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GEODESIC BEHAVIOR FOR FINSLER METRICS OF CONSTANT POSITIVE FLAG CURVATURE ON S^2

R. L. Bryant, P. Foulon, S. V. Ivanov, V. S. Matveev & W. Ziller

Abstract

We study non-reversible Finsler metrics with constant flag curvature 1 on S^2 and show that the geodesic flow of every such metric is conjugate to that of one of Katok's examples, which form a 1-parameter family. In particular, the length of the shortest closed geodesic is a complete invariant of the geodesic flow. We also show, in any dimension, that the geodesic flow of a Finsler metric with constant positive flag curvature is completely integrable.

Finally, we give an example of a Finsler metric on S^2 with positive flag curvature such that no two closed geodesics intersect and show that this is not possible when the metric is reversible or has constant flag curvature.

1. Introduction

In Riemannian geometry, metrics of constant sectional curvature play an important role. In Finsler geometry, an analog of the sectional curvature is the so-called flag curvature, which, in dimension 2, is a function on the unit tangent bundle that specializes to the Gauss curvature when the Finsler structure is Riemannian.

H. Akbar-Zadeh [1] showed that, if a Finsler metric on a compact surface has constant negative flag curvature, then it is Riemannian, and that, if it has zero flag curvature, then it is locally Minkowskian. By contrast, in [5] it was first shown that there exist non-Riemannian Finsler metrics with constant positive flag curvature on the 2-sphere. It is now known that there are many such metrics on S^2 and $\mathbb{RP}^2$ [5, 6, 7, 31, 9], but a full classification is still lacking. Note that all such examples are *not reversible* since, by [8], such a reversible Finsler metric must be Riemannian.

In fact, the simplest explicit examples of non-Riemannian Finsler metrics with constant positive flag curvature on the 2-sphere turn out

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to be a 1-parameter family of Finsler metrics constructed by A. B. Katok [17] (see also [32]), though it was not known at that time that they have constant positive flag curvature [31].

Katok's construction is a special case of *Zermelo deformation*. For a Finsler metric F and a vector field X on a manifold M such that $F(-X(p)) < 1$ for all $p \in M$, the *Zermelo deformation of F via X* is the Finsler metric on M whose unit sphere in each tangent space T_pM is the translation of the unit sphere of F at p by the vector $X(p)$. Zermelo deformations of Riemannian metrics are known as *Randers metrics* in the Finsler literature. In general, if a Finsler metric F has constant flag curvature K and X is a Killing vector field of F, then the Zermelo deformation of F via X also has constant flag curvature K [15, 16].

Katok's metrics are Zermelo deformations of the standard round metric on S^2 via its Killing vector fields. Since, up to isometry, there is a 1-parameter family of inequivalent Killing vector fields on the 2-sphere (distinguished by their maximum length), Katok's metrics form a 1-parameter family of Randers metrics all of which have constant flag curvature 1. This follows from the above-mentioned result about Zermelo deformations, but was first shown in [31] (see also [2] for the classification of Randers metrics of constant flag curvature). Katok's interest in this family of Finsler metrics was primarily in the dynamics of their geodesic flows.

It is known, see [30] and Section 2, that a compact connected Finsler manifold with constant positive flag curvature is covered by a sphere. In dimension $n = 2$, we show that geodesics of such metrics on the 2-sphere behave qualitatively like those of the Katok metrics. Our main result is:

Theorem 1. *Let F_1, F_2 be two Finsler metrics on S^2 with constant flag curvature* 1*. Then the geodesic flow of F_1 is conjugate to the geodesic flow of F_2 if and only if the lengths of their shortest closed geodesics are equal.*

Here 'conjugacy' means that there exists a diffeomorphism of the unit tangent bundles that takes the geodesic flow of F_1 to that of F_2. The conjugacy implies that the number of closed geodesics and their lengths are the same for F_1 and F_2. All details of these data are given in Theorem 2 below. For general metrics, a conjugacy of geodesic flows does not preserve intersections of geodesics and may take embedded geodesics to non-embedded ones. However, in our case, some of this information can be controlled, see Propositions 1, 2, 3.

We note that there are pairs of metrics satisfying the assumptions of Theorem 1 that have conjugate geodesic flows but are not isometric [6].

For the proof of Theorem 1, the following properties of geodesics are crucial.

Theorem 2. *Let F be a Finsler metric on S^2 with constant flag curvature 1. Then there exists an embedded shortest closed geodesic of length $2\pi\mu \in (\pi, 2\pi]$, and the following holds:*

(a) *If $\mu = 1$, all geodesics are closed and have the same length 2π.*
(b) *If μ is irrational, there exist two closed geodesics with the same image, and all other geodesics are not closed. The length of the second closed geodesic is $2\pi\mu/(2\mu-1)$. Moreover, the metric admits a Killing vector field.*
(c) *If $\mu = p/q \in (\frac{1}{2}, 1)$ with $p, q \in \mathbb{N}$ and $\gcd(p, q) = 1$, then all unit-speed geodesics have a common period $2\pi p$. Furthermore, there exist at most two closed geodesics with length less than $2\pi p$. A second one exists only if $2p-q > 1$, and its length is $2\pi\mu/(2\mu-1) = 2\pi p/(2p-q) \in (2\pi, 2p\pi)$.*

We will show that, for each $\mu \in (\frac{1}{2}, 1]$, there exists a unique Katok metric whose shortest closed geodesic has length $2\pi\mu$. Thus:

Corollary 1. *The geodesic flow of a Finsler metric on S^2 with flag curvature 1 is conjugate to that of a unique member of the family of Katok metrics.*

By Theorem 2, we see that the sum of the reciprocals of the lengths of the two shortest closed geodesics is always $1/\pi$ (cf. [**27**, Remark 1(b)]). This implies that, for Finsler metrics of constant positive flag curvature K on S^2, the sum of the reciprocals of the lengths of the two shortest closed geodesics is K/π. Hence we have:

Corollary 2. *The geodesic flows of two Finsler metrics on S^2 whose flag curvatures are different positive constants are not conjugate, even in the continuous sense.*

Combining Theorem 2 and Corollaries 1 and 2 gives us a complete classification of the geodesic flows of Finsler metrics on S^2 with constant flag curvature.

At the end of Section 3, we will show that Theorem 1, Corollary 1, and Corollary 2 also hold for Finsler metrics on $\mathbb{RP}^2$ with constant flag curvature. We note that the Katok metrics on S^2 naturally descend to Finsler metrics on $\mathbb{RP}^2$.

The examples in [**5**, **6**, **7**] all have $\mu = 1$. Some of these examples also admit Killing vector fields and hence, applying a Zermelo deformation to them, one obtains examples with $\mu < 1$. We do not know any other examples of Finsler metrics on S^2 with constant flag curvature 1 and $\mu < 1$.

We will also prove the following results in dimensions $n \geq 2$.

Theorem 3. *Let (S^n, F) be a Finsler manifold of dimension $n \geq 2$ of constant flag curvature 1. Then its geodesic flow is Liouville integrable and has zero topological entropy.*

Theorem 4. *Let F be a Finsler metric on S^n with constant flag curvature* 1 *that admits a Killing vector field. Then there exists an arbitrarily small Zermelo deformation of F to one with all of its geodesics closed. Further, when $n = 2$, there is a Zermelo deformation for which all geodesics are closed and of length 2π.*

In the proof of Theorem 2 we will show that, for metrics of constant positive flag curvature on S^2, any two geodesics intersect. (See Propositions 1 and 2 in Section 2.) Our final result concerns the question of what happens when the flag curvature is not a positive constant.

Recall that, in the Riemannian case, this question is answered by the famous theorem of Frankel [**12**], which states that, in an n-dimensional Riemannian manifold with positive curvature, two totally geodesic submanifolds of dimension n_1 and n_2 must meet if $n_1 + n_2 \geq n$.

Theorem 5. *Let F be a Finsler metric on S^2 with positive flag curvature. Then any two geodesically reversible closed geodesics meet. Meanwhile, there exists a Finsler metric on S^2 with positive flag curvature and exactly two closed geodesics, which are disjoint.*

Here, a geodesic is called *geodesically reversible* if the geodesic with reversed orientation is, up to parametrization, also a geodesic. In particular, for a reversible Finsler metric of positive flag curvature on S^2, any two closed geodesics intersect. Our counterexample in the non-reversible case is a small perturbation of a Katok example. A similar perturbation was constructed in [**25**].

We do not know whether an analogue of Frankel's theorem holds for reversible Finsler metrics in dimensions above 2. In [**18**] it was claimed that Frankel's theorem holds in all dimensions for reversible Finsler metrics, but the proof is not correct. At the end of Section 6, we discuss the difficulties involved in a straightforward generalization to Finlser metrics of the proof of Frankel's theorem in the Riemannian case.

Our paper is organized as follows. In Section 2, we first recall some well-known facts about Finsler metrics with constant positive flag curvature and then prove Theorem 2 as well as some other facts about properties of closed geodesics. In Section 3 we prove Theorem 1. In Section 4 we discuss and then prove Theorem 4. Theorem 3 is proved in Section 5 and Theorem 5 in Section 6.

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2. The isometry ψ and properties of closed geodesics

Let (M^n, F) be a compact Finlser manifold of dimension $n > 1$. For general facts and background about Finsler metrics, we refer to [10], see also [26].

For convenience, we occasionally take advantage of the *Binet-Legendre* metric g_F, a Riemannian metric that is naturally associated to the Finsler metric F [21]. Any isometry of F is an isometry of g_F, and hence the group of isometries of F is a compact Lie group, denoted by $\mathrm{Isom}(F)$. (Instead of the Binet-Legendre metric, one could use a different construction that naturally associates a Riemannian metric to a Finsler metric, for example the construction from [19, 20].)

For $v \in T_pM$ we denote by $\gamma_{p,v}(t)$, or, for short, $\gamma_v(t)$, the geodesic with $\gamma_{p,v}(0) = p$ and $\dot\gamma_{p,v}(0) = v$, and call it a *normal* geodesic if $F(p, v) = 1$. Furthermore, we denote by $\mathrm{U}(M)$ the unit sphere bundle, i.e., $\mathrm{U}(M) = \{v \in TM \mid F(v) = 1\}$.

We begin by recalling some well-known facts (see, e.g., [30]) about Finsler metrics of constant positive flag curvature, which, by scaling, we can assume to be 1. Let F be such a Finsler metric on a compact n-manifold M. Since the flag curvature is identically 1, all normal geodesics $\gamma_{p,v}$ have their first conjugate point at $t = \pi$ with multiplicity $n-1$. This implies that for a fixed p the exponential map sends the sphere of radius π into a fixed point, which we denote by $\psi(p)$. Thus, all geodesics starting at p meet again at $\psi(p)$, i.e., for all normal geodesics $\gamma_{p,v}(\pi) = \psi(p)$. The exponential map $\exp_p$ on the open ball $B_\pi(p) = \{\xi \in T_pM \mid F(p, \xi) < \pi\}$ is a local diffeomorphism. By identifying the boundary of $B_\pi(p)$ to a point and sending this point to $\psi(p)$, the local diffeomorphism extends to a continuous map $\sigma\colon S^n \to M$. It is also a local homeomorphism near $\psi(p)$ since, if two normal geodesics γ_{p,v_1} and γ_{p,v_2} meet at time close to π, they meet again at time π in $\psi(p)$, contradicting the fact that the exponential map is a local diffeomorphism on all balls of sufficiently small radius. Thus σ is a covering map. If M is simply connected, it follows that it is homeomorphic to a sphere, that all normal geodesics are minimizing up to time π, and that ψ has no fixed points. Thus, for every normal geodesic γ, we have $\psi(\gamma(t)) = \gamma(t + \pi)$ for all $t \in \mathbb{R}$.

We next show that ψ is a smooth isometry by first showing that it preserves the Finslerian distance d_F. Indeed, for two arbitrary points $x, y \in M$ take any normal minimizing geodesic γ with $\gamma(0) = x$ and

$\gamma(r) = y$. Then, by the construction of ψ, we have $\psi(x) = \gamma(\pi)$ and $\psi(y)\gamma(\pi + r)$ and, since $r \le \pi$, the geodesic γ is also minimizing from π to $\pi + r$. Thus, $d_F(x, y) = d_F(\psi(x), \psi(y))$ and, hence, ψ is an isometry. In particular, ψ is smooth by [**11**, **22**].

Restricting to $n = 2$ from now on, i.e., to a Finsler metric F of flag curvature 1 on S^2, we claim that ψ is an orientation-reversing isometry. Indeed, the orientation induced by the velocity vector of a geodesic and a Jacobi vector field along this geodesic changes sign when we pass the point where the Jacobi vector field vanishes.

Now, in the non-Riemannian case, ψ^2 need not be the identity. In fact, if $\psi^2 = \mathrm{Id}$ does hold, then, for any normal geodesic γ, we have $\gamma(t) = \psi^2\big(\gamma(t)\big) = \gamma(t + 2\pi)$ for all t, and hence γ is periodic with period 2π. Since there are no geodesics loops of length π, each geodesic is embedded and has length 2π. This is the case (a) of Theorem 2. We will set aside this special case in the following discussion, but note that there are many non-Riemannian examples for which $\psi^2 = \mathrm{Id}$, see [**5**, **6**, **7**].

We now recall the well-known fact that, for every compact Lie group G acting effectively on S^2, its action is conjugate to that of a subgroup of $\mathrm{O}(3)$ acting linearly on S^2. In particular, there exists a G-invariant Riemannian metric of constant Gauss curvature 1 on S^2. (This seems to be a folklore result, see e.g., [**29**] or [**3**, Theorem 12] for connected groups. For completeness, we recall its proof: Take any Riemannian metric on S^2 that is invariant with respect to G. By the uniformization theorem, it is conformal to a constant Gauss curvature metric on S^2 and is therefore a compact subgroup in the group of the conformal transformations of such a metric on S^2. Since any maximal compact subgroup of the group of conformal transformations is conjugate to $\mathrm{O}(3)$, the claim follows.)

Applying this result to the group of isometries of the Finsler metric F on S^2, which, as noted, is compact and contains ψ, it follows that, in particular, there is a constant curvature 1 Riemannian metric g on S^2 with respect to which ψ is an isometry, and hence the orientation-preserving g-isometry ψ^2 (which, by hypothesis, is not the identity) has exactly two fixed points, which are exchanged by ψ. Once such a g and a fixed point p of ψ^2 have been chosen, there exist p-centered, g-spherical polar coordinates (θ, ϕ) on S^2, where $\phi \in \mathbb{R}/(2\pi\mathbb{Z})$ is the longitude and $\theta \in [-\frac{1}{2}\pi, \frac{1}{2}\pi]$ is the latitude, and a constant $\lambda \in (0, \frac{1}{2}]$ such that

$$\psi(\theta, \phi) = (-\theta,\ \phi + 2\pi\lambda), \tag{1}$$

where p corresponds to $\theta = -\frac{1}{2}\pi$.

Indeed, one can choose coordinates such that $\lambda \in (0, \frac{1}{2}]$ since changing the orientation of ϕ replaces λ by $1 - \lambda$. Notice that λ cannot equal 0, since ψ has no fixed points. The value $\lambda = \frac{1}{2}$ corresponds to the case $\psi^2 = \mathrm{Id}$; thus, we assume that $\lambda < \frac{1}{2}$ in most of our arguments.

We will call λ the *rotation angle* of F and sometimes use the notation F_λ when F is a Finsler metric with rotation angle λ.

Remark 1. For use in Proposition 5 below, we note that in the case $\psi^2 \neq \mathrm{Id}$ the choices of g and p determine the coordinates (θ, ϕ) satisfying (1) uniquely up to the evident ambiguity $(\theta, \phi)_{\mathrm{new}} = (\theta, \phi + \mathrm{const})$. Replacing p by $\psi(p)$ replaces θ by $-\theta$.

In the case $\psi^2 = \mathrm{Id}$, there is no restriction on the choice of p. However, in this case, under the additional assumption that there exists precisely one nonzero Killing vector field up to constant multiples, one can further specify either of the two zeros of this Killing field to be p, and this is what we will usually do. (Among Finsler metrics on S^2, only a Riemannian metric of constant Gauss curvature can have more than one nonzero Killing field up to constant multiples anyway.)

Simultaneously replacing the 'pole' p by $\psi(p)$ and reversing the orientation of S^2 determined by (θ, ϕ) does not change the value of λ. Hence this value is an invariant of F. (Since the Katok examples on S^2 are invariant under the antipodal map of S^2, which switches the poles and reverses the orientation, it follows that one cannot always resolve the paired ambiguity of choice of pole and orientation.)

We distinguish two cases: When λ is rational and when λ is irrational. We start with the latter case.

Proposition 1. *Let $F = F_\lambda$ be a Finsler metric on S^2 with λ irrational. Then*

(a) *$(\theta, \phi) \mapsto (\theta, \phi + s)$ is an isometry for all s, i.e., $\frac{\partial}{\partial \phi}$ is a Killing vector field.*

(b) *The equator $\{\theta = 0\}$ is a geodesic that has length $\pi/\lambda > 2\pi$ when traversed in the positive direction and $\pi/(1-\lambda) \in (\pi, 2\pi)$ when traversed in the opposite direction.*

(c) *For any other geodesic there exists a number $\theta_0 > 0$ such that the geodesic oscillates between the parallels $\theta = \theta_0$ and $\theta = -\theta_0$. Furthermore, the θ-coordinate along the geodesic changes monotonically in between these parallels, while the ϕ coordinate increases by $2\pi\lambda$ or $2\pi(1-\lambda)$. In particular, the geodesic is not closed.*

In part (c) we allow $\theta_0 = \pm\frac{1}{2}\pi$ for the geodesics going through the poles.

Proof. Since λ is irrational, the set of the points $\{4k\lambda\pi\}_{k\in\mathbb{Z}}$ is dense in the circle $\mathbb{R}/(2\pi\mathbb{Z})$. Thus the closure of the subgroup of $\mathrm{Isom}(F)$ consisting of the powers of ψ contains all transformations of the form $(\theta, \phi) \mapsto (\theta, \phi + s)$, which implies (a). Notice also that $\mathrm{Isom}(F) \supseteq \{(\theta, \phi) \mapsto (\pm\theta, \pm\phi + s)\}$ and that any isometry preserves or switches the poles.

In order to prove (b), observe that composing ψ with the flow of the Killing vector field, we have that $(\theta, \phi) \to (-\theta, \phi)$ is an isometry

and thus its fixed point set $\{\theta = 0\}$ is a closed geodesic of F in either direction, but with possibly different parameterizations. Let $\gamma^{\pm}(t) = (0, \pm t)$ be the equator. The Killing vector field $\frac{\partial}{\partial\phi}$ is tangent to $\gamma^{\pm}$ and hence they are parametrized proportional to arc length. For $\gamma^{+}(t) = (0, t)$, the length of the curve, for $0 \le t \le 2\pi\lambda$, is equal to π by definition of ψ. Thus $F(\dot\gamma^{+}) = \frac{1}{2}\lambda$ and hence the length of γ^{+} is π/λ. Similarly, the length of γ^{-} is $\pi/(1-\lambda)$.

Let us now prove (c). Observe that the Killing vector field $X = \frac{\partial}{\partial\phi}$ is a Jacobi field along any normal geodesic γ. Since the flag curvature is positive and constant, this implies that for any $\gamma \neq \gamma^{\pm}$, there exists a time t_0 such that X is proportional to $\dot\gamma$ precisely when $t = t_0 + k\pi$ for some $k \in \mathbb{Z}$. Thus along γ the function θ reaches its maximum and minimum at $\theta = \pm\theta_0$ and θ is monotonic in between. The geodesic γ bounces back and forth between these parallels, increasing its ϕ coordinate by $4\lambda\pi$ or $4(1-\lambda)\pi$ (depending on its orientation) each time it returns. To see that it does not increase by $4\lambda\pi + 2\pi k$ or $4(1-\lambda)\pi + 2\pi k$ for some $k \neq 0$, we observe that this property would be the same for all geodesics, but does not hold for the geodesics on the equator. q.e.d.

For fixed $\theta_0 > 0$, the geodesics from Proposition 1(c) viewed as curves on $U(S^2)$ form two 2-tori in $U(S^2)$, depending on whether they rotate clockwise or counter clockwise. The geodesics through the poles form a 2-torus as well. The geodesic flow has constant rotation number λ on all of these 2-tori and degenerates for $\theta_0 \to 0$ to the two closed geodesics $\gamma^{\pm}$.

Before stating the next result, let us make the following definition: Given a curve $\alpha\colon [a, b] \to S^2$ not going through either pole, we define the change of the angular coordinate ϕ as follows: Assume, as we may, that, in our coordinates (θ, ϕ), we have $\phi\big(\alpha(t)\big) \equiv \hat\phi(t) \mod 2\pi$ for some continuous function $\hat\phi : [a, b] \to \mathbb{R}$. We call $\hat\phi(b) - \hat\phi(a)$ the *change* of ϕ along α, and we define the *winding number* (around the 'south' pole, $\psi(p)$) of α to be $\big(\hat\phi(b) - \hat\phi(a)\big)/(2\pi)$. The winding number of α is independent of the choice of $\hat\phi$ and is an integer when $\alpha(b) = \alpha(a)$.

If λ is rational, ψ has some finite order, which, since ψ reverses orientation, must be even. Thus $\psi^{2q} = \mathrm{Id}$ for some minimal $q \in \mathbb{N}$. We set $\lambda = p/(2q)$ with $\gcd(p, q) = 1$. Since we are assuming $\lambda < \frac{1}{2}$, we have $1 \le p < q$. For any normal geodesic, we have $\gamma(t) = \psi^{2q}(\gamma(t)) = \gamma(t + 2q\pi)$ for all t, and hence all normal geodesics are periodic with period $2q\pi$. We call a closed geodesic *exceptional* if its length is less than $2q\pi$. We start with the following observation.

Proposition 2. *Let F_λ be a Finsler metric on S^2 with constant flag curvature* 1. *Then any two closed geodesics intersect. If $\lambda = p/(2q) < \frac{1}{2}$*

in lowest terms, and one of the geodesics does not pass through the poles, then they intersect in at least $q > 1$ points.

Proof. If λ is not rational, then, by Proposition 1, there are only two closed geodesics, and they have the same image in S^2, so the claim follows.

Now suppose that $\lambda = p/(2q)$ in lowest terms. If two geodesics pass through the poles, then they intersect. Suppose that γ is a geodesic that does not pass through the poles. By the formula for ψ in local coordinates, the number of points in an orbit $\{\psi^k(x),\ k \in \mathbb{Z}\}$ other than the poles is equal to q or $2q$. Thus, since geodesics are invariant under ψ, to prove that any other geodesic intersects γ in at least q points, it suffices to show that it meets γ somewhere. Now, the complement of γ in S^2 is a countable disjoint union of open sets U_i, each homeomorphic to a disc, which ψ permutes. If a second closed geodesic did not meet γ, it would lie in one of these components, say U_k. But then $\psi(U_k) = U_k$, and hence ψ would have a fixed point, which it does not. q.e.d.

We now determine the length of all exceptional geodesics.

Proposition 3. *Let F_λ be a Finsler metric on S^2 with rotation angle $\lambda = p/2q$, $0 < p < q$ and $\gcd(p, q) = 1$. Then there are at most two exceptional geodesics, they are embedded, and we have:*

(a) *There exists a closed geodesic of length $\pi/(1-\lambda) = 2q\pi/(2q-p) \in (\pi, 2\pi)$, that has winding number -1 around the south pole.*
(b) *There exists a second exceptional closed geodesic if and only if $p > 1$. Its length is $\pi/\lambda = 2q\pi/p \in (2\pi, 2q\pi)$ and it has winding number $+1$ around the south pole.*
(c) *If p is even, the two exceptional geodesics have images on the equator.*

Proof. Since all geodesics are closed with period $2q\pi$, the geodesic flow generates a Seifert fibration on $U(S^2) \simeq \mathbb{RP}^3$. It is well known (see, e.g., [**24**, §5.3, §5.4]) that any Seifert fibration on a lens space has at most two singular leaves. Thus all geodesics, except possibly two, have the same length $2q\pi$. We first show that the winding number of an exceptional closed geodesics is well defined:

Lemma 1. *Every exceptional closed geodesic γ is embedded.*

Proof. If the length of a geodesic γ is not a multiple of π, then it equals $k\pi + r$ for some $0 < r < \pi$. Since ψ acts on γ by adding π to the parameter, ψ^{k+1} acts by adding $\pi - r$, which is positive and less than π. Hence every arc of γ of length $\pi - r$ is the unique minimizing geodesic from a point to its image by ψ^{k+1}. If γ has a self intersection at a point p_0, then there would exist two minimizing geodesics from p_0 to $\psi^{k+1}(p_0)$ which contradicts $\pi - r < \pi$. If the length is $k\pi$ with $k < 2q$, then ψ^k is the identity on γ and hence ψ^k has infinitely many

fixed points. Since $\psi^k \neq \mathrm{Id}$, this can only happen if $\psi^k(\theta,\phi) = (-\theta,\phi)$. It follows that, in this case, γ lies on the equator and is hence embedded. q.e.d.

Notice that the geodesics through the poles are not embedded, and hence every exceptional geodesic γ has a well defined winding number W around the south pole, and $W = 1, -1$, or 0 since it is embedded.

Lemma 2. *Let γ be an embedded closed geodesic. Then one of the following holds:*

(a) *γ has length π/λ and its winding number is $+1$.*
(b) *γ has length $\pi/(1-\lambda)$ and its winding number is -1.*

Proof. Recall that every geodesic is ψ-invariant since $\psi(\gamma(t)) = \gamma(t+\pi)$. Fix a point x_0 on γ. Let $x_0, x_1, ..., x_n = x_0$ be the points of the ψ-orbit of x_0, enumerated in the order of appearance on γ. Note that $n > 2$, since the case $\psi^2 = \mathrm{Id}$ is excluded.

Let γ_i denote the arc of γ from x_i to x_{i+1}. Each γ_i is the image of γ_0 under some power of ψ, so they all have the same length and the same change of angular coordinate ϕ. Hence the change of angular coordinate along γ_0, let us call it A, equals $2\pi W/n$.

One of the points x_i is $\psi(x_0)$. Let it be x_m with $1 \leq m \leq n-1$. The angular coordinate change from x_0 to x_m equals $mA = 2\pi W m/n$, which is a number between -2π and 2π since $|W| \leq 1$. Also this number is equivalent to $2\pi\lambda$ modulo 2π. Hence it is either $2\pi\lambda$ or $2\pi(\lambda-1)$. If it is $2\pi\lambda$, then $Wm/n = \lambda$, hence $W = 1$ and $m/n = \lambda$. If it is $2\pi(\lambda-1)$, then $Wm/n = \lambda-1$, hence $W = -1$ and $m/n = 1-\lambda$.

It remains to determine the length of γ. The length from x_0 to x_m is π, since $x_m = \psi(x_0)$. Thus the length of γ_0 is π/m, and hence the length of γ is $n\pi/m$. Substituting the above formulas for m/n in terms of λ, one obtains that the length of γ is π/λ (resp., $\pi/(1-\lambda)$). q.e.d.

We now prove the existence and uniqueness of the closed geodesics in Proposition 3.

First observe that if p is even, q must be odd, and hence $\psi^q(\theta,\phi) = (-\theta, \phi + p/2 \cdot 2\pi) = (-\theta,\phi)$. Thus the fixed point set of ψ^q, i.e., the equator, is a geodesic in both directions. Since they are embedded, Lemma 2 implies that their lengths are $\pi/(1-\lambda)$ and π/λ. Since $p > 1$, both lengths are less than $2q\pi$ and hence both closed geodesics are exceptional. This finishes Proposition 3 when p is even. From now on we can thus assume that p is odd.

Recall that if f is an isometry without fixed points and of finite order, then we obtain an f-invariant closed geodesic as the minimum of the displacement function $p \to d_F\big(p, f(p)\big)$: Let D be this minimum and let s be a minimal geodesic from p to $f(p)$. Then s and $f(s)$ form a geodesic, i.e., there is no angle at $f(p)$, since otherwise the distance

between the midpoints would be less than D. Thus $s, f(s), f^2(s), \ldots$ form a geodesic which eventually closes up, and $f(\gamma(t)) = \gamma(t+D)$. We call γ a *minimum displacement geodesic* of f.

We apply this construction to $f = \psi^k$, where k is odd and $k \not\equiv 1 \bmod 2q$. In this case we have $D < \pi$ since ψ is the only isometry with minimal displacement π. Furthermore, f has no fixed points since k and p are odd. This implies that a minimum displacement geodesic of ψ^k exists and is unique. Indeed, by Proposition 2, any two such geodesics γ_1, γ_2 intersect: $\gamma_1(t_1) = \gamma_2(t_2)$ for some t_1, t_2. Then $\gamma_1(t_1 + D) = \gamma_2(t_2 + D)$, which means there are two minimizing geodesics from $\gamma_1(t_1)$ to $\gamma_1(t_1 + D)$ with $D < \pi$, a contradiction. The same argument applies to self-intersections, proving that the minimum displacement geodesic of ψ^k is embedded.

We can use this to show that there exists an exceptional closed geodesic with length $< 2\pi$. Let γ_0 be the minimum displacement geodesic for ψ^{-1} and D the minimum displacement value. Then $\gamma_0(D) = \psi^{-1}(\gamma_0(0))$ and hence $\gamma_0(D+\pi) = \psi(\gamma_0(D)) = \gamma_0(0)$. Thus γ_0 closes up with length $\pi + D < 2\pi$ and is hence exceptional. We furthermore observe that γ_0 is the only closed geodesic of length $< 2\pi$. Indeed, suppose that γ_1 is another closed geodesic with length $< 2\pi$. Then by Lemma 2 both γ_0 and γ_1 have length $\pi/(1-\lambda)$, which is then equal to $\pi + D$. Hence the arc of γ_1 from $\gamma_1(\pi)$ to $\gamma_1(0) = \psi^{-1}(\gamma_1(\pi))$ has length D. Therefore γ_1 is also a minimum displacement geodesic for ψ^{-1} and thus agrees with γ_0. This proves part (a) in Proposition 3.

We now prove the existence of the second exceptional geodesic when $p > 1$. Since p is odd, there exists an odd k such that $kp \equiv 1 \bmod 2q$ and $k \not\equiv 1 \bmod 2q$. As shown above, there is a minimum displacement geodesic γ_1 for ψ^k, and γ_1 is embedded. Since $p > 1$, we have $\max\{\pi/\lambda, \pi/(1-\lambda)\} < 2q\pi$ and hence Lemma 2 implies that γ_1 is exceptional. We still need to show that γ_1 is different from the first exceptional geodesic γ_0. So assume that $\gamma_1 = \gamma_0$. Let $L = \pi/(1-\lambda) = 2q\pi/(2q-p)$ be the length of γ_0 and let α denote the arc of γ_0 from $x_0 = \gamma_0(0)$ to $\psi^k(x_0) = \gamma_0(k\pi)$. Since $(pk-1)/2q \in \mathbb{Z}$, it follows that

$$\left(k\pi - \frac{(2q-1)\pi}{2q-p}\right)/L = \frac{(2q-p)k - (2q-1)}{2q} \in \mathbb{Z}.$$

Furthermore, $0 < \frac{(2q-1)\pi}{2q-p} < L$ and hence the length of α equals $\frac{(2q-1)\pi}{2q-p} > \pi$. But then α is not minimizing and therefore γ_0 is not a minimum displacement geodesic for ψ^k. This proves that γ_1 is the second exceptional geodesic and by Lemma 2 it has length π/λ.

If $p = 1$, we have $\pi/\lambda = 2q\pi$. Hence by Lemma 2 all exceptional geodesics have length $\pi/(1-\lambda) < 2\pi$. But as we saw above, there can be only one such geodesic. This finishes the proof of Proposition 3.

q.e.d.

Combining Propositions 1 and 3 finishes the proof of Theorem 2 in the Introduction. Notice though that $\mu = \frac{1}{2(1-\lambda)}$.

We point out that the shortest closed geodesic γ has the following geometric interpretation: As follows from the proof of Theorem 2, γ is a minimum displacement geodesic for ψ^{-1} and, since $d(p, \psi^{-1}(p)) = d(\psi(p), p)$, it can be viewed as realizing the shortest return time from $\psi(p)$ to p.

3. The proof of Theorem 1

We now prove the conjugacy of the geodesic flow of two Finsler metrics F_λ with the same rotation angle λ.

We will first show that the geodesic flow g_t, regarded as an action of $\mathbb{R}$ on $\mathrm{U}(S^2) \simeq \mathbb{RP}^3$ is conjugate to an orthogonal action, i.e., a homomorphism $\mathbb{R} \to \mathrm{SO}(4)/\{\pm\mathrm{Id}\}$ where $\mathrm{SO}(4)$, acting orthogonally on S^3, descends to an action on $\mathbb{RP}^3$.

If $\lambda = p/2q$ is rational, then all geodesics are closed with common minimal period $2q\pi$ and hence g_t induces an effective action of S^1 on $\mathbb{RP}^3$. But a circle action on $\mathbb{RP}^3$ is conjugate to an orthogonal action, see [**28**], which can also be regarded as a homomorphism from $\mathbb{R}$ to $\mathrm{SO}(4)/\{\pm\mathrm{Id}\}$.

If λ is irrational, we saw in Proposition 1 that we also have a Killing vector field $X = \frac{\partial}{\partial\phi}$ with flow τ_t, which induces a vector field X^* on $\mathrm{U}(S^2)$ with flow $\tau_t^* = d(\tau_t)$ and $\tau_{2\pi}^* = \mathrm{Id}$. Clearly, g_t and τ_s^* commute, since τ_t acts by isometries and hence $[X^*, G] = 0$, where G is the geodesic flow vector field. From the definition of ψ, it also follows that $g_{2\pi} = \tau_{4\lambda\pi}^*$. Thus, the flow of the vector field $Y = G - 2\lambda X$ is 2π periodic. Hence X and Y commute and generate an action of T^2 on $\mathrm{U}(S^2)$. In [**23**] it was shown that, up to an automorphism of T^2 and a diffeomorphism of $\mathbb{RP}^3$, there exists a unique effective T^2 action on $\mathbb{RP}^3$ which can hence be thought of as an orthogonal action, i.e., a homomorphism $T^2 \to \mathrm{SO}(4)/\{\pm\mathrm{Id}\}$. Hence the same is true for the flow g_t of G.

Thus in both cases, the geodesic flow is conjugate to a homomorphism $\mathbb{R} \to \mathrm{SO}(4)/\{\pm\mathrm{Id}\}$. Such a homomorphism is furthermore conjugate, via an element of $O(4)$, to a block diagonal one of the form $t \mapsto \mathrm{diag}(R(at), R(bt))/\{\pm\mathrm{Id}\}$, with $a, b \in \mathbb{R}$, $a \geq b \geq 0$ and where $R(t)$ is a rotation of $\mathbb{R}^2$ with angle t. Furthermore, $b > 0$ since the geodesic flow is almost free. But such a homomorphism is uniquely characterized by its smallest minimal periods, i.e., π/a and π/b, corresponding to the orbits through the images of $(1, 0, 0, 0)$ and $(0, 0, 0, 1) \in S^3$ in $\mathbb{RP}^3$. As we saw in Proposition 1 and Proposition 3, the minimal periods of the geodesic flow are π/λ and $\pi/(1-\lambda)$. This implies that λ uniquely determines the geodesic flow up to conjugacy. Since $\pi/(1-\lambda)$ is also the length of the shortest closed geodesic, this finishes the proof of Theorem 1.

We now indicate how to show that the analogs of our results also hold for Finsler metrics on $\mathbb{RP}^2$ with constant flag curvature. First we can apply the previous results to the lift of the metric to S^2. In particular this defines the geometric parameter λ. Then, by [**28**] and [**23**], the geodesic flow is again conjugate to an orthogonal action since the unit tangent bundle of $\mathbb{RP}^2$ is the lens space $L(4,1) = S^3/\mathbb{Z}_4$ with $\mathbb{Z}_4$ generated by $\operatorname{diag}(R(\frac{\pi}{2}), R(\frac{\pi}{2}))$. The geodesic flow is thus conjugate to $t \to \operatorname{diag}(R(at), R(b\,t))/\mathbb{Z}_4$, with $a, b \in \mathbb{R}$. Note that the parameters a and b are the same, up to a sign, as the similar parameters for the covering metric on S^2 constructed in the proof of Theorem 1. Indeed, there exists only one two-sheeted covering of $L(4,1)$; it is the natural projection from $\mathbb{RP}^3 = S^3/\mathbb{Z}_2$ to $L(4,1) = S^3/\mathbb{Z}_4$. Hence, $|a| + |b| = 1$, and the two minimal periods on $L(4,1)$ are $\pi/(2|a|)$ and $\pi/(2|b|)$. In view of orthogonal conjugations in $L(4,1)$, we may assume that $a \geq |b| > 0$. However, unlike the case of $\mathbb{RP}^3$, orthogonal conjugations cannot change the signs of a and b individually.

To distinguish the actions with invariants (a, b) and $(a, -b)$, we compute the return map, i.e., the derivative of the flow orthogonal to the orbit, for the shortest periodic orbit $t \mapsto \operatorname{diag}(R(at), 0)$. The image of the orbit in $L(4,1)$ closes at time $t_0 = \pi/(2a)$ and hence the return map is given by $R\big(b\,\pi/(2a)\big) \circ R(\frac{1}{2}\pi)^{-1}$. It is hence a rotation whose angle, up to a choice of orientation on $L(4,1)$, lies in $[0, \frac{1}{2}\pi)$ if $b > 0$ and in $(\frac{1}{2}\pi, \pi]$ if $b < 0$. In particular, the action with $b > 0$ is not conjugate to the one with $b < 0$.

For a Finsler metric on $\mathbb{RP}^2$ with flag curvature 1, consider the closed geodesic γ corresponding to the above shortest orbit. It has length $\pi/(2a) = \pi/\big(2(1-\lambda)\big)$. The return map of the geodesic flow along γ can be computed in terms of Jacobi fields in a parallel basis. The deck transformation of the two-fold cover is orientation-reversing, and it preserves the lift of γ. Hence, the parallel translation along γ is a rotation by π in $T_{\dot\gamma(0)}\mathrm{U}(\mathbb{RP}^2)$. Looking at Jacobi fields, one sees that the rotation angle of the return map differs from that of parallel translation by $\pm\text{length}(\gamma)$. Hence, the rotation angle (normalized to the interval $[0, \pi]$) equals $\pi - \pi/\big(2(1-\lambda)\big) \in [0, \frac{\pi}{2})$. Therefore, $b > 0$.

Thus, as before, the geodesic flow of the Finsler metric is determined by λ and, hence, also by the length of the shortest closed geodesic.

Altogether, we see that the statements in Theorem 1, Corollary 1, and Corollary 2 also hold for Finsler metrics on $\mathbb{RP}^2$ with constant flag curvature.

4. Zermelo deformations and the proof of Theorem 4

In this section, we discuss Zermelo deformations of Finsler metrics with respect to a Killing vector field. Here we work in any dimension.

Let F be a Finsler metric on M^n and X a Killing vector field of F. Under the Legendre transform $D\left(\frac{1}{2}F^2\right) : TM \to T^*M$, we obtain the dual norm $F^* : T^*M \to \mathbb{R}$, whose Hamiltonian flow with respect to the canonical symplectic structure on T^*M corresponds to the geodesic flow of F via the Legendre transform.

Let $\alpha \in \mathbb{R}$ be a constant. The function $F^*_\alpha(\xi) := F^*(\xi) + \alpha\,\xi(X)$ defines a norm on T^*M if and only if $F(-\alpha X) < 1$. Applying to it the Legendre transform $D\left(\frac{1}{2}(F^*_\alpha)^2\right)$ with respect to $\frac{1}{2}(F^*_\alpha)^2$, we obtain a Finsler metric on M which we denote by F_α.

The Finsler metric F_α is the Zermelo deformation of F via the Killing field αX. It is easy to see that, for each p, its unit sphere in T_pM is the unit sphere of F in T_pM translated by $\alpha X(p)$. The surprising properties of such metrics were first discussed by Katok [**17**] (see also [**32**] for further discussions).

Denote by τ_t the flow of X and by $d^*\tau_t$ its natural lift to T^*M; it is generated by the Hamiltonian function $\xi \in T^*M \mapsto \xi(X) \in \mathbb{R}$. Denote by φ_t the geodesic flow of F on T^*M, which we view as the flow of the Hamiltonian F^*. Then, the geodesic flow of F_α is, by construction, the flow of the Hamiltonian F^*_α. Since τ_t acts by isometries of F, the flows $d^*\tau_t$ and φ_t commute. Hence, the flow of the Hamiltonian $F^*_\alpha = F^*(\xi) + \alpha\xi(X)$ is the composition $\varphi_t \circ d^*\tau_{\alpha t}$. Thus, if $c(t)$ is a normal geodesic of F, then $c_\alpha(t) = \tau_{\alpha t}(c(t))$ is a normal geodesic of F_α.

In [**15**, **16**], as already observed in [**14**], see also [**2**], it was shown that if F has constant flag curvature 1, then F_α does as well. If ψ is the isometry defined in Section 2 for F, and ψ_α is its analog for F_α, then $\psi_\alpha = \psi \circ \tau_{\alpha\pi}$.

Proposition 4. *Let F be a Finsler metric on a compact n-manifold M with constant flag curvature* 1 *that has a geodesic that is not closed. Then there exists a Killing vector field X such that the Zermelo deformation F_1 (i.e., $F^*_1(\xi) = F^*(\xi)+\xi(X)$) is a Finsler metric with constant flag curvature* 1 *and all of its geodesics are closed.*

Proof. Since there exists a geodesic that is not closed, the map ψ does not have finite order. Thus $\{\psi^k \mid k \in \mathbb{Z}\}$ is an abelian subgroup of the compact Lie group $\mathrm{Isom}(F)$ that is not closed. Its closure is a compact abelian group and hence has finitely many components, with its identity component being a compact torus T of positive dimension. Thus some positive power of ψ lies in T, and, hence, for some large k the isometry ψ^k is close to the identity in $\mathrm{Isom}(F)$. Hence there exists a Killing vector field X of small length and with flow τ_t such that $\tau_{-k\pi} = \psi^k$. For the Finsler metric $\tilde{F} := F_1$, the corresponding map $\tilde{\psi}$ satisfies $\tilde{\psi}^k = \mathrm{Id}$ since, as explained above, $\tilde{\psi}^k = \psi^k \circ \tau_{k\pi}$. q.e.d.

Notice that we can choose the vector field X such that it is arbitrary small, which implies that the metric $\tilde{F}$ can be chosen arbitrarily close

to F, and that Proposition 4 reduces the classification of Finsler metrics on S^n with constant flag curvature 1 to those for which all geodesics are closed.

In dimension 2 we can say more:

Proposition 5. *Let F be a Finsler metric on S^2 with constant flag curvature 1 and rotation angle λ, that, in addition, admits a nontrivial Killing vector field X. Assume that the flow generated by X is 2π-periodic. Then, the Zermelo deformation $F^*_\alpha(\xi) = F^*(\xi) + \alpha\xi(X)$ has the following properties:*

(a) *F_α is a Finsler metric for all $-2\lambda < \alpha < 2-2\lambda$.*
(b) *The rotation angle for F_α is $\lambda_\alpha = \min\{\lambda + \frac{\alpha}{2}, 1 - (\lambda + \frac{\alpha}{2})\}$.*
(c) *There exists a unique α such that F_α has all geodesics closed with the same prime length 2π.*

Proof. By Remark 1 we may assume that $X = \frac{\partial}{\partial\phi}$ in the coordinates (θ, ϕ) from Section 2. Since in the coordinates (θ, ϕ), the mapping ψ is given by (1) and X is proportional to $\frac{\partial}{\partial\phi}$ with a constant coefficient, the equator $\{\theta = 0\}$ is the fixed point set of an isometry and is therefore a reversible geodesic. Depending on the orientation, we denote this geodesic by $\gamma^\pm$.

In order to prove (a), consider the function on the cotangent bundle given by $\xi \mapsto \xi(X)$. Since X is a Killing vector field, it is an integral for the geodesic flow. Then, the differential of this function can be proportional to the differential of F only at the points $(x, \xi) \in TM$ (assuming $\xi \neq 0$) such that x lies over the geodesics $\gamma^\pm$. Indeed, otherwise the circle $\{\theta = \text{const} \neq 0\}$ is a geodesic, which is impossible since the points p and $\psi(p)$ lie on different sides of the equator $\{\theta = 0\}$.

Thus, the function $\xi \mapsto \xi(X)$ restricted to the unit cotangent bundle has zero differential only at the points corresponding to the geodesics $\gamma^\pm$. By direct calculation, we see that, at the points corresponding to γ^+, its value is $2-2\lambda$ and, at the points corresponding to γ^-, its value is 2λ. Here we use the convention that the geodesic γ^+ is oriented in the $\frac{\partial}{\partial\phi}$-direction, and γ^- in the $-\frac{\partial}{\partial\phi}$-direction. Thus, the function $\xi(X)$ has maximum $2-2\lambda$ and minimum -2λ. Hence F_α is positive for all $-2\lambda < \alpha < 2-2\lambda$ on nonzero vectors, which implies that F_α is a Finsler metric.

In order to prove (b), recall that $\psi_\alpha = \psi \circ \tau_{\alpha\pi}$. Thus ψ_α has the form (1) with λ replaced by $\lambda + \frac{\alpha}{2} \in (0, 1)$. If $\lambda + \frac{\alpha}{2} \leq \frac{1}{2}$, then it is the rotation angle. Otherwise we change the orientation of ϕ and obtain the rotation angle $1 - (\lambda + \frac{\alpha}{2})$.

Finally, for part (c) we simply choose $\alpha = 1-2\lambda$ and hence $\lambda_\alpha = \frac{1}{2}$.

q.e.d.

Thus any Finsler metric F on S^2 with constant flag curvature and irrational rotation angle λ can be viewed as a Zermelo deformation of

one for which all geodesics are closed with the same length and which, in addition, admits a Killing vector field. Meanwhile, we note that not all Finsler metrics on S^2 with constant flag curvature admit a non-trivial Killing field; in [5, 6], many examples were constructed with all geodesics closed of length 2π and finite isometry group.

As already mentioned above, if F is the round metric on S^2, then F_α is known as a Katok metric [17]. Note that in this case (S^2, F_α) and $(S^2, F_{-\alpha})$ are isometric, thus the family of Katok examples is parametrized by $\alpha \in [0, 1)$. Recall that the round metric has $\lambda = \frac{1}{2}$. Hence the rotation angle of the Katok metric F_α is $\lambda_\alpha = \frac{1}{2} - \frac{\alpha}{2}$; observe that it takes on each value in $(0, \frac{1}{2}]$ uniquely. This implies Corollary 1 in the Introduction.

5. The proof of Theorem 3

First observe that the π-shift along the geodesic flow coincides with the diffeomorphism of $\mathrm{U}(M)$ induced by ψ, which implies that it is an isometry of the natural metric on $\mathrm{U}(M)$ induced by the Binet-Legendre metric. But any isometry of a compact metric space has zero topological entropy.

Next, we prove that the geodesic flow of F, viewed now as the Hamiltonian system on the cotangent bundle to M without the zero section, which we denote by $T^*M \setminus \{0\}$, and equipped with the canonical symplectic form and whose Hamiltonian F^* is the Legendre transform of F with respect to $\frac{1}{2}F^2$, is Liouville integrable, i.e., that there exist n integrals that are functionally independent almost everywhere and are in involution.

As in the proof of Proposition 4, we consider the closure of the group generated by ψ in the group $\mathrm{Isom}(F)$. Its Id component is a torus T^k of dimension k, where we also allow $k = 0$. The action of the torus T^k on M induces a Hamiltonian action of T^k on the cotangent space T^*M. The generators of this action, which we denote by $\xi_1, ..., \xi_k$, are the vector fields that generate the action of T^k on M, viewed as functions on T^*M. We assume that the Hamiltonian flows of ξ_i are periodic with minimal period 1.

Because of compactness of the isometry group of M, there exists $\ell \in \mathbb{N}$ such that ψ^ℓ lies in the torus T^k. This implies that the Arnold-Poisson action of $\mathbb{R}^{k+1}$ on $T^*M \setminus \{0\}$ generated by the $k+1$ functions $\xi_1, ..., \xi_k, F^*$, is a Hamiltonian action of a $k+1$-dimensional torus T^{k+1}. The generators of this action are the functions $\xi_1, ..., \xi_k$ and an appropriate linear combination of F^* and $\xi_1, ..., \xi_k$ constructed arguing as in Section 4, see the proof of Proposition 4 there. Note that F^* is functionally independent from $\xi_1, ..., \xi_k$. Indeed, observe that in the proof of Proposition 4, we have shown that there exists a linear combination $\xi = c_1\xi_1 + ... + c_k\xi_k$ with constant coefficients such that the flows of the

Hamiltonian systems generated by ξ and by F^* satisfy $\phi_\pi^\xi = \phi_{\ell\pi}^{F^*}$. If F^* were functionally dependent on $\xi_1, ..., \xi_k$ in some open subset, then the above observation implies that, on this open set, the Hamiltonian vector field of F^* is proportional to a linear combination of $\xi_1, ..., \xi_k$ with constant coefficients, which is clearly impossible.

We denote by Reg the open dense set of regular points of the action, which is defined as the set of the points of $T^*M \setminus \{0\}$ where the differentials of the functions $\xi_1, ..., \xi_k, F^*$ are linearly independent. Next, consider the symplectic reduction of Reg with respect to T^{k+1}, which is a smooth symplectic manifold of dimension $2n-2k-2$. It is known that one can choose $n-k-1$ integrals on any symplectic manifold of dimension $2n-2k-2$, such that they are in involution and are functionally independent almost everywhere, see [**13**, p. 145] or [**4**]. The pullback of these functions to Reg gives us $n-k-1$ functions $I_{k+1}, ..., I_{n-1}$ on Reg, such that $\xi_1, ..., \xi_k, I_{k+1}, ..., I_{n-1}, F^*$ are in involution and functionally independent almost everywhere. Furthermore, examining the proofs of [**13**, p.145] or [**4**], we see that these functions can be extended to smooth functions on all of $T^*M \setminus \{0\}$. Hence the geodesic flow of F is Liouville-integrable.

6. The proof of Theorem 5

We first discuss a generalization of Theorem 5 for Finsler metrics that are not necessarily reversible. For this, we make the following definition. Let γ_1 and γ_2 be embedded closed geodesics on S^2 that do not intersect. They divide the sphere into three components, one of which is homeomorphic to $S^1 \times (0, 1)$. The closure of this component is an embedded cylinder $C \subset S^2$, $C \simeq S^1 \times [0, 1]$, whose boundary is the union of γ_1 and γ_2. We say that γ_1 and γ_2 *have the same orientation* if they are freely homotopic as oriented loops in C.

Proposition 6. *Let F be a Finsler metric on S^2 with positive flag curvature. If γ_1 and γ_2 are two closed geodesics that are geodesically reversible, or if they are embedded and have the same orientation, then they intersect.*

Proof. We first discuss the case when the closed geodesics γ_i are embedded with the same orientation. Let $\mathfrak{A}$ be the set of embedded closed curves in C that have the same orientation as γ_i. By Arzela-Ascoli, there exists an embedded closed curve α that realizes the minimum length among all curves in $\mathfrak{A}$; we assume that α is arc-length parametrized.

Let us show that either α coincides with one of the boundary geodesics or is itself a geodesic that lies in the interior of C. In order to do this, let us first observe that its parts lying inside the interior of C is a union of nonintersecting geodesic segments. Further observe that, at the points lying on the boundary of C, the curve α has a velocity vector,

and this velocity vector is tangent to the boundary, i.e., proportional to the velocity vector of the corresponding γ_i at this point. Indeed, were α to have a 'corner' at this point, one could make it shorter by 'shortcutting the corner' by a geodesic within C, which would contradict its minimality.

Suppose α is tangent at a point to one of the boundary geodesics, say, to the geodesic γ_1. Then α cannot be tangent to γ_1 in such a way that the velocity vectors are proportional at the point of tangency with a negative coefficient, since this would imply the existence of a self intersection of α, which contradicts the minimality. Thus, the velocity vectors of γ_1 and α are proportional at the point of tangency with a positive coefficient. Then α coincides with γ_1 because it is globally and therefore locally minimal.

Thus α is either a geodesic in the interior of C or agrees, including the orientation, with one of the boundary geodesics. Now, we can repeat the proof of Synge's theorem: There exists a smooth vector field along α, pointing into the interior of C if α is a boundary geodesic, and parallel in the Chern connection. From the second variation formula, it follows that there exist shorter curves in $\mathfrak{A}$, which is a contradiction.

The same proof works in the case where γ_i are geodesically reversible, but not necessarily embedded. Indeed, in that case, there exists a cylinder C in S^2 such that its boundary forms two closed piecewise smooth curves with each segment a geodesic segment of one of γ_i. Indeed, γ_1 has finitely many self-intersections, and it divides S^2 into several discs, one of which contains γ_2. Let β_1 be the boundary of that disc. Similarly, let β_2 be the boundary of the component of $S^2 \setminus \gamma_2$ that contains γ_1. Then C is the closure of the cylindrical component of $S^2 \setminus (\beta_1 \cup \beta_2)$ and the boundary of C is $\beta_1 \cup \beta_2$. Furthermore, at each vertex of $\beta_1 \cup \beta_2$ the two adjacent segments are convex towards the interior of C. We can then repeat the same argument as above using a shortest non-contractible loop $\alpha \subset C$ that is homotopic to a generator of the fundamental group of the interior of C. The first statement of Theorem 5 is now proved.

We now prove the second statement of Theorem 5 by constructing a Finsler metric on S^2 with two non intersecting closed geodesics.

Take a Katok metric $F = F_\lambda$ such that λ is irrational. In the spherical coordinates (θ, ϕ) constructed in Section 2, the closed geodesics $\gamma^\pm$ lie on the equator $\{\theta = 0\}$ and are, in fact, the only closed geodesics. Though the image of these geodesics coincide, the corresponding trajectories on the cotangent bundles, which we denote by $\bar{\gamma}^\pm \colon \mathbb{R} \to T^*S^2$, are disjoint. They are trajectories of the geodesic flow of the Hamiltonian $F^* \colon T^*S^2 \to \mathbb{R}$, and we may assume that they lie on the energy level $E := (F^*)^{-1}(1)$.

Let p_θ, p_ϕ be the canonical momenta corresponding to the coordinates θ, ϕ. Consider a function $H : T^*M \to \mathbb{R}$ such that, in a small

neighborhood of $\bar{\gamma}^+$, it coincides with the function p_θ, and, in a small neighborhood of $\bar{\gamma}^-$, it coincides with the function $-p_\theta$, and is smooth. Denote by Φ_t the Hamiltonian flow of the function H. Near the curves $\bar{\gamma}^\pm$, the Hamiltonian vector field X_H has coordinates

$$\left(\frac{\partial H}{\partial p_\theta}, \frac{\partial H}{\partial p_\phi}, -\frac{\partial H}{\partial \theta}, -\frac{\partial H}{\partial \phi}\right) = (\pm 1, 0, 0, 0),$$

which implies that, for small t and near the points of $\gamma^\pm$, the flow is given by $\Phi_t(\theta, \phi, p_\theta, p_\phi) = (\theta \pm t, \phi, p_\theta, p_\phi)$. This means that the projections of the curves $\Phi_t(\bar{\gamma}^\pm)$ to the sphere are the curves $\{(\theta, \phi) \mid \theta = \pm t\}$.

Let $F_t^* = F^* \circ \Phi_t$ be a new Hamiltonian function for a fixed small value of t. Since Φ_t preserves the symplectic form, the orbits of the Hamiltonian system generated by F_t^* are Φ_t applied to the orbits of the Hamiltonian flow of F^*. In particular, it has precisely two closed orbits, which are the images of $\bar{\gamma}^\pm$ under Φ_t. Next, we define $\bar{F}_t \colon T^*S^2 \to \mathbb{R}$ by the condition that it agrees with F_t^* on $\Phi_{-t}(E) = (F_t^*)^{-1}(1)$ and is extended to be homogeneous of degree 1. For small t, $\bar{F}_t^*$ is smooth on the cotangent bundle minus the zero section and strictly convex, so it defines a Finsler metric on S^2 with flag curvature close to 1 and therefore positive.

The orbits of $\bar{F}_t^*$ lying on $\Phi_{-t}(E)$ are, up to a reparametrization, the orbits of the flow generated by F_t^*, since the energy level 1 of $\bar{F}_t^*$ and of F_t^* coincide. Thus all the geodesics of $\bar{F}_t$ other than two are not closed, and these two are projected to the disjoint closed curves $\{(\phi, \theta) \mid \theta = \pm t\}$. q.e.d.

Remark 2. Notice that in the above example we can choose H such that the Finsler metric $\bar{F}_t$ has the same isometries as F, and hence admits a Killing vector field, or such that its isometry group is finite.

We note that recently similar examples appeared in [**25**], where the author constructs a positively curved Finsler metric on S^2 carrying a pair of simple closed geodesics that do not intersect. However, the construction does not guarantee that the metric has only two closed geodesics.

Let us briefly explain what the difficulty is in higher dimensions of proving an analogue of Frankel's theorem. For any Finsler metric F and any $v \in T_pM$, we have the Riemannian inner product $g^v(x, y) = D_\xi^2(F^2)_v(x, y)$ on T_pM, where D_ξ denotes the fiber derivative. We furthermore have the Chern connection D^v associated to F, that depends on a choice of v as well. If γ is a geodesic of F and X is a vector field along γ, then we have the second variation formula

$$E''(0) = \int_a^b g^v(D_v^v X, D_v^v X) - g^v(R^v(X, v)v, X)dt + g^v(D_X^v X, v)|_a^b,$$

where $v = \dot\gamma$. In the proof of Frankel's theorem, one constructs a parallel vector field X along a minimal geodesic connecting the disjoint totally geodesic submanifolds N_1 and N_2 that is normal to them at the endpoints. This defines a 1-parameter variation $c(t,s) = \exp_{\gamma(t)}(sX(t))$ of γ connecting N_1 and N_2. One would then like to use the second variation formula to conclude that $E''(0) < 0$ to obtain a contradiction. Although the integral is negative, in the Finsler case, the boundary term does not necessarily vanish, as it does in the Riemannian case, since $\eta(s) = c(a,s)$ is a geodesic, which implies that $\frac{D^w\dot\eta}{dt} = 0$ for $w = \dot\eta$, but we need $\frac{D^v\dot\eta}{dt} = 0$ for $v = \dot\gamma$ instead.

References

[1] H. Akbar-Zadeh, *Sur les espaces de Finsler à courbures sectionnelles constantes.* Acad. Roy. Belg. Bull. Cl. Sci. (5) **74** (1988), 281–322. MR1052466, Zbl 0686.53020.

[2] D. Bao, C. Robles, Z. Shen, *Zermelo navigation on Riemannian manifolds.* J. Diff. Geom. **66** (2004), 377–435. MR2106471, Zbl 1078.53073.

[3] A. V. Bolsinov, V. S. Matveev, A. T. Fomenko, *Two-dimensional Riemannian metrics with integrable geodesic flows. Local and global geometry.* Mat. Sb. **189** (1998), no. 10, 5–32. English translation in Sb. Math. **189** (1998), no. 9–10, 1441–1466. MR1691292, Zbl 0917.58031.

[4] A. V. Bolsinov, B. Jovanovic, *Noncommutative integrability, moment map and geodesic flows.* Ann. Global Anal. Geom. **23** (2003), no. 4, 305–322. MR1972543, Zbl 1022.37038.

[5] R. L. Bryant, *Finsler structures on the* 2*-sphere satisfying* $K = 1$. In: Finsler geometry (Seattle, WA, 1995), 27–41. Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996. MR1403574, Zbl 0864.53054.

[6] R. L. Bryant, *Projectively flat Finsler* 2*-spheres of constant curvature.* Sel. Math. **3** (1997), 161–203. MR1466165, Zbl 0897.53052.

[7] R. L. Bryant, *Some remarks on Finsler manifolds with constant flag curvature.* Houston J. Math. **28** (2002), 221–262. MR1898190, Zbl 1027.53086.

[8] R. L. Bryant, *Geodesically reversible Finsler* 2*-spheres of constant curvature.* In: Inspired by S. S. Chern—A Memorial Volume in Honor of a Great Mathematician, 95–111. Nankai Tracts in Mathematics **11**, edited by P. A. Griffiths, World Sci. Publ., Hackensack, NJ, 2006. MR2313331, Zbl 1155.53041.

[9] R. L. Bryant, L. Huang, X. Mo, *On Finsler surfaces of constant flag curvature with a Killing field.* Journal of Geometry and Physics **116** (2017), 345–357. MR3623665, Zbl 1373.58008.

[10] S.-S. Chern, Z. Shen, *Riemann-Finsler geometry.* Nankai Tracts in Mathematics **6** (2005). World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ. x+192 pp. MR2169595, Zbl 1085.53066.

[11] S. Deng, Z. Hou, *The group of isometries of a Finsler space.* Pacific J. Math. **207** (2002), 149–155. MR1974469, Zbl 1055.53055.

[12] T. Frankel, *Manifolds of positive curvature.* Pacific J. Math. **11** (1961), 165–174. MR0123272, Zbl 0107.39002.

[13] A. T. Fomenko, *Symplectic geometry.* Translated from the Russian by R. S. Wadhwa. Advanced Studies in Contemporary Mathematics **5** (1988). Gordon and Breach Science Publishers, New York, 1988. xiv+387 pp. MR0994805, Zbl 0716.53005.

[14] P. Foulon, *Ziller-Katok deformations of Finsler metrics.* 2004 International Symposium on Finsler Geometry, Tianjin, PRC, 2004, 22–24.

[15] P. Foulon, V. S. Matveev, *Zermelo deformation of Finsler metrics by Killing vector fields.* Electron. Res. Announc. Math. Sci. **25** (2018), 1–7. MR3808237.

[16] L. Huang, X. Mo, *On the flag curvature of a class of Finsler metrics produced by the navigation problem.* Pac. J. Math. **277** (2015), 149–168. MR3393685, Zbl 1322.58007.

[17] A. B. Katok, *Ergodic perturbations of degenerate integrable Hamiltonian systems.* Izv. Akad. Nauk SSSR. **37** (1973), 539–576. English translation in Math. USSR-Isv. **7** (1973), 535–571. MR0331425, Zbl 0316.58010.

[18] L. Kozma, I. R. Peter, *Intersection theorems for Finsler manifolds.* Publ. Math. Debrecen **57** (2000), 193–201. MR1771685, Zbl 1062.53064.

[19] V. S. Matveev, *Riemannian metrics having common geodesics with Berwald metrics.* Publ. Math. Debrecen **74** (2009), 405–416. MR2521384, Zbl 1199.53160.

[20] V. S. Matveev, H.-B. Rademacher, M. Troyanov, A. Zeghib, *Finsler conformal Lichnerowicz-Obata conjecture.* Annales de l'Institut Fourier **59** (2009), 937–949. MR2543657, Zbl 1179.53075.

[21] V. S. Matveev, M. Troyanov, *The Binet-Legendre metric in Finsler geometry.* Geom. Topol. **16** (2012), 2135–2170. MR3033515, Zbl 1258.53080.

[22] V. S. Matveev, M. Troyanov, *The Myers-Steenrod theorem for Finsler manifolds of low regularity.* Proc. Amer. Math. Soc. **145** (2017), 2699–2712. MR3626522, Zbl 1362.53083.

[23] W. Neumann, *3-dimensional G-manifolds with 2-dimensional orbits.* In: Proceedings of the Conference on Transformation Groups (New Orleans, La., 1967), 220–222. Springer, New York, 1968. MR0245043, Zbl 0177.52101.

[24] P. Orlik, *Seifert manifolds.* Lecture Notes in Mathematics **291**. Springer-Verlag, Berlin-New York, 1972. viii+155 pp. MR0426001, Zbl 0263.57001.

[25] H.-B. Rademacher, *Bumpy metrics on spheres and minimal index growth.* J. Fixed Point Theory Appl. **19** (2017), 289–298. MR3625073, Zbl 1378.53053.

[26] H.-B. Rademacher, *Nonreversible Finsler metrics of positive flag curvature.* In: A sampler of Riemann-Finsler geometry, 261–302. Math. Sci. Res. Inst. Publ. **50**, Cambridge Univ. Press, Cambridge, 2004. MR2132661, Zbl 1074.53062.

[27] H.-B. Rademacher, *Existence of closed geodesics on positively curved Finsler manifolds.* Ergodic Theory Dynam. Systems **27** (2007), 251–260. MR2322187, Zbl 1124.53018.

[28] F. Raymond, *Classification of the actions of the circle on 3-manifolds.* Trans. Amer. Math. Soc. **13** (1968), 51–78. MR0219086, Zbl 0157.30602.

[29] R. Schultz, *Nonlinear analogs of linear group actions on spheres.* Bull. Amer. Math. Soc. **11** (1984), 263–285. MR0752788, Zbl 0564.57001.

[30] Z. Shen, *Finsler manifolds of constant positive curvature.* In: Finsler geometry (Seattle, WA, 1995), 83–93. Contemp. Math. **196**, Amer. Math. Soc., Providence, RI, 1996. MR1403580, Zbl 0864.53057.

[31] Z. Shen, *Two-dimensional Finsler metrics with constant flag curvature.* Manuscripta Mathematica **109** (2002), 349–366. MR1948019, Zbl 1027.53093.

[32] W. Ziller, *Geometry of the Katok examples.* Ergodic Theory Dynam. Systems **3** (1982), 135–157. MR0743032, Zbl 0559.58027.

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