and \( \text{R}_{\text{adm}} \) a matrix reflecting the desired accuracy. The dependency of the optimal spectrum \( \Phi_{u, \text{opt}} \) on the unknown true parameter vector \( \theta_0 \) is the so-called chicken-and-egg issue encountered in optimal experiment design. This issue is generally circumvented by replacing \( \theta_0 \) by an initial estimate \( \hat{\theta}_{\text{init}} \) of \( \theta_0 \) since, in this case, the optimal experiment design problem boils down to a convex optimization problem (see e.g. [10,5]). However, this approach has the drawback that the optimal spectrum is not guaranteed to yield the desired accuracy and that the experiment cost computed with \( \hat{\theta}_{\text{init}} \) and \( \Phi_{u, \text{opt}} \) can underestimate the actual experiment cost. These observations are at the root of the research area on robust optimal experiment design (see [19] for a good survey).

In robust experiment design, different lines of research have been considered. In [18], a spectrum that yields good accuracy for a very broad set of systems (also of different orders) is discussed. However, in the engineering literature, the most widely used approach is the one that consists in considering an uncertainty set \( U \) containing the unknown true parameter vector \( \theta_0 \) (the so-called min-max design [19,16]). The optimal experiment design problem can then be formulated as determining the spectrum \( \Phi_u \) minimizing the value of a scalar \( \gamma \) under the constraints that \( J(\theta, \Phi_u) \leq \gamma \forall \theta \in U \) and that \( P^{-1}(\theta, \Phi_u) \geq \text{R}_{\text{adm}} \forall \theta \in U \). If we denote by \( \gamma_{\text{opt}} \) and \( \Phi_{u, \text{opt}} \) the solution of this optimization problem, we have the guarantee that \( P^{-1}(\theta_0, \Phi_{u, \text{opt}}) \geq \text{R}_{\text{adm}} \) and that \( \gamma_{\text{opt}} \) is an upper bound for the actual experiment cost \( J(\theta_0, \Phi_{u, \text{opt}}) \). However, finding a tractable approach to deal with such a robustified optimal experiment design...
problem is still an open research question. While this optimization problem can be exactly solved in very particular and simple situations (see e.g., [19,1]), the general approach when facing more complex systems is to replace the initial uncertainty set (containing an infinite number of elements) by a number \( n \) of grid points of this uncertainty set \( U \) (see e.g., [10,5,19]). Consequently, the cost constraint and the accuracy constraint over the set \( U \) in the robustified optimal experiment design problem are replaced by \( n \) constraints (one for each grid point). Even though it is obviously better from a robustification point-of-view than just replacing \( \theta_0 \) by one grid point i.e. \( \theta_{\text{init}} \), this relaxation of the original robustified optimal experiment design problem cannot yield the guarantees linked to the original problem and it can become computationally heavy for large values of \( n \).

In [12,7,11], approaches are presented to uniquely tackle the robustified cost constraint \( J(\theta, \Phi_u) \leq \gamma \) for \( \forall \theta \in U \) (i.e. in the accuracy constraint, \( \theta_0 \) is replaced by \( \theta_{\text{init}} \)). However, these approaches all entail some approximation: a first-order approximation in [7], a second-order approximation in [12] and an approximation based on the unscented transform in [11].

Our main contribution in this paper is to present an approach in order to tackle the robust optimal experiment design problem without approximation. For this purpose, we observe that, except for its dependence on the to-be-determined spectrum, the robustified cost constraint and the robustified accuracy constraint are similar to constraints that are treated in robustness analysis. Based on this observation and on the separation of graph framework [20,9,14], we derive constraints that are linear in the decision variables of the optimal experiment design problem and that imply the original robustified cost and accuracy constraints. We do that for one of the most commonly used parametrization of the to-be-determined spectrum \( \Phi_u \), i.e. the one corresponding to a multirate [10]. We however restrict attention to Box-Jenkins (BJ) model structures and to accuracy constraints on the parameters of the plant transfer functions (we can indeed not robustify the noise part of the covariance matrix using the tools presented in this paper).

Notations. The matrix

\[
\begin{pmatrix}
X_1 & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & X_N
\end{pmatrix}
\]

will be denoted \( \text{diag}(X_1, \ldots, X_N) \) if the elements \( X_i \) (\( i = 1, \ldots, N \)) are scalar quantities while it will be denoted \( \text{bdiag}(X_1, \ldots, X_N) \) if the elements \( X_i \) (\( i = 1, \ldots, N \)) are matrices. In addition, the Fourier transform of a signal \( x(t) \) is denoted \( x(e^{j\omega}) \), \( I_n \) represents the identity matrix of dimension \( n \) and \( \otimes \), the Kronecker product. Finally, \( 0 \) represents a vector or a matrix containing only zeros and \( A^* \) is the conjugate transpose of the complex matrix \( A \).

2 Identification

We consider a single-input single-output true system with input \( u \) and output \( y \):

\[
y(t) = G_0(z)u(t) + H_0(z)\epsilon(t) = v(t)
\]

where \( v(t) = H_0(z)\epsilon(t) \) is the disturbance acting on the system. In (1), \( \epsilon(t) \) is a white noise with variance \( \sigma_v^2 \) and \( G_0(z) \) and \( H_0(z) \) are stable transfer functions. In addition, \( H_0(z) \) is also assumed to be inversely stable and monic.

The true system (1) will be identified in a BJ model structure i.e., \( \{G(z, \theta) = G(z, \rho), \ H(z, \theta) = H(z, \zeta) \mid \theta = (\rho^T, \zeta^T)^T \in \mathbb{R}^k \} \). The orders of \( G(z, \theta) \) and \( H(z, \theta) \) are chosen in such a way that there exists \( \theta_0 = (\rho_0^T, \zeta_0^T)^T \) such that \( G_0(z) = G(z, \theta_0) = G(z, \rho_0) \) and \( H_0(z) = H(z, \theta_0) = H(z, \zeta_0) \). We will denote by \( k_G \) (resp. \( k_H \)) the dimension of \( \rho_0 \) (resp. \( \zeta_0 \)) and we have thus \( k = k_G + k_H \).

If we apply a sequence \( \{u(t) \mid t = 1, \ldots, N\} \) of spectrum \( \Phi_u \) to (1) and collect the corresponding output \( \{y(t) \mid t = 1, \ldots, N\} \), an estimate \( \hat{\theta}_N \) of \( \theta_0 \) can be deduced using prediction error identification [13]:

\[
\hat{\theta}_N = \arg \min_{\theta} \frac{1}{N} \sum_{t=1}^{N} e^2(t, \theta)
\]

\[
e(t, \theta) = H^{-1}(z, \theta)(y(t) - G(z, \theta)u(t))
\]

The estimate \( \hat{\theta}_N \) is (asymptotically) normally distributed around \( \theta_0 \) with a covariance matrix \( P_{\theta}(\theta_0, \Phi_u) \) whose known expression is a function of \( \theta_0 \) and of the input spectrum \( \Phi_u \) used during the identification experiment [13]. It is important to note that \( P_{\theta}^{-1}(\theta_0, \Phi_u) \) is a measure of the accuracy of \( \hat{\theta}_N \) [10]. In general, we are mainly interested in the accuracy of the part \( \rho_N \) of the vector \( \hat{\theta}_N = (\hat{\rho}_N^T, \hat{\zeta}_N^T)^T \) (the part that defines the model \( G(z, \rho_N) \) of \( G(z, \rho_0) \)). The covariance matrix \( P_{\rho} \) of \( \hat{\rho}_N \) can be deduced from \( P_{\theta} \) as follows:

\[
P_{\rho}(\theta_0, \Phi_u) = (I_{k_G} \ 0) \ P_{\theta}(I_{k_G} \ 0)^T,
\]

and its inverse has the following expression as a function of \( \Phi_u \) and \( \theta_0 \) [10,3]:

\[
P_{\rho}^{-1}(\theta_0, \Phi_u) = \frac{N}{\sigma_v^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{F_u(e^{j\omega}, \theta_0)F_u^*(e^{j\omega}, \theta_0)\Phi_u(\omega) \ d\omega}{(\partial G(z, \rho) / \partial \rho, H^{-1}(z, \theta)(y(t) - G(z, \theta)u(t)))}
\]

with

\[
F_u(z, \theta) = H^{-1}(z, \zeta) \partial G(z, \rho) / \partial \rho
\]

In the sequel, we will suppose that an initial identification experiment has delivered an initial estimate \( \hat{\theta}_{\text{init}} = (\hat{\rho}_{\text{init}}^T, \hat{\zeta}_{\text{init}}^T)^T \) with covariance matrix \( P_{\theta,\text{init}} \).
Consequently, the following ellipsoid $U_{\text{init}}$ is a $\eta\%$-confidence region for the unknown true parameter vector $\theta_0$:

\[
U_{\text{init}} = \left\{ \theta \mid (\theta - \hat{\theta}_{\text{init}})^T P^{-1}_{\theta,\text{init}} (\theta - \hat{\theta}_{\text{init}}) \leq \chi \right\}
\]

(5)

with $\chi$ such that $P \chi^2(k) = \chi$ $(\eta \text{ say } 95 \%)$. The significance of this set is that, if the initial experiment and the estimation of $\hat{\theta}_{\text{init}}$ is repeated, the true parameter $\theta_0$ will belong to $U_{\text{init}}$ in the fraction $\eta$ of these essays. For this reason, $U_{\text{init}}$ can be used as a description of the uncertainty of the initial estimate. From now on, we will assume that $\theta_0$ indeed belongs to the ellipsoid $U_{\text{init}}$ constructed based on the initial estimate $\hat{\theta}_{\text{init}}$ and its covariance matrix $P_{\theta,\text{init}}$.

We will suppose that the accuracy $P^{-1}_{\theta,\text{init}}$ is not sufficient for the purpose of the identified model $(P_{\rho,\text{init}} = (U_{\theta,\text{init}}^T) P_{\theta,\text{init}} (U_{\theta,\text{init}})^T)$. In the sequel, we will deem an estimate of $\rho_0$ sufficiently accurate when the inverse of its covariance matrix satisfies $P^{-1}_\rho \geq R_{U_{\theta,\text{init}}}$ for a given positive-definite matrix $R_{U_{\theta,\text{init}}}$ in $R^{h \times k_{\theta}}$.

In order to obtain a sufficiently accurate estimate of $\theta_0$, we need to perform a second experiment that yields an estimate $\hat{\theta}_N = (\hat{\rho}_N^T, \hat{\zeta}_N^T)^T$ having the property that the covariance matrix $P_\rho(\theta_0, \Phi_u)$ of $\hat{\rho}_N$ is such that $P^{-1}_\rho(\theta_0, \Phi_u) \geq R_{U_{\theta,\text{init}}}$ $\Phi_u$ is the spectrum of the input signal used during the second experiment). Note that the initial and the second identification experiments can also be combined [13] and the inverse $P^{-1}_\rho(\theta_0, \Phi_u)$ of the covariance matrix corresponding to the second experiment must then satisfy: $P^{-1}_\rho(\theta_0, \Phi_u) + P^{-1}_{\rho,\text{init}} \geq R_{U_{\theta,\text{init}}}$. For the sequel, it will be important to make the following assumptions:

**Assumption 1** We restrict attention to parametrizations $G(z, \theta)$ and $H(z, \theta)$ that are rational functions of the parameter vector $\theta$:

\[
G(z, \theta) = \frac{Z_N(z) \theta}{1 + Z_D(z) \theta}
\]

(6)

\[
H(z, \theta) = \frac{1 + Z_{N,H}(z) \theta}{1 + Z_{D,H}(z) \theta}
\]

(7)

where $Z_N(z), Z_D(z), Z_{N,H}(z)$ and $Z_{D,H}(z)$ are row vectors of transfer functions.

**Assumption 2** The uncertainty $U_{\text{init}}$ defined in (5) is small enough to guarantee that, like $G(z, \theta_0)$ and $H^{-1}(z, \theta_0)$, $G(z, \theta)$ and $H^{-1}(z, \theta)$ are stable transfer functions for all $\theta \in U_{\text{init}}$. Due to (4), this also implies that $F_u(z, \theta)$ is a vector of stable transfer functions for all $\theta \in U_{\text{init}}$.

Note that the classical BJ parametrization used in prediction error identification satisfy Assumption 1 (see Appendix A). Moreover, we can easily verify whether a given $U_{\text{init}}$ satisfies the property mentioned in Assumption 2 using the results in [4].

### 3 Optimal experiment design

As done in the least costly experiment design framework [5], we will design the spectrum $\Phi_u$ of the excitation signal $u$ of the second experiment in such a way that the accuracy constraint is met with the least perturbation on the system $i.e.$ with the least identification cost. The perturbation on the system induced by $u$ will be here measured by a linear combination of the power of the input signal and of the power of the output signal:

\[
\mathcal{J}(\theta_0, \Phi_u) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + \beta |G(e^{j\omega}, \theta_0)|^2) \Phi_u(\omega) \, d\omega
\]

(8)

where $\beta$ is an user-chosen constant that weighs the two terms in $\mathcal{J}$. We observe that the cost $\mathcal{J}(\theta_0, \Phi_u)$ is a function of the unknown $\theta_0$ and of the chosen spectrum $\Phi_u$. Based on the above expression, the optimal experiment design can be formulated as the problem of determining the power spectrum $\Phi_u$ of the second experiment which guarantees that $P^{-1}_\rho(\theta_0, \Phi_u) + P^{-1}_{\rho,\text{init}} \geq R_{U_{\theta,\text{init}}}$ with the smallest cost $\mathcal{J}(\theta_0, \Phi_u)$. Like all other optimal experiment design problems, the above optimization problem unfortunately depends on the unknown true parameter vector. As mentioned in the introduction, we will here robustify this optimal experiment design problem using $U_{\text{init}}$. The robustified optimal experiment design problem is therefore:

\[
\min_{\Phi_u} \gamma \quad \text{s.t.} \quad \mathcal{J}(\theta_0, \Phi_u) \leq \gamma \quad \forall \theta \in U_{\text{init}}
\]

and $P^{-1}_\rho(\theta, \Phi_u) + P^{-1}_{\rho,\text{init}} \geq R_{U_{\theta,\text{init}}} \quad \forall \theta \in U_{\text{init}}$

(10)

(11)

If we denote by $\Phi_u^{\text{orig}}$ and $\gamma^{\text{orig}}$ the solution of this optimization problem, we have that $\gamma^{\text{opt}} = \sup_{\Phi_u \in \Phi_{\text{init}}} \mathcal{J}(\theta_0, \Phi_u^{\text{orig}})$. Moreover, the spectrum $\Phi_u^{\text{orig}}$ is, by construction, the spectrum $\Phi_u$ leading to the smallest value of $\sup_{\Phi_u \in \Phi_{\text{init}}} \mathcal{J}(\theta_0, \Phi_u)$ while guaranteeing the robustified accuracy constraint (11). Since we assume that $\theta_0 \in U_{\text{init}}$, this robustified formulation ensures 1) that the a-priori unknown cost $\mathcal{J}(\theta_0, \Phi_u^{\text{orig}})$ is smaller than $\gamma^{\text{opt}}$ and 2) that $P^{-1}_\rho(\theta, \Phi_u^{\text{orig}})$ is guaranteed to satisfy $P^{-1}_\rho(\theta_0, \Phi_u^{\text{orig}}) + P^{-1}_{\rho,\text{init}} \geq R_{U_{\theta,\text{init}}}$.

The above optimization problem will be a convex optimization problem if (10) and (11) can be transformed into two constraints that are linear in the decision variables $\Phi_u$ and $\gamma$. In the sequel, we will show that, as very often in robustness analysis theory, we cannot find tractable linear constraints that are equivalent to (10) and (11), but we can find one that implies (10) and another one that implies (11). This entails a certain conservatism. However, if we solve the optimization problem...
with these alternative constraints and if we denote its solution by $\gamma_{opt}$ and $\Phi_{u,opt}$, we still have the guarantee that

$$P_{p}^{-1}(\theta, \Phi_{u,opt}) + P_{p,\text{init}}^{-1} \geq R_{adm} \quad \forall \theta \in U_{\text{init}}$$

and thus

$$P_{p}^{-1}(\theta, \Phi_{u,opt}) + P_{p,\text{init}}^{-1} \geq R_{adm}$$

(2) $J(\theta_{0}, \Phi_{u,opt}) \leq \sup_{\theta \in U_{\text{init}}} J(\theta, \Phi_{u,opt}) \leq \gamma_{opt}$.

In addition, we have also that $\gamma_{opt}$ is an upper bound for $\gamma$, the solution of the original optimization problem (9)-(11).

We will derive the tractable alternative constraints discussed in the previous paragraph in the case of a commonly used parametrization of the to-be-determined spectrum $\Phi_u$ [10]. This spectrum parametrization corresponds to the spectrum of a multisine signal at fixed frequencies $\omega_m$ ($m = 1, \ldots, L$) but with arbitrary amplitudes $^1$:

$$\Phi_u(\omega) = \pi \sum_{m=1}^{L} c_m (\delta(\omega - \omega_m) + \delta(\omega + \omega_m)) \geq 0 \forall \omega$$

(12)

The positivity of $\Phi_u(\omega)$ for all $\omega$ can be imposed by the constraints $c_m \geq 0$ ($m = 1, \ldots, L$). Using (12), the constraint (10) can be rewritten successively as follows:

$$\sum_{m=1}^{L} c_m (1 + \beta |G(e^{j\omega_m}, \theta)|^2) \leq \gamma \quad \forall \theta \in U_{\text{init}}$$

(13)

with $G(\theta) = (G(e^{j\omega_1}, \theta), G(e^{j\omega_2}, \theta), \ldots, G(e^{j\omega_L}, \theta))^T$ and $\bar{C} = \text{diag}(c_1, c_2, \ldots, c_L)$. Using (3) and (12), the term $P_{p}^{-1}(\theta, \Phi_u)$ in (11) can also be rewritten as follows:

$$P_{p}^{-1}(\theta, \Phi_u) = \frac{N}{2 \sigma_2^2} \sum_{m=1}^{L} c_m$$

$$\left( F_u(e^{j\omega_m}, \theta) F_u^*(e^{j\omega_m}, \theta) + F_u(e^{-j\omega_m}, \theta) F_u^*(-e^{-j\omega_m}, \theta) \right)$$

$$\left( \sum_{m=1}^{L} c_m (1 + \beta |G(e^{j\omega_m}, \theta)|^2) \right)^{\frac{1}{2}}$$

(15)

$$P_{p}^{-1}(\theta, \Phi_u) = \frac{N}{2 \sigma_2^2} \sum_{m=1}^{L} (e_i \otimes F_i(\theta))^{*} \bar{C} (e_i \otimes F_i(\theta))$$

where $e_i$ ($i = 1, \ldots, k_G$) is a unit vector of dimension $1 \times k_G$ whose entries are all zero except the $i^{th}$ entry which is equal to 1 and where $F_i(\theta)$ ($i = 1, \ldots, k_G$) is a complex vector of dimension $L \times 1$ defined as:

$$F_i(\theta) = (F_{u,1}(e^{i\omega_1}, \theta), F_{u,1}(e^{i\omega_2}, \theta), \ldots, F_{u,1}(e^{i\omega_L}, \theta))^T$$

(16)

with $F_{u,i}(z, \theta)$ the $i^{th}$ entry of the vector $F_u(z, \theta)$.

Remark. In (15), the actual value of the variance $\sigma_e^2$ of the white noise $e$ (see (1)) is generally unknown. However, we can also robustify the optimal experiment design problem against this uncertainty. For this purpose, we can replace $\sigma_e^2$ in (15) by $\sigma_{e,\text{max}}^2$ where $\sigma_{e,\text{max}}^2$ is the maximal value of the $\gamma\%$-confidence interval $[\sigma_{e,\text{min}}^2, \sigma_{e,\text{max}}^2]$ for $\sigma_e^2$ that can be constructed using the initial identification experiment (the one yielding $\theta_{\text{init}}$) [13].

In the next two sections, we will derive, using robustness analysis tools, tractable alternatives for both (10) and (11) when the parametrization (12) is used for $\Phi_u$. These tractable alternatives will be under the form of Linear Matrix Inequality (LMI) constraints [6] that are tractable alternative constraints for (13)-(14). The constraints (13)-(14) are indeed equivalent to (10) when the parametrization (12) is used for $\Phi_u$. In the paper [15], we have proposed the following tractable alternative for (13)-(14):

$$\sum_{m=0}^{L} c_m (1 + \beta \alpha_G(\omega_m)) \leq \gamma$$

(17)

with $\alpha_G(\omega_m) = \sup_{\theta \in U_{\text{init}}} |G(e^{j\omega_m}, \theta)|^2$ (a computable quantity for each $\omega_m$ [4]). It is clear that (17) implies (13) and that it is linear in $\gamma$ and in $c_m$ ($m = 1, \ldots, L$). However, the constraint (17) is rather conservative since $\alpha_G(\omega_m) = \sup_{\theta \in U_{\text{init}}} |G(e^{j\omega_m}, \theta)|^2$ can be obtained for different $\theta$ at different frequencies $\omega_m$ ($m = 1, \ldots, L$). In this paper, we propose an alternative approach which, as will be shown in the example section, will generally be less conservative since it will explicitly take into account the dependency on $\theta$ of the frequency response elements $G(e^{j\omega_m}, \theta)$ in $G(\theta)$ (see (14)).

4 Tackling the robustified cost constraint using robustness analysis tools

4.1 Introduction

We will start by deriving a tractable alternative for (13)-(14). The constraints (13)-(14) are indeed equivalent to (10) when the parametrization (12) is used for $\Phi_u$. In the paper [15], we have proposed the following tractable alternative for (13)-(14):
Recall now that the Fourier transform $\hat{y}(e^{j\omega})$ of $y(t) = G(z, \theta)u(t)$ is equal to $G(e^{j\omega}, \theta)$ when $u(t)$ is equal to a pulse signal $\delta(t)$ (i.e. $u(e^{j\omega}) = 1$). Consequently, the frequency response $G(e^{j\omega}, \theta)$ of $G(z, \theta)$ at one given frequency $\omega$ can also be deduced by solving for $\hat{y}(e^{j\omega})$ in the following system of equations:

$$
\begin{align*}
q(e^{j\omega}) &= \theta \, p(e^{j\omega}) \\
\hat{y}(e^{j\omega}) &= M_G(e^{j\omega}) \begin{pmatrix} p(e^{j\omega}) \\ 1 \end{pmatrix}
\end{align*}
$$

(19)

Note that, in this system of equations, all Fourier transforms are well defined for all $\theta \in U_{init}$ due to Assumption 2.

Using the same reasoning, the vector $G(\theta)$, containing the frequency response of $G(z, \theta)$ at the frequencies present in the spectrum (12), can be determined by solving for $\hat{y}$ in the system of equations (20) derived using (19):

$$
\begin{align*}
\bar{p} &= (I_L \otimes \theta) \, \bar{q} \\
\begin{pmatrix} \bar{q} \\ \bar{y} \end{pmatrix} &= \begin{pmatrix} M_{11,G} & M_{12,G} \\ M_{21,G} & M_{22,G} \end{pmatrix} \begin{pmatrix} \bar{p} \\ 1 \end{pmatrix}
\end{align*}
$$

(20)

with $\bar{p} = (p^T(e^{j\omega_1}), ..., p^T(e^{j\omega_L}))^T$, $\bar{q} = (q(e^{j\omega_1}), ..., q(e^{j\omega_L}))^T$ and

$$
\begin{align*}
M_{11,G} &= -bdiag(Z_D(e^{j\omega_1}), ..., Z_D(e^{j\omega_L})) \\
M_{12,G} &= (1, ..., 1)^T \\
M_{21,G} &= bdiag(Z_N(e^{j\omega_1}), ..., Z_N(e^{j\omega_L})) \\
M_{22,G} &= 0
\end{align*}
$$

Remark. It is important to note that the relation between $\bar{p}$ and $\bar{q}$ in (20) is in $I_L \otimes \theta = bdiag(\theta, ..., \theta)$ i.e. a repetition of the same value. This is due to the dependency on $\theta$ of the frequency response elements $G(e^{j\omega}, \theta)$ in $G(\theta)$. The approach in [15] (see Section 4.1) would in fact correspond to a relation between $\bar{p}$ and $\bar{q}$ of the form
dag$(\theta(\omega_1), \theta(\omega_2), ..., \theta(\omega_L))$ where $\theta(\omega_m) \in U_{init}$ for all $\omega_m$ ($m = 1, ..., L$), but can be different for each frequency $\omega_m$ ($m = 1, ..., L$). Indeed, $\alpha_G(\omega_m)$ can be rewritten as $\sup_{(\omega_m) \in U_{init}} |G(e^{j\omega_m}, \theta(\omega_m)) |^2$ ($m = 1, ..., L$).

4.3 Set of multipliers related to the uncertainty set $U_{init}$

Since we consider here (14), the parameter vector $\theta$ in the LFT for $G(\theta)$ is restricted to be in the uncertainty set $U_{init}$ (see (5)). In our approach, a necessary ingredient to find a tractable alternative for (14) is to associate, with the set $U_{init}$, a so-called set of multipliers. In a nutshell, the set of multipliers $A_n$ that we will consider in this paper is an explicit and affine parametrization of the quadratic constraint satisfied by the graphs of the signals $q_n$ and $p_n$ when $p_n(t) = (I_n \otimes \theta)q_n(t)$ with $\theta \in U_{init}$ ($n$ is an arbitrary integer such that $n \geq 1$) [20,9,14].

**Definition 1** Consider the set $U_{init}$ defined in (5) satisfying Assumption 2 and an arbitrary integer $n \geq 1$. For each value of $n$, we define the set of multipliers $A_n$ as a set of affinely parametrized Hermitian matrices $A_n$ (of dimension $n(k+1) \times n(k+1)$) that all have the following property:

$$
\begin{pmatrix} I_n \\ I_n \otimes \theta \end{pmatrix} A_n \begin{pmatrix} I_n \\ I_n \otimes \theta \end{pmatrix}^T \geq 0 \forall \theta \in U_{init}
$$

(21)

In other words, $A_n \in A_n \implies (21)$.

It is important to stress that the more extensive the parametrization of the set of multipliers, the smaller the conservatism discussed in Section 3 will be [20,9,14]. The set of multipliers $A_n$ corresponding to $U_{init}$ can be easily derived from our previous contribution (see Proposition 2 of [2]).

In [2], the set of multipliers $A_n$ is in fact developed for an uncertainty set of the form $U_{init} = \{ \delta \theta | \delta \theta^T P^{-1} \delta \theta \leq 1 \}$. As shown in these papers, the use of this set of multipliers therefore entails the straightforward transformation of the LFT (18) (i.e. an LFT in $\theta$) into an LFT in $\delta \theta = \theta - \theta_{init}$. Another option is to adapt the multipliers of [2] to an uncertainty set $U_{init}$ that is not centered at zero.

Remark. It is also to be noted that the set of multipliers in Definition 1 can also be derived for uncertainty sets $U_{init}$ that are not ellipsoidal. In other words, the approach presented in this paper is valid for all uncertainty sets $U_{init}$ for which the sets of multipliers $A_n$ of Definition 1 can be constructed.

4.4 Robustified cost constraint

Taking a set of multipliers $A_n$ with $n$ equal to the number of frequencies in the to-be-determined spectrum (i.e., $n = L$) and considering the LFT representation (20) of $G(\theta)$, we have now all the ingredients to derive a tractable alternative constraint for (14).

**Proposition 1** Consider an initial uncertainty set $U_{init}$ (see (5)) satisfying Assumption 2 and the robust cost constraint (14) obtained when the spectrum $\Phi_u$ is parametrized as in (12). Consider the LFT representation (20) for $G(\theta)$ as well as the set of multipliers $A_n$ associated with $U_{init}$ (see Definition 1 with $n = L$). Then, the constraint (14) holds for a given $\gamma$ if we can find a matrix $A_L \in A_L$ such that

$$
V^* A_L \, V + L^* \, C \, L \leq \begin{pmatrix} 0 \\ 0 \\ \gamma \sum_{\beta=1}^{n} c_{\beta} \end{pmatrix}
$$

(22)

Note that, in [2], the notations $A$ (resp. $\tilde{n}$) are used instead of $A_n$ (resp. $n$).
where \( \mathcal{L} = \left( \begin{array}{cc} M_{21, \theta} & M_{22, \theta} \end{array} \right) \) and \\
\( \mathcal{V} = \left( \begin{array}{cc} \bar{M}_{11, \theta} & \bar{M}_{12, \theta} \\
I & 0 \end{array} \right) \)

We observe that the matrix inequality (22) is linear in \( \gamma, A_L \) and in the coefficients \( c_m \) (\( m = 1, \ldots, L \)) present in \( \hat{C} \).

Proof. Let us consider (20) for a given \( \theta \in U_{\text{init}} \) and let us consider the corresponding signals \( \bar{p}, \bar{q} \) and \( \bar{\gamma} = \hat{G}(\theta) \). Let us then pre- and post-multiply the LMI constraint (22) with \( (\bar{p}^T, 1) \) and \( (\bar{q}^T, 1)^T \), respectively. Using (20) and \( \bar{\gamma} = \hat{G}(\theta) \), this yields:

\[
\left( \begin{array}{c}
\bar{q} \\
\bar{p}
\end{array} \right)^* A_L \left( \begin{array}{c}
\bar{q} \\
\bar{p}
\end{array} \right) + \hat{G}(\theta) \hat{C} \hat{G}(\theta) \leq \gamma - \frac{\sum_{m=1}^L c_m}{\beta} \left( \begin{array}{c}
\bar{q} \\
\bar{p}
\end{array} \right)
\]

(23)

Since \( \bar{p} = (I_L \otimes \theta) \bar{q} \), we can rewrite (23) as follows:

\[
\bar{q}^* \left( \begin{array}{c}
I_L \\
I_L \otimes \theta
\end{array} \right) A_L \left( \begin{array}{c}
I_L \\
I_L \otimes \theta
\end{array} \right) \bar{q} + \hat{G}(\theta) \hat{C} \hat{G}(\theta) \leq \gamma - \frac{\sum_{m=1}^L c_m}{\beta} \left( \begin{array}{c}
\bar{q} \\
\bar{p}
\end{array} \right)
\]

(24)

The above reasoning can be done for any value of \( \theta \in U_{\text{init}} \). In other words, for the matrix \( A_L \in \mathcal{A}_L \) found by the optimization problem, (24) holds true for all \( \theta \in U_{\text{init}} \). Consequently, using Definition 1 with \( n = L \), we have therefore also that \( \hat{G}(\theta) \hat{C} \hat{G}(\theta) \leq \gamma - \frac{\sum_{m=1}^L c_m}{\beta} \) for each \( \theta \in U_{\text{init}} \); which is the desired result. \( \square \)

In Proposition 1 (and also later in this paper), when we speak of finding a matrix \( A_L \in \mathcal{A}_L \), we more precisely mean finding the free parameters in the affine structure of the matrix \( A_L \).

5 Robustified accuracy constraint

Taking inspiration from what has been done for the robustified cost constraint, we will now derive a tractable alternative for the accuracy constraint (11) when the parametrization (12) is used for \( \Phi_u \). Recall the expression (15) for \( P_{\rho^{-1}}(\theta, \Phi_u) \) and let us observe that \( F_u(z, \theta) \) (see (4)) is a rational function of \( \theta \) due to Assumption 1. Consequently, we can find signals \( p_F \) and \( q_F \) such that \( s(t) = F_u(z, \theta)u(t) \) can be expressed as:

\[
p_F = (I_f \otimes \theta) \quad q_F \\
\left( \begin{array}{c}
q_F \\
\bar{s}
\end{array} \right) = \left( \begin{array}{c}
M_{11,F} & M_{12,F} \\
M_{21,F} & M_{22,F}
\end{array} \right) \left( \begin{array}{c}
p_F \\
\bar{s}
\end{array} \right) \left( \begin{array}{c}
u \\
M_F(z)
\end{array} \right)
\]

(25)

where \( f = 3 \) as shown in Appendix A. Note that \( f = 2 \) in the case where \( H(z, \theta) = 1 \) (OE model structure).

Using a similar reasoning as in Section 4.2, we can derive from (25) an LFT expression for \( \bar{s} = F_l(\theta) \) (\( i = 1, \ldots, k_G \)) defined in (16). If we denote by \( \bar{s} = (s^T_1, s^T_2, \ldots, s^T_{k_G})^T \), we have indeed:

\[
\left( \begin{array}{c}
\bar{q}_F \\
\bar{s}
\end{array} \right) = \left( \begin{array}{c}
M_{11,F} & M_{12,F} \\
M_{21,F} & M_{22,F}
\end{array} \right) \left( \begin{array}{c}
p_F \\
\bar{s}
\end{array} \right) \left( \begin{array}{c}
u \\
M_F(z)
\end{array} \right)
\]

(26)

\[
M_{11,F} = \text{bdia}(M_{11,F}(e^{j\omega_1}), \ldots, M_{11,F}(e^{j\omega_L})) \\
M_{12,F} = (M_{12,F}(e^{j\omega_1}), \ldots, M_{12,F}(e^{j\omega_L}))^T \\
M_{21,F} = (H^T_1, \ldots, H^T_{k_G})^T \\
M_{22,F} = (K^T_1, \ldots, K^T_{k_G})^T \\
M_F(z) = \text{bdia}(M_{21,F}^i(e^{j\omega_1}), \ldots, M_{22,F}^i(e^{j\omega_L})) \quad (i = 1, \ldots, k_G)
\]

where \( M_{21,F}^i \) (resp. \( M_{22,F}^i \)) denotes the \( i \)th line of \( M_{21,F} \) (resp. the \( i \)th entry of \( M_{22,F} \)). Note that, in this system of equations, all elements are well defined for all \( \theta \in U_{\text{init}} \) due to Assumption 2.

We have then the following result that gives a tractable alternative for the robustified accuracy constraint (11).

**Proposition 2** Consider an initial uncertainty set \( U_{\text{init}} \) (see (5)) satisfying Assumption 2 and the robustified accuracy constraint (11) when the spectrum \( \Phi_u \) is parameterized as in (12) and where, for this reason, \( P_{\rho^{-1}}(\theta, \Phi_u) \) has the expression given in (15). Consider the LFT (26) in \( f \) \( \otimes \) \( \theta \) which is the LFT representation for \( \bar{s} = (F_l(\theta))^T, F_2(\theta)^T, \ldots, F_{k_G}(\theta)^T \) and consider the set of multipliers \( \eta_{G,f} \) associated with \( U_{\text{init}} \) (see Definition 1 with \( n = k_G f L \)). Then, the constraint (11) holds for a given \( R_{\text{adm}} \) and a given \( P_{\rho, \text{init}} \) if we can find a matrix \( \eta_{G,f} \in \eta_{G,f} \) such that \( ^{3}\)

\[
\frac{N}{2} \sigma^2 \sum_{i=1}^{k_G} \sum_{j=1}^{k_G} (e_i \otimes X_i) C(e_j \otimes X_j) + (e_i \otimes X_j) C(e_j \otimes X_i)
\]

\[
+ \left( (P_{\rho, \text{init}}^{-1} - R_{\text{adm}}) \otimes \left( \begin{array}{cc}
0 & 0 \\
0 & 1 \end{array} \right) \right) - M^* \eta_{G,f} M \geq 0
\]

(27)

where \( X_i = \left( H_i, K_i \right) \quad (i = 1, \ldots, k_G) \) and

\[
M = \left( I_{k_G} \otimes \left( \begin{array}{c}
M_{11,F} & M_{12,F} \\
M_{21,F} & M_{22,F}
\end{array} \right) \right)
\]

We observe that the matrix inequality (27) is linear in \( \eta_{G,f} \) and in the coefficients \( c_m \) (\( m = 1, \ldots, L \)) present

---

3 The matrix made of zeros and a one in the second line of (27) has dimension \((f L k + 1) \times (f L k + 1)\).
Proof. Let us consider (26) for a given \( \theta \in U_{\text{init}} \) and let us consider the corresponding signals \( \tilde{p}, \tilde{q} \) and \( \bar{s} = (F_1(\theta)^T, F_2(\theta)^T, \ldots, F_K(\theta)^T)^T \). Let us then pre- and post-multiply the LMI constraint (27) with \( (I_{K_{G}} \otimes (\bar{p}_{F}^T,1)^T)^* \) and \( I_{K_{G}} \otimes (\bar{p}_{F}^T,1)^T \), respectively. Using (26), (15) and the lemma in Appendix B, this yields:

\[
P_{\rho}^{-1}(\theta, \Phi_u) + P_{\rho,\text{init}}^{-1} - R_{\text{adm}} - X(\theta) \geq 0 \quad (28)
\]

where \( P_{\rho}^{-1}(\theta, \Phi_u) \) is given by (15) and \( X(\theta) \) is a matrix given by:

\[
X(\theta) = \begin{pmatrix} I_{K_{G}} \otimes \tilde{q} \tilde{r} & I_{K_{G}} \otimes \tilde{q} \tilde{r} \end{pmatrix} A_{k_{G}fL} \begin{pmatrix} I_{K_{G}} \otimes \tilde{q} \tilde{r} & I_{K_{G}} \otimes \tilde{p} \tilde{r} \end{pmatrix}^T
\]

Since \( \tilde{p}_F = (I_{fL} \otimes \theta) \tilde{q}_F \), \( X(\theta) \) can be rewritten as:

\[
X(\theta) = (I_{k_{G}} \otimes \tilde{q})^* Y(\theta) (I_{k_{G}} \otimes \tilde{q})
\]

\[
Y(\theta) = \begin{pmatrix} I_{k_{G}fL} & I_{k_{G}fL} \end{pmatrix}^T A_{k_{G}fL} \begin{pmatrix} I_{k_{G}fL} & I_{k_{G}fL} \end{pmatrix}
\]

The above reasoning can be done for any value of \( \theta \in U_{\text{init}} \). In other words, for the matrix \( A_{k_{G}fL} \in A_{k_{G}fL} \) found by the optimization problem, (28) holds true for all \( \theta \in U_{\text{init}} \). Using Definition 1 with \( n = k_{G}fL \) and the Lemma in Appendix B, note also that:

\[
X(\theta) \geq 0 \quad \forall \theta \in U_{\text{init}}
\]

Consequently, we have that \( P_{\rho}^{-1}(\theta, \Phi_u) + P_{\rho,\text{init}}^{-1} \geq R_{\text{adm}} \) for each \( \theta \in U_{\text{init}} \); which is the desired result. \( \blacksquare \)

6 Convex formulation of the optimal experiment design problem

Using Propositions 1 and 2, we can now straightforwardly derive a convex formulation for the robust optimal experiment design problem (9)-(11). This formulation will be under the form of a LMI optimization problem [6].

LMI formulation Consider the parametrization (12) for the to-be-designed spectrum \( \Phi_u \). The LMI optimization problem has as decision variables a scalar \( \gamma > 0 \), coefficients \( c_m \geq 0 \) (\( m = 1, \ldots, L \)), a matrix \( A_L \in A_L \), and a matrix \( A_{k_{G}fL} \in A_{k_{G}fL} \) (see Definition 1) and consists in determining the smallest value of \( \gamma \) for which the LMI constraints (22) and (27) hold.

As mentioned in Section 3, if we denote the solution of the above LMI optimization problem by \( \gamma_{\text{opt}} \) and \( \Phi_{u,\text{opt}} \), \( \gamma_{\text{opt}} \) is an upper bound for the solution \( \gamma_{\text{orig}} \) of the original robustified optimization problem (9)-(11) i.e. the minimal value of the cost that is required to guarantee the robust accuracy constraint (11). Moreover, \( \gamma_{\text{opt}} \) is an upper bound for \( \sup_{\theta \in U_{\text{init}}} J(\theta, \Phi_{u,\text{opt}}) \) and, since we assume \( \theta_0 \in U_{\text{init}} \), we have the guarantees 1) that \( P_{\rho}^{-1}(\theta_0, \Phi_{u,\text{opt}}) + P_{\rho,\text{init}} \geq R_{\text{adm}} \) and 2) that \( J(\theta_0, \Phi_{u,\text{opt}}) \leq \gamma_{\text{opt}} \).

Remark. In this paper, we have considered the least costly optimal experiment framework and we have supposed that the desired accuracy is represented by a given matrix \( R_{\text{adm}} \). As mentioned in the introduction, the results of this paper can also be used for other optimal experiment design frameworks. For example, let us consider the classical E-optimality framework that, when robustified, consists in:

\[
\max_{\Phi_u, \varepsilon} \varepsilon
\]

such that \( \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi_u(\omega) d\omega \leq \gamma \)

and \( P_{\rho}^{-1}(\theta, \Phi_u) + P_{\rho,\text{init}}^{-1} \geq \varepsilon I_{k_{G}} \quad \forall \theta \in U_{\text{init}} \) (29)

where \( \varepsilon \) is a scalar decision variable and \( \gamma \) is a given bound on the power of the input signal. It is clear that (29) can be tackled with the tools of Section 5. Note that, here, \( R_{\text{adm}} = \varepsilon I_{k_{G}} \) is not fixed, but a decision variable. This however does not pose a problem since the constraint (27) is also linear in \( R_{\text{adm}} \).

7 Numerical illustration

In this section we present simulation results in order to show the effectiveness of our approach. We consider the following system \( y(t) = G_0(z)u(t) + e(t) \) with \( G_0(z) = \frac{1}{1-z^{-1}} \) and \( e(t) \) a white noise having variance \( \sigma^2 = 1 \). The true parameter vector \( \theta_0 \) is thus \( \theta_0 = \rho_0 = (1, -0.7)^T \). An initial estimate \( \hat{\theta}_{\text{init}} \) of \( \theta_0 \) and its covariance matrix \( P_{\theta,\text{init}} \) have been obtained using an experiment of duration \( N = 1000 \) with a white noise input signal of variance \( 0.1 \): \( \hat{\theta}_{\text{init}} = (0.904, -0.7161)^T \). This allows to build the initial uncertainty \( U_{\text{init}} \) (see (5)) that is a confidence region for \( \theta_0 \) with probability level \( \eta = 95\% \) (\( \chi = 5.99 \)). This initial estimate does not satisfy the desired accuracy which is here that the standard deviation of the two parameters is smaller than two percent of their exact value. Based on this requirement, \( R_{\text{adm}} \) is chosen as the inverse of the following diagonal matrix \( \text{diag}(0.02)^2, (0.014)^2 \) (see e.g., [8]).

We use the LMI formulation of Section 6 to determine the spectrum \( \Phi_u \) of a second identification experiment of duration \( N = 1000 \) that will minimize a robustified version of the cost (10) under the robust accuracy constraint (11). Note that we here choose \( \beta = 1 \) in the expression of the cost (8). In (12), we choose \( L = 9 \) with \( \omega_{\text{in}} \) (\( m = 1, \ldots, 9 \)) covering the interval [0.1 3]. With these settings, the LMI optimization problem of Section 6 yields \( \gamma_{\text{opt}} = 5.4948 \) and an optimal multisine of spectrum \( \Phi_{u,\text{opt}} \) for which all the
amplitudes are negligible except the ones at $\omega = 0.5$ and $\omega = 1$. By construction, we know that, if an excitation signal of spectrum $\Phi_{u,g,\text{opt}}$ is used in the second identification experiment, the obtained accuracy will be satisfactory (i.e. $P^{-1}(\theta_i, \Phi_{u,g,\text{opt}}) + P^{-1}_{\rho,\text{init}} \geq R_{\text{adm}}$) and the cost $J(\theta_i, \Phi_{u,\text{opt}})$ of this second identification experiment will be such that $J(\theta_i, \Phi_{u,\text{opt}}) \leq \gamma_{\text{opt}} = 5.4948$. The value $\gamma_{\text{opt}}$ is indeed an upper bound for $\sup_{\theta \in U_{\text{init}}} J(\theta, \Phi_{u,\text{opt}})$ as discussed in Section 3 and thus also an upper bound for the a-priori unknown cost $J(\theta_i, \Phi_{u,\text{opt}})$ which is here equal to 4.7.

In order to check the conservatism linked to the proposed LMI formulation, we will compare the above result with the one that is obtained using the gridding approach for robust optimal experiment design. This approach considers the following optimization problem:

$$\min_{\Phi_{u,g}, \gamma_g} \gamma_g$$

such that $J(\theta_i, \Phi_{u,g}) \leq \gamma_g \ \forall \theta_i \in \Theta_n$ and $P^{-1}(\theta_i, \Phi_{u,g}) + P^{-1}_{\rho,\text{init}} \geq R_{\text{adm}} \ \forall \theta_i \in \Theta_n$

where $\Theta_n$ is a set containing $n$ grid points $\theta_i$ ($i = 1, \ldots, n$) such that $\theta_i \in U_{\text{init}}$. We have solved the above optimization problem for $n = 25$ using the same spectrum parametrization as above (same number $L$ of frequencies and same frequencies $\omega_n$) and we have obtained $\gamma_{g,\text{opt}} = 5.4866$ and a spectrum $\Phi_{u,g,\text{opt}}$ which has also contributions at two frequencies $\omega = 0.5$ and $\omega = 1$, but with (slightly) different amplitudes.

As mentioned in Sections 3 and 6, $\gamma_{g,\text{opt}}$ (obtained with the approach proposed in this paper) is an upper bound for the solution $\gamma_{\text{orig}}$ of the original robustified optimization problem (9)-(11). It is also clear that the value $\gamma_{g,\text{opt}}$ obtained with the gridding approach is a lower bound for the a-priori unknown cost $J(\theta_i, \Phi_{u,\text{opt}})$ of the second experiment.

To verify this property, we have generated 1000 grid points $\theta_i$ in $U_{\text{init}}$ and we have computed $J(\theta_i, \Phi_{u,\text{opt}})$ and $P^{-1}(\theta_i, \Phi_{u,\text{opt}})$ for these 1000 grid points. For all these grid points, we have indeed observed that $P^{-1}(\theta_i, \Phi_{u,\text{opt}}) + P^{-1}_{\rho,\text{init}} \geq R_{\text{adm}}$. We have also observed that the smallest eigenvalue of $P^{-1}(\theta_i, \Phi_{u,\text{opt}}) + P^{-1}_{\rho,\text{init}} - R_{\text{adm}}$ is, for one of these grid points $\theta_i$, equal to 0.0004 (and thus very close to zero) and that the cost $J(\theta_i, \Phi_{u,\text{opt}})$ is equal to 5.4948 for one of these grid points. This once again confirms that the conservatism of our approach is very limited in this example. Let us now compute, for these 1000 grid points, $P^{-1}(\theta_i, \Phi_{u,g,\text{opt}})$ with the spectrum $\Phi_{u,g,\text{opt}}$ obtained with the gridding approach. Here, 82 of the $\theta_i$ led to a matrix $P^{-1}(\theta_i, \Phi_{u,g,\text{opt}})$ for which $P^{-1}(\theta_i, \Phi_{u,g,\text{opt}}) + P^{-1}_{\rho,\text{init}} \geq R_{\text{adm}}$ is not satisfied. This shows the clear advantage of the approach of this paper upon the gridding approach.

Finally, let us illustrate the discussion in Section 4.1. For this purpose, we will compare the upper bound for $\sup_{\theta \in U_{\text{init}}} J(\theta, \Phi_{u})$ given by the left-hand side of (17) and the one corresponding to the LMI formulation proposed in this paper. We will do that for $\Phi_{u,\text{opt}}$. We know that the upper bound for $\sup_{\theta \in U_{\text{init}}} J(\theta, \Phi_{u,\text{opt}})$ obtained using the tools proposed in this paper is $\gamma_{\text{opt}} = 5.4948$. Let us now compute the upper bound $\gamma_{\text{a}} = \sum_{\alpha} c_{\alpha,\text{opt}} (1 + \alpha \tilde{\alpha}(\tilde{\omega} \in))$ for $\sup_{\theta \in U_{\text{init}}} J(\theta, \Phi_{u,\text{opt}})$ (see (17)). This yields $\gamma_{\text{a}} = 5.74108$, which shows that the approach discussed in Section 4.1 is more conservative.

8 Conclusion

In this paper, we have presented a convex relaxation that allows to robustify the least costly optimal experiment design problem using an initial uncertainty set for the unknown true parameter vector $\theta_0$. This robustification is obtained using tools from robustness analysis. In this paper, we have restricted attention to multisine excitation signals. In the future, we will investigate whether the least costly optimal experiment design problem can also be robustified when the to-be-determined excitation spectrum is the one of a filtered white noise. This spectrum parametrization is indeed also a commonly used parametrization in optimal experiment design [10]. We will also investigate how the results presented in this paper can be extended to tackle the robustification of more complex accuracy constraints (such as the ones presented in [5]).

References


A LFT representation of $F_u(z, \theta)$

The classical BJ parametrization is as follows:

$$G(z, \theta) = \frac{z^{-n_k}(b_0 + b_1 z^{-1} + \ldots + b_{n_k} z^{-n_k})}{1 + f_1 z^{-1} + \ldots + f_{n_f} z^{-n_f}}$$

$$H(z, \theta) = \frac{1 + c_1 z^{-1} + \ldots + c_{n_c} z^{-n_c}}{1 + d_1 z^{-1} + \ldots + d_{n_d} z^{-n_d}}$$

with $\theta = (\rho^T, \zeta^T)^T \in \mathbb{R}^p$ with $\rho = (b_0, \ldots, b_{n_k}, f_1, \ldots, f_{n_f})^T$ and $\zeta = (c_1, \ldots, c_{n_c}, d_1, \ldots, d_{n_d})^T$ ($k_G = n_b + n_f + 1$,

$$k_H = n_c + n_d, \ k = k_G + k_H$$. This parametrization satisfies Assumption 1. As an example, if $n_k = n_f = n_c = n_d = 1$ and $n_k = 0$ (i.e., $\theta = (b_0, f_1, c_1, d_1)^T$), we have $Z_N(z) = (z^{-1} 0 0 0)$, $Z_D(z) = (0 z^{-1} 0)$, $Z_{N,H}(z) = (0 0 z^{-1} 0)$ and $Z_{D,H}(z) = (0 0 0 z^{-1})$. Moreover, we have also that $s(t) = F_u(z, \theta) u(t)$ is given by:

$$s(t) = \begin{pmatrix} s_1(t) \\ \vdots \\ s_{n_k+1} \\ \vdots \\ s_{k_G}(t) \end{pmatrix} = \begin{pmatrix} \frac{z^{-n_k}}{1 + Z_{D,H}(z) \theta} \\ \vdots \\ \frac{z^{-n_k+n_f}}{1 + Z_{D,H}(z) \theta} \\ \vdots \\ \frac{z^{-n_k+n_c}}{1 + Z_{N,H}(z) \theta} \end{pmatrix} \frac{1 + Z_{D,H}(z) \theta}{1 + Z_{N,H}(z) \theta} u(t)$$

which is a rational function in $\theta$. This rational function can be expressed as in (25) with $f = 3$ (see Figure A.1):
B  Useful lemma for the proof of Proposition 2

Lemma 1  Consider an Hermitian matrix $A = A^*$ of dimension $n \times n$ and a (complex) matrix $B$ of dimension $n \times \tilde{n}$. Then, we have that

$$A \geq 0 \implies B^* A B \geq 0$$

Proof. $B^* A B \geq 0$ is equivalent to the fact that, for all complex vector $x$ of dimension $\tilde{n}$,

$$x^* B^* A B x \geq 0$$

Denoting $y$ the complex vector $B x$ of dimension $n$, the latter matrix inequality is equivalent to:

$$y^* A y \geq 0$$

which always holds since $A \geq 0$.  \[\blacksquare\]