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ON GENUS ONE MIRROR SYMMETRY IN HIGHER DIMENSIONS AND THE BCOV CONJECTURES

DENNIS ERIKSSON, GERARD FREIXAS I MONTPLET, AND CHRISTOPHE MOURougANE

ABSTRACT. The mathematical physicists Bershadsky–Cecotti–Ooguri–Vafa (BCOV) proposed, in a seminal article from '94, an extension of genus zero mirror symmetry to higher genera. We offer a mathematical treatment of the BCOV conjecture at genus one, based on the usage of the arithmetic Riemann–Roch theorem. As an application of our previous results on the BCOV invariant, we establish this conjecture for Calabi–Yau hypersurfaces in projective space. This seems to be the first example of higher dimensional mirror symmetry, of BCOV type, at genus one. The case of quintic threefolds was studied by Fang–Lu–Yoshikawa. Our contribution takes place on the B -side, and the relation to the A -side is provided by Zinger.

Our approach also lends itself to arithmetic considerations of the BCOV invariant, and we study a Chowla–Selberg type theorem expressing it in terms of special Γ values for certain Calabi–Yau manifolds with complex multiplication. Finally, we put forward a variant of the BCOV program at genus one, as a conjectured functorial Grothendieck–Riemann–Roch relationship.

CONTENTS

1. Introduction	2
1.1. The BCOV conjecture at genus one	2
1.2. Main result	3
1.3. Overview of proof	5
1.4. Further results	6
1.5. Conventions and notations	7
2. The BCOV invariant and the arithmetic Riemann–Roch theorem	8
2.1. The BCOV invariant	8
2.2. The arithmetic Riemann–Roch theorem	9
2.3. Kronecker limit formulas for families of Calabi–Yau hypersurfaces	12
3. The Dwork and mirror families, and their Hodge bundles	13
3.1. The mirror family and its crepant resolution	13
3.2. Comparison of Hodge bundles	15
3.3. Sections of the middle degree Hodge bundles	18
4. The degeneration of Hodge bundles of the mirror family	20
4.1. Behaviour of η_k at the MUM point	21
4.2. Behaviour of η_k at the ODP points	23
4.3. Triviality of the Hodge bundles outside the middle degree	26
5. The BCOV invariant of the mirror family	26
5.1. The Kronecker limit formula for the mirror family	26

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5.2. Canonical trivializations of the Hodge bundles at the MUM point	29
5.3. Generating series of Gromov–Witten invariants and Zinger’s theorem	33
5.4. Genus one mirror symmetry and the BCOV invariant	34
6. A Chowla–Selberg formula for the BCOV invariant	36
7. The BCOV conjecture and functorial Grothendieck–Riemann–Roch	38
References	41

1. INTRODUCTION

The purpose of this article is to establish higher dimensional cases of genus one mirror symmetry, as envisioned by mathematical physicists Bershadsky–Cecotti–Ooguri–Vafa (henceforth abbreviated BCOV) in their influential paper [BCOV94]. Precisely, we relate the generating series of genus one Gromov–Witten invariants on Calabi–Yau hypersurfaces to an invariant of a mirror family, built out of holomorphic analytic torsions. The invariant, whose existence was conjectured in *loc. cit.*, was mathematically defined and studied in our previous paper [EFiMM18a]. We refer to it as the BCOV invariant τ_{BCOV} . It generalizes from dimension 3 to higher dimensions previous constructions of Fang–Lu–Yoshikawa [FLY08], where the 3-dimensional case of our main result was also settled.

1.1. The BCOV conjecture at genus one. Let X be a Calabi–Yau manifold of dimension n . In this article, this will mean a complex projective connected manifold with trivial canonical sheaf. The mirror symmetry program of BCOV relates two types of invariants of X (known as A -model) and a mirror family of Calabi–Yau manifolds, whose members we denote X^\vee (known as B -model). We now briefly describe the genus one case of their conjecture.

On the one hand, on the A -side, one forms a generating series of genus one Gromov–Witten invariants on X . For this, for every curve class β in $H_2(X, \mathbb{Z})$, we consider the proper Deligne–Mumford stack of stable maps from genus one curves to X , whose fundamental class is β :

$$\overline{\mathcal{M}}_1(X, \beta) = \{f : C \rightarrow X \mid g(C) = 1, f \text{ stable and } f_*[C] = \beta\}.$$

Under the above assumptions, the stack $\overline{\mathcal{M}}_1(X, \beta)$ is of virtual dimension 0, and the Gromov–Witten invariant is the degree of its virtual fundamental class

$$\text{GW}_1(X, \beta) = \deg [\overline{\mathcal{M}}_1(X, \beta)]^{\text{vir}} \in \mathbb{Q}.$$

One then defines the formal power series

$$(1.1) \quad F_1^A(\tau) = \frac{-1}{24} \int_X c_{n-1}(X) \cap 2\pi i \tau + \sum_{\beta > 0} \text{GW}_1(X, \beta) e^{2\pi i \langle \tau, \beta \rangle},$$

where τ belongs to the complexified Kähler cone \mathcal{H}_X , i.e. $\tau \in H_{\mathbb{R}}^{1,1}(X) / H_{\mathbb{Z}}^{1,1}(X) + i\mathcal{K}_X$ and \mathcal{K}_X is the Kähler cone of X . On the other hand, on the B -side, BCOV introduced a spectral quantity \mathcal{F}_1^B built out of holomorphic Ray–Singer analytic torsions of X^\vee . It depends on an auxiliary choice of a Kähler structure ω on X^\vee , and can be recasted as

$$\mathcal{F}_1^B(X^\vee, \omega) = \prod_{0 \leq p, q \leq n} (\det \Delta_{\bar{\partial}}^{p,q})^{(-1)^{p+q} pq},$$

2

where $\det \Delta_{\frac{p,q}{\partial}}$ is the ζ -regularized determinant of the Dolbeault Laplacian acting on $A^{p,q}(X^\vee)$.¹ In our previous work [EFiMM18a] we renormalized this quantity to make it independent of the choice of ω :

$$\tau_{BCOV}(X^\vee) = C(X^\vee, \omega) \cdot \mathcal{F}_1^B(X^\vee, \omega).$$

There is a closed expression for $C(X^\vee, \omega)$ (see Section 2.1 for a precise formulation), which is not relevant for the introduction. Thus $\tau_{BCOV}(X^\vee)$ only depends on the complex structure of the Calabi–Yau manifold, in accordance with the philosophy that the B -model only depends on variations of the complex structure on X^\vee .

The mirror symmetry expectation predicts that given X , there is a mirror family of Calabi–Yau manifolds over a punctured multi-disc $\varphi: \mathcal{X}^\vee \rightarrow \mathbf{D}^\times = (\mathbb{D}^\times)^d$, with maximal unipotent monodromies and $d = h^{1,1}(X) = h^{1,n-1}(X^\vee)$. As above, we denote by X^\vee any member of the mirror family.² Also, denote by q a parameter of the disc \mathbf{D} centered at the origin. The A -model and the B -model should be related by a biholomorphism onto its image $\mathbf{D}^\times \rightarrow \mathcal{H}_X$, which is referred to as the mirror map and is denoted $q \mapsto \tau(q)$. Via the mirror map, the origin of the multi-disc is sent to infinity. Fixing a basis of ample classes on X , we can think of the mirror map as a change of coordinates on \mathbf{D}^\times . In the special case of $h^{1,n-1}(X^\vee) = 1$, one such a map is constructed as a quotient of carefully selected periods in [Mor93].

BCOV conjecture at genus one. *Suppose that X and $\varphi: \mathcal{X}^\vee \rightarrow \mathbf{D}^\times$ are Calabi–Yau mirrors as above. Then:*

- (1) *as $q \rightarrow 0$, the function $\log \tau_{BCOV}(\mathcal{X}_q^\vee)$ has a Taylor expansion in $\tau(q)$ and $\overline{\tau(q)}$, whose holomorphic part is well-defined and can be isolated into a function $F_1^B(q)$.*
- (2) *the functions F_1^A and F_1^B are related via the mirror map by*

$$F_1^A(\tau(q)) = F_1^B(q).$$

The procedure of extracting a holomorphic function from $\log \tau_{BCOV}(\mathcal{X}_q^\vee)$ as $q \rightarrow 0$ is called *passing to the holomorphic limit*.

1.2. Main result. The principal purpose of this article is to establish the BCOV conjecture at genus one for the mirror families of Calabi–Yau hypersurfaces in $\mathbb{P}_{\mathbb{C}}^n$. Let us discuss the framework and statement of our theorem.

The mirror family is often realized using a crepant resolution of the quotient of the Dwork pencil

$$(1.2) \quad x_0^{n+1} + \dots + x_n^{n+1} - (n+1)\psi x_0 \dots x_n = 0, \quad \psi \in U = \mathbb{C} \setminus \mu_{n+1},$$

by the subgroup of $\mathrm{GL}_{n+1}(\mathbb{C})$ given by $G = \{g \cdot [x_0 : \dots : x_n] = [\xi_0 x_0 : \dots : \xi_n x_n], \xi_i^{n+1} = 1, \prod \xi_i = 1\}$. The family, denoted $f: \mathcal{Z} \rightarrow U$ and with fibers Z_ψ , has maximal unipotent monodromy at $\psi = \infty$. Therefore, restricting to a punctured disc \mathbb{D}^\times centered at infinity with parameter $q = \psi^{-1}$, it fulfills the properties of a mirror family as in the previous subsection. Moreover, $f: \mathcal{Z} \rightarrow U$ can be naturally extended across μ_{n+1} to a degeneration with ordinary double point singularities, sometimes referred to as a conifold degeneration.

¹Actually BCOV work with $\frac{1}{2} \log \mathcal{F}_1^B$.

²Such families are also called *large complex structure limits* of Calabi–Yau manifolds.

Consider the following multivalued holomorphic functions, defined for $|\psi| \gg 0$:

$$I_{0,0}(\psi) = \sum_{d=0}^{\infty} \frac{1}{((n+1)\psi)^{d(n+1)}} \frac{((n+1)d)!}{(d!)^{n+1}},$$

$$I_{0,1}(\psi) = -(n+1) \log((n+1)\psi) I_{0,0}(\psi) + \sum_{d=1}^{\infty} \frac{1}{((n+1)\psi)^{d(n+1)}} \left(\frac{((n+1)d)!}{(d!)^{n+1}} \sum_{r=d+1}^{(n+1)d} \frac{n+1}{r} \right).$$

The recipe in [Mor84] for the construction of the mirror map, gives, in the variable ψ ,

$$\psi \mapsto \tau(\psi) = \frac{1}{2\pi i} \frac{I_{0,1}(\psi)}{I_{0,0}(\psi)}.$$

A key step in our approach to the BCOV conjecture is an understanding of the holomorphic limit procedure in terms of limiting mixed Hodge structures. Using this, we can extract natural holomorphic trivializations $\tilde{\eta}_k$, in a neighborhood of $\psi = \infty$, of the primitive Hodge bundles $(R^k f_* \Omega_{\mathcal{Z}/U}^{n-1-k})_{\text{prim}}$. In this article, these are line bundles with fibers at ψ given by $H^{n-1-k,k}(Z_\psi)_{\text{prim}}$, and the implicit primitivity notion is actually independent of the choice of polarization. They have natural L^2 (or, up to constant, Hodge) norms given by

$$\|\tilde{\eta}_k\|_{L^2}^2 = \frac{1}{(2\pi)^{n-1}} \left| \int_{Z_\psi} \tilde{\eta}_k \wedge \overline{\tilde{\eta}_k} \right|.$$

Finally, let $F_1^A(\tau(\psi))$ be the generating series defined as in (1.1), for a general Calabi–Yau hypersurface $X \subset \mathbb{P}_{\mathbb{C}}^n$ (notice that X has now dimension $n-1$ instead of n). Then our main result (Theorem 5.9 *infra*) can be stated as follows:

Main Theorem. *The BCOV conjecture at genus one is true for the mirror family of Calabi–Yau hypersurfaces in $\mathbb{P}_{\mathbb{C}}^n$. More precisely, in a neighborhood of infinity, the BCOV invariant of Z_ψ factors as*

$$\tau_{BCOV}(Z_\psi) = C \left| \exp((-1)^{n-1} F_1^B(\psi)) \right|^4 \frac{\|\tilde{\eta}_0\|_{L^2}^{\chi(Z_\psi)/6}}{\left(\prod_{k=0}^{n-1} \|\tilde{\eta}_k\|_{L^2}^{2(n-1-k)} \right)^{(-1)^{n-1}}},$$

where $F_1^B(\psi)$ is a multivalued holomorphic function with $F_1^B(\psi) = F_1^A(\tau(\psi))$ as formal series in $\tau(\psi)$, and C is a positive constant.

To sum up, by comparing the BCOV invariant and the L^2 norms of the natural sections, we can canonically produce a multivalued holomorphic function as in the theorem. We refer to this function as the *holomorphic limit* of the BCOV invariant. Our understanding is that this provides a mathematical framework for thinking about holomorphic limits in the context of the BCOV conjectures.

By construction, the L^2 norms in the statement are independent of the crepant resolution. We expect the independence of the constant C as well, in agreement with the birational invariance of τ_{BCOV} conjectured in [EFiMM18a, Conj. B].

The Main Theorem extends to arbitrary dimensions previous work of Fang–Lu–Yoshikawa [FLY08, Thm. 1.3] in dimension 3 (see Remark 5.10 below for the comparison of both statements). In their approach, all the Hodge bundles have geometric meaning in terms of Weil–Peterson geometry and Kuranishi space. The lack thereof is an additional complication in our setting. To our knowledge, our theorem is the first example of higher dimensional mirror symmetry, of BCOV type at genus one, established in the mathematics literature. It confirms various

instances that had informally been utilized for computational purposes, *e.g.* [KP08, Sec. 6] in dimension 4.

1.3. Overview of proof.

Arithmetic Riemann–Roch. In the algebro-geometric setting, the arithmetic Riemann–Roch theorem from Arakelov theory allows us to compute the BCOV invariant of a family of Calabi–Yau varieties in terms of L^2 norms of auxiliary sections of Hodge bundles (cf. §2.2). This bypasses some analytical arguments in former approaches, such as [FLY08], based on the holomorphic anomaly equation (cf. [EFiMM18a, Proposition 5.9]). The theorem determines the BCOV invariant up to a meromorphic function, in fact a rational function.³ The divisor of this rational function is encapsulated in the asymptotics of the L^2 norms and the BCOV invariant. In the special case when the base is a Zariski open set of $\mathbb{P}_{\mathbb{C}}^1$, as for the Dwork pencil (1.2) and the mirror family, this divisor is determined by all but one point. Hence so is the function itself, up to constant.

Hodge bundles of the mirror family. The construction of the auxiliary sections is first of all based on a comparison of the Hodge bundles of the mirror family with the G -invariant part of the Hodge bundles on the Dwork pencil (1.2), explained in Section 3. Using the residue method of Griffiths we construct algebraic sections of the latter. These are then transported into sections η_k of the Hodge bundles of the crepant resolution, *i.e.* the mirror family. This leads us to a systematic geometric study of these sections in connection with Deligne extensions and limiting Hodge structures at various key points, notably at μ_{n+1} where ordinary double point singularities arise. We rely heavily on knowledge of the Yukawa coupling and our previous work in [EFiMM18a, Sec. 2] on logarithmic Hodge bundles and semi-stable reduction. The arguments are elaborated in Section 4.

Asymptotics of L^2 norms and the BCOV invariant. The above arithmetic Riemann–Roch reduction leads us to study the norm of the auxiliary sections outside of the maximal unipotent monodromy point, enabling us to focus on ordinary double points. Applying our previous result [EFiMM18a, Thm. 4.4] to the auxiliary sections, we find that the behaviour of their L^2 norms is expressed in terms of monodromy eigenvalues, and the possible zeros or poles as determined by the geometric considerations of the preceding paragraph. The monodromy is characterized by the Picard–Lefschetz theorem. As for the asymptotics for the BCOV invariant, they were already accomplished in [EFiMM18a, Thm. B]. This endeavor results in Theorem 5.1, which is a description of the rational function occurring in the arithmetic Riemann–Roch theorem.

Connection to enumerative geometry. The BCOV conjecture suggests that we need to study the BCOV invariant close to $\psi = \infty$. However, the formula in Theorem 5.1 is not adapted to the mirror symmetry setting, for example the sections η_k do not make any reference to H_{lim}^{n-1} . We proceed to divide the η_k by holomorphic periods, for a fixed basis of the weight filtration on the homology $(H_{\text{lim}}^{n-1})^{\vee}$, to obtain the sections $\tilde{\eta}_k$ of the Main Theorem. These new sections are characterized by having unipotent lower triangular period matrix. Rephrasing Theorem 5.1 with these sections, we thus arrive at an expression for the *holomorphic limit* F_1^B in the theorem. Combined with results of Zinger [Zin08, Zin09], this yields the relation to the generating series of Gromov–Witten invariants in the mirror coordinate.

³This rational function compares to the so-called *holomorphic ambiguity* in the physics literature.

1.4. Further results.

Kronecker limit formulas. The simplest Calabi–Yau varieties are elliptic curves, which can conveniently be presented as $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, for τ in the Poincaré upper half-plane. The generating series (1.1) of Gromov–Witten invariants is then given by $-\frac{1}{24}\log\Delta(\tau)$, where $\Delta(\tau) = q\prod(1 - q^n)^{24}$ and $q = e^{2\pi i\tau}$. The corresponding function \mathcal{F}_1^B is computed as $\exp(\zeta'_\tau(0))$, where

$$\zeta_\tau(s) = (2\pi)^{-2s} \sum_{(m,n) \neq (0,0)} \frac{(\operatorname{Im} \tau)^s}{|m + n\tau|^{2s}}.$$

The BCOV conjecture at genus one is deduced from the equality

$$(1.3) \quad \exp(-\zeta'_\tau(0)) = \frac{1}{(2\pi)^2} \operatorname{Im}(\tau) |\Delta(\tau)|^{1/6}.$$

This is a formulation of the first Kronecker limit formula, see *e.g.* [Yos99, Intro.]. In the mirror symmetry interpretation, the correspondence $\tau \mapsto q$ is the (inverse) mirror map. Equation (1.3) can be recovered from a standard application of the arithmetic Riemann–Roch theorem. In this vein, we will interpret all results of this shape as generalizations of the Kronecker limit formula. This includes the Theorem 5.1 cited above, as well as a Theorem 2.6 for Calabi–Yau hypersurfaces in Fano manifolds.

Chowla–Selberg formula. While being applicable to algebraic varieties over \mathbb{C} , the Riemann–Roch theorem in Arakelov geometry has the further advantage of providing arithmetic information when the varieties are defined over \mathbb{Q} . For the mirror family, this is reflected in some additional structure for the constant in the Main Theorem, as stated in Theorem 5.9.

The arithmetic Riemann–Roch theorem is also suited to evaluating the BCOV invariant of certain arithmetically defined Calabi–Yau varieties with additional automorphisms. As an example, for the special fibre Z_0 of our mirror family (1.2), Theorem 6.2 computes the BCOV invariant as a product of special values of the Γ function. This is reminiscent of the Chowla–Selberg theorem [SC67], which derives from (1.3) an expression of the periods of a CM elliptic curve as a product of special Γ values. Assuming deep conjectures of Gross–Deligne [Gro78], we would be able to write any BCOV invariant of a CM Calabi–Yau manifold in such terms.

Functorial Grothendieck–Riemann–Roch. During the preparation of this article, we were lead to the question of whether one can formulate the nature of the holomorphic function F_1^B without any reference to *e.g.* spectral theory or holomorphic anomaly equations. A model that served as inspiration in this quest is the Yukawa coupling in genus zero mirror symmetry. The key ingredients going into the Yukawa coupling construction are the Kodaira–Spencer mappings between Hodge bundles, and canonical trivializations of those (cf. [Mor93]).

As our scheme shows, the spectral theory is subsumed by the arithmetic Riemann–Roch theorem. Actually, the holomorphic anomaly equation can also be derived from this. An influential program by Deligne [Del87] suggests that the arithmetic Riemann–Roch theorem can in fact be lifted to a natural isometry of line bundles. A parallel aim of the program is to lift the degree one part of the Grothendieck–Riemann–Roch theorem and it is collectively referred to as functorial Grothendieck–Riemann–Roch. We thus propose the following conjecture (see (2.4) for the definition of the BCOV bundle λ_{BCOV}).

Conjecture. Let X and $\varphi: \mathcal{X}^\vee \rightarrow \mathbf{D}^\times$ be Calabi–Yau mirrors as in §1.1. Then:

(1) there exists a natural isomorphism of line bundles,

$$(1.4) \quad \mathbf{GRR}: \lambda_{\text{BCOV}}(\mathcal{X}^\vee/\mathbf{D}^\times)^{\otimes 12\kappa} \xrightarrow{\sim} \varphi_*(K_{\mathcal{X}^\vee/\mathbf{D}^\times})^{\otimes \chi\kappa}.$$

Here χ is the Euler characteristic of any fiber of φ and κ only depends on the relative dimension of φ .

(2) There exists natural trivializing sections of both sides of (1.4).

(3) The isomorphism **GRR** can thus be realized as a holomorphic function. In the mirror coordinate $\tau(q)$, this can be written as

$$\exp\left((-1)^n F_1^A(\tau(q))\right)^{24\kappa}.$$

In other words, the holomorphic limit function F_1^B should naturally appear as functorial Grothendieck–Riemann–Roch and should relate to the BCOV conjecture at genus one. The existence of some isomorphism in (1.4) is provided by the Grothendieck–Riemann–Roch theorem, the key point is the naturality requirement. Notice that the conjecture has a structure similar to the Yukawa coupling construction, and notably **GRR** plays a role analogous to the Kodaira–Spencer mappings. As supporting evidence, we observe that the conjecture holds for the mirror family of Calabi–Yau hypersurfaces, up to a constant. This stems from a combination of the arithmetic Riemann–Roch theorem and the Main Theorem. We refer the reader to Section 7 for a detailed discussion and further motivation.

1.5. Conventions and notations.

Kähler manifolds. Let X be a complex manifold. In this article, a hermitian metric on X means a smooth hermitian metric on the holomorphic vector bundle T_X . Let h be a hermitian metric on X . The Arakelov theoretic Kähler form attached to h is given in local holomorphic coordinates by

$$(1.5) \quad \omega = \frac{i}{2\pi} \sum_{j,k} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) dz_j \wedge d\bar{z}_k.$$

Notice that the standard normalization in Hodge theory would rather be

$$(1.6) \quad \tilde{\omega} = i \sum_{j,k} h\left(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial z_k}\right) dz_j \wedge d\bar{z}_k.$$

The complex hermitian manifold (X, h) is said to be Kähler if, as usual, the differential form ω is closed. We assume this from now on.

Suppose furthermore that X is compact. The hermitian metric h induces hermitian metrics on the \mathcal{C}^∞ vector bundles of differential forms of type (p, q) , that we still denote h . Then, on the spaces $A^{p,q}(X)$ of global sections, we define the L^2 hermitian inner product

$$(1.7) \quad \langle \alpha, \beta \rangle_{L^2} = \int_X h(\alpha, \beta) \frac{\omega^n}{n!}.$$

Our normalization for ω in (1.5) is such that the L^2 product on $A^{p,q}(X)$ is $1/(2\pi)^n$ times the usual one in Hodge theory, defined instead using $\tilde{\omega}$. The coherent cohomology groups $H^q(X, \Omega_X^p)$ can be computed as Dolbeault cohomology. Dolbeault cohomology groups can be embedded in $A^{p,q}(X)$ by taking $\bar{\partial}$ -harmonic representatives. Via this identification, $H^q(X, \Omega_X^p)$ inherits a

L^2 metric. Similarly, the hermitian metric h also induces hermitian metrics on the vector bundles and spaces of complex differential forms of degree k . The complex de Rham cohomology $H^k(X, \mathbb{C})$ has an induced L^2 metric by taking d -harmonic representatives. The canonical Hodge decomposition

$$H^k(X, \mathbb{C}) \simeq \bigoplus_{p,q}^{\perp} H^q(X, \Omega_X^p)$$

is an isometry for the L^2 metrics.

Calabi–Yau manifolds. A Calabi–Yau manifold is a compact connected Kähler manifold with trivial canonical sheaf. In this paper we will only deal with projective Calabi–Yau manifolds, and from now on this algebraicity condition will be implicit. A Calabi–Yau variety over a field k is a smooth projective variety over k , with trivial canonical sheaf. Finally, we say that a Calabi–Yau variety X of dimension n is strict if the Hodge numbers $h^{0,p}$ vanish for $0 < p < n$.

2. THE BCOV INVARIANT AND THE ARITHMETIC RIEMANN–ROCH THEOREM

2.1. The BCOV invariant. We briefly recall the construction of the BCOV invariant [EFiMM18b, Sec. 5].

Let X be a Calabi–Yau manifold of dimension n . Fix a Kähler metric h on X , with Kähler form ω . The vector bundle of holomorphic differential forms Ω_X^p inherits a hermitian metric. Let $T(\Omega_X^p, \omega)$ be the holomorphic analytic torsion of Ω_X^p endowed with this metric, and with respect to the Kähler form ω . The *BCOV torsion* of (X, ω) is

$$T(X, \omega) = \prod_{0 \leq p \leq n} T(\Omega_X^p, \omega)^{(-1)^p p}.$$

Let $\Delta_{\bar{\partial}}^{p,q}$ be the Dolbeault Laplacian acting on $A^{p,q}(X)$, and $\det \Delta_{\bar{\partial}}^{p,q}$ its ζ -regularized determinant (excluding the zero eigenvalue). Unraveling the definition of holomorphic analytic torsion, we find for the BCOV torsion

$$T(X, \omega) = \prod_{0 \leq p, q \leq n} (\det \Delta_{\bar{\partial}}^{p,q})^{(-1)^{p+q} pq}.$$

The BCOV torsion depends on the choice of the Kähler metric. A suitable normalization makes it independent of choices. For this purpose, we introduce two real valued quantities. For the first one, let η be a basis of $H^0(X, K_X)$, and define

$$(2.1) \quad A(X, \omega) = \exp\left(-\frac{1}{12} \int_X (\log \varphi) c_n(T_X, h)\right), \quad \text{with} \quad \varphi = \frac{i^{n^2} \eta \wedge \bar{\eta}}{\|\eta\|_{L^2}^2} \frac{n!}{(2\pi\omega)^n}.$$

For the second one, we consider the largest torsion free quotient of the cohomology groups $H^k(X, \mathbb{Z})$, denoted by $H^k(X, \mathbb{Z})_{\text{nt}}$. These are lattices in the real cohomology groups $H^k(X, \mathbb{R})$. The latter have Euclidean structures induced from the L^2 metrics on the $H^k(X, \mathbb{C})$, depending on ω . We define $\text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)$ to be the square of the covolume of the lattice $H^k(X, \mathbb{Z})_{\text{nt}}$ with respect to this Euclidean structure, and we put

$$(2.2) \quad B(X, \omega) = \prod_{0 \leq k \leq 2n} \text{vol}_{L^2}(H^k(X, \mathbb{Z}), \omega)^{(-1)^{k+1} k/2}.$$

The *BCOV invariant* of X is then defined to be

$$(2.3) \quad \tau_{BCOV}(X) = \frac{A(X, \omega)}{B(X, \omega)} T(X, \omega) \in \mathbb{R}_{>0}.$$

By [EFiMM18a, Prop. 5.8], the BCOV invariant depends only on the complex structure of X .

Let now $f: X \rightarrow S$ be a projective morphism of complex manifolds, whose fibers are Calabi–Yau manifolds. Then the function $s \mapsto \tau_{BCOV}(X_s)$ is \mathcal{C}^∞ on S . The complex Hessian $dd^c \log \tau_{BCOV}$ of the smooth function $\log \tau_{BCOV}$ can be expressed in terms of Griffiths–Hodge forms (curvatures of Griffiths–Hodge bundles endowed with L^2 metrics) and the Weil–Petersson form (curvature of the L^2 metric on $f_* K_{X/S}$); see [EFiMM18a, Prop. 5.9]. The equation is called the holomorphic anomaly equation for the BCOV invariant. In this paper we won’t need it.

2.2. The arithmetic Riemann–Roch theorem. In this section, we work over an arithmetic ring. This means an excellent regular domain A together with a finite set Σ of embeddings $\sigma: A \hookrightarrow \mathbb{C}$, closed under complex conjugation. For example, A could be a number field with the set of all complex embeddings, or the complex field \mathbb{C} . Denote by K the field of fractions of A .

Let X be an arithmetic variety, *i.e.* a regular, integral, flat and quasi-projective scheme over A . For every embedding $\sigma: A \hookrightarrow \mathbb{C}$, the base change $X_\sigma = X \times_{A, \sigma} \mathbb{C}$ is a quasi-projective and smooth complex variety, whose associated analytic space X_σ^{an} is therefore a quasi-projective complex manifold. It is convenient to define X^{an} as the disjoint union of the X_σ^{an} . For instance, when A is a number field, then X^{an} is the complex analytic space associated to X as an arithmetic variety over \mathbb{Q} . Differential geometric objects on X^{an} such as line bundles, differential forms, metrics, etc. may equivalently be seen as collections of corresponding objects on the X_σ^{an} , indexed by σ . The complex conjugation induces an anti-holomorphic involution on X^{an} , and it is customary in Arakelov geometry to impose some compatibility of analytic data with this action. Let us now recall the definitions of the arithmetic Picard and first Chow groups of X .

Definition 2.1. *A smooth hermitian line bundle on X consists in a pair (L, h) , where*

- L is a line bundle on X .
- h is a smooth hermitian metric on the holomorphic line bundle L^{an} on X^{an} deduced from L , invariant under the action of the complex conjugation. Hence, h is a conjugation invariant collection $\{h_\sigma\}_{\sigma: A \rightarrow \mathbb{C}}$, where h_σ is a smooth hermitian metric on the holomorphic line bundle L_σ^{an} on X_σ^{an} deduced from L by base change and analytification.

The set of isomorphism classes of hermitian line bundles (L, h) , with the natural tensor product operation, is a commutative group denoted by $\widehat{\text{Pic}}(X)$ and called the arithmetic Picard group of X .

Definition 2.2. *The first arithmetic Chow group $\widehat{\text{CH}}^1(X)$ of X is the commutative group*

- *generated by arithmetic divisors, i.e. couples (D, g_D) , where D is a Weil divisor on X and g_D is a Green current for the divisor D^{an} , compatible with complex conjugation. Hence, by definition g_D is a degree 0 current on X^{an} that is a dd^c -potential for the current of integration $\delta_{D^{\text{an}}}$*

$$dd^c g_D + \delta_{D^{\text{an}}} = [\omega_D],$$

up to some smooth differential $(1, 1)$ form ω_D on X^{an} .

- *with relations $(\text{div}(\phi), [-\log|\phi|^2])$, for non-zero rational functions ϕ on X .*

The arithmetic Picard and first Chow groups are related via the first arithmetic Chern class

$$\widehat{c}_1 : \widehat{\text{Pic}}(X) \rightarrow \widehat{\text{CH}}^1(X),$$

which maps a hermitian line bundle (L, h) to the class of the arithmetic divisor $(\text{div}(\ell), [-\log \|\ell\|^2])$, where ℓ is any non-zero rational section of L . This is in fact an isomorphism. We refer the reader to [GS90b, Sec. 2] for a complete discussion.

More generally, Gillet–Soulé developed a theory of arithmetic cycles and Chow rings [GS90a], an arithmetic K -theory and characteristic classes [GS90b, GS90c], and an arithmetic Riemann–Roch theorem [GS92]. While for the comprehension of the theorem below only $\widehat{\text{CH}}^1$, $\widehat{\text{Pic}}$ and \widehat{c}_1 are needed, the proof uses all this background, for which we refer to the above references.

Let now $f: X \rightarrow S$ be a smooth projective morphism of arithmetic varieties of relative dimension n , with generic fiber X_∞ . To simplify the exposition, we assume that S is geometrically connected, so that S_σ^{an} is connected for every embedding σ . More importantly, we suppose that the fibers X_s are Calabi–Yau, hence they satisfy $K_{X_s} = \mathcal{O}_{X_s}$. We define the BCOV line bundle on S as the determinant of cohomology of the virtual vector bundle $\sum_p (-1)^p p \Omega_{X/S}^p$, that is, in additive notation for the Picard group of S

$$(2.4) \quad \lambda_{\text{BCOV}}(X/S) = \sum_{p=0}^n (-1)^p p \lambda(\Omega_{X/S}^p) = \sum_{p,q} (-1)^{p+q} p \det R^q f_* \Omega_{X/S}^p.$$

If there is no possible ambiguity, we will sometimes write λ_{BCOV} instead of $\lambda_{\text{BCOV}}(X/S)$. For the following statement, we fix an auxiliary conjugation invariant Kähler metric h on $T_{X^{\text{an}}}$. We denote by ω the associated Kähler form, normalized according to the conventions in Arakelov theory as in (1.5). We assume that the restriction of ω to fibers (still denoted by ω) has rational cohomology class. All the L^2 metrics below are computed with respect to ω as in (1.7). Depending on the Kähler metric ω , the line bundle λ_{BCOV} carries a Quillen metric h_Q . Following [EFiMM18b, Def. 4.1] and [EFiMM18a, Def. 5.2], the Quillen-BCOV metric on λ_{BCOV} is defined by multiplying h_Q by the correcting factor A in (2.1): for every $s \in S^{\text{an}}$, we put

$$h_{Q,\text{BCOV},s} = A(X_s, \omega) \cdot h_{Q,s}.$$

It is shown in *loc. cit.* that the Quillen-BCOV is actually a smooth hermitian metric, independent of the choice of ω .

Theorem 2.3. *Under the above assumptions, there is an equality in $\widehat{\text{CH}}^1(S)_{\mathbb{Q}} = \widehat{\text{CH}}^1(S) \otimes \mathbb{Q}$*

$$(2.5) \quad \widehat{c}_1(\lambda_{\text{BCOV}}, h_{Q,\text{BCOV}}) = \frac{\chi(X_\infty)}{12} \widehat{c}_1(f_* K_{X/S}, h_{L^2}).$$

Hence, for any complex embedding σ we have an equality of functions on S_σ^{an}

$$(2.6) \quad \log \tau_{\text{BCOV},\sigma} = \log |\Delta|_\sigma^2 + \frac{\chi(X_\infty)}{12} \log \|\eta\|_{L^2,\sigma}^2 - \sum_{0 \leq p,q \leq n} (-1)^{p+q} p \log \|\eta_{p,q}\|_{L^2,\sigma}^2 + \log C_\sigma,$$

where:

- $\Delta \in K(S)^\times \otimes_{\mathbb{Z}} \mathbb{Q}$.
- η is any rational section of $f_* K_{X/S}$.
- $\eta_{p,q}$ is any rational section of $\det R^q f_* \Omega_{X/S}^p$.
- $C_\sigma \in \pi^r \mathbb{Q}_{>0}$, where $r = \frac{1}{2} \sum (-1)^{k+1} k^2 b_k$ and b_k is the k -th Betti number of X_∞ .

Proof. The proof is a routine application of the arithmetic Riemann–Roch theorem of Gillet–Soulé [GS92, Thm. 7]. We give the details for the convenience of the reader. Consider the virtual vector bundle $\sum(-1)^p p\Omega_{X/S}^p$, with virtual hermitian structure deduced from the metric h , and denoted h^\bullet . Its determinant of cohomology λ_{BCOV} carries the Quillen metric h_Q . The theorem of Gillet–Soulé provides an equality in $\widehat{\text{CH}}^1(S)_\mathbb{Q}$

$$\begin{aligned}\widehat{c}_1(\lambda_{BCOV}, h_Q) &= f_* \left(\widehat{\text{ch}}(\sum(-1)^p p\Omega_{X/S}^p, h^\bullet) \widehat{\text{Td}}(T_{X/S}, h) \right)^{(1)} \\ &\quad - a(\text{ch}(\sum(-1)^p p\Omega_{X^{\text{an}}/S^{\text{an}}}^p) \text{Td}(T_{X^{\text{an}}/S^{\text{an}}}) R(T_{X^{\text{an}}/S^{\text{an}}}))^{(1)} \\ &= \frac{1}{12} f_* (\widehat{c}_1(K_{X/S}, h^\bullet) \widehat{c}_n(T_{X/S}, h)),\end{aligned}$$

where $h^\bullet = (\det h)^{-1}$ is the hermitian metric on $K_{X/S}$ induced from h . Notice that the topological factor containing the R -genus in *loc. cit.* vanishes in our situation, since

$$\text{ch}(\sum(-1)^p p\Omega_{X^{\text{an}}/S^{\text{an}}}^p) \text{Td}(T_{X^{\text{an}}/S^{\text{an}}}) = -c_{n-1} + \frac{n}{2}c_n - \frac{1}{12}c_1c_n + \text{higher degree terms}$$

and R has only odd degree terms and $c_1(T_{X^{\text{an}}/S^{\text{an}}}) = 0$. Now, the evaluation map $f^* f_* K_{X/S} \rightarrow K_{X/S}$ is an isomorphism, but it is in general not an isometry if we equip $f_* K_{X/S}$ with the L^2 metric and $K_{X/S}$ with the metric h^\bullet . Comparing both metrics yields a relation in $\widehat{\text{CH}}^1(X)$

$$(2.8) \quad \widehat{c}_1(K_{X/S}, h^\bullet) = f^* \widehat{c}_1(f_* K_{X/S}, h_{L^2}) + [(0, -\log \varphi)].$$

Here φ is the smooth function on X^{an} given by

$$\varphi = \frac{i^{n^2} \eta \wedge \bar{\eta}}{\|\eta\|_{L^2}^2} \frac{n!}{(2\pi\omega)^n},$$

where η denotes a local trivialization of $f_* K_{X^{\text{an}}/S^{\text{an}}}$, thought of as a section of $K_{X^{\text{an}}/S^{\text{an}}}$ via the evaluation map. Multiplying (2.8) by $\widehat{c}_n(T_{X/S}, h)$ and applying f_* and the projection formula for arithmetic Chow groups, we find

$$\begin{aligned}f_* (\widehat{c}_1(K_{X/S}, h^\bullet) \widehat{c}_n(T_{X/S}, h)) &= f_* (f^* \widehat{c}_1(f_* K_{X/S}, h_{L^2}) \widehat{c}_n(T_{X/S}, h)) + f_* [(0, -\log \varphi)] \widehat{c}_n(T_{X/S}, h) \\ &= \chi(X_\infty) \widehat{c}_1(f_* K_{X/S}, h_{L^2}) + \left[\left(0, - \int_{X^{\text{an}}/S^{\text{an}}} (\log \varphi) c_n(T_{X^{\text{an}}/S^{\text{an}}}, h) \right) \right],\end{aligned}$$

where $c_n(T_{X/S}, h)$ is the n -th Chern–Weil differential form of $(T_{X^{\text{an}}/S^{\text{an}}}, h)$. Together with (2.7), this shows that the metric

$$h_{Q,BCOV} = h_Q \cdot \exp \left(-\frac{1}{12} \int_{X^{\text{an}}/S^{\text{an}}} (\log \varphi) c_n(T_{X^{\text{an}}/S^{\text{an}}}, h) \right)$$

indeed satisfies (2.5).

The outcome (2.6) is a translation of the meaning of the equality (2.5) in $\widehat{\text{CH}}^1(S)_\mathbb{Q}$, combined with our construction of the BCOV invariant [EFiMM18a, Sec. 5] and (2.3). For this we write $\tau_{BCOV} = h_{Q,BCOV}/h_{L^2,BCOV}$, where the L^2 -BCOV metric equals the usual L^2 metric up to the normalizing factor B defined in (2.2). Under the rationality assumption on ω , the normalizing factor B is constant on each connected manifold S_σ^{an} . To compute it up to rational number, we apply [EFiMM18a, Prop. 4.2]. This reference provides a rationality statement for the volumes of cohomology lattices with respect to a rational Kähler form. However, the Kähler form in *loc. cit.* is the Hodge theoretic one, while our rational Kähler form is now the Arakelov theoretic one. Hence, we need to take into account that the Arakelov theoretic normalization (1.5) is

$1/2\pi$ times the Hodge theoretic one (1.6). For this we apply [EFiMM18a, Prop. 4.2] to the metric $h/2\pi$, whose Hodge theoretic Kähler form is precisely ω , thus rational. With this understood, we find

$$(2.9) \quad \text{vol}_{L^2}(H^k(X_s, \mathbb{Z}), \omega) \in (2\pi)^{-kb_k} \mathbb{Q}_{>0}^\times$$

for any $s \in S_\sigma^{\text{an}}$. Together with the definition of B (2.2), this is responsible for the constants C_σ . \square

Remark 2.4. (1) The use of the arithmetic Riemann–Roch theorem requires an algebraic setting, but directly yields the existence of the rational function Δ . By contrast, previous techniques (cf. e.g. [FLY08, Sections 7 & 10]) rely on subtle integrability estimates of the functions in (2.6), in order to ensure that the *a priori* pluriharmonic function $\log|\Delta|_\sigma^2$ is indeed the logarithm of a rational function. The arithmetic Riemann–Roch theorem further provides the field of definition of Δ and the constants C_σ .

(2) In the case of a Calabi–Yau 3-fold defined over a number field, similar computations were done by Maillot–Rössler [MR12, Sec. 2].

2.3. Kronecker limit formulas for families of Calabi–Yau hypersurfaces. In this section we give an example of use of Theorem 2.3 and we determine the BCOV invariant for families of Calabi–Yau hypersurfaces in Fano manifolds. The argument provides a simplified model for the later computation of the BCOV invariant of the family of the mirror Calabi–Yau hypersurfaces.

Let V be a complex Fano manifold, with very ample anti-canonical bundle $-K_V$. We consider the anti-canonical embedding of V into $|-K_V| = \mathbb{P}(H^0(V, -K_V)) \simeq \mathbb{P}^N$, whose smooth hyperplane sections are Calabi–Yau manifolds. The dual projective space $\check{\mathbb{P}} = \mathbb{P}(H^0(V, -K_V)^\vee) \simeq \check{\mathbb{P}}^N$ parametrizes hyperplane sections, and contains an irreducible subvariety $\Delta \subseteq \check{\mathbb{P}}$ which corresponds to singular such sections [GKZ08, Chap. 1, Prop. 1.3]. We assume that Δ is a hypersurface in $\check{\mathbb{P}}$. This is in general not true, and a necessary condition is proven in [GKZ08, Chap. 1, Cor. 1.2]. Denote by U the quasi-projective complement $U := \check{\mathbb{P}} \setminus \Delta$. Denote by $f: \mathcal{X} \rightarrow \check{\mathbb{P}}$ the universal family of hyperplane sections. Therefore f is smooth on U , and the corresponding BCOV line bundle λ_{BCOV} is thus defined on U .

Lemma 2.5. *For some positive integer m , the line bundles $(f_*K_{\mathcal{X}/U})^{\otimes m}$ and $\lambda_{BCOV}^{\otimes m}$ have trivializing sections. These are unique up to constants.*

Proof. A standard computation shows that $\text{Pic}(U) = \mathbb{Z}/\deg \Delta$, providing the first claim of the lemma. For the second assertion, for any of the line bundles under consideration, let θ and θ' be two trivializations on U . Therefore, $\theta = h\theta'$ for some invertible function h on U . The previous description of $\text{Pic}(U)$ shows that the divisor of h , as a rational function on $\check{\mathbb{P}}$, is supported on Δ . As Δ is irreducible, in the projective space $\check{\mathbb{P}}$ this is only possible if the divisor vanishes. We conclude that h is necessarily constant. \square

For the following statement, we need a choice of auxiliary Kähler metric on \mathcal{X} (restricted to U), whose Arakelov theoretic Kähler form has fiberwise rational cohomology class. We compute L^2 norms on Hodge bundles and on λ_{BCOV} with respect to this choice.

Theorem 2.6. *For some integer $m > 0$ as in the lemma, let β be a trivialization of $\lambda_{BCOV}^{\otimes m}$ and η a trivialization of $(f_*K_{\mathcal{X}/U})^{\otimes m}$. Then there is a global constant C such that, for any Calabi–Yau hyperplane section $X_H = V \cap H$, we have*

$$\tau_{BCOV}(X_H) = C \|\eta\|_{L^2}^{\chi/6m} \|\beta\|_{L^2}^{-2/m}.$$

Proof. We apply Theorem 2.3 to $f: \mathcal{X} \rightarrow U$ (over \mathbb{C}), which in terms of β and η becomes

$$m \log \tau_{BCOV}(X_H) = \log |g|^2 + \frac{\chi}{12} \log \|\eta\|_{L^2}^2 - \log \|\beta\|_{L^2}^2 + \log C$$

for some regular invertible function g on U and some constant C . By construction, as a rational function on $\check{\mathbb{P}}$, g must have its zeros or poles along Δ . Since Δ is irreducible this forces g to be constant. \square

Remark 2.7. (1) When V is a toric variety with very ample anti-canonical class, all of the constructions can in fact be done over the rational numbers. The sections β and η can be taken to be defined over \mathbb{Q} , and unique up to a rational number. With this choice, the constant C takes the form stated in Theorem 2.3.

(2) In the case when the discriminant Δ has higher codimension, we have $\text{Pic}(U) \simeq \text{Pic}(\check{\mathbb{P}})$. In particular, λ_{BCOV} uniquely extends to a line bundle $\check{\mathbb{P}}$. The existence of the canonical (up to constant) trivializations β and η is no longer true. However, one can propose a variant of the theorem where β and η are trivializations outside a chosen ample divisor in $\check{\mathbb{P}}$.

3. THE DWORK AND MIRROR FAMILIES, AND THEIR HODGE BUNDLES

3.1. The mirror family and its crepant resolution. We review general facts on the Dwork pencil of Calabi–Yau hypersurfaces, and the construction of the mirror of Calabi–Yau hypersurfaces in projective space. Initially, we work over the field of complex numbers. Rationality refinements will be made along the way.

Let $n \geq 3$ be an integer. The Dwork pencil $\mathcal{X} \rightarrow \mathbb{P}^1$ is defined by the hypersurface of $\mathbb{P}^n \times \mathbb{P}^1$ of equation

$$F_\psi(x_0, \dots, x_n) := \sum_{j=0}^n x_j^{n+1} - (n+1)\psi x_0 \dots x_n = 0, \quad [x_0 : x_1 : \dots : x_n] \in \mathbb{P}^n, \quad \psi \in \mathbb{P}^1.$$

The smooth fibers of this family are strict Calabi–Yau manifolds of dimension $n-1$. The singular fibers are:

- fiber at $\psi = \infty$, given by the divisor with normal crossings $x_0 \cdot \dots \cdot x_n = 0$.
- the fibers where $\psi^{n+1} = 1$. These fibers have ordinary double point singularities. The singular points have projective coordinates (x_0, \dots, x_n) with $x_0 = 1$ and $x_j^{n+1} = 1$ for all $j \geq 1$, and $\prod_j x_j = \psi^{-1}$.

Denote by μ_{n+1} the group of the $(n+1)$ -th roots of unity. Let K be the kernel of the multiplication map $\mu_{n+1}^{n+1} \rightarrow \mu_{n+1}$, given by elements (ξ_0, \dots, ξ_n) such that $\prod_j \xi_j = 1$. Let also Δ be the diagonal embedding of μ_{n+1} in K . The group $G := K/\Delta$ acts naturally on the fibers X_ψ of $\mathcal{X} \rightarrow \mathbb{P}^1$ by multiplication of the projective coordinates, and we denote the quotient space by $\mathcal{Y} \rightarrow \mathbb{P}^1$.

We notice that the above construction can be done over \mathbb{Q} . Indeed, F_ψ is already defined over \mathbb{Q} , and the groups K, Δ are finite algebraic groups over \mathbb{Q} , and hence so does the quotient G . The action of G on F_ψ is defined over \mathbb{Q} as well, as one can see by examining the compatibility with the action of $\text{Aut}(\mathbb{C}/\mathbb{Q})$ on the \mathbb{C} points of \mathcal{X} , or alternatively by writing the co-action at the level of algebras. Therefore, the quotient $\mathcal{Y} = \mathcal{X}/G$ is defined over \mathbb{Q} , and so does the projection map $\mathcal{Y} \rightarrow \mathbb{P}^1$.

Lemma 3.1. *The total space of the restricted family $\mathcal{Y} \rightarrow \mathbb{A}^1$ has rational Gorenstein singularities. Consequently, it has an invertible relative dualizing sheaf $K_{\mathcal{Y}/\mathbb{A}^1}$.*

Proof. To lighten notations, let us write in this proof \mathcal{X} and \mathcal{Y} for the corresponding restrictions to \mathbb{A}^1 . The total space \mathcal{X} is non-singular, and \mathcal{Y} is a quotient of it by the action of a finite group. Therefore, \mathcal{Y} has rational singularities.

Now for the Gorenstein property. We introduce \mathcal{X}° the complement of the fixed locus of G , and $\mathcal{Y}^\circ = \mathcal{X}^\circ/G$. Denote by \mathcal{X}^* the smooth locus of $\mathcal{X} \rightarrow \mathbb{A}^1$, and $\mathcal{Y}^* = \mathcal{X}^*/G$. Then, since the ordinary double points in the fibers of $\mathcal{X} \rightarrow \mathbb{A}^1$ are disjoint from the fixed point locus of G , \mathcal{Y}° and \mathcal{Y}^* form an open cover of \mathcal{Y} . Thus it is enough to show that they are both Gorenstein.

Since G acts freely on \mathcal{X}° , the quotient \mathcal{Y}° is non-singular, and is therefore Gorenstein. For \mathcal{Y}^* , it is enough to observe that G preserves a relative holomorphic volume form on \mathcal{X}^* . Indeed, in affine coordinates $z_k = \frac{x_k}{x_j}$ on the open set $x_j \neq 0$, and where $\partial F_\psi / \partial z_i \neq 0$, the expression

$$(3.1) \quad \theta_0 = \frac{(-1)^{i-1} dz_0 \wedge \dots \widehat{dz_i} \wedge \dots \wedge \widehat{dz_j} \wedge \dots \wedge dz_n}{\partial F_\psi / \partial z_i} \Big|_{F_\psi=0}$$

provides such an invariant relative volume form. □

Because the BCOV invariant has not been fully developed for Calabi–Yau orbifolds (see nevertheless [Yos17] for some three-dimensional cases), we need crepant resolutions of the varieties Y_ψ . This needs to be done in families, so that the results of §2.2 apply. The family of crepant resolutions $\mathcal{Z} \rightarrow \mathbb{P}^1$ that we exhibit will be called the mirror family, although it is not unique. We also have to address the rationality of the construction.

Lemma 3.2. *There is a projective birational morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ of algebraic varieties over \mathbb{Q} , such that*

- (1) \mathcal{Z} is smooth.
- (2) If $\psi^{n+1} = 1$, the fiber Z_ψ has a single ordinary double point singularity.
- (3) If $\psi = \infty$, Z_∞ is a simple normal crossings divisor in \mathcal{Z} .
- (4) Otherwise, $Z_\psi \rightarrow Y_\psi$ is a crepant resolution of singularities. In particular, Z_ψ is a smooth Calabi–Yau variety.
- (5) The smooth complex fibers Z_ψ are mirror to the X_ψ , in that their Hodge numbers satisfy $h^{p,q}(Z_\psi) = h^{n-1-p,q}(X_\psi)$. In particular, the smooth Z_ψ are strict Calabi–Yau with $\chi(Z_\psi) = (-1)^{n-1} \chi(X_\psi)$.

Proof. The proof of (1)–(4) is based on [DHZ98, Sec. 8 (v)], [DHZ06] and [BG14, Prop. 3.1], together with Hironaka’s resolution of singularities. We recall the strategy, in order to justify the existence of a model over \mathbb{Q} .

Introduce $W = \mathbb{P}^n/G$. We claim this is a split toric variety over \mathbb{Q} . First of all, it can be realized as the hypersurface in $\mathbb{P}_{\mathbb{Q}}^{n+1}$ of equation

$$W: y_0^{n+1} = \prod_{j=1}^{n+1} y_j.$$

Second, the associated torus is split over \mathbb{Q} . It is actually given by $\mathbb{G}_m \times \mathbb{T}$, where \mathbb{T} is the kernel of the multiplication map $\mathbb{G}_m^{n+1} \rightarrow \mathbb{G}_m$. Finally, the action of the torus on W is defined over \mathbb{Q} :

$$\left((t_0, t_1, \dots, t_{n+1}), (y_0, y_1, \dots, y_{n+1}) \right) \mapsto (t_0 y_0, t_0 t_1 y_1, \dots, t_0 t_{n+1} y_{n+1}).$$

Once we know that W is a split toric variety over \mathbb{Q} , with same equation as in [DHZ06, Application 5.5], the toric and crepant projective resolution exhibited in *loc. cit.* automatically works over \mathbb{Q} as well. We write \widetilde{W} for this resolution of W .

We now consider \mathcal{Y} as a closed integral \mathbb{Q} -subscheme of $\widetilde{W} \times \mathbb{P}^1$. Let $\widetilde{\mathcal{Y}}$ be the strict transform of \mathcal{Y} in $\widetilde{W} \times \mathbb{P}^1$. By [DHZ98, Sec. 8 (v)], the fibers of $\widetilde{\mathcal{Y}}$ at $\psi \in \mathbb{C} \setminus \mu_{n+1}$ are projective crepant resolutions of the fibers Y_ψ . In particular, $\widetilde{\mathcal{Y}}$ is smooth over $\mathbb{C} \setminus \mu_{n+1}$, and in turn this implies smoothness over the complement U of the closed subscheme $V(\psi^{n+1} - 1)$ of $\mathbb{A}_{\mathbb{Q}}^1$. Necessarily, the fibers of $\widetilde{\mathcal{Y}}$ over U have trivial canonical bundle as well. For the fibers at $\psi^{n+1} = 1$, the claim of the lemma requires two observations:

- (1) the ordinary double points of X_ψ are permuted freely and transitively by G , and get identified to a single point in the quotient Y_ψ . This entails that the total space \mathcal{Y} is non-singular in a neighborhood of these points, and that they remain ordinary double points of $\mathcal{Y} \rightarrow \mathbb{P}^1$.
- (2) the center of the toric resolution is disjoint from the ordinary double points, since it is contained in the locus of \mathbb{P}^n/G where two or more projective coordinates vanish. Therefore, the morphism $\mathcal{Z} \rightarrow \mathcal{Y}$ is an isomorphism in a neighborhood of these points. Finally, on the complement, Z_ψ is a resolution of singularities of Y_ψ . Indeed, this is a local question in a neighborhood of the fixed points of G , so that the above references [DHZ98, DHZ06] still apply.

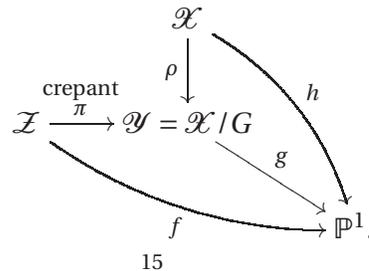
Finally, $\widetilde{\mathcal{Y}}$ is by construction smooth on the complement of the fiber $\psi = \infty$. After a resolution of singularities given by blowups with smooth centers in \widetilde{Y}_∞ (defined over \mathbb{Q}), we obtain a smooth algebraic variety \mathcal{Z} over \mathbb{Q} , such that Z_∞ is a simple normal crossings divisor in \mathcal{Z} . This sets items (1)–(4).

For (5), we refer for instance to [BD96, Thm. 6.9, Conj. 7.5 & Ex. 8.7]. This is specific to the Dwork pencil. More generally, we can cite work of Yasuda, who proves an invariance property of orbifold Hodge structures (and hence orbifold Hodge numbers) under crepant resolutions, for quotient Gorenstein singularities [Yas04, Thm. 1.5]. Orbifold Hodge numbers coincide with stringy Hodge numbers of global (finite) quotient orbifolds, whose underlying group respects a holomorphic volume form [BD96, Thm. 6.14]. Finally, by [BB96, Thm. 4.15], stringy Hodge numbers satisfy the expected mirror symmetry property for the mirror pairs constructed by Batyrev [Bat94]. \square

Definition 3.3. *The point $\infty \in \mathbb{P}^1$ is called the MUM point of the family $f: \mathcal{Z} \rightarrow \mathbb{P}^1$. The points $\xi \in \mathbb{P}^1$ with $\xi^{n+1} = 1$ are called the ODP points.*

The terminology MUM stands for *Maximal Unipotent Monodromy*, and it will be justified later in Lemma 4.1. The terminology ODP stands for *Ordinary Double Point*.

3.2. Comparison of Hodge bundles. Recall from the previous subsection the families \mathcal{X} , \mathcal{Y} and \mathcal{Z} , fibred over \mathbb{P}^1 :



We denote by U the maximal Zariski open subset of \mathbb{P}^1 where f (resp. h) is smooth. When it is clear from the context, we will still write \mathcal{X} , \mathcal{Y} and \mathcal{Z} for the total spaces of the fibrations restricted to U . Otherwise, we add an index U to mean the restriction to U . We let \mathcal{Y}° be the non-singular locus of \mathcal{Y}_U . It is the étale quotient of \mathcal{X}° , the complement in \mathcal{X}_U of the fixed point set of G . They are both open subsets whose complement has codimension ≥ 2 .

We begin our considerations by working complex analytically. Our discussion is based on a minor adaptation of [Ste77, Sec. 1] to the relative setting. First of all, we observe that the higher direct images $R^k g_* \mathbb{C}$ are locally constant sheaves, and actually $R^k g_* \mathbb{C} \simeq (R^k h_* \mathbb{C})^G$. Indeed, we have the equality $\mathbb{C}_{\mathcal{Y}} = (\rho_* \mathbb{C}_{\mathcal{X}})^G$. Moreover, since G is finite, so is ρ and taking G -invariants is an exact functor in the category of sheaves of $\mathbb{C}[G]$ -modules. A spectral sequence argument allows us to conclude. Similarly, one has $R^k g_* \mathbb{Q} \simeq (R^k h_* \mathbb{Q})^G$.

Let now $\tilde{\Omega}_{\mathcal{Y}/U}^\bullet$ be the relative holomorphic de Rham complex of $\mathcal{Y} \rightarrow U$, in the orbifold sense. It is constructed as follows. If $j: \mathcal{Y}^\circ \hookrightarrow \mathcal{Y}_U$ is the open immersion, then we let $\tilde{\Omega}_{\mathcal{Y}_U}^\bullet := j_* \Omega_{\mathcal{Y}^\circ}^\bullet$, and we derive the relative version $\tilde{\Omega}_{\mathcal{Y}/U}^\bullet$ out of it in the usual manner. An equivalent presentation is

$$\tilde{\Omega}_{\mathcal{Y}/U}^\bullet = (\rho_* \Omega_{\mathcal{X}/U}^\bullet)^G.$$

The complex $\tilde{\Omega}_{\mathcal{Y}/U}^\bullet$ is a resolution of $g^{-1} \mathcal{O}_U$. Hence its k -th relative hypercohomology computes $(R^k g_* \mathbb{C}) \otimes \mathcal{O}_U$, and satisfies

$$(3.2) \quad R^k g_* \tilde{\Omega}_{\mathcal{Y}/U}^\bullet \simeq (R^k h_* \Omega_{\mathcal{X}/U}^\bullet)^G,$$

compatibly with $R^k g_* \mathbb{C} \simeq (R^k h_* \mathbb{C})^G$. It has a Hodge filtration and Gauss–Manin connection defined in the usual way, satisfying a relationship analogous to (3.2). Equipped with this extra structure, $R^k g_* \mathbb{Q}$ defines a variation of pure rational Hodge structures of weight k .

The canonical identification $\tilde{\Omega}_{\mathcal{Y}_U}^\bullet = \pi_* \Omega_{\mathcal{Z}_U}^\bullet$ established in [Ste77, Lemma 1.11] induces a natural morphism

$$(3.3) \quad \tilde{\Omega}_{\mathcal{Y}/U}^\bullet \longrightarrow \pi_* (\Omega_{\mathcal{Z}/U}^\bullet).$$

The restriction of (3.3) to \mathcal{Y}° is given by pull-back of differential forms. We derive a natural map

$$(3.4) \quad (R^k h_* \Omega_{\mathcal{X}/U}^\bullet)^G \simeq R^k g_* \tilde{\Omega}_{\mathcal{Y}/U}^\bullet \longrightarrow R^k f_* \Omega_{\mathcal{Z}/U}^\bullet,$$

which is an injective morphism of variations of pure Hodge structures of weight k , cf. [Ste77, Cor. 1.5]. It is in particular compatible with restriction to the fibers, and remains injective on those. It can be checked to be compatible with the \mathbb{Q} -structures.

The following lemma summarizes the compatibility of (3.4) with the cup-product between Hodge bundles of complementary bi-degree. Before the statement, we recall that \mathcal{Y}_U is Gorenstein, and hence the morphism $g: \mathcal{Y} \rightarrow U$ has an invertible relative dualizing sheaf $K_{\mathcal{Y}/U}$.

Lemma 3.4. (1) $\tilde{\Omega}_{\mathcal{Y}/U}^n$ is the relative dualizing sheaf $K_{\mathcal{Y}/U}$.

(2) The natural morphism $R^k g_* \tilde{\Omega}_{\mathcal{Y}/U}^\bullet \rightarrow R^k f_* \Omega_{\mathcal{Z}/U}^\bullet$ induces a commutative diagram

$$\begin{array}{ccc}
R^q g_* \tilde{\Omega}_{\mathcal{Y}/U}^p \otimes R^{n-q} g_* \tilde{\Omega}_{\mathcal{Y}/U}^{n-p} & \longrightarrow & R^n g_* K_{\mathcal{Y}/U} \\
\downarrow & & \parallel \\
R^q f_* \Omega_{\mathcal{Z}/U}^p \otimes R^{n-q} f_* \Omega_{\mathcal{Z}/U}^{n-p} & \longrightarrow & R^n f_* K_{\mathcal{Z}/U}
\end{array}
\begin{array}{c}
\searrow \text{tr} \\
\mathcal{O}_U \\
\swarrow \text{tr}
\end{array}$$

(3) The natural isomorphism $R^k g_* \tilde{\Omega}_{\mathcal{Y}/U}^\bullet \simeq (R^k h_* \Omega_{\mathcal{X}/U}^\bullet)^G$ induces a commutative diagram

$$\begin{array}{ccc}
R^q g_* \tilde{\Omega}_{\mathcal{Y}/U}^p \otimes R^{n-q} g_* \tilde{\Omega}_{\mathcal{Y}/U}^{n-p} & \longrightarrow & R^n g_* K_{\mathcal{Y}/U} & \xrightarrow{\text{tr}} & \mathcal{O}_U \\
\downarrow & & \downarrow & & \downarrow |G| \\
R^q h_* \Omega_{\mathcal{X}/U}^p \otimes R^{n-q} h_* \Omega_{\mathcal{X}/U}^{n-p} & \longrightarrow & R^n h_* K_{\mathcal{X}/U} & \xrightarrow{\text{tr}} & \mathcal{O}_U
\end{array}$$

Proof. For the first property, we notice that $\rho^* K_{\mathcal{Y}/U} = K_{\mathcal{X}/U}$, since both coincide outside a codimension ≥ 2 closed subset and \mathcal{X}/U is smooth. Then we have the string of equalities

$$\tilde{\Omega}_{\mathcal{Y}/U}^n = \rho_*(K_{\mathcal{X}/U})^G = (K_{\mathcal{Y}/U} \otimes \rho_* \mathcal{O}_{\mathcal{X}/U})^G = K_{\mathcal{Y}/U} \otimes (\rho_* \mathcal{O}_{\mathcal{X}/U})^G = K_{\mathcal{Y}/U}.$$

For the first diagram, only the commutativity of the triangle requires a justification. It is a consequence of the three following facts: i) the transitivity of trace maps with respect to composition of morphisms [Har66, Thm. 10.5 (TRA1)]; ii) the crepant resolution property $\pi^* K_{\mathcal{Y}/U} = K_{\mathcal{Z}/U}$ and iii) \mathcal{Y}/U has rational singularities, so that $R\pi_* \mathcal{O}_{\mathcal{X}/U} = \mathcal{O}_{\mathcal{Y}/U}$. The argument is similar for the second diagram. Briefly, one combines: i) the transitivity of trace maps; ii) the duality $\rho_* K_{\mathcal{X}/U/\mathcal{Y}/U} = \text{Hom}_{\mathcal{O}_{\mathcal{Y}/U}}(\rho_* \mathcal{O}_{\mathcal{X}/U}, \mathcal{O}_{\mathcal{Y}/U})$ and iii) the trace $\text{tr}: \rho_* K_{\mathcal{X}/U/\mathcal{Y}/U} \rightarrow \mathcal{O}_{\mathcal{Y}/U}$ is given by $\varphi \mapsto \varphi(1)$ [Har66, proof of Prop. 6.5], and the composite map

$$K_{\mathcal{Y}/U} \longrightarrow \rho_* K_{\mathcal{X}/U} = K_{\mathcal{Y}/U} \otimes \rho_* K_{\mathcal{X}/U/\mathcal{Y}/U} \xrightarrow{\text{id} \otimes \text{tr}} K_{\mathcal{Y}/U}$$

is the multiplication by $|G|$. This is clear over \mathcal{Y}° , since it is the étale quotient of \mathcal{X}° by G . It is then necessarily true everywhere. \square

In the case of direct images of relative canonical sheaves, the discussion above specializes to the chain of isomorphism of line bundles

$$(3.5) \quad (h_* K_{\mathcal{X}/U})^G \xrightarrow{\sim} g_* K_{\mathcal{Y}/U} \xrightarrow{\sim} f_* K_{\mathcal{Z}/U}.$$

Moreover, these are the natural morphisms already defined over \mathbb{Q} . We leave the details to the reader.

Remark 3.5. In contrast to (3.5), it is in general not true that the injective morphism $R^k g_* \tilde{\Omega}_{\mathcal{Y}/U}^\bullet \rightarrow R^k f_* \Omega_{\mathcal{Z}/U}^\bullet$ is an isomorphism of variations of Hodge structures. The fibers of $R^k g_* \tilde{\Omega}_{\mathcal{Y}/U}^\bullet$ are but a piece of so-called orbifold cohomology groups, which also includes the cohomology of the so-called twisted sectors. It is known that the orbifold cohomology of a proper variety with quotient Gorenstein singularities is isomorphic, as a Hodge structure, to the cohomology of a crepant resolution [Yas04, Thm. 1.5]. The isomorphism is however not explicit. In any event, *loc. cit.* relates the Hodge numbers of both structures (see the proof of Lemma 3.2).

For later use, we record the following lemma.

Lemma 3.6. *Let $h^{p,q}$ be the rank of the Hodge bundle $R^q f_* \Omega_{\mathcal{Z}/U}^p$. Then:*

- $h^{p,q} = 1$ if $p + q = n - 1$ and $p \neq q$.
- $h^{p,p} = \sum_{j=0}^p (-1)^j \binom{n+1}{j} \binom{(p+1-j)n+p}{n} + \delta_{2p, n-1}$.
- $h^{p,q} = 0$ otherwise.

In particular,

$$\chi(Z_\psi) = (-1)^{n-1} \chi(X_\psi) = (-1)^{n-1} \left(\frac{(-n)^{n+1} - 1}{n+1} + n + 1 \right).$$

Moreover, in even relative dimension $n - 1 = 2d$ and for any choice of polarization, the sheaf $(R^d f_* \Omega_{\mathcal{Z}/U}^d)_{\text{prim}}$ has rank one.

Proof. The items are a consequence of the mirror symmetry property for the Hodge numbers in Lemma 3.2, and the computation of the cohomology of a hypersurface in projective space (cf. [BD96, Ex. 8.7]). The claim on primitive cohomology follows from the primitive decomposition together with the formula for all the $h^{p,p}$, $p \leq d$. \square

3.3. Sections of the middle degree Hodge bundles. We maintain the setting and notations of the previous subsection. We now further compare the middle degree Hodge bundles of the Dwork family $h: \mathcal{X} \rightarrow U$ and that of its mirror $f: \mathcal{Z} \rightarrow U$, by constructing explicit sections via Griffiths' residue method [Gri69]. We introduce primitivity notions for the relative Hodge bundles, induced by any projective factorization of f and the natural projective embedding of h . Observe the latter is G -equivariant and defined over \mathbb{Q} . We also require the polarization for $\mathcal{Z} \rightarrow U$ to be defined over \mathbb{Q} . Then the primitive Hodge bundles are defined in the category of \mathbb{Q} -schemes.

Our reasoning starts in the complex analytic category. Denote by $H = x_0 \cdot x_1 \cdot \dots \cdot x_n$ and $\Omega = \sum (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_n \in H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}^n(n+1))$. For $\psi \in U(\mathbb{C})$, the residue along X_ψ

$$\theta_k = \text{res}_{X_\psi} \left(\frac{k! H^k \Omega}{F_\psi^{k+1}} \right)$$

defines a G -invariant element of $H^{n-1}(X_\psi)$, still denoted θ_k . For $k = 0$, this indeed agrees with the holomorphic volume form (3.1). By [Gri69, Thm. 8.3], we actually have for $k = 0, \dots, n-1$

$$\theta_k \in F^{n-1-k} H^{n-1}(X_\psi)_{\text{prim}}^G \setminus F^{n-k} H^{n-1}(X_\psi)_{\text{prim}}^G.$$

Taking into account the injective morphism (3.4) and the Hodge numbers computed in Lemma 3.6, we see that $H^{n-1-k, k}(X_\psi)_{\text{prim}}^G$ is necessarily one-dimensional and θ_k projects to a basis element of it. Let us call σ_k this projected element. In the families setting, θ_k and σ_k define sections of the corresponding Hodge bundles. In a nutshell, the collection of sections θ_k forms a basis of $(R^{n-1} h_* \Omega_{\mathcal{X}/U}^\bullet)_{\text{prim}}^G$ adapted to the Hodge filtration, and each individual σ_k is a trivialization of $(R^k h_* \Omega_{\mathcal{X}/U}^{n-1-k})_{\text{prim}}^G$.

Now let ∇ be the Gauss–Manin connection acting on $R^{n-1} h_* \Omega_{\mathcal{X}/U}^\bullet$. It is compatible with the G -action after §3.2, and it preserves primitive classes as well. From the definition of the sections

θ_k , one can check the following recurrence:

$$(3.6) \quad \nabla_{d/d\psi} \theta_k = \text{res}_{X_\psi} \left(\frac{\partial}{\partial \psi} \left(\frac{k! H^k \Omega}{F_\psi^{k+1}} \right) \right) = (n+1) \theta_{k+1}.$$

Notice that θ_0 lies in the G -invariant primitive cohomology for obvious reasons, since it belongs to $(h_* K_{\mathcal{X}/U})^G$. Therefore this recurrence explains that all the θ_k are G -invariant and primitive as well. This argument is at the basis of the following proposition:

Proposition 3.7. *The natural morphism (3.4) induces an isomorphism of variations of Hodge structures*

$$(3.7) \quad (R^{n-1} h_* \Omega_{\mathcal{X}/U}^\bullet)_{\text{prim}}^G \xrightarrow{\sim} (R^{n-1} f_* \Omega_{\mathcal{Z}/U}^\bullet)_{\text{prim}}.$$

Proof. By the Hodge numbers computed in Lemma 3.6, it is enough to check that the θ_k are mapped into primitive classes. Let θ'_k be the image of θ_k under (3.4). As (3.4) is compatible with Gauss–Manin connections, the θ'_k satisfy the analogous recurrence to (3.6). Because $f_* K_{\mathcal{Z}/U}$ is primitive and the Gauss–Manin connection preserves primitive cohomology, we see that the θ'_k land in the primitive cohomology. \square

Remark 3.8. (1) Notice that there was no *a priori* compatibility between the primitivity notions for $\mathcal{X} \rightarrow U$ and $\mathcal{Z} \rightarrow U$. The proposition depends crucially on the concrete geometrical setting and Hodge structures.

(2) Although (3.7) is an isomorphism of variations of Hodge structures, the intersection pairings do not match. According to Lemma 3.4, they differ by the order of the group G .

Definition 3.9. *We define η_k° as the trivializing section of $(R^k f_* \Omega_{\mathcal{Z}/U}^{n-1-k})_{\text{prim}}$, deduced from θ_k via the isomorphism (3.7) and by projecting to the Hodge bundle. It corresponds to the section σ_k above. We also define $\eta_k = -(n+1)^{k+1} \psi^{k+1} \eta_k^\circ$.*

Remark 3.10. By construction, the section η_k vanishes at order $k+1$ at $\psi = 0$.

Let now $\text{KS}^{(q)}$ be the cup product with the Kodaira–Spencer class, induced by the Gauss–Manin connection:

$$(3.8) \quad \text{KS}^{(q)} : T_U \longrightarrow \text{Hom}_{\mathcal{O}_U} (R^q f_* \Omega_{\mathcal{Z}/U}^{n-1-q}, R^{q+1} f_* \Omega_{\mathcal{Z}/U}^{n-2-q}).$$

The algebraic theory of the Gauss–Manin connection [KO68] ensures that $\text{KS}^{(q)}$ is already defined in the category of \mathbb{Q} -schemes. Somewhat abusively, we refer to $\text{KS}^{(q)}$ as a Kodaira–Spencer morphism.

Lemma 3.11. *The sections η_k° satisfy the recurrence*

$$(3.9) \quad \text{KS}^{(k)} \left(\frac{d}{d\psi} \right) \eta_k^\circ = (n+1) \eta_{k+1}^\circ.$$

Consequently, η_k° is defined over \mathbb{Q} and

$$(3.10) \quad \text{KS}^{(k)} \left(\psi \frac{d}{d\psi} \right) \eta_k = \eta_{k+1}.$$

Proof. The recurrence follows from (3.6), Proposition 3.7, the link between the Gauss–Manin connection ∇ and the Kodaira–Spencer maps $\text{KS}^{(q)}$, and the definition of η_k° . For the rationality statement, we first claim it for η_0° . Indeed, the residue construction defining θ_0 makes sense

in the algebraic category. The claim follows, since η_0° is the image of θ_0 under (3.5), which is defined in the category of \mathbb{Q} -schemes. For the rest of sections, we apply the recurrence (3.9), together with the algebraicity property of $\text{KS}^{(k)}$ and the fact that $d/d\psi$ is a section of $T_{U/\mathbb{Q}}$. Equation (3.10) follows from (3.9) by the definition $\eta_k = -(n+1)^{k+1}\psi^{k+1}\eta_k^\circ$ and the \mathcal{O}_U -linearity of the Kodaira–Spencer maps. \square

An analogous argument can be carried out with the sections θ_k and θ'_k , thus proving the following statement.

Lemma 3.12. *The isomorphism (3.7) already exists in the category of \mathbb{Q} -schemes.*

4. THE DEGENERATION OF HODGE BUNDLES OF THE MIRROR FAMILY

In the previous section we exhibited explicit trivializing sections of the middle degree Hodge bundles of the mirror family $\mathcal{Z} \rightarrow U$. The next goal is to extend these sections to the whole compactification \mathbb{P}^1 . We also address the trivialization of the Hodge bundles outside the middle degree. For this purpose, we exploit the approach to degenerating Hodge structures via relative logarithmic de Rham cohomology.

Before embarking on our task, we recall some background from Steenbrink [Ste76, Ste77] and our previous work [EFiMM18a, Sec. 2 & Sec. 4]. Let $f: X \rightarrow \mathbb{D}$ be a projective morphism of reduced analytic spaces, over the unit disc \mathbb{D} . We suppose that the fibers X_t with $t \neq 0$ are smooth and connected. We consider the variation of Hodge structures defined by $R^k f_* \mathbb{C}$ over the punctured disc \mathbb{D}^\times . Let T be its monodromy operator and ∇ the Gauss–Manin connection. Recall that T is a quasi-unipotent transformation of the cohomology of the general fiber. The flat vector bundle $((R^k f_* \mathbb{C}) \otimes \mathcal{O}_{\mathbb{D}^\times}, \nabla)$ has a unique extension to a vector bundle with regular singular connection on \mathbb{D} , whose residue is an endomorphism with eigenvalues in $[0, 1) \cap \mathbb{Q}$. This is the Deligne lower extension of $R^k f_* \mathbb{C}$, and somewhat loosely we refer to it by ${}^\ell R^k f_* \mathbb{C}$. The analytic construction of this extension involves the choice of the lower branch of the logarithm ${}^\ell \log: \mathbb{C}^\times \rightarrow \mathbb{R} + 2\pi i(-1, 0]$, and the corresponding logarithm of the monodromy ${}^\ell \log T$. An equivalent algebraic approach goes through the logarithmic de Rham cohomology. To this end, fix $f': X' \rightarrow \mathbb{D}$ any normal crossings model of f . Hence, f' and f coincide over \mathbb{D}^\times , but X' is non-singular and the special fiber X'_0 is a normal crossings divisor. Then one can define the associated relative logarithmic de Rham complex $\Omega_{X'/\mathbb{D}}^\bullet(\log)$, and its relative hypercohomology sheaf $R^k f' \Omega_{X'/\mathbb{D}}^\bullet(\log)$. It is locally free, and the Gauss–Manin connection extends to a regular singular connection on it, whose residue has eigenvalues in $[0, 1) \cap \mathbb{Q}$. It thus realizes the Deligne lower extension above. In particular, the construction does not depend on the chosen normal crossings model. The Hodge filtration extends to a filtration by vector sub-bundles, with locally free graded quotients of the form $R^{k-p} f' \Omega_{X'/\mathbb{D}}^p(\log)$. If the monodromy operator is unipotent, then the fiber of $R^k f' \Omega_{X'/\mathbb{D}}^\bullet(\log)$ at 0, together with the restricted Hodge filtration, can be identified with the cohomology of the generic fiber H_{lim}^k with the limiting Hodge filtration F_∞^\bullet . The identification depends on the choice of a holomorphic coordinate on \mathbb{D} . There is also the monodromy weight filtration W_\bullet on H_{lim}^k , attached to the nilpotent operator N , the residue of the Gauss–Manin connection.⁴ The triple $(H_{\text{lim}}^k, F_\infty^\bullet, W_\bullet)$ is called the limiting mixed Hodge structure. It is isomorphic to Schmid’s limiting mixed Hodge structure [Sch73]. In the general quasi-unipotent

⁴The standard convention for N is actually $-2\pi i$ times the residue of the Gauss–Manin connection. The normalization we choose here is however more convenient for the purposes of this article.

case, one first performs a semi-stable reduction and then constructs the limiting mixed Hodge structure. The precise behaviour of the Hodge bundles $R^{k-p} f' \Omega_{X'/\mathbb{D}}^p(\log)$ with respect to semi-stable reduction will be fundamental below. Succinctly, the formation of logarithmic Hodge bundles does not commute with semi-stable reduction, and the default of commutativity is encoded in the action of the semi-simple part of the monodromy on $(H_{\lim}^k, F_\infty^\bullet)$ [EFiMM18a, Cor. 2.8]. This translates into the vanishing order of the L^2 metric on $\det R^{k-p} f' \Omega_{X'/\mathbb{D}}^p(\log)$ [EFiMM18a, Thm. 4.4].

Analogously, for a normal crossings degeneration $f: X \rightarrow S$ between complex projective manifolds, there are algebraic counterparts of the logarithmic de Rham cohomology, Gauss–Manin connection, Hodge filtration, etc. This is compatible with the analytic theory after localizing to a holomorphic coordinate neighborhood of a given point $p \in S$. We will in particular speak of the limiting mixed Hodge structure at p , and simply write H_{\lim}^k if there is no danger of confusion.

The foregoing discussion can be carried out in the polarized setting and for primitive cohomology. We will only consider polarizations induced by projective factorizations of our morphisms.

In the sequel, we specialize to the mirror family $f: \mathcal{Z} \rightarrow \mathbb{P}^1$. We fix the normal crossings model $f': \mathcal{Z}' \rightarrow \mathbb{P}^1$ obtained by blowing-up the ordinary double points in the fibers Z_ξ , where $\xi \in \mathbb{C}$ and $\xi^{n+1} = 1$. Given a polarization, say induced by a projective factorization of f' , we study the limiting mixed Hodge structures on the middle primitive cohomology. To lighten notations, we write H_{\lim}^{n-1} instead of $H_{\text{prim}, \lim}^{n-1}$.

4.1. Behaviour of η_k at the MUM point. For the mirror family $f: \mathcal{Z} \rightarrow \mathbb{P}^1$, let \mathbb{D}_∞ be a holomorphic disc neighborhood at infinity, with parameter $t = 1/\psi$. To lighten notations, we still denote by $f: \mathcal{Z} \rightarrow \mathbb{D}_\infty$ the restricted family. Also, following the previous conventions, we write H_{\lim}^{n-1} for the limiting mixed Hodge structure at infinity of the middle primitive cohomology.

Lemma 4.1. (1) *The monodromy T of $(R^{n-1} f_* \mathbb{C})_{\text{prim}}$ at ∞ is maximal unipotent. In particular, the nilpotent operator N on H_{\lim}^{n-1} satisfies $N^{n-1} \neq 0$.*

(2) *The graded pieces $\text{Gr}_k^W H_{\lim}^{n-1}$ are one-dimensional if k is even, and trivial otherwise. For all $1 \leq k \leq n-1$, N induces isomorphisms*

$$\text{Gr}_k^W N: \text{Gr}_k^W H_{\lim}^{n-1} \xrightarrow{\sim} \text{Gr}_{k-2}^W H_{\lim}^{n-1}.$$

(3) *For all $1 \leq p \leq n-1$, N induces isomorphisms*

$$\text{Gr}_{F_\infty}^p N: \text{Gr}_{F_\infty}^p H_{\lim}^{n-1} \xrightarrow{\sim} \text{Gr}_{F_\infty}^{p-1} H_{\lim}^{n-1}.$$

Proof. In odd relative dimension, the maximal unipotent property for $R^{n-1} f_* \mathbb{C} = (R^{n-1} h_* \mathbb{C})_{\text{prim}}^G$ is [HSBT10, Cor. 1.7]. In even relative dimension, exactly the same argument as in *loc. cit.* yields the claim for $(R^{n-1} h_* \mathbb{C})_{\text{prim}}^G$. The property is inherited by $(R^{n-1} f_* \mathbb{C})_{\text{prim}}$ thanks to Proposition 3.7. In particular $N^{n-1} \neq 0$. This settles the first point. Because moreover N^{n-1} induces an isomorphism $\text{Gr}_{2(n-1)}^W H_{\lim}^{n-1} \xrightarrow{\sim} \text{Gr}_0^W H_{\lim}^{n-1}$ we deduce that $\text{Gr}_0^W H_{\lim}^{n-1} \neq 0$. Since H_{\lim}^{n-1} is n -dimensional, the second item follows for dimension reasons. Finally, we use that $\text{Gr}_{F_\infty}^p H_{\lim}^{n-1}$ is one-dimensional by Lemma 3.6 and then necessarily $\text{Gr}_{F_\infty}^p H_{\lim}^{n-1} = \text{Gr}_{F_\infty}^p \text{Gr}_{2p}^W H_{\lim}^{n-1} = \text{Gr}_{2p}^W H_{\lim}^{n-1}$. Hence the second point implies the third one. \square

By the maximal unipotent monodromy and for dimension reasons, the T -invariant classes of the primitive cohomology of a general fiber span a rank one trivial sub-system of $(R^{n-1} f_* \mathbb{C})_{\text{prim}}$

on \mathbb{D}_∞^\times . We fix a basis γ' of this trivial system. It extends to a nowhere vanishing holomorphic section of the Deligne extension ${}^\ell(R^{n-1}f_*\mathbb{C})_{\text{prim}}$. The fiber at 0 is then a basis for $W_0 = \ker N$ (this is not a general fact, but a special feature of the weight filtration under consideration). We still write γ' for this limit element. Similarly, $(R^{n-1}f_*\mathbb{C})_{\text{prim}}^\vee$ has a rank one trivial sub-system, spanned by the class of a T -invariant homological cycle γ . We may choose γ to correspond to γ' by Poincaré duality. Hence, for any $\eta \in H^{n-1}(Z_t)$, $t \in \mathbb{D}_\infty^\times$, the period $\langle \gamma, \eta \rangle$ equals the intersection pairing $S(\gamma', \eta)$. It is possible to explicitly construct an invariant cycle. Although we will need this in a moment, we postpone the discussion to §5.2, where a broader study of homological cycles is delivered.

Lemma 4.2. *Let η be a holomorphic trivialization of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$. Then the period $\langle \gamma, \eta \rangle$ defines a holomorphic function on \mathbb{D}_∞ , non-vanishing at the origin.*

Proof. The argument is well-known (see e.g. [Mor93, Prop.] and [Voi99, Lemma 3.10]), but we sketch it due to its relevance.

The pairing $\langle \gamma, \eta \rangle = S(\gamma', \eta)$ is clearly a holomorphic function on \mathbb{D}_∞^\times , since both γ' and η are holomorphic sections of $(R^{n-1}f_*\mathbb{C}) \otimes \mathcal{O}_{\mathbb{D}_\infty^\times}$. Moreover, they are both global sections of the Deligne extension. This ensures that $|S(\gamma', \eta)|$ has at most a logarithmic singularity at 0. It follows that $S(\gamma', \eta)$ is actually a holomorphic function.

For the non-vanishing property, we make use of the interplay between the intersection pairing seen on H_{lim}^{n-1} and the monodromy weight filtration [Sch73, Lemma 6.4], together with Lemma 4.1. Let $\eta' \in H_{\text{lim}}^{n-1}$ be the fiber of η at 0. We need to show that $S(\gamma', \eta') \neq 0$. Suppose the contrary. Since γ' is a basis of $W_0 = \ker N = \text{Im } N^{n-1}$, we have $\eta' \in (\text{Im } N^{n-1})^\perp$. The intersection pairing is non-degenerate and satisfies $S(Nx, y) + S(x, Ny) = 0$. Therefore, we find that $\eta' \in (\text{Im } N^{n-1})^\perp = \ker N^{n-1} = W_{2n-3}$. But η' is a basis of $F^{n-1}H_{\text{lim}}^{n-1} = F^{n-1}\text{Gr}_{2n-2}^W H_{\text{lim}}^{n-1}$, and therefore $\eta' \notin W_{2n-3}$. We thus have reached a contradiction. \square

Before the next theorem, we consider the logarithmic extension of the Kodaira–Spencer maps (3.8): if D is the divisor $[\infty] + \sum_{\xi^{n+1}=1} [\xi]$, then

$$(4.1) \quad \text{KS}^{(q)} : T_{\mathbb{P}^1}(-\log D) \longrightarrow \text{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(R^q f_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^{n-1-q}(\log), R^{q+1} f_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^{n-2-q}(\log)).$$

They preserve the primitive components.

Theorem 4.3. *The section η_k is a holomorphic trivialization of $R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_\infty}^{n-1-k}(\log)_{\text{prim}}$.*

Proof. First of all, we prove that η_0 is a meromorphic section of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$. Indeed, η_0 is an algebraic section of $f_*K_{\mathcal{Z}/U}$ (see Lemma 3.11), hence a rational section of $f_*K_{\mathcal{Z}'/\mathbb{P}^1}(\log)$ and thus a meromorphic section of $f_*K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$.

Second, we establish the claim of the theorem for η_0 . By Lemma 4.2, we need to show that the holomorphic function $\langle \gamma, \eta_0 \rangle$ on \mathbb{D}_∞^\times extends holomorphically to \mathbb{D}_∞ , and does not vanish at the origin. This property can be checked by a standard explicit computation reproduced below (5.6).

Finally, for the sections η_k , we use the recurrence (3.10) and the logarithmic extension of the Kodaira–Spencer maps (4.1). It follows that the sections η_k are global sections of the sheaves $R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_\infty}^{n-1-k}(\log)_{\text{prim}}$. Let us denote by η'_k the fiber at 0 of the sections η_k . Specializing (3.10) at 0, we find $(\text{Gr}_{F_\infty}^{n-1-k} N)\eta'_k = \eta'_{k+1}$. By Lemma 4.1 (3) and because $\eta'_0 \neq 0$, we see that $\eta'_k \neq 0$ for all k . This concludes the proof. \square

4.2. Behaviour of η_k at the ODP points. Recall the normal crossings model $f': \mathcal{Z}' \rightarrow \mathbb{P}^1$. We restrict it to a disc neighborhood \mathbb{D}_ξ of some $\xi \in \mu_{n+1}$. Concretely, we fix the coordinate $t = \psi - \xi$. We write $f': \mathcal{Z}' \rightarrow \mathbb{D}_\xi$ for the restricted family. We now deal with the limiting mixed Hodge structure H_{lim}^{n-1} at ξ of the middle primitive cohomology. Since the monodromy around ξ is not unipotent in general, the construction of H_{lim}^{n-1} requires a preliminary semi-stable reduction. This can be achieved as follows:

$$(4.2) \quad \begin{array}{ccccc} \tilde{\mathcal{Z}} & \xrightarrow{\text{normalization}} & \mathcal{Z}'' & \xrightarrow{r} & \mathcal{Z}' \\ & \searrow \tilde{f} & \downarrow & \square & \downarrow f' \\ & & \mathbb{D}_\xi & \xrightarrow{\rho(u)=u^2=t} & \mathbb{D}_\xi \end{array}$$

Hence $\tilde{f}: \tilde{\mathcal{Z}} \rightarrow \mathbb{D}_\xi$ is the normalized base change of f' by ρ . An explicit computation in local coordinates shows it is indeed semi-stable. The special fiber $\tilde{f}^{-1}(0)$ consists of two components intersecting transversally. One is the strict transform \tilde{Z} of Z_ξ . We denote by E the other component. Then E is a non-singular quadric of dimension $n-1$, and $\tilde{Z} \cap E$ is a non-singular quadric of dimension $n-2$. In terms of this data, the monodromy weight filtration is computed as follows, cf. [Ste77, Ex. 2.15].

Lemma 4.4. *The graded pieces of the weight filtration on H_{lim}^{n-1} are given by:*

- if $n-1$ is odd, then

$$\text{Gr}_k^W H_{\text{lim}}^{n-1} = \begin{cases} \mathbb{Q}(-\frac{n-2}{2}), & \text{if } k = n-2, \\ H^{n-1}(\tilde{Z}), & \text{if } k = n-1, \\ \mathbb{Q}(-\frac{n}{2}), & \text{if } k = n, \\ 0, & \text{otherwise.} \end{cases}$$

- if $n-1$ is even, then

$$\text{Gr}_k^W H_{\text{lim}}^{n-1} = \begin{cases} H(H^{n-3}(\tilde{Z} \cap E)(-1) \rightarrow H^{n-1}(\tilde{Z}) \oplus H^{n-1}(E) \rightarrow H^{n-1}(\tilde{Z} \cap E)), & \text{if } k = n-1, \\ 0, & \text{if } k \neq n-1. \end{cases}$$

Hence, H_{lim}^{n-1} is a pure Hodge structure of weight $n-1$.

We will need the comparison of the middle degree Hodge bundles between before and after semi-stable reduction. We follow [EFiMM18a, Sec. 2 & Prop. 3.10]. There are natural morphisms

$$(4.3) \quad \varphi^{p,q}: \rho^* R^q f'_* \Omega_{\mathcal{Z}'/\mathbb{D}_\xi}^p(\log)_{\text{prim}} \hookrightarrow R^q \tilde{f}_* \Omega_{\tilde{\mathcal{Z}}/\mathbb{D}_\xi}^p(\log)_{\text{prim}}.$$

Lemma 4.5. *Suppose that $p+q = n-1$. Let $Q^{p,q}$ be the cokernel of $\varphi^{p,q}$ in (4.3).*

- If $p \neq q$, then $Q^{p,q} = 0$.
- If $p = q = \frac{n-1}{2}$, then $Q^{p,p} = \mathcal{O}_{\mathbb{D}_\xi,0}/u\mathcal{O}_{\mathbb{D}_\xi,0}$.

Proof. The results in [EFiMM18a, Sec. 2 & Prop. 3.10] are explicitly stated for the whole Hodge bundles. For their primitive components, see however Remark 2.7 (iii) in *loc. cit.*, or notice an easy compatibility with the primitive decomposition. \square

The last fact we need is the computation of the *Yukawa coupling*. A repeated application of the Kodaira–Spencer maps gives a morphism

$$(4.4) \quad Y: \text{Sym}^{n-1} T_U \longrightarrow \text{Hom}_{\mathcal{O}_U}(f_* K_{\mathcal{Z}/U}, R^{n-1} f_* \mathcal{O}_{\mathcal{Z}}) \simeq (f_* K_{\mathcal{Z}/U})^{\otimes -2}.$$

Using the section $\psi d/d\psi$ of T_U and the section η_0 of $f_* K_{\mathcal{Z}/U}$, we obtain a holomorphic function on U , denoted $Y(\psi)$. Working with $(R^{n-1} h_* \Omega_{\mathcal{X}/U}^\bullet)^G$ instead, one similarly defines a function $\tilde{Y}(\psi)$. Via the isomorphism of Proposition 3.7, the functions $\tilde{Y}(\psi)$ and $Y(\psi)$ can be compared. The only subtle point to bear in mind is the use of Serre duality in the definition of the Yukawa coupling. For Hodge bundles of complementary bi-degree, Serre duality is induced by the cup-product and the trace morphism. Hence, an application of Lemma 3.4 shows that $Y(\psi)$ and $\tilde{Y}(\psi)$ are equal up to the order of G . With this understood, we can invoke the computation of the Yukawa coupling in [BvS95, Cor. 4.5.6 & Ex. 4.5.7], and conclude

$$(4.5) \quad Y(\psi) = c \frac{\psi^{n+1}}{1 - \psi^{n+1}}$$

for some constant $c \neq 0$. With notations as in *loc. cit.*, their factor $\lambda_{\mathcal{Z}}$ is $1/\psi^{n+1}$, thus explaining the formal discrepancy of both formulas.

We are now fully equipped for the proof of:

Theorem 4.6. *The sections η_k extend to rational sections of the logarithmic Hodge bundles $R^k f'_* \Omega_{\mathcal{Z}'/\mathbb{A}^1}^{n-1-k}(\log)_{\text{prim}}$. Furthermore, if $\text{ord}_\xi \eta_k$ is the order of zero or pole of η_k at ξ , as a rational section of $R^k f'_* \Omega_{\mathcal{Z}'/\mathbb{A}^1}^{n-1-k}(\log)_{\text{prim}}$, then:*

- if $n-1$ is odd, then $\text{ord}_\xi \eta_k = 0$ for $k \leq n/2 - 1$ and $\text{ord}_\xi \eta_k = -1$ otherwise.
- if $n-1$ is even, then $\text{ord}_\xi \eta_k = 0$ for $k \leq \frac{n-3}{2}$ and $\text{ord}_\xi \eta_k = -1$ otherwise.

Proof. Throughout the proof, we write \mathcal{X} , \mathcal{Y} and \mathcal{Z} for the respective total spaces over \mathbb{A}^1 . We begin by showing that η_0 extends to a global section of $f'_* K_{\mathcal{Z}'/\mathbb{A}^1}(\log)$, non-vanishing at ξ . Since the singular fibers of $\mathcal{Z} \rightarrow \mathbb{A}^1$ present only ordinary double points, there is an equality

$$f_* K_{\mathcal{Z}/\mathbb{A}^1} = f'_* K_{\mathcal{Z}'/\mathbb{A}^1}(\log).$$

This can be seen as the coincidence of the upper and lower extensions of $f_* K_{\mathcal{Z}/U}$ to \mathbb{A}^1 (apply [EFiMM18a, Cor. 2.8 & Prop. 2.10] and the Picard–Lefschetz formula for the monodromy). Since \mathcal{Y} has rational singularities (cf. Lemma 3.1), the natural morphism $g_* K_{\mathcal{Y}/\mathbb{A}^1} \rightarrow f_* K_{\mathcal{Z}/\mathbb{A}^1}$ is an isomorphism. Also $g_* K_{\mathcal{Y}/\mathbb{A}^1} = (h_* K_{\mathcal{X}/\mathbb{A}^1})^G$. Indeed, let \mathcal{X}° be the complement of the fixed point locus of G in \mathcal{X} and similarly for \mathcal{Y}° , so that $\mathcal{Y} \setminus \mathcal{Y}^\circ$ has codimension ≥ 2 . Then, because \mathcal{Y} is normal Gorenstein and $\mathcal{Y}^\circ = \mathcal{X}^\circ/G$ is an étale quotient, and \mathcal{X} is non-singular, we find

$$g_* K_{\mathcal{Y}/\mathbb{A}^1} = g_* K_{\mathcal{Y}^\circ/\mathbb{A}^1} = (h_* K_{\mathcal{X}^\circ/\mathbb{A}^1})^G = (h_* K_{\mathcal{X}/\mathbb{A}^1})^G.$$

By construction of η_0 (cf. Definition 3.9), it is enough to prove that θ_0 defines a trivialization of $h_* K_{\mathcal{X}/\mathbb{A}^1}$ around ξ . Denote by \mathcal{X}^* the complement in \mathcal{X} of the ordinary double points, so that $\mathcal{X} \setminus \mathcal{X}^*$ has codimension ≥ 2 . Because \mathcal{X} is non-singular, we have $h_* K_{\mathcal{X}/\mathbb{A}^1} = h_* K_{\mathcal{X}^*/\mathbb{A}^1}$. Now, the expression (3.1) for θ_0 defines a relative holomorphic volume form on the whole \mathcal{X}^* , and hence a trivialization of $h_* K_{\mathcal{X}^*/\mathbb{A}^1}$ as desired.

That the sections η_k define rational sections of the sheaves $R^k f'_* \Omega_{\mathcal{Z}'/\mathbb{A}^1}^{n-1-k}(\log)_{\text{prim}}$ follows from the corresponding property for η_0 , plus the recurrence (3.10) and the existence of the logarithmic extension of the Kodaira–Spencer maps (4.1). From the same recurrence, we reduce

the computation of $\text{ord}_\xi \eta_k$ to the computation of the orders at ξ of the rational morphisms $\text{KS}^{(j)}(\psi d/d\psi)$, with respect to the logarithmic extension of the Hodge bundles:

$$\text{ord}_\xi \eta_k = \text{ord}_\xi \eta_0 + \sum_{j=0}^{k-1} \text{ord}_\xi \text{KS}^{(j)} \left(\psi \frac{d}{d\psi} \right) = \sum_{j=0}^{k-1} \text{ord}_\xi \text{KS}^{(j)} \left(\psi \frac{d}{d\psi} \right).$$

Let us define $M^{(j)} = \text{ord}_\xi \text{KS}^{(j)} \left(\psi \frac{d}{d\psi} \right)$. Because η_0 trivializes $f_* K_{\mathcal{Z}/\mathbb{A}^1}$ at ξ , formula (4.5) shows that

$$(4.6) \quad \sum_{j=0}^{n-2} M^{(j)} = \text{ord}_\xi Y(\psi) = -1.$$

We argue that all but one of the $M^{(j)}$ are in fact zero. For this, we relate $M^{(j)}$ to the action of the nilpotent operator N on the limiting mixed Hodge structure at ξ . Recall we defined the coordinate $t = \psi - \xi$ on a disc neighborhood \mathbb{D}_ξ of ξ . The first observation is

$$\text{ord}_{t=0} \text{KS}^{(j)} \left(t \frac{d}{dt} \right) = \text{ord}_\xi \text{KS}^{(j)} \left((\psi - \xi) \frac{d}{d\psi} \right) = M^{(j)} + 1.$$

We now need to distinguish two cases, depending on the parity of $n - 1$.

Odd case: If $n - 1$ is odd, then the monodromy is unipotent and the fiber of $\text{KS}^{(p)}(td/dt)$ at $t = 0$ is already $\text{Gr}_{F_\infty}^p N: \text{Gr}_{F_\infty}^p H_{\text{lim}}^{n-1} \rightarrow \text{Gr}_{F_\infty}^{p-1} H_{\text{lim}}^{n-1}$. From Lemma 4.4, we deduce that unless $p = n/2$, $\text{Gr}_{F_\infty}^p N = 0$ so that $\text{ord}_{t=0} \text{KS}^{(p)}(td/dt) > 0$ and hence $M^{(p)} \geq 0$. By (4.6) we necessarily have $M^{(n/2)} = -1$ and the other $M^{(j)} = 0$.

Even case: If $n - 1$ is even, the nilpotent operator N is in fact trivial, but the monodromy is no longer unipotent. The construction of the limiting mixed Hodge structure thus involves a semi-stable reduction. Choose a square root u of t as in (4.2). Then since $u \frac{d}{du} = 2t \frac{d}{dt}$ and $\text{ord}_{u=0} = 2 \text{ord}_{t=0}$ we get

$$(4.7) \quad \text{ord}_{u=0} \varphi^{p,q} + \text{ord}_{u=0} \text{KS}^{(q)} \left(u \frac{d}{du} \right) = \text{ord}_{u=0} (\varphi^{p-1,q+1}) + 2 \text{ord}_{t=0} \text{KS}^{(q)} \left(t \frac{d}{dt} \right).$$

By Lemma 4.5, $\text{ord}_{u=0} (\varphi^{p,q}) = 0$ except for the case $(p, q) = ((n-1)/2, (n-1)/2)$, where in fact $\text{ord}_{u=0} (\varphi^{p,q}) = 1$. From (4.7) we then conclude that

$$(4.8) \quad \text{ord}_{u=0} \text{KS}^{((n-3)/2)} \left(u \frac{d}{du} \right) = 1 + 2 \text{ord}_{t=0} \text{KS}^{((n-3)/2)} \left(t \frac{d}{dt} \right)$$

$$(4.9) \quad 1 + \text{ord}_{u=0} \text{KS}^{((n-1)/2)} \left(u \frac{d}{du} \right) = 2 \text{ord}_{t=0} \text{KS}^{((n-1)/2)} \left(t \frac{d}{dt} \right).$$

In both cases (4.8)–(4.9) the order of vanishing of Kodaira–Spencer along the vector field $u \frac{d}{du}$ is strictly positive, since the restriction to 0 is the nilpotent operator $N = 0$. It follows that

$$\text{ord}_{t=0} \text{KS}^{((n-3)/2)} \left(t \frac{d}{dt} \right) \geq 0, \quad \text{i.e.} \quad M^{((n-3)/2)} \geq -1,$$

and

$$\text{ord}_{t=0} \text{KS}^{((n-1)/2)} \left(t \frac{d}{dt} \right) \geq 1, \quad \text{i.e.} \quad M^{((n-1)/2)} \geq 0.$$

Since all other $M^{(j)} \geq 0$ as in the odd case, we conclude from (4.6) that all these inequalities are in fact equalities. \square

4.3. Triviality of the Hodge bundles outside the middle degree. Recall the normal crossings model $f': \mathcal{Z}' \rightarrow \mathbb{P}^1$, obtained by blowing-up the ordinary double points in \mathcal{Z} . Notice that f' is actually defined over \mathbb{Q} , and hence so are the corresponding logarithmic Hodge bundles. By Lemma 3.6 we have $R^d f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^\bullet(\log) = 0$ for $d \neq n-1$ odd, while $R^{2p} f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^\bullet(\log) = R^p f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^p(\log)$ if $2p \neq n-1$.

Theorem 4.7. *For $2p \neq n-1$, the Hodge bundle $R^p f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^p(\log)$ is trivial in the category of sheaves on \mathbb{Q} -schemes.*

Proof. It is enough to prove the claim over the complex numbers. Indeed, let E be a vector bundle over $\mathbb{P}_{\mathbb{Q}}^1$, which is trivial after base change to \mathbb{C} . Then the natural morphism $H^0(\mathbb{P}_{\mathbb{Q}}^1, E) \otimes \mathcal{O}_{\mathbb{P}_{\mathbb{Q}}^1} \rightarrow E$ is necessarily an isomorphism, since it is an isomorphism after a flat base change.

For the triviality of $R^p f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^p(\log) = R^{2p} f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^\bullet(\log)$ over \mathbb{C} , we shall show that the local system $R^{2p} f'_* \mathbb{C}$ over $U(\mathbb{C}) = \mathbb{C} \setminus \mu_{n+1}$ is trivial. Then the Deligne extension, realized by $R^{2p} f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^\bullet(\log)$, will be trivial as well. Take a base point $b \in U(\mathbb{C})$, and let $\rho: \pi_1(U(\mathbb{C}), b) \rightarrow \text{GL}(H^{2p}(Z_b, \mathbb{C}))$ be the monodromy representation determining the local system. The fundamental group $\pi_1(U(\mathbb{C}), b)$ is generated by loops γ_ξ circling around $\xi \in \mu_{n+1}$, and a loop γ_∞ circling around ∞ , with a relation $\prod_\xi \gamma_\xi = \gamma_\infty$. Because the singularities of $\mathcal{Z} \rightarrow \mathbb{P}^1$ at the points ξ are ordinary double points, and $2p \neq n-1$, the local monodromies $\rho(\gamma_\xi)$ are trivial. Therefore $\rho(\gamma_\infty)$ is trivial as well, and so is ρ . \square

5. THE BCOV INVARIANT OF THE MIRROR FAMILY

5.1. The Kronecker limit formula for the mirror family. For the mirror family $f: \mathcal{Z} \rightarrow U$, we proceed to prove an expression for the BCOV invariant $\tau_{BCOV}(Z_\psi)$ in terms of the L^2 norms of the sections η_k (cf. Definition 3.9). The strategy follows the same lines as for families of Calabi–Yau hypersurfaces §2.3.

We fix a polarization and a projective factorization of f , defined over \mathbb{Q} . We denote by L the corresponding *algebraic* Lefschetz operator, that is the cup-product against the algebraic cycle class of a hyperplane section. We will abusively confound L with the algebraic cycle class of a hyperplane section. With this choice of L , the primitive decomposition of the Hodge bundles $R^p f_* \Omega_{\mathcal{Z}/U}^q$ holds over \mathbb{Q} . Let h be a Kähler metric and ω the Kähler form normalized as in (1.5), and assume that the fiberwise cohomology class is in the topological hyperplane class. Hence, under the correspondence between algebraic and topological cycle classes, L is sent to $(2\pi i)[\omega] \in R^2 f_* \mathbb{Q}(1)$.

Below, all the L^2 norms are computed with respect to ω as in (1.7).

Theorem 5.1. *There exists a real positive constant $C \in \pi^c \overline{\mathbb{Q}}^\times$ such that*

$$\tau_{BCOV}(Z_\psi) = C \left| \frac{(\psi^{n+1})^a}{(1 - \psi^{n+1})^b} \right|^2 \frac{\|\eta_0\|_{L^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\eta_k\|_{L^2}^{2(n-1-k)} \right)^{(-1)^{n-1}}}$$

where $\chi = \chi(Z_\psi)$ and

$$\begin{aligned} a &= (-1)^{n-1} \frac{n(n-1)}{6} - \frac{\chi}{12(n+1)}, \\ b &= (-1)^{n-1} \frac{n(3n-5)}{24} \\ c &= \frac{1}{2} \sum_k (-1)^{k+1} k^2 b_k. \end{aligned}$$

Proof. We apply the version of the arithmetic Riemann–Roch theorem formulated in Theorem 2.3, to the family $f: \mathcal{Z} \rightarrow U$ as being defined over \mathbb{Q} . We need to specify the section η and the sections $\eta_{p,q}$ in equation (2.6). The section η is chosen to be η_0 as defined in Definition 3.9. We next describe our choices of $\eta_{p,q}$:

- If $p+q \neq n-1$ and $p \neq q$, then the corresponding Hodge bundle vanishes by Lemma 3.6, and thus gives no contribution.
- For $2p \neq n-1$, Theorem 4.7 guarantees that $\det R^p f'_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^p(\log)$ is trivial, in the category of \mathbb{Q} -schemes. We choose $\eta_{p,p}$ to be any trivialization defined over \mathbb{Q} , and then restrict it to U . Notice that the L^2 norm $\|\eta_{p,p}\|_{L^2}$ is constant.
- For $p+q = n-1$ and $p \neq q$, the (p, q) Hodge bundle is primitive and has rank one. Then we take $\eta_{p,q} = \eta_q$ in Definition 3.9. By Lemma 3.11, η_q is defined over \mathbb{Q} .
- For $p+q = n-1$ and $p = q$, which can only occur when $n-1$ is even, the (p, q) Hodge bundle is no longer primitive. We first employ the algebraic primitive decomposition:

$$(5.1) \quad \begin{aligned} \det R^p f_* \Omega_{\mathcal{Z}/U}^p &= \det(R^p f_* \Omega_{\mathcal{Z}/U}^p)_{\text{prim}} \otimes \det LR^{p-1} f_* \Omega_{\mathcal{Z}/U}^{p-1} \\ &\simeq \det(R^p f_* \Omega_{\mathcal{Z}/U}^p)_{\text{prim}} \otimes \det R^{p-1} f_* \Omega_{\mathcal{Z}/U}^{p-1}. \end{aligned}$$

We define $\eta_{\frac{n-1}{2}, \frac{n-1}{2}}$ as the element corresponding to $\eta_{\frac{n-1}{2}} \otimes \eta_{\frac{n-3}{2}, \frac{n-3}{2}}$ under this isomorphism. Again, this element is defined over \mathbb{Q} .

To establish the theorem we need to specify the element $\Delta \in \mathbb{Q}(\psi)^\times \otimes \mathbb{Q}$ in (2.6) (formal rational power of a rational function), which satisfies:

$$(5.2) \quad \log \tau_{BCOV} = \log |\Delta|^2 + \frac{\chi}{12} \log \|\eta\|_{L^2}^2 - \sum_{p,q} (-1)^{p+q} p \log \|\eta_{p,q}\|_{L^2}^2 + \log C_\sigma.$$

We will determine Δ up to an algebraic number. To this end, it suffices to know its divisor. Unless $\psi = 0$ or $\psi = \xi$ where $\xi^{n+1} = 1$, Δ has no zeroes nor poles by construction, since the sections $\eta_{p,q}$ are holomorphic and non-vanishing, and $\log \tau_{BCOV}$ is smooth. Hence we are lead to consider the logarithmic behaviour of the right hand side of (5.2) at these points. Since for $2p \neq n-1$ the sections $\eta_{p,p}$ have constant L^2 norm, we only need to examine the functions $\log \|\eta_{p,q}\|_{L^2}$ with $p+q = n-1$.

Behaviour at $\psi = 0$. This corresponds to a smooth fiber of $f: \mathcal{Z} \rightarrow U$. Hence $\log \tau_{BCOV}$ is smooth at $\psi = 0$, as are the L^2 metrics. However, the sections $\eta_{p,q}$ with $p+q = n-1$ admit zeros at $\psi = 0$ (see Remark 3.10), with $\text{ord}_0 \eta_{p,q} = q+1 = n-p$. This means that a in the theorem is given by

$$(n+1)a = (-1)^{n-1} \sum_{p=0}^{n-1} p(n-p) - \frac{\chi}{12} = (-1)^{n-1} \frac{(n-1)n(n+1)}{6} - \frac{\chi}{12}.$$

Behaviour at $\psi = \xi \in \mu_{n+1}$. This corresponds to a singular fiber of $f: \mathcal{Z} \rightarrow \mathbb{P}^1$, which has a unique ordinary double point. By Theorem 4.6 we control $\text{ord}_\xi \eta_k$ according to the parity of $n-1$. Here we encounter the additional problem that the L^2 norms might have contributions from the semi-simple part of the monodromy. More precisely, consider the local parameter $t = \psi - \xi$ around ξ , and write $\eta_{p,q} = t^{b_{p,q}} \sigma_{p,q}$ where $\sigma_{p,q}$ trivializes $\det R^q f_* \Omega_{\mathcal{Z}'/\mathbb{P}^1}^p(\log)$. Then by [EFiMM18a, Thm. C], we have

$$\log \|\eta_{p,q}\|_{L^2}^2 = (b_{p,q} + \alpha_{p,q}) \log |t|^2 + o(\log |t|^2)$$

with

$$\alpha_{p,q} = -\frac{1}{2\pi i} \text{tr} \left(\ell \log T_s \mid \text{Gr}_{F_\infty}^p H_{\text{lim}}^{n-1} \right) \in \mathbb{Q}.$$

In the case at hand, this exponent can be determined from Lemma 4.5. Let us combine all this information:

Odd case: If $n-1$ is odd, according to Theorem 4.6, if $k \leq \frac{n}{2} - 1$, $\text{ord}_\xi \eta_k = 0$ and $\text{ord}_\xi \eta_k = -1$ otherwise. In this case the monodromy is unipotent, so that $\alpha_{p,q} = 0$ for all $p+q = n-1$. Moreover, by [EFiMM18a, Thm. B], we have that $\log \tau_{BCOV} = \frac{n}{24} \log |t|^2 + o(\log |t|^2)$. Putting all these contributions together we find that

$$b = \frac{n}{24} + (-1)^{n-1} \sum_{k=n/2}^{n-1} (n-1-k) \cdot (-1) = \frac{n(3n-5)}{24}.$$

Even case: If $n-1$ is even, according to Theorem 4.6, if $k \leq \frac{n-3}{2}$, $\text{ord}_\xi \eta_k = 0$ and $\text{ord}_\xi \eta_k = -1$ otherwise. Also, unless $p = q = (n-1)/2$, $\alpha_{p,q} = 0$. In the remaining case $p = q = (n-1)/2$, Lemma 4.5 implies that $\alpha_{p,p} = 1/2$. Finally, from [EFiMM18a, Thm. B], we have that $\log \tau_{BCOV} = \frac{3-n}{24} \log |t|^2 + o(\log |t|^2)$. Putting all these contributions together we find that

$$b = \frac{3-n}{24} + (-1)^{n-1} \left((n-1)/2(-1+1/2) + \sum_{k=(n+1)/2}^{n-1} (n-1-k) \cdot (-1) \right) = -\frac{n(3n-5)}{24}.$$

To complete the proof of the theorem, we still need to tackle the constant C . There are two sources that contribute: i) for $2p \neq n-1$, the L^2 norms $\|\eta_{p,p}\|_{L^2}$ are constant and ii) if $n-1 = 2p$, then $\eta_{p,p}$ was taken to correspond to $\eta_p \otimes \eta_{p-1,p-1}$ through (5.1), so that there might be extra contributions from L and from $\|\eta_{p-1,p-1}\|_{L^2}$.

First for $2p \neq n-1$. Let $\psi \in \mathbb{Q}$, so that we have the period isomorphism

$$H^{2p}(Z_\psi, \Omega_{Z_\psi/\mathbb{Q}}^\bullet) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^{2p}(Z_\psi, \mathbb{Q}) \otimes \mathbb{C}.$$

Taking rational bases on both sides, the determinant can be defined in $\mathbb{C}^\times / \mathbb{Q}^\times$. It equals $(2\pi)^{pb^{2p}}$. Since $\|\eta_{p,p}\|_{L^2}$ is constant, it can be evaluated at any $\psi \in \mathbb{Q}$. We find

$$\|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} (2\pi)^{2pb^{2p}} \text{vol}_{L^2}(H^{2p}(Z_\psi, \mathbb{Z}), \omega),$$

where $\sim_{\mathbb{Q}^\times}$ means equality up to a non-zero rational number. Now recall from (2.9) that with the Arakelov theoretic normalization of the Kähler form, and under the integrality assumption on its cohomology class, we have $\text{vol}_{L^2}(H^{2p}(Z_\psi, \mathbb{Z}), \omega) \sim_{\mathbb{Q}^\times} (2\pi)^{-2pb^{2p}}$. All in all, we arrive at the pleasant

$$(5.3) \quad \|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} 1.$$

If $2p = n-1$, $\eta_{p,p}$ corresponds to $\eta_p \otimes \eta_{p-1,p-1}$ through (5.1). We bring together several facts. The first one is that the Lefschetz decomposition is orthogonal for the L^2 metrics, regardless

of the normalization of the Kähler forms. The second one is that the algebraic cycle class of L corresponds to $(2\pi i)[\omega]$ in analytic de Rham cohomology. The last fact is that the operator $[2\pi\omega] \wedge \cdot$ is an isometry up to a rational constant, since $2\pi\omega$ is the Hodge theoretic Kähler form (see for instance [Huy05, Prop. 1.2.31]). All these remarks together lead to

$$\|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} \|\eta_p\|_{L^2}^2 \|\eta_{p-1,p-1}\|_{L^2}^2.$$

Applying (5.3) to $\|\eta_{p-1,p-1}\|_{L^2}$, we find again

$$(5.4) \quad \|\eta_{p,p}\|_{L^2}^2 \sim_{\mathbb{Q}^\times} 1.$$

Now plug (5.3)–(5.4) into (5.2), introduce as well the value of C_σ (cf. Theorem 2.3) and recall that Δ was determined only up to algebraic number. We conclude that C has the asserted shape. \square

Corollary 5.2. *As $\psi \rightarrow \infty$, $\log \tau_{BCOV}(Z_\psi)$ behaves as*

$$(5.5) \quad \log \tau_{BCOV}(Z_\psi) = \kappa_\infty \log |\psi|^{-2} + \varrho_\infty \log \log |\psi|^{-2} + \text{continuous},$$

where

$$\begin{aligned} \kappa_\infty &= (-1)^n \frac{n+1}{12} \left(\frac{(n-1)(n+2)}{2} + \frac{1 - (-n)^{n+1}}{(n+1)^2} \right), \\ \varrho_\infty &= (-1)^{n-1} \frac{(n-1)(n+1)}{12} \left(\frac{(-n)^{n+1} - 1}{(n+1)^2} - 2n + 1 \right). \end{aligned}$$

Proof. The general shape (5.5) was proven in [EFiMM18a, Prop. 6.8]. The precise value of κ_∞ is $(n+1)(b-a)$ entirely due to the term $\left| \frac{(\psi^{n+1})^a}{(1-\psi^{n+1})^b} \right|$ in Theorem 5.1. Indeed, by Theorem 4.3 the sections η_k trivialize $R^k f_* \Omega_{\mathcal{Z}/\mathbb{P}^1}^{n-1-k}(\log)$ at infinity, and moreover the monodromy is unipotent there (Lemma 4.1). This entails that the functions $\log \|\eta_k\|_{L^2}^2$ are $O(\log \log |\psi|^{-2})$ at infinity, and hence do not contribute to κ_∞ . For the subdominant term, the expression of [EFiMM18a, Prop. 6.8.] can be explicitly evaluated for the mirror family, thanks to the complete understanding of the limiting Hodge structure at infinity (again Lemma 4.1), and the known value of χ (Lemma 3.6). \square

5.2. Canonical trivializations of the Hodge bundles at the MUM point.

The Picard–Fuchs equation of the mirror. For the mirror family $f: \mathcal{Z} \rightarrow U$, we review classical facts on the Picard–Fuchs equation of the local system of middle degree cohomologies. The discussion serves as the basis for the construction of canonical trivializing sections of the middle degree Hodge bundles, close to the MUM point, which differ from the η_k by some periods.

The starting point is the construction of an invariant $(n-1)$ -homological cycle at infinity for the mirror family $f: \mathcal{Z} \rightarrow \mathbb{P}^1$. Recall the Dwork pencil $h: \mathcal{X} \rightarrow \mathbb{P}^1$, which comes with a natural embedding in $\mathbb{P}^n \times \mathbb{P}^1$. We obtain a "physical" n -cycle Γ in \mathbb{P}^n as follows: we place ourselves in the affine piece $x_0 \neq 0$ and define Γ by the condition $|x_i/x_0| = 1$ for all i . If $\psi \in \mathbb{C}$ and $|\psi|^{-1}$ is small, then the fiber X_ψ does not encounter Γ . Therefore, Γ induces a constant family of cycles in $H_n(\mathbb{P}^n \setminus X_\psi, \mathbb{Z})$. Notice that these are clearly G -invariant cycles. Under the tube isomorphism $H_n(\mathbb{P}^n \setminus X_\psi, \mathbb{Z}) \simeq H_{n-1}(X_\psi, \mathbb{Z})$, which is G -equivariant, we find a T -invariant cycle $\tilde{\gamma}_0 \in H_{n-1}(X_\psi, \mathbb{Z})^G$. Finally, through $H_{n-1}(X_\psi, \mathbb{Q})^G \hookrightarrow H_{n-1}(Z_\psi, \mathbb{Q})$ (cf. §3.2), $|G| \cdot \tilde{\gamma}_0$ maps to a T -invariant cycle on Z_ψ , denoted γ_0 . The convenience of multiplication by $|G|$ will be clear in a moment.

The period integral $I_0(\psi) := \int_{\gamma_0} \eta_0$ can be written as a convergent power series in ψ^{-1} . Indeed, taking into account the relationship between the cup-product on X_ψ and Z_ψ (see e.g. Lemma 3.4), and the definition of η_0 (cf. Definition 3.9) we find

$$I_0(\psi) = \int_{\gamma_0} \eta_0 = -\frac{(n+1)\psi}{|G|} \int_{|G|\cdot\tilde{\gamma}_0} \theta_0 = -(n+1)\psi \int_{\tilde{\gamma}_0} \theta_0.$$

For the computation of the latter integral, we use that the residue map and the tube map are mutual adjoint, and then perform an explicit computation:

$$(5.6) \quad I_0(\psi) = \frac{1}{(2\pi i)^n} \int_{\Gamma} \frac{-(n+1)\psi dz_1 \wedge \dots \wedge dz_n}{F_\psi(1, z_1, \dots, z_n)} = \sum_{k \geq 0} \frac{1}{((n+1)\psi)^{(n+1)k}} \frac{((n+1)k)!}{(k!)^{n+1}},$$

where the $z_i = x_i/x_0$ are affine coordinates. This is the period integral used in Theorem 4.3, to prove that η_0 trivializes $f_* K_{\mathcal{Z}/\mathbb{D}_\infty}(\log)$.

To the local system $(R^{n-1} f_* \mathbb{C})_{\text{prim}}$ there is an associated Picard–Fuchs equation. We make the change of variable $z = \psi^{-(n+1)}$, so that I_0 becomes

$$I_0(z) = \sum_{k \geq 0} \frac{z^k}{(n+1)^{(n+1)k}} \frac{((n+1)k)!}{(k!)^{n+1}}.$$

Define the differential operators $\delta = z \frac{d}{dz}$ and

$$(5.7) \quad D = \delta^n - z \prod_{j=1}^n \left(\delta + \frac{j}{n+1} \right).$$

Differentiating $I_0(z)$ term by term and repeatedly, one checks $DI_0(z) = 0$. Now, on the one hand $D = 0$ is a degree n irreducible differential equation of hypergeometric type [Kat90, Cor. 3.2.1]. On the other hand, $(R^{n-1} f_* \mathbb{C})_{\text{prim}}$ is a local system of rank n . It follows that $D = 0$ is necessarily the Picard–Fuchs equation satisfied by the periods of η_0 .

We now exhibit all the solutions of the Picard–Fuchs equation. For dimension reasons, these will determine a multivalued basis of homology cycles. Following Zinger (see e.g. [Zin08, pp. 1214–1215]), for $q = 0, \dots, n-1$ we define an *a priori* formal series $I_{0,q}$ by

$$\sum_{q=0}^{\infty} I_{0,q}(t) w^q = e^{wt} \sum_{d=0}^{\infty} e^{dt} \frac{\prod_{r=1}^{(n+1)d} ((n+1)w+r)}{\prod_{r=1}^d (w+r)^{n+1}} =: R(w, t).$$

Let us also define $F(w, t)$ for the infinite sum on the right hand side, so that $R(w, t) = e^{wt} F(w, t)$. Under the change of variable

$$(5.8) \quad e^t = (n+1)^{-(n+1)} z = ((n+1)\psi)^{-(n+1)},$$

the series $I_{0,0}(t)$ becomes $I_0(z) = I_0(\psi)$ [Zin08, eq. (2–17)].

Proposition 5.3. *Under the change of variable (5.8), the functions $I_{0,q}(z)$, $q = 0, \dots, n-1$, define a basis of multivalued holomorphic solutions of the Picard–Fuchs equation for the local system $(R^{n-1} f_* \mathbb{C})_{\text{prim}}$ on $0 < |z| < 1$.*

Proof. After the change of variable, one checks that $F(w, z)$ is absolutely convergent on compact subsets in the region $|w| < 1$ and $|z| < 1$. This implies that the functions $I_{0,q}(z)$ are multivalued holomorphic functions on $0 < |z| < 1$. Again taking into account the change of variable, it is

formal to verify that $R(w, t)$ solves the Picard-Fuchs equation (5.7), and hence so do the functions $I_{0,q}(z)$. To see that they form a basis of solutions, it is enough notice that each $I_{0,q}(z)$ has a singularity of the form $(\log z)^q$ as $z \rightarrow 0$. \square

By the proposition, and because $(R^{n-1}f_*\mathbb{C})_{\text{prim}}$ has rank n , the functions $I_{0,q}(z)$ determine a flat multivalued basis of sections γ_q of $(R^{n-1}f_*\mathbb{C})_{\text{prim}}^\vee$ on $0 < |z| < 1$, by the recipe

$$I_{0,q}(z) = \int_{\gamma_q(z)} \eta_0.$$

See for instance [Voi99, Sec. 3.4 & Lemme 3.12] for a justification. The notation is compatible with the invariant cycle γ_0 constructed above, as we already observed that $I_{0,0}(z) = I_0(z)$. The flat multivalued basis elements $\gamma_q(z)$ provide a basis of $(H_{\text{lim}}^{n-1})^\vee$, the limiting Hodge structure on the homology, at infinity. We still write $\gamma_0, \dots, \gamma_{n-1}$ for this limit basis. We next prove it is adapted to the weight filtration.

Proposition 5.4. *Let W'_\bullet be the weight filtration of the limiting mixed Hodge structure on $(H_{\text{lim}}^{n-1})^\vee$. Then $\gamma_q \in W'_{2q} \setminus W'_{2q-1}$.*

Proof. It is enough to establish the analogous property for the Poincaré duals $\gamma'_q \in H_{\text{lim}}^{n-1}$, similarly defined as the limits of the Poincaré duals $\gamma'_q(z)$ of the $\gamma_q(z)$. On each fiber Z_z , the Hodge decomposition and the Cauchy–Schwarz inequality imply

$$|I_{0,q}(z)| = \left| \int_{Z_z} \gamma'_q(z) \wedge \eta_0 \right| \leq (2\pi)^{n-1} \|\gamma'_q(z)\|_{L^2} \|\eta_0\|_{L^2}.$$

Now $|I_{0,q}(z)|$ grows like $(\log|z|^{-1})^q$ as $z \rightarrow 0$ along angular sectors (cf. proof of Proposition 5.3). Because the monodromy is maximal unipotent at infinity and η_0 is a basis of $f_*K_{\mathcal{X}/\mathbb{D}_\infty}(\log)$, the L^2 norm $\|\eta_0\|_{L^2}$ grows like $(\log|z|^{-1})^{(n-1)/2}$ (see [EFiMM18b, Thm. A] or the more general [EFiMM18a, Thm. 4.4]). We infer that as $z \rightarrow 0$, along angular sectors,

$$\|\gamma'_q(z)\|_{L^2} \gtrsim (\log|z|^{-1})^{\frac{2q-(n-1)}{2}}.$$

By Schmid's metric characterization of the limiting Hodge structure [Sch73, Thm. 6.6], we then see that $\gamma'_q \notin W_{2q-1}$.

It remains to show that $\gamma'_q \in W_{2q}$. First of all, starting with $q = n-1$, we already know $\gamma'_{n-1} \in W_{2n-2} \setminus W_{2n-3}$. We claim that $\gamma'_{n-2} \in W_{2n-4}$. Otherwise $\gamma'_{n-2} \in W_{2n-2} \setminus W_{2n-4}$. But the weight filtration has one-dimensional graded pieces in even degrees, and zero otherwise (cf. Lemma 4.1). It follows that $W_{2n-4} = W_{2n-3}$ and $\gamma'_{n-1} = \lambda\gamma'_{n-2} + \beta$, for some constant λ and some $\beta \in W_{2n-4}$. Integrating against η_0 , this relation entails

$$I_{0,n-1}(z) = \lambda I_{0,n-2}(z) + \int_{Z_z} \beta(z) \wedge \eta_0,$$

where $\beta(z)$ is the flat multivalued section corresponding to β . Let us examine the asymptotic behaviour of the right hand side of this equality, as $z \rightarrow 0$, along angular sectors. We know that $|I_{0,n-2}(z)|$ grows like $(\log|z|^{-1})^{n-2}$. By the Hodge decomposition and the Cauchy–Schwarz inequality, and Schmid's theorem, the integral grows at most like $(\log|z|^{-1})^{n-2}$. This contradicts that $|I_{0,n-1}(z)|$ grows like $(\log|z|^{-1})^{n-1}$. Hence $\gamma'_{n-2} \in W_{2n-4}$. Continuing inductively in this fashion, we conclude that $\gamma'_q \in W_{2q}$ for all q , as desired. \square

A normalized basis of ${}^\ell(R^{n-1}f_*\mathbb{C})_{\text{prim}}$. We construct a new basis of holomorphic sections of ${}^\ell(R^{n-1}f_*\mathbb{C})_{\text{prim}}$ close to infinity, and corresponding period integrals $I_{p,q}(z)$. We proceed inductively:

- (1) set $\tilde{\vartheta}_0 = \eta_0$;
- (2) for $p \geq 1$, suppose that $\tilde{\vartheta}_0, \dots, \tilde{\vartheta}_{p-1}$ have been constructed. Define

$$I_{p-1,q}(z) = \int_{\gamma_q(z)} \tilde{\vartheta}_{p-1}.$$

This notation is consistent with the previous definition of $I_{0,q}$;

- (3) assuming for the time being that $I_{p-1,p-1}(z)$ is holomorphic and non-vanishing at $z = 0$ (see the proof of Proposition 5.5 below), we define $\tilde{\vartheta}_p$ by

$$(5.9) \quad \tilde{\vartheta}_p = \nabla_{z d/dz} \left(\frac{\tilde{\vartheta}_{p-1}}{I_{p-1,p-1}(z)} \right);$$

One verifies integrating (5.9) over $\gamma_q(z)$ that the period integrals $I_{p,q}(z) := \int_{\gamma_q(z)} \tilde{\vartheta}_p$ satisfy the following recursion:

$$(5.10) \quad I_{p,q}(z) = z \frac{d}{dz} \left(\frac{I_{p-1,q}(z)}{I_{p-1,p-1}(z)} \right).$$

Taking into account the change of variable (5.8), we see that this is the same recurrence relation as in [Zin08, eq. (2–18)] (see also [Zin09, eq. (0.16)]). Hence the $I_{p,q}(z)$ above coincides with the $I_{p,q}(t)$ in *loc. cit.* We further normalize:

$$\vartheta_p = \frac{\tilde{\vartheta}_p}{I_{p,p}(z)}.$$

Proposition 5.5. (1) For all k , the sections $\{\vartheta_j\}_{j=0,\dots,k}$, constitute a holomorphic basis of the filtered piece $\mathcal{F}^{n-1-k} R^{n-1}f_*\Omega_{\mathcal{Z}/\mathbb{D}_\infty}^\bullet(\log)_{\text{prim}}$.

- (2) The periods of ϑ_k satisfy

$$\int_{\gamma_k} \vartheta_k = 1 \quad \text{and} \quad \int_{\gamma_q} \vartheta_k = 0 \quad \text{if } q < k.$$

- (3) The projection of ϑ_k to $R^k f_*\Omega_{\mathcal{Z}/\mathbb{D}_\infty}^{n-1-k}(\log)_{\text{prim}}$ relates to η_k by

$$(\vartheta_k)^{n-1-k,k} = \frac{\eta_k}{\prod_{p=0}^k I_{p,p}(z)}.$$

- (4) The sections $\{\vartheta_j\}_{j=0,\dots,n-1}$ are uniquely determined by properties (1)–(2) above.

Proof. We notice that the period integrals $I_{p,p}(z)$ are holomorphic in z and non-vanishing at $z = 0$. This is [Zin09, Prop. 3.1], in turn based on [ZZ08]. With this observation at hand, the claims (1)–(3) then follow from properties of the Gauss–Manin connection and Kodaira–Spencer maps, Lemma 3.11 and Theorem 4.3. The details are left to the reader. The uniqueness property is obtained by comparing two such bases adapted to the Hodge filtration as in (1), and then imposing the period relations (2). \square

Actually, the basis $\vartheta_\bullet = \{\vartheta_j\}_{j=0,\dots,n-1}$ is determined by the limiting Hodge structure H_{lim}^{n-1} , up to constant, as we now show:

Proposition 5.6. (1) Let γ'_\bullet be an adapted basis of the weight filtration on $(H_{\lim}^{n-1})^\vee$, as in Proposition 5.4. Then there exists a unique holomorphic basis ϑ'_\bullet of $R^{n-1}f_*\Omega_{\mathcal{Z}/\mathbb{D}_\infty}^\bullet(\log)_{\text{prim}}$ satisfying the conditions analogous to (1)–(2) with respect to γ'_\bullet .
(2) There exist non-zero constants $c_k \in \mathbb{C}$ such that $\vartheta'_k = c_k \vartheta_k$.

Proof. We prove both assertions simultaneously. We write γ_\bullet and γ'_\bullet as column vectors. Since the graded pieces of the weight filtration on $(H_{\lim}^{n-1})^\vee$ are all one-dimensional, there exists a lower triangular matrix $A \in \text{GL}_n(\mathbb{C})$ with $\gamma'_\bullet = A\gamma_\bullet$. If we decompose $A = D + L$, where D is diagonal and L is lower triangular, we see that the entries of the column vector $\vartheta'_\bullet := D^{-1}\vartheta_\bullet$ fulfill the requirements. \square

Definition 5.7. We define the canonical trivializing section of $R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_\infty}^{n-1-k}(\log)_{\text{prim}}$ to be

$$\tilde{\eta}_k = (\vartheta_k)^{n-1-k,k} = \frac{\eta_k}{\prod_{p=0}^k I_{p,p}(z)}.$$

By the previous proposition, the sections $\tilde{\eta}_k$ depend only on H_{\lim}^{n-1} , up to constants. For a general discussion about distinguished sections, we refer the reader to [Mor97, Section 6.3].

5.3. Generating series of Gromov–Witten invariants and Zinger’s theorem. In order to state Zinger’s theorem on generating series of Gromov–Witten invariants of genus one, and for coherence with the notations of this author, it is now convenient to work in the t variable instead of z . The mirror map is the change of variable

$$t \mapsto T = \frac{I_{0,1}(t)}{I_{0,0}(t)} = \frac{\int_{\gamma_1(t)} \eta_0}{\int_{\gamma_0(t)} \eta_0}.$$

In particular, the Jacobian of the mirror map is computed from (5.10)

$$\frac{dT}{dt} = I_{1,1}(t).$$

Let us introduce some last notations:

- X_{n+1} denotes a general degree $n+1$ hypersurface in \mathbb{P}^n .
- $N_1(0) = -\left(\frac{(n-1)(n+2)}{48} + \frac{1-(-n)^{n+1}}{24(n+1)^2}\right) = \frac{1}{24}\left(-\frac{n(n+1)}{2} + \frac{\chi(X_{n+1})}{n+1}\right)$.
- $N_1(d)$ is the genus 1 and degree d Gromov-Witten invariant of X_{n+1} ($d \geq 1$).

From these invariants we build a generating series:

$$F_1^A(T) = N_1(0)T + \sum_{d=1}^{\infty} N_1(d)e^{dT}.$$

It follows from [Zin08, Thm. 2] that this generating series satisfies

$$F_1^A(T) = N_1(0)t + \frac{(n+1)^2 - 1 + (-n)^{n+1}}{24(n+1)} \log I_{0,0}(t) - \begin{cases} \frac{n}{48} \log(1 - (n+1)^{n+1} e^t) + \sum_{p=0}^{(n-2)/2} \frac{(n-2p^2)}{8} \log I_{p,p}(t), & \text{if } n \text{ even} \\ \frac{n-3}{48} \log(1 - (n+1)^{n+1} e^t) + \sum_{p=0}^{(n-3)/2} \frac{(n+1-2p)(n-1-2p)}{8} \log I_{p,p}(t) & \text{if } n \text{ odd.} \end{cases}$$

As an application of relations between the hypergeometric series $I_{p,p}(t)$, studied in detail in [ZZ08], the following identity holds (for a version of this particular identity, see [Zin09, eq.

(3.2)]):

$$\begin{aligned} \frac{n(3n-5)}{48} \log(1 - (n+1)^{n+1} e^t) + \frac{1}{2} \sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p} = \\ \begin{cases} \frac{n}{48} \log(1 - (n+1)^{n+1} e^t) + \sum_{p=0}^{(n-2)/2} \frac{(n-2p^2)}{8} \log I_{p,p}(t), & \text{if } n \text{ even} \\ \frac{n-3}{48} \log(1 - (n+1)^{n+1} e^t) + \sum_{p=0}^{(n-3)/2} \frac{(n+1-2p)(n-1-2p)}{8} \log I_{p,p}(t) & \text{if } n \text{ odd} \end{cases} \end{aligned}$$

Consequently, Zinger's theorem takes the following pleasant form, that we will use to simplify the task of recognizing $F_1^A(T)$ in our expression for the BCOV invariant (cf. Theorem 5.1).

Theorem 5.8 (Zinger). *Under the change of variables $t \mapsto T$, the series $F_1^A(T)$ takes the form*

$$\begin{aligned} F_1^A(T) = N_1(0) t + \frac{\chi(X_{n+1})}{24} \log I_{0,0}(t) \\ - \frac{n(3n-5)}{48} \log(1 - (n+1)^{n+1} e^t) - \frac{1}{2} \sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t). \end{aligned}$$

5.4. Genus one mirror symmetry and the BCOV invariant. We are now in position to show that the BCOV invariant of the mirror family $f: \mathcal{Z} \rightarrow U$ realizes genus one mirror symmetry for Calabi–Yau hypersurfaces in projective space. That is, one can extract the generating series $F_1^A(T)$ from the function $\psi \mapsto \tau_{BCOV}(Z_\psi)$. The precise recipe by which this is accomplished goes through expressing τ_{BCOV} in terms of the L^2 norms of the canonical sections $\tilde{\eta}_k$ (cf. Definition 5.7). But first we need to make $\tau_{BCOV}(Z_\psi)$ and $F_1^A(T)$ depend on the same variable. To this end, we let

$$(5.11) \quad F_1^B(\psi) = F_1^A(T), \quad \text{for } T = \frac{I_{0,1}(t)}{I_{0,0}(t)} \quad \text{and} \quad e^t = ((n+1)\psi)^{-(n+1)}.$$

Theorem 5.9. *In a neighborhood of $\psi = \infty$, there is an equality*

$$\tau_{BCOV}(Z_\psi) = C \left| \exp((-1)^{n-1} F_1^B(\psi)) \right|^4 \frac{\|\tilde{\eta}_0\|_{L^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\tilde{\eta}_k\|_{L^2}^{2(n-1-k)} \right)^{(-1)^{n-1}}},$$

where $\chi = \chi(Z_\psi)$ and $C \in \pi^c \mathbb{Q}_{>0}^\times$, $c = \frac{1}{2} \sum_k (-1)^{k+1} k^2 b^k$.

Proof. The proof is a simple computation, which consists in changing the variable T to ψ , using (5.11), in the expression for $F_1^A(T)$ provided by Theorem 5.8. Modulo log of rational numbers,

we find

$$\begin{aligned}
4F_1^A(T) &= \left(-\frac{n(n+1)}{12} + \frac{\chi(X_{n+1})}{6(n+1)} \right) t + \frac{\chi(X_{n+1})}{6} \log I_{0,0}(t) \\
&\quad - \frac{n(3n-5)}{12} \log(1 - (n+1)^{n+1} e^t) - 2 \sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t) \\
&= \left(\frac{n(n+1)}{12} - \frac{\chi(X_{n+1})}{6(n+1)} + \frac{n(3n-5)}{12} \right) \log(\psi^{n+1}) \\
&\quad - \frac{n(3n-5)}{12} \log(\psi^{n+1} - 1) + \frac{\chi(X_{n+1})}{6} \log I_{0,0}(t) - 2 \sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t) \\
&= (-1)^{n-1} \log \frac{(\psi^{n+1})^{2a}}{(\psi^{n+1} - 1)^{2b}} + \frac{\chi(X_{n+1})}{6} \log I_{0,0}(t) - 2 \sum_{p=0}^{n-2} \binom{n-p}{2} \log I_{p,p}(t).
\end{aligned}$$

Now, in terms of the normalized forms $\tilde{\eta}_k$ given in Definition 5.7, Theorem 5.1 becomes:

$$\tau_{BCOV}(Z_\psi) = C \left| \frac{(\psi^{n+1})^a}{(1 - \psi^{n+1})^b} \right|^2 \frac{|I_{0,0}(t)|^{\chi/6}}{\left(\prod_{p=0}^{n-2} |I_{p,p}(t)|^{2 \binom{n-p}{2}} \right)^{(-1)^{n-1}}} \frac{\|\tilde{\eta}_0\|_{L^2}^{\chi/6}}{\left(\prod_{k=0}^{n-1} \|\tilde{\eta}_k\|_{L^2}^{2(n-1-k)} \right)^{(-1)^{n-1}}}.$$

□

Remark 5.10. (1) In relative dimension 3, we recover the main theorem of Fang–Lu–Yoshikawa [FLY08, Thm 1.3]. Their result is presented in a slightly different form. The first formal discrepancy is in the choice of the trivializing sections. Their trivializations can be related to ours via Kodaira–Spencer maps. The second discrepancy is explained by a different normalization of F_1^A : they work with two times Zinger’s generating series. This justifies why their expression for the BCOV invariant contains $|\exp(-F_1^B(\psi))|^2$, while our formula in dimension 3 specializes to $|\exp(-F_1^B(\psi))|^4$.

(2) Let us consider the \mathbb{Q} -line bundle on \mathbb{D}_∞

$$\begin{aligned}
(f_* K_{\mathcal{Z}/\mathbb{D}_\infty}(\log))^{\otimes \chi/12} \otimes \lambda_{BCOV}(\mathcal{Z}/\mathbb{D}_\infty)^{-1} &= \\
&= (f_* K_{\mathcal{Z}/\mathbb{D}_\infty}(\log))^{\otimes \chi/12} \otimes \bigotimes_{k=0}^{n-1} (R^k f_* \Omega_{\mathcal{Z}/\mathbb{D}_\infty}^{n-1-k}(\log))_{\text{prim}}^{\otimes (-1)^n (n-1-k)},
\end{aligned}$$

with trivialization

$$\Theta = \tilde{\eta}_0^{\otimes \chi/12} \otimes \bigotimes_{k=0}^{n-1} \tilde{\eta}_k^{\otimes (-1)^n (n-1-k)}.$$

The L^2 norm of Θ asymptotically behaves like a power of $\log|\psi|$ close to infinity:

$$\|\Theta\|_{L^2}^2 \asymp (\log|\psi|)^{\rho_\infty},$$

where ρ_∞ was determined in Corollary 5.2. This L^2 norm has to be seen as an analogue of a power of $\text{Im } \tau$ on the upper half plane. The expression for the BCOV invariant

$$\tau_{BCOV}(Z_\psi) = C \left| \exp((-1)^{n-1} F_1^B(\psi)) \right|^4 \|\Theta\|_{L^2}^2$$

is then to be compared with the Kronecker limit formula (1.3) In this comparison, $\exp((-1)^{n-1} F_1^B(\psi))^{24}$ is the analogue of the discriminant modular form Δ and $\|\Theta\|_{L^2}^2$ is the analogue of $\text{Im } \tau = \frac{1}{2\pi} \log|q|^{-1}$. See also Conjecture 2 below.

- (3) The section Θ depends only on H_{lim}^{n-1} , up to constant, by Proposition 5.6.
- (4) The norms of the sections $\tilde{\eta}_k$ are independent of the choice of crepant resolution. It follows that the expression on the right hand side in Theorem 5.9 is independent of the crepant resolution, except possibly for the constant C . In [EFiMM18a, Conj. B] we conjectured that the BCOV invariant is a birational invariant. Thus C should in fact be independent of the choice of crepant resolution.

For $d \geq 1$ the invariants $N_1(d)$ are defined as enumerative invariants on the general degree $n+1$ hypersurface X_{n+1} in \mathbb{P}^n . The invariant $N_1(0)$ actually has an intersection theoretic interpretation on X_{n+1} as well. Accordingly, the leading term in the asymptotic behaviour of $\log \tau_{\text{BCOV}}(Z_\psi)$ as $\psi \rightarrow \infty$ (cf. Corollary 5.2) can be expressed in terms of X_{n+1} . This is in agreement with the genus one mirror symmetry expectations (cf. [EFiMM18a, Sec. 1.4] for a discussion).

Corollary 5.11. (1) *The invariant $N_1(0)$ satisfies*

$$N_1(0) = \frac{-1}{24} \int_{X_{n+1}} c_{n-2}(X_{n+1}) \wedge H,$$

where H is the hyperplane class in \mathbb{P}^n .

(2) *As $\psi \rightarrow \infty$, $\log \tau_{\text{BCOV}}(Z_\psi)$ behaves as*

$$(5.12) \quad \log \tau_{\text{BCOV}}(Z_\psi) = \left(\frac{(-1)^n}{12} \int_{X_{n+1}} c_{n-2}(X_{n+1}) \wedge H \right) \log |\psi^{-(n+1)}|^2 + O(\log \log |\psi|).$$

Proof. The sought for interpretation of $N_1(0)$, or equivalently for the coefficient κ_∞ in Corollary 5.2, is obtained by an explicit computation of, and comparison to $\int_{X_{n+1}} c_{n-1}(\Omega_{X_{n+1}}) \wedge H$. Indeed, by the cotangent exact sequence for the immersion of X_{n+1} into \mathbb{P}^n , this reduces to

$$\int_{X_{n+1}} c_{n-2}(\Omega_{X_{n+1}}) \wedge H = \frac{(-1)^{n-1}}{n+1} \chi(X_{n+1}) - \int_{\mathbb{P}^n} c_{n-1}(\Omega_{\mathbb{P}^n}) \wedge H,$$

and we have explicit formulas for both terms on the right. \square

Remark 5.12. The asymptotic expansion (5.12) has been written in the variable $\psi^{-(n+1)}$ on purpose, since this is the natural parameter in a neighborhood of the MUM point in the moduli space. In this form, the equation indeed agrees with the mirror symmetry predictions.

6. A CHOWLA–SELBERG FORMULA FOR THE BCOV INVARIANT

In this short section we would like to convince the reader that statements such as Theorem 5.1 and Theorem 5.9 can indeed be seen as analogues of the classical Kronecker limit formula a (cf. (1.3)). To strengthen this point of view, we now address the facet of complex multiplication.

For the sake of motivation, recall that the Kronecker limit formula computes the derivative at $s = 0$ of the non-holomorphic Eisenstein series $E(\tau, s)$ of $\text{SL}_2(\mathbb{Z})$ in terms of the Petersson norm of the discriminant modular form $\Delta(\tau)$. As we indicated in the introduction, this is tantamount to an expression for the analytic torsion of the elliptic curve $\mathbb{C}/(\mathbb{Z} + \tau\mathbb{Z})$, endowed with a suitable flat hermitian metric (see equation (1.3)). The Chowla–Selberg formula can be obtained by specializing the Kronecker limit formula to quadratic irrationalities in the upper half-plane, and can be recasted as an evaluation of the analytic torsion of a CM elliptic curve as a product of special values of the Γ function. Analogously, we expect that the BCOV invariant of a Calabi–Yau manifold with complex multiplication should be a combination of Γ values at rational points.

We won't provide a general conjectural picture here, but we will treat the relevant case of the CM mirror to Calabi–Yau hypersurfaces of prime degree, when the value of the BCOV invariant adopts a specially pleasant form.

Let $p \geq 5$ be a prime number, and define $n = p - 1$. We consider the mirror family $f: \mathcal{Z} \rightarrow U$ to Calabi–Yau hypersurfaces of degree p in \mathbb{P}^n . The special fiber Z_0 is a crepant resolution of X_0/G , where X_0 is now the Fermat hypersurface

$$x_0^p + \dots + x_n^p = 0.$$

The quotient X_0/G has an extra action of $\mu_p \subset \mathbb{C}$: a p -th root of unity $\xi \in \mathbb{C}$ sends a point $(x_0: \dots: x_n)$ to $(x_0: \dots: x_{n-1}: \xi x_n)$. This action induces a \mathbb{Q} -linear action of $K = \mathbb{Q}(\mu_p) \subset \mathbb{C}$ on $H^{n-1}(X_0, \mathbb{Q})^G$. As a rational Hodge structure, the latter is isomorphic to $H^{n-1}(Z_0, \mathbb{Q})$ (cf. §3.2 and Proposition 3.7; all the cohomology is primitive now). Hence $H^{n-1}(Z_0, \mathbb{Q})$ inherits a \mathbb{Q} -linear action of K . Observe that $[K: \mathbb{Q}] = p - 1$, which is exactly the dimension of $H^{n-1}(Z_0, \mathbb{Q})$. We say that Z_0 has complex multiplication by K . Similarly, the algebraic de Rham cohomology $H^{n-1}(Z_0, \Omega_{Z_0/\mathbb{Q}}^\bullet)$ affords a \mathbb{Q} -linear action of K . Indeed, this is clear for $H^{n-1}(X_0, \Omega_{X_0/\mathbb{Q}}^\bullet)^G$, since the action of μ_p on X_0 by automorphisms can actually be defined over \mathbb{Q} and commutes with the G action. Then, we transfer this to Z_0 via Proposition 3.7 and Lemma 3.12.

Let us fix a non-trivial $\xi \in \mu_p$. If we base change $H^{n-1}(Z_0, \mathbb{Q})$ to K , we have an eigenspace decomposition

$$H^{n-1}(Z_0, K) = \bigoplus_{k=0}^{p-1} H^{n-1}(Z_0, K)_{\xi^k}.$$

Hence, ξ acts by multiplication by ξ^k on $H^{n-1}(Z_0, K)_{\xi^k}$. Similarly, for algebraic de Rham cohomology:

$$H^{n-1}(Z_0, \Omega_{Z_0/K}^\bullet) = \bigoplus_{k=0}^{p-1} H^{n-1}(Z_0, \Omega_{Z_0/K}^\bullet)_{\xi^k}.$$

If we compare with $H^{n-1}(X_0, \Omega_{X_0/K}^\bullet)^G$, and we recall the construction of the sections θ_k and η_k° (cf. §3.3), we see by inspection that ξ acts on η_k° by multiplication by ξ^{k+1} . Therefore, we infer that the non-trivial eigenspaces only occur when $1 \leq k \leq p - 1$ and

$$H^{n-1}(Z_0, \Omega_{Z_0/K}^\bullet)_{\xi^k} = K\eta_{k-1}^\circ = H^{k-1}(Z_0, \Omega_{Z_0/K}^{n-k}).$$

Hence, the eigenspace $H^{n-1}(Z_0, \Omega_{Z_0/K}^\bullet)_{\xi^k}$ has Hodge type $(n - k, k - 1)$.

The period isomorphism relating algebraic de Rham and Betti cohomologies decomposes into eigenspaces as well. We obtain refined period isomorphisms

$$\text{per}_k: H^{n-1}(Z_0, \Omega_{Z_0/K}^\bullet)_{\xi^k} \otimes_K \mathbb{C} \xrightarrow{\sim} H^{n-1}(Z_0, K)_{\xi^k} \otimes_K \mathbb{C}.$$

Evaluating the isomorphism on K -bases of both sides, we obtain a period, still denoted $\text{per}_k \in \mathbb{C}^\times / K^\times$.

Lemma 6.1. *Fix an algebraic closure $\overline{\mathbb{Q}}$ of \mathbb{Q} in \mathbb{C} . Then there is an equality in $\mathbb{C}^\times / \overline{\mathbb{Q}}^\times$*

$$\text{per}_k = \frac{1}{\pi} \Gamma\left(\frac{k+1}{p}\right)^p.$$

Proof. The claim is equivalent to the analogous computation on X_0 . Hidden behind this phrase is the comparison of cup products on X_0 and Z_0 accounted for by Lemma 3.4. On X_0 , the formula for the period is well-known, and given for instance in Gross [Gro78, Sec. 4, p. 206]. Notice

that the author would rather work with the Fermat hypersurface $x_0^p + \dots + x_{n-1}^p = x_n^p$. However, as we compute periods up to algebraic numbers, by applying the obvious isomorphism of varieties defined over $\overline{\mathbb{Q}}$, the result is the same. Also, we have used standard properties of the Γ -function to transform *loc. cit.* in our stated form. \square

Theorem 6.2. *For Z_0 of dimension $p - 2$, with $p \geq 5$ prime, the BCOV invariant satisfies*

$$\tau_{BCOV}(Z_0) = \frac{1}{\pi^\sigma} \left(\Gamma\left(\frac{1}{p}\right)^{\chi/12} \prod_{k=1}^{p-1} \Gamma\left(\frac{k}{p}\right)^{p-k-1} \right)^{2p} \quad \text{in } \mathbb{R}^\times / \mathbb{R} \cap \overline{\mathbb{Q}}^\times,$$

where

$$\sigma = 3(p-2) \left(\frac{\chi}{12} + \frac{(p-1)(p-2)}{2} \right) + \frac{1}{4} \sum_k (-1)^k k^2 b_k.$$

Proof. We apply Theorem 5.1, written in terms of the sections η_k° instead of η_k (which vanish at 0). Up to rational number, this has the effect of letting down the term $(\psi^{n+1})^a$ in that statement. We are thus lead to evaluated the L^2 norms of the sections η_k° . By [MR04, Lemma 3.4], the L^2 norms satisfy

$$\|\eta_k^\circ\|_{L^2}^2 = (2\pi)^{-(p-2)} |\text{per}_k|^2.$$

It is now enough to plug this expression in Theorem 5.1, as well as the value of per_k provided by Lemma 6.1. \square

Remark 6.3. (1) Without any restriction on the dimension of Z_0 , it is possible to write down an expression for $\tau_{BCOV}(Z_0)$ in the same lines. For all the periods of the complex multiplication, eigenvectors on Fermat hypersurfaces are known [DMOS82, Chap. I, Sec. 7]. Nevertheless, the result is cumbersome and there is no conceptual gain for our purpose. (2) Combining Theorem 2.3 and the conjecture of Gross–Deligne (cf. [Fre17, MR04] for up to date discussions and positive results), one can propose a general conjecture for the values of the BCOV invariants of some Calabi–Yau varieties with complex multiplication. For this to be plausible, it seems however necessary to impose further conditions on the Hodge structure. A conjectural statement would be unwieldy, and we prefer not to elaborate on it here.

7. THE BCOV CONJECTURE AND FUNCTORIAL GROTHENDIECK–RIEMANN–ROCH

Classical genus 0 mirror symmetry as envisioned in [CdLOGP91] suggests a correspondence between some variations of Hodge structures of Calabi–Yau 3-folds, and the enumeration of rational curves on a mirror manifold. Roughly, one departs from a maximal unipotent degeneration of Calabi–Yau 3-folds. From the variation of Hodge structures on the middle degree cohomology, one builds the Yukawa coupling. This is a repeated application of Kodaira–Spencer maps, as we recalled in §4.2 and (4.4) for our mirror family. Using the maximal unipotent condition, one introduces canonical coordinates on the parameter space (mirror map) and canonical trivializations of the Hodge bundles, akin to the constructions in §5.2. The Yukawa coupling is then identified with a holomorphic function in the canonical coordinates. The conjecture states that the coefficients of its series expansion are Gromov–Witten invariants of genus 0 on a mirror. In higher dimensions similar results are known to hold in special cases, see e.g. [Zin14].

Higher genus mirror symmetry is the theme of the BCOV conjectural program [BCOV94]. As we saw in the introduction, in genus one, it is expected that the BCOV invariant of a degeneration of Calabi–Yau manifolds, with maximal unipotent monodromy, encodes genus one

Gromov–Witten invariants. A rigorous prototype of this phenomenon is our Theorem 5.9. However, while in genus zero the Yukawa coupling is a holomorphic function, in genus one the BCOV invariant is a \mathcal{C}^∞ function involving spectral theory. The ultimate goal consists in extracting a holomorphic function out of it, whose series expansion in the mirror coordinates has genus one Gromov–Witten invariants as coefficients.

In this section we propose an alternative approach to genus one mirror symmetry for Calabi–Yau manifolds, which bypasses spectral theory and is closer in spirit to the genus zero picture. The genus one counterpart of the Yukawa coupling will now be a functorial Grothendieck–Riemann–Roch isomorphism (GRR) of line bundles, build out of Hodge bundles. As for the Yukawa coupling, these Hodge bundles should be trivialized in a canonical way for maximal unipotent degenerations of Calabi–Yau manifolds, and the GRR isomorphism would then give rise to a holomorphic function of the mirror coordinates. This is the function that should encapsulate the genus one Gromov–Witten invariants of a mirror manifold.

Let $f: X \rightarrow S$ be a projective morphism of connected complex manifolds, whose fibers are Calabi–Yau manifolds. As in Section 2 we define the BCOV bundle (cf. (2.4)) as a combination of determinants of Hodge bundles:

$$\lambda_{BCOV}(X/S) = \bigotimes_p \det(R^\bullet f_* \Omega_{X/S}^p)^{(-1)^p p} = \bigotimes_{p,q} \det(R^q f_* \Omega_{X/S}^p)^{(-1)^{p+q} p}.$$

The formation of the BCOV bundle commutes with arbitrary base change.

Conjecture 1. *For every projective family of Calabi–Yau manifolds $f: X \rightarrow S$ as above, there exists a natural isomorphism of line bundles,*

$$\mathbf{GRR}(X/S): \lambda_{BCOV}(X/S)^{\otimes 12\kappa} \xrightarrow{\sim} f_*(K_{X/S})^{\otimes \chi\kappa}.$$

Here natural refers to the fact that the isomorphism commutes with arbitrary base change $S' \rightarrow S$, χ is the Euler characteristic of any fiber of f and κ only depends on the relative dimension of f .

Let us present arguments in favour of the conjecture:

- applying this to the universal elliptic curve, the right hand side becomes trivial in view of $\chi = 0$. This suggests that the left hand side is trivial. It is indeed trivialized by the discriminant modular form Δ , with $\kappa = 1$. For higher dimensional abelian varieties both sides are trivial and the identity provides a natural isomorphism.
- for $K3$ surfaces both sides are identical, and the identity provides a natural isomorphism. For Enriques surfaces a result similar to that of elliptic curves exist, see [Pap08]. This can probably also be realized by a Borcherds product [Bor96].
- an application of the arithmetic Riemann–Roch theorem (cf. Theorem 2.3) shows that a weak version of the conjecture holds if we restrict to the category of complex algebraic varieties. In this case, an isomorphism $\mathbf{GRR}(X/S)$ of \mathbb{Q} -line bundles exists, compatible with base change up to a complex number of modulus one, and with the additional property of being an isometry for the Quillen-BCOV metric on $\lambda_{BCOV}(X/S)$, and the L^2 metric on $f_*(K_{X/S})$. In the category of algebraic varieties defined over \mathbb{Q} , this can be strengthened to an isomorphism of \mathbb{Q} -line bundles up to sign.
- in the category of schemes, a natural isomorphism of \mathbb{Q} -line bundles up to sign exists by work of the first author [Eri08] and [Fra92]. It is compatible with the arithmetic Riemann–Roch theorem, but is far more general and stronger.

- a form of the conjecture, with a precise value of κ depending only on the relative dimension as predicted, can probably be established as a refinement of both the arithmetic Riemann–Roch theorem and Pappas’ integral GRR theorem [Pap07]. This seems within reach with the methods of [BGFIML12, BGFIML14]. A particular instance has been established by Rössler [Rös19].
- it is plausible that all these positive results, *a priori* valid in an algebraic setting, can be adapted to the category of complex manifolds, via Hilbert scheme arguments.

Our second conjecture suggests that for degenerating families of Calabi–Yau manifolds, with maximal unipotent monodromy, **GRR** realizes genus one mirror symmetry.

Conjecture 2. *Let $f: X \rightarrow \mathbf{D}^\times = (\mathbb{D}^\times)^d$ be a projective morphism of Calabi–Yau n -folds, with $d = h^{1,n-1}$ the dimension of the deformation space of the fibers, effectively parametrized with maximal unipotent monodromy. Then there exist*

- (1) *canonical holomorphic coordinates $\mathbf{q} = (q_1, \dots, q_d)$ on \mathbf{D} (exponential mirror map),*
- (2) *canonical trivializations of the Hodge bundles $R^q f_* \Omega_{X/\mathbf{D}^\times}^p$,*

such that $\mathbf{GRR}(X/\mathbf{D}^\times)$ computed in this trivializations and in coordinate \mathbf{q} becomes

$$\mathbf{GRR}(\mathbf{q}) = \exp\left((-1)^n F_1(\mathbf{q})\right)^{24\kappa},$$

where

$$F_1(\mathbf{q}) = -\frac{1}{24} \sum_{k=1}^d \left(\int_{X^\vee} c_{n-1}(X^\vee) \wedge \omega_k \right) \log q_k + \sum_{\beta \in H_2(X^\vee, \mathbb{Z})} \text{GW}_1(X^\vee, \beta) \mathbf{q}^{\langle \underline{\omega}, \beta \rangle}$$

is a generating series of genus one Gromov–Witten invariants on a mirror Calabi–Yau manifold X^\vee :

- $\underline{\omega} = (\omega_1, \dots, \omega_d)$ *is some basis of $H^{1,1}(X^\vee) \cap H^2(X^\vee, \mathbb{Z})$ formed by ample classes.*
- $\text{GW}_1(X^\vee, \beta)$ *is the genus one Gromov–Witten invariant on X^\vee associated to the class β .*
- $\mathbf{q}^{\langle \underline{\omega}, \beta \rangle} = \prod_k q_k^{\langle \omega_k, \beta \rangle}$.

The conjecture might be too optimistic in such generality; further restrictions on the Hodge structures are possibly necessary. Also, the precise way in which the coordinates and trivializations are meant to be canonical is not clear in general, but this should be modeled on propositions 5.4, 5.5 and 5.6. We refer to Morrison for a general discussion [Mor97, Section 6.3].

A variant of the conjecture would consist in rather predicting the logarithmic derivative of $\mathbf{GRR}(\mathbf{q})$. This has the advantage of requiring the existence of **GRR** only up to a constant. By the arithmetic Riemann–Roch theorem and Theorem 5.9, this version of the conjecture is satisfied for the mirror family $f: \mathcal{Z} \rightarrow U$. This is the best available evidence.

It is also tempting to ask whether $\mathbf{GRR}(\mathbf{q})$ can be written as a product of the form

$$(7.1) \quad \mathbf{GRR}(\mathbf{q}) = \mathbf{q}^{(-1)^n \mathbf{c}_0} \prod_{k \geq 1} (1 - \mathbf{q}^{\mathbf{a}_k})^{c_k}$$

for $\mathbf{c}_0, \mathbf{a}_k \in \mathbb{N}^d$ and $c_k \in \mathbb{N}$. Accepting Conjecture 2, in dimension 3 this is related to the integrality of the genus one Gopakumar–Vafa invariants. This is also compatible with the corresponding integrality conjectures in dimension 4 in [KP08]. Other sources of motivation for a product expansion as in (7.1) is the first Kronecker limit formula, and the product expansion of the discriminant modular form, as well as extensive work of Yoshikawa on equivariant analytic torsions for K3 surfaces with involution in connection with Borcherds products [Yos04, Yos09, Yos12, Yos13a, Yos13b, Yos17].

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