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TOPOLOGY OF TENSOR RANKS

PIERRE COMON, LEK-HENG LIM, YANG QI, AND KE YE

ABSTRACT. We study path-connectedness and homotopy groups of sets of tensors defined by tensor rank, border rank, multilinear rank, as well as their symmetric counterparts for symmetric tensors. We show that over \mathbb{C} , the set of rank- r tensors and the set of symmetric rank- r symmetric tensors are both path-connected if r is not more than the complex generic rank; these results also extend to border rank and symmetric border rank over \mathbb{C} . Over \mathbb{R} , the set of rank- r tensors is path-connected if it has the expected dimension but the corresponding result for symmetric rank- r symmetric d -tensors depends on the order d : connected when d is odd but not when d is even. Border rank and symmetric border rank over \mathbb{R} have essentially the same path-connectedness properties as rank and symmetric rank over \mathbb{R} . When r is greater than the complex generic rank, we are unable to discern any general pattern: For example, we show that border-rank-three tensors in $\mathbb{R}^2 \otimes \mathbb{R}^2 \otimes \mathbb{R}^2$ fall into four connected components. For multilinear rank, the manifold of d -tensors of multilinear rank (r_1, \dots, r_d) in $\mathbb{C}^{n_1} \otimes \dots \otimes \mathbb{C}^{n_d}$ is always path-connected, and the same is true in $\mathbb{R}^{n_1} \otimes \dots \otimes \mathbb{R}^{n_d}$ unless $n_i = r_i = \prod_{j \neq i} r_j$ for some $i \in \{1, \dots, d\}$. Beyond path-connectedness, we determine, over both \mathbb{R} and \mathbb{C} , the fundamental and higher homotopy groups of the set of tensors of a fixed small rank, and, taking advantage of Bott periodicity, those of the manifold of tensors of a fixed multilinear rank. We also obtain analogues of these results for symmetric tensors of a fixed symmetric rank or a fixed symmetric multilinear rank.

1. INTRODUCTION

Let V_1, \dots, V_d be vector spaces over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} and let $\mathbb{N}_0 := \{0, 1, 2, \dots\} = \mathbb{N} \cup \{0\}$ denote the set of nonnegative integers. For a d -tensor $A \in V_1 \otimes \dots \otimes V_d$, its *tensor rank* [29, 20, 33] is

$$(1.1) \quad \text{rank}(A) := \min \left\{ r \in \mathbb{N}_0 : A = \sum_{i=1}^r v_{1,i} \otimes \dots \otimes v_{d,i}, v_{j,i} \in V_j \right\},$$

and its *multilinear rank* [29, 20, 33] is the d -tuple

$$(1.2) \quad \mu\text{rank}(A) := \min \left\{ (r_1, \dots, r_d) \in \mathbb{N}_0^d : A \in W_1 \otimes \dots \otimes W_d, W_j \subseteq V_j, \dim_{\mathbb{F}}(W_j) = r_j \right\},$$

well-defined since the set on the right is a directed subset of \mathbb{N}_0^d . When $d = 2$, the multilinear rank in (1.2) reduces to row and column ranks of a matrix, which are of course equal to each other and to (1.1), the minimal number of rank-one summands required to decompose the matrix. Thus (1.2) and (1.1) are both generalizations of matrix rank although for $d \geq 3$, these numbers are in general all distinct.

For a symmetric d -tensor $A \in \mathcal{S}^d(V)$, there is also a corresponding notion of *symmetric tensor rank* [19, 33], given by

$$(1.3) \quad \text{rank}_{\mathcal{S}}(A) := \min \left\{ r \in \mathbb{N}_0 : A = \sum_{i=1}^r v_i^{\otimes d}, v_i \in V \right\},$$

and *symmetric multilinear rank*, given by

$$(1.4) \quad \mu\text{rank}_{\mathcal{S}}(A) := \min \left\{ r \in \mathbb{N}_0 : A \in \mathcal{S}^d(W), W \subseteq V, \dim_{\mathbb{F}}(W) = r \right\}.$$

It is now known that $\text{rank}(A) \neq \text{rank}_{\mathcal{S}}(A)$ in general [43] although it is easy to see that one always has $\mu\text{rank}(A) = (r, \dots, r)$ where $r = \mu\text{rank}_{\mathcal{S}}(A)$.

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When $d \geq 3$, the sets $\{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) \leq r\}$ and $\{A \in \mathbf{S}^d(V) : \text{rank}_{\mathcal{S}}(A) \leq r\}$ are in general not closed (whether in the Euclidean or Zariski topology) [33], giving rise to the notions of *border rank* and *symmetric border rank*

$$(1.5) \quad \overline{\text{rank}}(A) := \min\{r \in \mathbb{N}_0 : A \in \overline{\{B \in V_1 \otimes \cdots \otimes V_d : \text{rank}(B) \leq r\}}\},$$

$$(1.6) \quad \overline{\text{rank}}_{\mathcal{S}}(A) := \min\{r \in \mathbb{N}_0 : A \in \overline{\{B \in \mathbf{S}^d(V) : \text{rank}_{\mathcal{S}}(B) \leq r\}}\}.$$

The closures here are in the Euclidean topology. Although over \mathbb{C} , the Euclidean and Zariski topologies give the same closure for these sets [40, Theorem 2.33]. This ‘border rank’ phenomenon does not happen with multilinear rank and symmetric multilinear rank.

In this article we will study (i) path-connectedness, (ii) fundamental group, and (iii) higher homotopy groups of the sets:

$$\begin{aligned} \textcircled{1} \quad \{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) = r\}, & \quad \textcircled{2} \quad \{A \in \mathbf{S}^d(V) : \text{rank}_{\mathcal{S}}(A) = r\}, \\ \textcircled{3} \quad \{A \in V_1 \otimes \cdots \otimes V_d : \overline{\text{rank}}(A) = r\}, & \quad \textcircled{4} \quad \{A \in \mathbf{S}^d(V) : \overline{\text{rank}}_{\mathcal{S}}(A) = r\}, \\ \textcircled{5} \quad \{A \in V_1 \otimes \cdots \otimes V_d : \mu\text{rank}(A) = (r_1, \dots, r_d)\}, & \quad \textcircled{6} \quad \{A \in \mathbf{S}^d(V) : \mu\text{rank}_{\mathcal{S}}(A) = r\}, \end{aligned}$$

for arbitrary $d \geq 3$ and for a vast range of (although not all) values of r and (r_1, \dots, r_d) . These topological properties will in general depend on whether the vector spaces involved are over \mathbb{R} or \mathbb{C} and the two cases will often require different treatments. $\textcircled{1}$ and $\textcircled{2}$ are semialgebraic sets; $\textcircled{3}$ and $\textcircled{4}$ are locally closed semialgebraic sets; $\textcircled{5}$ and $\textcircled{6}$ are smooth manifolds. One common feature of $\textcircled{1}$ – $\textcircled{6}$ is that they all contain a nonempty Euclidean open subset of their closures, implying that each of these sets has the same dimension as its closure.

Throughout this article, ‘rank- r ’ will mean ‘rank exactly r ’ and likewise for ‘border-rank r ,’ ‘symmetric rank- r ,’ ‘multilinear rank- (r_1, \dots, r_d) ,’ etc. Statements such as ‘path-connectedness of border rank’ or ‘homotopy groups of symmetric multilinear rank’ will be understood to mean (respectively) path-connectedness of the set in $\textcircled{3}$ or homotopy groups of the set in $\textcircled{6}$.

Outline. Our results for the three topological properties of the six notions of tensor ranks over two base fields are too lengthy to reproduce in the introduction. Instead we provide Table 1 to serve as a road map to these results. As is evident, one notable omission is the homotopy groups of border ranks, which accounts for the empty cells in the table. The reason is that the approaches we used to obtain homotopy groups for ranks do not directly apply to border ranks (e.g., Proposition 5.2 does not have a counterpart for border rank) because of the more subtle geometry of border ranks and at this point we are unable to go beyond path-connectedness for border ranks.

TABLE 1. Road map to results.

	Connectedness	Fundamental group	Higher homotopy
X -rank over \mathbb{C}	Thm 3.7	Prop 5.2	Prop 5.2
border X -rank over \mathbb{C}	Thm 3.1		
rank over \mathbb{C}	Cor 3.8	Thm 6.1	Thm 6.2, Thm 6.3
rank over \mathbb{R}	Thm 4.7, Cor 4.8	Thm 6.4	Thm 6.5, Thm 6.6
border rank over \mathbb{C}	Cor 3.2		
border rank over \mathbb{R}	Thm 4.7, Cor 4.8		
symmetric rank over \mathbb{C}	Cor 3.8	Thm 7.1	Thm 7.2, Thm 7.3
symmetric rank over \mathbb{R}	Thm 4.4	Thm 7.4	Thm 7.5, Thm 7.6
symmetric border rank over \mathbb{C}	Cor 3.3		
symmetric border rank over \mathbb{R}	Thm 4.5		
multilinear rank over \mathbb{C}	Thm 8.3	Thm 8.5	Thm 8.5
multilinear rank over \mathbb{R}	Thm 8.2	Thm 8.4	Thm 8.4
symmetric multilinear rank over \mathbb{C}	Thm 9.3	Thm 9.5	Thm 9.5
symmetric multilinear rank over \mathbb{R}	Thm 9.2	Thm 9.4	Thm 9.4

Coordinates. All notions of rank in this article, and in particular the tensor ranks (1.1)–(1.6), are independent of bases, i.e., they are indeed defined on the respective tensor spaces — $V_1 \otimes \cdots \otimes V_d$ or $S^d(V)$ where V_1, \dots, V_d and V are \mathbb{F} -vector spaces. We will therefore state our results in this article in a coordinate-free manner. Nevertheless some practitioners tend to view tensors in terms of *hypermatrices*, i.e., d -dimensional matrices that are coordinate representations of tensors with respect to some choices of bases. These are usually denoted

$$\mathbb{F}^{n_1 \times \cdots \times n_d} := \{(a_{i_1 \dots i_d}) : a_{i_1 \dots i_d} \in \mathbb{F}, 1 \leq k_1 \leq n_k, k = 1, \dots, d\}.$$

All results in this article may be applied to hypermatrices by choosing bases and setting $V_1 = \mathbb{F}^{n_1}, \dots, V_d = \mathbb{F}^{n_d}$, with $n_i = \dim_{\mathbb{F}}(V_i)$, and identifying tensors with hypermatrices:

$$\mathbb{F}^{n_1} \otimes \cdots \otimes \mathbb{F}^{n_d} = \mathbb{F}^{n_1 \times \cdots \times n_d},$$

or symmetric tensors with symmetric hypermatrices

$$S^d(\mathbb{F}^n) = \{(a_{i_1 \dots i_d}) \in \mathbb{F}^{n \times \cdots \times n} : a_{i_{\sigma(1)} \dots i_{\sigma(d)}} = a_{i_1 \dots i_d} \text{ for all } \sigma \in \mathfrak{S}_d\}.$$

Note that when we said the sets ①–⑥ have semialgebraic, locally closed, or manifold structures, these statements are coordinate independent.

Application impetus. The primary goal of this article is to better understand the topological properties of various tensor ranks, an aspect that has been somewhat neglected in existing studies. However, the results on path-connectedness and simple-connectedness of tensor rank, multilinear rank, and their symmetric counterparts have useful practical implications.

One of the most basic and common problems involving tensors in applications is to find low-rank approximations [20] with respect to one of these notions of rank: Given $A \in V_1 \otimes \cdots \otimes V_d$ and $r \in \mathbb{N}$ or $(r_1, \dots, r_d) \in \mathbb{N}^d$, find a best rank- r or best multilinear rank- (r_1, \dots, r_d) approximation:

$$\inf_{\text{rank}(B) \leq r} \|A - B\| \quad \text{or} \quad \inf_{\mu\text{rank}(B) \leq (r_1, \dots, r_d)} \|A - B\|;$$

or, given $A \in S^d(V)$ and $r \in \mathbb{N}$, find the best symmetric rank- r approximation or best symmetric multilinear rank- r approximation:

$$\inf_{\text{rank}_S(B) \leq r} \|A - B\| \quad \text{or} \quad \inf_{\mu\text{rank}_S(B) \leq r} \|A - B\|.$$

Riemannian manifold optimization techniques [21, 2] were first used for the best multilinear rank approximations of tensors and symmetric tensors in [22, 42]. Numerous variants have appeared since, mostly dealing with different objective functions, e.g., for the so-called ‘tensor completion’ problems. In one of these works [31], the authors considered approximation by tensors of a *fixed* multilinear rank, i.e.,

$$X_{r_1, \dots, r_d}(V_1, \dots, V_d) := \{A \in V_1 \otimes \cdots \otimes V_d : \mu\text{rank}(A) = (r_1, \dots, r_d)\},$$

as opposed to those *not more than* a fixed multilinear rank, i.e.,

$$\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d) := \{A \in V_1 \otimes \cdots \otimes V_d : \mu\text{rank}(A) \leq (r_1, \dots, r_d)\}.$$

The advantages of using $\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d)$, called a *subspace variety*, are well-known: The set is topologically well-behaved, e.g., closed in the Euclidean (and Zariski) topology and therefore guaranteeing the existence of a best approximation [20]; connected in the Euclidean (and Zariski) topology and therefore ensuring that path-following optimization methods that start from any initial point could in principle arrive at the optimizer [33]. However $\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d)$ suffers from one defect — it is not a smooth manifold, e.g., any point in $\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d)$ with multilinear rank strictly less than (r_1, \dots, r_d) is singular, and this prevents the use of Riemannian optimization techniques. With this in mind, the authors of [31] formulated their optimization problem over $X_{r_1, \dots, r_d}(V_1, \dots, V_d)$, which is a smooth Riemannian manifold [47]. But this raises the question of whether $X_{r_1, \dots, r_d}(V_1, \dots, V_d)$ is path-connected. If not, then the path-following algorithms in [31] that begin from an initial point in one component will never converge to an optimizer located in

another. For example, when $d = 2$, it is well-known that the set of $n \times n$ real matrices of rank n has two components given by the sign of the determinant but that the set of $n \times n$ complex matrices of rank n is connected. More generally, the set of $n_1 \times n_2$ real matrices of rank r is connected unless $n_1 = n_2 = r$ [36, 48].

Homotopy continuation techniques [5] have also made a recent appearance [28] in tensor decomposition problems over \mathbb{C} . In general, a tensor of a given rank may have several rank decompositions and such techniques have the advantage of being able to find all decompositions with high probability. The basic idea is that for a given general complex rank- r tensor $A \in W_1 \otimes \cdots \otimes W_d$ with a known rank- r decomposition, one may construct a random loop $\tau: [0, 1] \rightarrow W_1 \otimes \cdots \otimes W_d$ with $\tau(0) = \tau(1) = A$, the endpoint of this loop gives a rank- r decomposition of A , repeat this process a considerable number of times by choosing random loops, and one may expect to obtain all rank- r decompositions. The consideration of loops naturally leads us to questions of simple-connectedness.

We expect our results on the path-connectedness of sets of d -tensors of various ranks to be useful to practitioners applying Riemannian optimization algorithms to tensor approximations problems by allowing them to ascertain if the case they are interested in is path-connected or not. Likewise, we expect our simply-connectedness results to be useful to practitioners applying homotopy continuation methods.

2. X -RANK, TENSOR RANK, SYMMETRIC RANK, AND BORDER RANK

Our results in this section are relatively straightforward to state but their proofs will be technical and require an algebraic geometric view of tensor rank. We start by providing some relevant background in Section 2.1. Even those already conversant with the standard treatment of these materials may nevertheless benefit from going over Section 2.1 because of the subtleties that arise when one switches between \mathbb{R} and \mathbb{C} . The standard treatment, say as in [25, 33], invariably assumes that everything is carried out over \mathbb{C} .

2.1. Rank and border rank. Let V be a finite-dimensional real vector space, and $W = V \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Let $\mathbb{P}W$ be the corresponding projective space¹ with quotient map

$$(2.1) \quad p: W \setminus \{0\} \rightarrow \mathbb{P}W, \quad v \mapsto [v],$$

where $[v]$ denotes the projective equivalence class of $v \in W \setminus \{0\}$. For any subset $X \subseteq \mathbb{P}W$, the *affine cone* over X is the set $\widehat{X} := p^{-1}(X) \cup \{0\}$. Note that $\widehat{X} \subseteq W$. A complex projective variety $X \subseteq \mathbb{P}W$ is called *nondegenerate* if X is not contained in any hyperplane, and X is called *irreducible* if it is not a union of two proper subvarieties. If X is defined by homogeneous polynomials with real coefficients, then $X(\mathbb{R})$, the set of *real points* of X , is the zero locus of these polynomials in $\mathbb{P}V$. In fact, $X(\mathbb{R}) = X \cap \mathbb{P}V$. If $X \subseteq \mathbb{P}W$ is an irreducible nondegenerate projective variety defined by real homogeneous polynomials, then $X(\mathbb{R})$ is Zariski dense in X if and only if X has a nonsingular real point [7, 44].

Let $s_r(X) := \text{im}(s_r)$ be the image of the morphism²

$$(2.2) \quad s_r: (\widehat{X} \setminus \{0\})^r \rightarrow W, \quad (x_1, \dots, x_r) \mapsto x_1 + \cdots + x_r.$$

The r th *secant variety* $\sigma_r(X)$ is the projective subvariety whose affine cone is the Zariski closure of $s_r(X)$. Henceforth we will write $\overline{s_r(X)} := \overline{s_r(X)}$ for the Euclidean closure of $s_r(X)$ and $\widehat{\sigma}_r(X) := \widehat{\sigma_r(X)}$ for the affine cone of $\sigma_r(X)$. For a complex irreducible projective variety X ,

$$\overline{s_r(X)} = \widehat{\sigma}_r(X).$$

¹We will also write $\mathbb{R}\mathbb{P}^n$ and $\mathbb{C}\mathbb{P}^n$ for $\mathbb{P}(\mathbb{R}^n)$ and $\mathbb{P}(\mathbb{C}^n)$ respectively.

²As usual, throughout this article, a set raised to a power r denotes the set theoretic product of r copies of the set. So $(\widehat{X} \setminus \{0\})^r := \widehat{X} \setminus \{0\} \times \cdots \times \widehat{X} \setminus \{0\}$ (r copies).

Let $x \in W$. We say that x has X -rank r if $x \in s_r(X) \setminus s_{r-1}(X)$; in notation, $\text{rank}_X(x) = r$. We say that x has X -border rank r if $x \in \overline{s_r}(X) \setminus \overline{s_{r-1}}(X)$; in notation, $\overline{\text{rank}}_X(x) = r$. In summary,

$$\begin{aligned} s_r(X) &= \{x \in W : \text{rank}_X(x) \leq r\}, & s_r(X) \setminus s_{r-1}(X) &= \{x \in W : \text{rank}_X(x) = r\}, \\ \overline{s_r}(X) &= \{x \in W : \overline{\text{rank}}_X(x) \leq r\}, & \overline{s_r}(X) \setminus \overline{s_{r-1}}(X) &= \{x \in W : \overline{\text{rank}}_X(x) = r\}. \end{aligned}$$

Let $\mathcal{A}(\mathbb{P}W)$ denote the set of all complex projective varieties $X \subseteq \mathbb{P}W$ that are (i) irreducible, (ii) nondegenerate, (iii) defined by real homogeneous polynomials, and (iv) whose real points $X(\mathbb{R})$ are Zariski dense. Given $X \in \mathcal{A}(\mathbb{P}W)$, consider the real analogue of the map in (2.2),

$$s_r : (\widehat{X}(\mathbb{R}) \setminus \{0\})^r \rightarrow V, \quad (x_1, \dots, x_r) \mapsto x_1 + \dots + x_r,$$

also denoted s_r by a slight abuse of notation. It follows from [41, 14] that

$$\sigma_r(X(\mathbb{R})) = (\sigma_r(X))(\mathbb{R}).$$

Thus if $X \in \mathcal{A}(\mathbb{P}W)$, then $\sigma_r(X) \in \mathcal{A}(\mathbb{P}W)$. However, $\overline{s_r}(X(\mathbb{R}))$ may not be equal to $\widehat{\sigma_r}(X(\mathbb{R}))$.

An important point to note is that the values of X -rank and border X -rank depend on the choice of base field. For $x \in V$, it is entirely possible [19, 20, 37] that

$$\text{rank}_X(x) \neq \text{rank}_{X(\mathbb{R})}(x) \quad \text{or} \quad \overline{\text{rank}}_X(x) \neq \overline{\text{rank}}_{X(\mathbb{R})}(x).$$

As such we will have to treat the real and complex cases separately.

The smallest r so that $\overline{s_r}(X) = W$, or equivalently, $\sigma_r(X) = \mathbb{P}W$, is called *complex generic X -rank*, and is denoted by $r_g(X)$. Note that the notion of generic rank is only defined over \mathbb{C} . If $s_r(X(\mathbb{R})) \setminus s_{r-1}(X(\mathbb{R}))$ contains a Euclidean open subset of V , then r is called a *typical X -rank*. Note that the notion of typical rank is only defined over \mathbb{R} . The two notions are related in that the complex generic X -rank $r_g(X)$ is the smallest typical X -rank [7].

2.2. Secant, Segre, and Veronese varieties. Our discussions will be framed in terms an arbitrary variety $X \in \mathcal{A}(\mathbb{P}W)$ for greatest generality. However, when we apply these results to tensor rank, the variety in question is the *Segre variety* $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$, the manifold of projective equivalence classes of rank-one d -tensors, where each W_i is the complexification of some real vector space V_i , with $W = W_1 \otimes \dots \otimes W_d$ and $V = V_1 \otimes \dots \otimes V_d$. In this case, $X(\mathbb{R}) = \text{Seg}(\mathbb{P}V_1 \times \dots \times \mathbb{P}V_d)$, which is Zariski dense in $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$. Similarly, when we apply these results to symmetric tensor rank, the variety in question is the *Veronese variety* $X = \nu_d(\mathbb{P}U)$, the manifold of projective equivalence classes of symmetric rank-one d -tensors, where U is the complexification of some real vector space T , with $W = \mathbb{S}^d(U)$ and $V = \mathbb{S}^d(T)$. In this case, $X(\mathbb{R}) = \nu_d(\mathbb{P}T)$, which is Zariski dense in $X = \nu_d(\mathbb{P}U)$.

When $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$, we write

$$\text{rank}(A) = \text{rank}_{\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)}(A) \quad \text{and} \quad \overline{\text{rank}}(A) = \overline{\text{rank}}_{\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)}(A)$$

for the tensor rank and border rank of a tensor $A \in W_1 \otimes \dots \otimes W_d$. When $X = \nu_d(\mathbb{P}U)$, we write

$$\text{ranks}(A) = \text{rank}_{\nu_d(\mathbb{P}U)}(A) \quad \text{and} \quad \overline{\text{ranks}}(A) = \overline{\text{rank}}_{\nu_d(\mathbb{P}U)}(A)$$

for the symmetric tensor rank and symmetric border rank of a symmetric tensor $A \in \mathbb{S}^d(U)$.

Note that if, say, W_1 is one-dimensional, then $W_1 \otimes W_2 \otimes \dots \otimes W_d \cong W_2 \otimes \dots \otimes W_d$. So for $W_1 \otimes \dots \otimes W_d$ to be faithfully a space of order- d tensors, the dimensions of W_1, \dots, W_d must all be at least two. Throughout this article, we will assume that all vector spaces that appear in tensor product spaces such as $W_1 \otimes \dots \otimes W_d$ or $\mathbb{S}^d(U)$ are of dimensions at least two. The same assumption will apply to real vector spaces as well for the same reason.

3. PATH-CONNECTEDNESS OF COMPLEX TENSOR RANKS

The notions of path-connectedness and connectedness are equivalent for all spaces that we will consider in this article and henceforth, except in section headings, we will write “connected” or “connectedness” for brevity. We start by establishing the connectedness of border X -rank over \mathbb{C} , which is a straightforward consequence of the following fact [3]: For any complex irreducible nondegenerate projective variety $X \subsetneq \mathbb{P}W$, we have a strict inclusion $\sigma_{r-1}(X) \subsetneq \sigma_r(X)$ whenever $r \leq r_g(X)$. By [40, Corollary 4.16], $\sigma_r(X) \setminus \sigma_{r-1}(X)$ is connected. Given any nonempty subset $S \subseteq \mathbb{P}W$, let

$$(3.1) \quad \mathcal{O}_S^\circ(-1) = \{(x, v) \in \mathbb{P}W \times W : x \in S, v \in \widehat{x} \setminus \{0\}\}$$

be a fiber bundle³ over S . Let $p_1: \mathcal{O}_S^\circ(-1) \rightarrow \mathbb{P}W$ and $p_2: \mathcal{O}_S^\circ(-1) \rightarrow W$ be the projections onto the first and second factor respectively. For any $x \in S$, the fiber $p_1^{-1}(x) = \widehat{x} \setminus \{0\} \cong \mathbb{C} \setminus \{0\}$ is connected. So if S is connected, $p_1^{-1}(S)$ is connected, which implies $p_2(p_1^{-1}(S))$ is connected. In our case, $S = \sigma_r(X) \setminus \sigma_{r-1}(X)$. Hence $p_2(p_1^{-1}(S)) = \widehat{\sigma}_r(X) \setminus \widehat{\sigma}_{r-1}(X)$ is connected, or, in other words, the set of border X -rank- r points,

$$\{x \in W : \overline{\text{rank}}_X(x) = r\} = \widehat{\sigma}_r(X) \setminus \widehat{\sigma}_{r-1}(X),$$

is connected. We state this formally below.

Theorem 3.1 (Connectedness of X -border rank- r points). *Let W be a complex vector space and $X \subsetneq \mathbb{P}W$ be any complex irreducible nondegenerate projective variety. If $r \leq r_g(X)$, then the set $\{x \in W : \overline{\text{rank}}_X(x) = r\}$ is a connected set.*

Let W_1, \dots, W_d and W be finite-dimensional complex vector spaces. Applying Theorem 3.1 to the special cases $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$ and $X = \nu_d(\mathbb{P}W)$, we obtain the connectedness of tensor border rank and symmetric border rank over \mathbb{C} .

Corollary 3.2 (Connectedness of border rank- r complex tensors). Let r be not more than the complex generic tensor rank. The set of border rank- r complex tensors

$$\{A \in W_1 \otimes \dots \otimes W_d : \overline{\text{rank}}(A) = r\}$$

is a connected set.

Corollary 3.3 (Connectedness of symmetric border rank- r complex symmetric tensors). Let r be not more than the complex generic symmetric rank. The set of symmetric border rank- r complex symmetric tensors

$$\{A \in S^d(W) : \overline{\text{rank}}_S(A) = r\}$$

is a connected set.

We next move on to the connectedness of X -rank (as opposed to border X -rank) over \mathbb{C} . For the following discussions, one should bear in mind that every complex variety is naturally a real semi-algebraic set; and every complex nonsingular variety of complex dimension n is a complex smooth manifold of complex dimension n , which is naturally a real smooth manifold of real dimension $2n$. Throughout this article, whenever we refer to the k th homotopy group of a semialgebraic set X , we mean the k th topological homotopy group of X under its Euclidean topology.

In the sequel, we write $\text{codim}_{\mathbb{R}}(X, M)$ for the real codimension of an appropriate subset X of an ambient manifold M if both are defined over \mathbb{R} ; and similarly $\text{codim}_{\mathbb{C}}(X, M)$ for the complex codimension if both are defined over \mathbb{C} . Recall the following well-known fact.

Theorem 3.4. *If M is a smooth manifold and X is a union of finitely many embedded submanifolds of M with $\text{codim}_{\mathbb{R}}(X, M) \geq n$, then $\pi_k(M) \cong \pi_k(M \setminus X)$ for all $k = 0, \dots, n - 2$.*

³Note that $\mathcal{O}_S^\circ(-1)$ differs from the tautological line bundle $\mathcal{O}_S(-1) = \{(x, v) \in \mathbb{P}W \times W : x \in S, v \in \widehat{x}\}$ in that its fiber over $x \in S$ is $\widehat{x} \setminus \{0\}$ and not \widehat{x} .

By [9, Proposition 2.9.10], any semialgebraic subset $X \subsetneq \mathbb{R}^m$ is a disjoint union of finitely many submanifolds of \mathbb{R}^m . This yields the following corollary of Theorem 3.4, which will be an important tool for us.

Theorem 3.5. *If M is a smooth manifold and X is a semialgebraic subset of M of real codimension $\text{codim}_{\mathbb{R}}(X, M) \geq n$, then $\pi_k(M) \cong \pi_k(M \setminus X)$ for $k = 0, \dots, n - 2$.*

Another standard fact that we will use repeatedly is the following well-known result [27], stated here for easy reference.

Theorem 3.6. *Let $F \rightarrow E \xrightarrow{p} B$ be a fiber bundle and B be connected. For any $x \in F$, $b = p(x)$, there is a long exact sequence*

$$\cdots \rightarrow \pi_{i+1}(F, x) \rightarrow \pi_{i+1}(E, x) \xrightarrow{p_*} \pi_{i+1}(B, b) \rightarrow \pi_i(F, x) \rightarrow \cdots \rightarrow \pi_0(E, x) \rightarrow 0.$$

Let $X \subsetneq \mathbb{P}W$ be a complex irreducible nondegenerate nonsingular projective variety. When $r \leq r_g(X)$, the aforementioned fact that $s_{r-1}(X) \subsetneq s_r(X)$ implies that the complex codimension of $s_{r-1}(X)$ in $s_r(X)$ is at least one. So the preimage $s_r^{-1}(s_{r-1}(X))$ has complex codimension at least one in $(\widehat{X} \setminus \{0\})^r$, i.e., the real codimension of $s_r^{-1}(s_{r-1}(X))$ in $(\widehat{X} \setminus \{0\})^r$ is at least two. Let $\mathcal{O}_X^{\circ}(-1)$ be the bundle in (3.1) with $S = X$. Let $p_1: \mathcal{O}_X^{\circ}(-1) \rightarrow \mathbb{P}W$ and $p_2: \mathcal{O}_X^{\circ}(-1) \rightarrow W$ be the projections. For any $x \in X$, the fiber $p_1^{-1}(x) = \widehat{x} \setminus \{0\} \cong \mathbb{C} \setminus \{0\}$ is connected. Since X is irreducible, X is connected. Thus $p_1^{-1}(X)$ is connected, which implies $\widehat{X} \setminus \{0\} = p_2(p_1^{-1}(X))$ is connected. By Theorem 3.5, the semialgebraic subset

$$(\widehat{X} \setminus \{0\})^r \setminus s_r^{-1}(s_{r-1}(X))$$

is connected. Therefore $s_r(X) \setminus s_{r-1}(X)$ is also connected, being the image of a connected set under a continuous map. We have thus deduced the connectedness of complex X -rank.

Theorem 3.7 (Connectedness of X -rank- r points). *Let W be a complex vector space and $X \subsetneq \mathbb{P}W$ be any complex irreducible nondegenerate projective variety. If $r \leq r_g(X)$, then the set $\{x \in W : \text{rank}_X(x) = r\}$ is a connected set.*

Let W_1, \dots, W_d and W be finite-dimensional complex vector spaces. Applying Theorem 3.7 to the special cases $X = \text{Seg}(\mathbb{P}W_1 \times \cdots \times \mathbb{P}W_d)$ and $X = \nu_d(\mathbb{P}W)$, we obtain the connectedness of tensor rank and symmetric tensor rank over \mathbb{C} .

Corollary 3.8 (Connectedness of rank- r complex tensors). (i) Let r be not more than the complex generic tensor rank. The set of rank- r complex tensors

$$\{A \in W_1 \otimes \cdots \otimes W_d : \text{rank}(A) = r\}$$

is a connected set.

(ii) Let r be not more than the complex generic symmetric rank. The set of symmetric rank- r complex symmetric tensors

$$\{A \in S^d(W) : \text{rank}_S(A) = r\}$$

is a connected set.

4. PATH-CONNECTEDNESS OF REAL TENSOR RANKS

We will now establish results similar to those in Section 3 but over \mathbb{R} ; these will however require quite different techniques. The marked difference between real tensor rank and complex tensor rank will not come as too much of a surprise to those familiar with tensor rank, which depends very much on the base field.

Let W be a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . Let $X \subseteq \mathbb{P}W$ be an irreducible nondegenerate nonsingular projective variety. In particular $\widehat{X} \setminus \{0\}$ is naturally a smooth \mathbb{F} -manifold. As usual,

we will denote the tangent space of a smooth manifold M at a nonsingular point $x \in M$ by $\mathbb{T}_x M$. Let x_1, \dots, x_{r-1} be general points in $\widehat{X} \setminus \{0\}$. We define

$$(4.1) \quad Z := s_r^{-1}(s_{r-1}(X)) \quad \text{and} \quad Y := \{x \in \widehat{X} : (x_1, \dots, x_{r-1}, x) \in Z\}.$$

Pick a general $x_r \in Y$. Since $s_r : Z \rightarrow s_{r-1}(X)$ is surjective [in an open neighborhood of \$\(x_1, \dots, x_r\)\$, which is in \$Z\$](#) , its differential

$$s_{r*} : \mathbb{T}_{(x_1, \dots, x_r)} Z \rightarrow \mathbb{T}_{x_1 + \dots + x_r} s_{r-1}(X)$$

is also surjective.

Because x_1, \dots, x_{r-1} are general in \widehat{X} ,

$$\mathbb{T}_{x_1 + \dots + x_{r-1}} s_{r-1}(X) = \mathbb{T}_{x_1} \widehat{X} + \dots + \mathbb{T}_{x_{r-1}} \widehat{X}$$

by the semialgebraic Terracini's lemma [41, Lemma 12]. On the other hand,

$$\begin{aligned} \mathbb{T}_{x_1 + \dots + x_r} s_{r-1}(X) &= s_{r*}(\mathbb{T}_{(x_1, \dots, x_r)} Z) \\ &= s_{r*}(\mathbb{T}_{x_1} \widehat{X} \oplus \dots \oplus \mathbb{T}_{x_{r-1}} \widehat{X} \oplus \mathbb{T}_{x_r} Y) \\ &= \mathbb{T}_{x_1} \widehat{X} + \dots + \mathbb{T}_{x_{r-1}} \widehat{X} + \mathbb{T}_{x_r} Y \\ &\supseteq \mathbb{T}_{x_1} \widehat{X} + \dots + \mathbb{T}_{x_{r-1}} \widehat{X} \\ &= \mathbb{T}_{x_1 + \dots + x_{r-1}} s_{r-1}(X), \end{aligned}$$

which, by a dimension count, implies that

$$(4.2) \quad \mathbb{T}_{x_r} Y \subseteq \mathbb{T}_{x_1} \widehat{X} + \dots + \mathbb{T}_{x_{r-1}} \widehat{X}.$$

Let $\dim_{\mathbb{F}}(X) := n - 1$ and the codimension of Y in \widehat{X} , $\text{codim}_{\mathbb{F}}(Y, \widehat{X}) := k$. Then

$$\text{codim}_{\mathbb{F}}(Z, (\widehat{X} \setminus \{0\})^r) = k,$$

and (4.2) implies

$$(4.3) \quad \dim_{\mathbb{F}}(\mathbb{T}_{x_1} \widehat{X} + \dots + \mathbb{T}_{x_r} \widehat{X}) \leq k + \dim_{\mathbb{F}}(\mathbb{T}_{x_1} \widehat{X} + \dots + \mathbb{T}_{x_{r-1}} \widehat{X}).$$

To establish the connectedness of tensor rank and symmetric tensor rank over \mathbb{R} , we will need (4.3) and the following notion of defectivity.

Definition 4.1. Let W be a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and $X \subsetneq \mathbb{P}W$ be an irreducible projective variety of dimension $m - 1$. We say that X is *not r -defective* if

$$\dim_{\mathbb{F}}(\sigma_r(X)) = \min\{rm - 1, \dim_{\mathbb{F}}(W) - 1\}$$

and r -defective otherwise.

We will address the connectedness of symmetric tensor rank over \mathbb{R} before addressing that of (nonsymmetric) tensor rank over \mathbb{R} as we have more detailed results for the former. The reason being that our approach requires knowledge of r -defectivity. For symmetric tensors, the r -defectivity of $\sigma_r(\nu_d(\mathbb{P}U))$ is completely known due to the work of Alexander and Hirschowitz but for nonsymmetric tensors, the r -defectivity of $\sigma_r(\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d))$ has not been completely determined.

4.1. Path-connectedness of real symmetric tensor rank and real symmetric border rank.

Let W be the complexification of a real vector space V . Recall that if $X = \nu_d(\mathbb{P}W)$, then $X(\mathbb{R}) = \nu_d(\mathbb{P}V)$. We first address the symmetric rank-one case, i.e., the connectedness of $\widehat{X}(\mathbb{R}) \setminus \{0\}$, and later generalize it to arbitrary symmetric rank.

Proposition 4.2. Let V be a real vector space.

(i) When d is odd, the set of symmetric rank-one real symmetric tensors

$$\{A \in \mathbb{S}^d(V) : \text{rank}_{\mathbb{S}}(A) = 1\},$$

is a connected set.

(ii) When d is even, the set of symmetric rank-one real symmetric tensors

$$\{A \in \mathbb{S}^d(V) : \text{rank}_{\mathbb{S}}(A) = 1\}$$

has two connected components.

Proof. Let $\dim_{\mathbb{R}}(V) = n$, and $X(\mathbb{R}) = \nu_d(\mathbb{P}V)$. Fix a basis $\{e_1, \dots, e_n\}$ and a norm $\|\cdot\|$ for V . Let $\{e_1^*, \dots, e_n^*\}$ be the dual basis of V^* .

(i) Let $u, v \in V$ and $\lambda, \mu \in \mathbb{R}$ with $\|u\| = \|v\| = 1$ and $\lambda, \mu \neq 0$. As d is odd, the signs of λ and μ can be absorbed into u and v respectively, so we may assume that $\lambda > 0, \mu > 0$. Since $u, v \in \mathbb{S}^{n-1}$, the connectedness of \mathbb{S}^{n-1} implies the existence of a curve $\beta(t)$ on \mathbb{S}^{n-1} connecting u and v . Then $\gamma : [0, 1] \rightarrow \mathbb{S}^d(V)$,

$$\gamma(t) := (t\mu + (1-t)\lambda) \cdot \beta(t)^{\otimes d},$$

is curve of constant rank connecting $\lambda u^{\otimes d}$ and $\mu v^{\otimes d}$.

(ii) Consider the map

$$\varphi: \widehat{X}(\mathbb{R}) \setminus \{0\} \rightarrow \mathbb{R}, \quad A \mapsto (e_1^*)^{\otimes d}(A) + \dots + (e_n^*)^{\otimes d}(A).$$

Given any symmetric rank-one tensor A , since d is even, $\varphi(A) \neq 0$. Therefore $\widehat{X}(\mathbb{R}) \setminus \{0\}$ is a disjoint union of $\varphi^{-1}((-\infty, 0))$ and $\varphi^{-1}((0, +\infty))$; we will show these two sets are connected, which implies the set of symmetric rank-one real tensors has two connected components. **First observe that** for any nonzero $u \in V$, $\varphi(u^{\otimes d}) > 0$ as d is even. Thus if $A \in \varphi^{-1}((0, +\infty))$, then A is of the form $\lambda u^{\otimes d}$ for some $u \neq 0$ and $\lambda > 0$. If $A \in \varphi^{-1}((-\infty, 0))$, then A is of the form $\lambda u^{\otimes d}$ for some $u \neq 0$ and $\lambda < 0$. **Hence we may identify $\varphi^{-1}((0, +\infty))$ with $\varphi^{-1}((-\infty, 0))$ by the map $\lambda u^{\otimes d} \mapsto -\lambda u^{\otimes d}$ and it suffices to demonstrate connectedness of $\varphi^{-1}((0, +\infty))$, which is given by the same curve constructed in the proof of (i). \square**

A celebrated result due to Alexander and Hirschowitz [4] (see also [12] for a simplified proof) shows that if $r < \binom{n+d-1}{d}/n$, then $X = \nu_d(\mathbb{P}W)$ is not r -defective. Since $\sigma_r(X) \in \mathcal{A}(\mathbb{P}\mathbb{S}^d(W))$, $X(\mathbb{R})$ is not r -defective either. This allows us to deduce the following about $Z = s_r^{-1}(s_{r-1}(X(\mathbb{R})))$ in (4.1).

Proposition 4.3. Let $n > 2$ and $r < \binom{n+d-1}{d}/n$. Then

$$\text{codim}_{\mathbb{R}}(s_r^{-1}(s_{r-1}(X)(\mathbb{R})), (\widehat{X}(\mathbb{R}) \setminus \{0\})^r) > 1.$$

Proof. In fact we will show that $\text{codim}_{\mathbb{R}}(s_r^{-1}(\widehat{\sigma}_{r-1}(X)(\mathbb{R})), (\widehat{X}(\mathbb{R}) \setminus \{0\})^r) > 1$, which clearly implies the required result. Suppose not, then $s_r^{-1}(\widehat{\sigma}_{r-1}(X)(\mathbb{R}))$ is a hypersurface. For given general $v_1, \dots, v_{r-1} \in V$, the set Y in (4.1) takes form

$$Y = \{v \in V : v_1^{\otimes d} + \dots + v_{r-1}^{\otimes d} + v^{\otimes d} \in \widehat{\sigma}_{r-1}(X)(\mathbb{R})\},$$

which is an affine variety. If $s_r^{-1}(\widehat{\sigma}_{r-1}(X)(\mathbb{R}))$ is a hypersurface, then Y is a hypersurface⁴ in V , and therefore defined by the vanishing of a single real homogeneous polynomial h . Let $Y(\mathbb{C}) \subseteq W$ be the complex hypersurface defined by h . Since $r < r_g(X)$, and v_1, \dots, v_{r-1} are general, $Y(\mathbb{C})$ is contained in

$$\widetilde{Y} := \{v \in W : v_1^{\otimes d} + \dots + v_{r-1}^{\otimes d} + v^{\otimes d} \in \widehat{\sigma}_{r-1}(X)\},$$

⁴Note that $Y \neq V$, since otherwise $\dim s_r^{-1}(\widehat{\sigma}_{r-1}(X)(\mathbb{R})) = r \dim V$ as v_1, \dots, v_{r-1} are general, implying that $s_r^{-1}(\widehat{\sigma}_{r-1}(X)(\mathbb{R}))$ is not a hypersurface.

and thus \tilde{Y} must have codimension at most one. We will see that this leads to a contradiction.

Given a nonzero vector $w \in W$, let

$$\mathfrak{m}_{[w]} := \left\{ f \in \bigoplus_{k=0}^{\infty} \mathbb{S}^k(W^*) : f(w) = 0 \right\}$$

be the maximal ideal of $[w] \in \mathbb{P}W$, the point corresponding to w in projective space. Recall [6, 26] that a scheme is called a *double point* if it is defined by the ideal $\mathfrak{m}_{[w]}^2$ for some w , and we denote such a double point by $[w]^2$.

For a vector subspace $Q \subseteq \mathbb{S}^d(W)$, its dual space is given by

$$Q^\perp := \{f \in \mathbb{S}^d(W^*) : f(u) = 0 \text{ for all } u \in Q\}.$$

A classical result [35] stated in modern language says that

$$(\mathbb{T}_{[v^{\otimes d}]}\hat{X})^\perp = \mathbb{S}^d(W^*) \cap \mathfrak{m}_{[v]}^2.$$

Let $\mathcal{C} = \{[v_1]^2, \dots, [v_r]^2\}$ be a set of double points. Then by Terracini's lemma [45], the degree- d piece of the ideal of \mathcal{C} , denoted by $I_{\mathcal{C}}(d)$, equals $(\mathbb{T}_{[v_1^{\otimes d}]}\hat{X} + \dots + \mathbb{T}_{[v_r^{\otimes d}]}\hat{X})^\perp$. Thus

$$\text{codim}_{\mathbb{C}}(I_{\mathcal{C}}(d), \mathbb{S}^d(W^*)) = \dim_{\mathbb{C}}(\mathbb{T}_{[v_1^{\otimes d}]}\hat{X} + \dots + \mathbb{T}_{[v_r^{\otimes d}]}\hat{X}).$$

The codimension $\text{codim}_{\mathbb{C}}(I_{\mathcal{C}}(d), \mathbb{S}^d(W^*))$ is in fact the *Hilbert function* of \mathcal{C} evaluated at d , and is denoted by $h_{\mathbb{P}W}(\mathcal{C}, d)$. The result of Alexander and Hirschowitz [4] then implies that for $r < \binom{n+d-1}{d}/n$ general double points, we have $h_{\mathbb{P}W}(\mathcal{C}, d) = nr$. In our case, since $[v_1]^2, \dots, [v_{r-1}]^2$ are general, and v_r is on a hypersurface \tilde{Y} , we get that

$$(4.4) \quad h_{\mathbb{P}W}(\mathcal{C}, d) = \deg(\mathcal{C}) = n(r-1) + \deg([v_r]^2) \geq n(r-1) + (n-1).$$

By (4.3), we obtain

$$\begin{aligned} h_{\mathbb{P}W}(\mathcal{C}, d) &= \dim_{\mathbb{C}}(\mathbb{T}_{[v_1^{\otimes d}]}\hat{X} + \dots + \mathbb{T}_{[v_r^{\otimes d}]}\hat{X}) \\ &\leq \text{codim}_{\mathbb{C}}(\tilde{Y}, W) + \dim_{\mathbb{C}}(\mathbb{T}_{[v_1^{\otimes d}]}\hat{X} + \dots + \mathbb{T}_{[v_{r-1}^{\otimes d}]}\hat{X}) \leq 1 + n(r-1), \end{aligned}$$

which contradicts (4.4). □

We are in a position to address the connectedness of symmetric tensor rank over \mathbb{R} .

Theorem 4.4 (Connectedness of symmetric rank- r real symmetric tensors). *Let V be a real vector space of dimension $n > 2$ and $r < \binom{n+d-1}{d}/n$.*

(i) *When d is odd, the set of symmetric rank- r real tensors*

$$\{A \in \mathbb{S}^d(V) : \text{rank}_{\mathbb{S}}(A) = r\}$$

is a connected set.

(ii) *When d is even, the set of symmetric rank- r real tensors*

$$\{A \in \mathbb{S}^d(V) : \text{rank}_{\mathbb{S}}(A) = r\}$$

has $r+1$ connected components.

Proof. (i) Since the set of symmetric rank- r tensors is the image of $(\hat{X}(\mathbb{R}) \setminus \{0\})^r \setminus s_r^{-1}(s_{r-1}(X)(\mathbb{R}))$ under the continuous map s_r , it suffices to show that this set is connected. Now note that the required connectedness follows from Proposition 4.3 and Theorem 3.5.

(ii) For each $i \in \{0, \dots, r\}$, let

$$P_i := \{A \in S^d(V) : A = v_1^{\otimes d} + \dots + v_i^{\otimes d} - v_{i+1}^{\otimes d} - \dots - v_r^{\otimes d}, \text{rank}_S(A) = r\}.$$

Note that the pair of numbers⁵ $(i, r-i)$ associated to P_i is $\text{GL}(V)$ -invariant. Hence $P_i \cap P_j = \emptyset$ when $i \neq j$. For each $i \in \{0, \dots, r\}$, define the map Σ_i by

$$\Sigma_i : (V \setminus \{0\})^r \rightarrow S^r(V), \quad (v_1, \dots, v_r) \mapsto v_1^{\otimes d} + \dots + v_i^{\otimes d} - v_{i+1}^{\otimes d} - \dots - v_r^{\otimes d}.$$

Let $D_r := \{A \in S^r(V) : \text{rank}_S(A) < r\}$. By a similar argument of Proposition 4.3,

$$\text{codim}_{\mathbb{R}}(\Sigma_i^{-1}(\text{im}(\Sigma_i) \cap D_r), (V \setminus \{0\})^i) > 1.$$

Thus by Theorem 3.5 and the fact that the image of a connected set under a continuous map is connected, P_i is connected. Since

$$\{A \in S^d(V) : \text{rank}_S(A) = r\} = \bigcup_{i=0}^r P_i,$$

the set $\{A \in S^d(V) : \text{rank}_S(A) = r\}$ has $r + 1$ connected components. □

Since for any symmetric border rank- r tensor $B \in S^d(V)$, there is a continuous curve $\gamma : [0, 1] \rightarrow S^d(V)$ with $\gamma(0) = B$ and $\gamma(t) \subseteq \{A \in S^d(V) : \text{rank}_S(A) = r\}$ for $t \in (0, 1]$, we obtain the border rank analogue of Theorem 4.4.

Theorem 4.5 (Connectedness of symmetric border rank- r real symmetric tensors). *Let V be a real vector space of dimension $n > 2$ and $r < \binom{n+d-1}{d}/n$.*

(i) *When d is odd, the set of symmetric border rank- r real tensors*

$$\{A \in S^d(V) : \overline{\text{rank}}_S(A) = r\}$$

is a connected set.

(ii) *When d is even, the set of symmetric border rank- r real tensors*

$$\{A \in S^d(V) : \overline{\text{rank}}_S(A) = r\}$$

has $r + 1$ connected components.

4.2. Path-connectedness of real tensor rank and real border rank. We next turn our attention to [tensors that are not necessarily symmetric](#), i.e., $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$ and $X(\mathbb{R}) = \text{Seg}(\mathbb{P}V_1 \times \dots \times \mathbb{P}V_d)$. As in the case of symmetric tensors, we first address the rank-one case, i.e., the connectedness of $\widehat{X}(\mathbb{R}) \setminus \{0\}$, and later generalize it to arbitrary rank. Note that the set of rank-one tensors and the set of border rank-one tensors are equal.

Proposition 4.6. The set of rank-one real tensors

$$\{A \in V_1 \otimes \dots \otimes V_d : \text{rank}(A) = 1\} = \{A \in V_1 \otimes \dots \otimes V_d : \overline{\text{rank}}(A) = 1\}$$

is connected.

Proof. The idea is the same as in the proof of Proposition 4.2(i). Upon absorbing signs and scaling, it suffices to show that for any $A = \lambda u_1 \otimes \dots \otimes u_d$ and $B = \mu v_1 \otimes \dots \otimes v_d$ with $\lambda, \mu > 0$ and $\|u_j\| = \|v_j\| = 1$, $j = 1, \dots, d$, there exists a curve $\gamma(t)$ in $V_1 \otimes \dots \otimes V_d$ connecting A and B such that $\gamma(t)$ has rank one for any $t \in [0, 1]$. For each $j = 1, \dots, d$, we pick a curve $\tau_j(t)$ on the unit sphere $S^{n_j-1} \subseteq V_j$ connecting u_j and v_j , where $n_j = \dim V_j$. Then the curve $\gamma : [0, 1] \rightarrow V_1 \otimes \dots \otimes V_d$ defined by

$$\gamma(t) = (t\lambda + (1-t)\mu) \cdot \tau_1(t) \otimes \dots \otimes \tau_d(t),$$

has the desired property. □

⁵When $d = 2$, the set P_i may be regarded as $n \times n$ symmetric matrices of rank r and signature $(i, r-i)$. Here we may view these pairs of numbers as a generalization of *signature* to arbitrary $d \geq 3$.

Now we address the connectedness of the set of rank- r tensors and the set of border-rank- r tensors. Here the condition that X is not r -defective in the symmetric case can be slightly weakened and replaced by a condition on the codimension plus the requirement that $r < r_g(X)$.

Theorem 4.7 (Connectedness of rank- r and border-rank- r real tensors). *Let V_1, \dots, V_d be real vector spaces of real dimensions n_1, \dots, n_d , where $2 \leq n_1 \leq \dots \leq n_d$. Let r be strictly smaller than the complex generic rank. If*

$$(4.5) \quad \text{codim}_{\mathbb{C}}(\sigma_{r-1}(X), \sigma_r(X)) > n_1 + \dots + n_{d-1} - d + 2,$$

then the set of real rank- r tensors

$$\{A \in V_1 \otimes \dots \otimes V_d : \text{rank}(A) = r\}$$

and the set of real border rank- r tensors

$$\{A \in V_1 \otimes \dots \otimes V_d : \overline{\text{rank}}(A) = r\}$$

are connected sets. Equivalently, in coordinates, the following sets of hypermatrices are connected:

$$\{A \in \mathbb{R}^{n_1 \times \dots \times n_d} : \text{rank}(A) = r\} \quad \text{and} \quad \{A \in \mathbb{R}^{n_1 \times \dots \times n_d} : \overline{\text{rank}}(A) = r\}.$$

Proof. As in the proofs of Theorems 4.4 and 4.5, it suffices to show that

$$\text{codim}_{\mathbb{R}}(s_r^{-1}(\widehat{\sigma}_{r-1}(X)(\mathbb{R})), (\widehat{X}(\mathbb{R}) \setminus \{0\})^r) > 1.$$

Suppose not. Let $x_1, \dots, x_{r-1} \in \widehat{X}(\mathbb{R})$ be general points and $v_1 \in V_1, \dots, v_{d-1} \in V_{d-1}$ be general vectors. We set

$$Y := \{v \in V_d : v_1 \otimes \dots \otimes v_{d-1} \otimes v + x_1 + \dots + x_{r-1} \in \widehat{\sigma}_{r-1}(X)(\mathbb{R})\}.$$

As in the proof of Proposition 4.3, we have $\text{codim}_{\mathbb{R}}(Y, V_d) = 1$. Choose a general $v_d \in Y$ and a general $v \in V_d$. Let $x_r = v_1 \otimes \dots \otimes v_d$ and $x = v_1 \otimes \dots \otimes v_{d-1} \otimes v$. Since the vector space $v_1 \otimes \dots \otimes v_{d-1} \otimes V_d$ is contained in both $\mathbb{T}_{x_r} \widehat{X}(\mathbb{R})$ and $\mathbb{T}_x \widehat{X}(\mathbb{R})$, by (4.3), we get

$$\begin{aligned} \dim_{\mathbb{R}}(\widehat{\sigma}_r(X)(\mathbb{R})) &= \dim_{\mathbb{R}}(\mathbb{T}_{x_1} \widehat{X}(\mathbb{R}) + \dots + \mathbb{T}_{x_{r-1}} \widehat{X}(\mathbb{R}) + \mathbb{T}_x \widehat{X}(\mathbb{R})) \\ &\leq \dim_{\mathbb{R}}(\mathbb{T}_{x_1} \widehat{X}(\mathbb{R}) + \dots + \mathbb{T}_{x_r} \widehat{X}(\mathbb{R})) + (n_1 + \dots + n_{d-1} - d + 1) \\ &\leq 1 + \dim_{\mathbb{R}}(\mathbb{T}_{x_1} \widehat{X}(\mathbb{R}) + \dots + \mathbb{T}_{x_{r-1}} \widehat{X}(\mathbb{R})) + (n_1 + \dots + n_{d-1} - d + 1) \\ &= 1 + \dim_{\mathbb{R}}(\widehat{\sigma}_{r-1}(X)(\mathbb{R})) + (n_1 + \dots + n_{d-1} - d + 1), \end{aligned}$$

which contradicts the assumption that $\text{codim}_{\mathbb{C}}(\sigma_{r-1}(X), \sigma_r(X)) > n_1 + \dots + n_{d-1} - d + 2$ as $\dim_{\mathbb{R}}(\widehat{\sigma}_j(X)(\mathbb{R})) = \dim_{\mathbb{C}}(\widehat{\sigma}_j(X))$ for all $j = 1, \dots, r_g(X)$. \square

Note that the condition on codimension (4.5) in Theorem 4.7 is guaranteed whenever $\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$ is not r -defective, i.e.,

$$\dim_{\mathbb{C}}(\sigma_r(\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d))) = \dim_{\mathbb{C}}(\sigma_{r-1}(\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d))) + n_1 + \dots + n_d - d + 1.$$

Corollary 4.8 (Connectedness of rank- r and border-rank- r real tensors). Let W_1, \dots, W_d be complexifications of the real vector spaces V_1, \dots, V_d . If $\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$ is not r -defective, then the sets

$$\{A \in V_1 \otimes \dots \otimes V_d : \text{rank}(A) = r\} \quad \text{and} \quad \{A \in V_1 \otimes \dots \otimes V_d : \overline{\text{rank}}(A) = r\}$$

are connected sets.

We would like to point out that determining r -defectivity of $\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$, or more generally, the dimension of $\sigma_r(\text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d))$ is a problem that has not been completely resolved (unlike the case of symmetric tensors, where the r -defectivity of $\nu_d(\mathbb{P}U)$ is completely known thanks to the work of Alexander and Hirschowitz). However, there has been remarkable progress in recent years [1, 8, 17, 18] and we know the dimensions (and therefore r -defectivity) in

many cases. In particular, when $n_d > 3$, all known cases satisfy condition (4.5) of Theorem 4.7. It is possible that the condition (4.5) is always satisfied and may be dropped from the theorem.

We conclude this section by showing that the condition $r < r_g(X)$ cannot be omitted. The reason being that when $r \geq r_g(X)$, we have $\dim s_r(X(\mathbb{R})) = \dim s_{r+1}(X(\mathbb{R}))$, and the set of real (border) rank- r points may have several connected components. We illustrate this with a specific example.

Proposition 4.9. The set of real border rank-three $2 \times 2 \times 2$ hypermatrices, i.e.,

$$\{A \in \mathbb{R}^{2 \times 2 \times 2} : \overline{\text{rank}}(A) = 3\},$$

has four connected components.

Proof. In fact, this result is not coordinate dependent and we will give a coordinate-free proof. Let U, V, W be real two-dimensional vector spaces. Pick any bases $\{u_1, u_2\}$ on U , $\{v_1, v_2\}$ on V , and $\{w_1, w_2\}$ on W . It is known [20] that the space $U \otimes V \otimes W$ has the set of border rank-three tensors $\{A \in U \otimes V \otimes W : \overline{\text{rank}}(A) = 3\}$ is the orbit of

$$B = u_1 \otimes v_1 \otimes w_1 + u_2 \otimes v_2 \otimes w_1 - u_1 \otimes v_2 \otimes w_2 + u_2 \otimes v_1 \otimes w_2$$

under the action of the group $G = \text{GL}(U) \times \text{GL}(V) \times \text{GL}(W)$. For $(g_1, g_2, g_3) \in G$ and $A \in U \otimes V \otimes W$, we write $(g_1, g_2, g_3) \cdot A$ for the action of (g_1, g_2, g_3) on A .

Let H be the stabilizer of B in G . Let H_0 be the connected component of H containing the identity element. The Lie algebra \mathfrak{h} of H_0 takes the form

$$\mathfrak{h} = \left\{ \left(\begin{bmatrix} \alpha_1 & -\alpha_2 \\ \alpha_2 & \alpha_1 \end{bmatrix}, \begin{bmatrix} \beta_1 & -\beta_2 \\ \beta_2 & \beta_1 \end{bmatrix}, \begin{bmatrix} \gamma_1 & -\gamma_2 \\ \gamma_2 & \gamma_1 \end{bmatrix} \right) \in \mathfrak{gl}(U) \oplus \mathfrak{gl}(V) \oplus \mathfrak{gl}(W) : \right. \\ \left. \alpha_1 + \beta_1 + \gamma_1 = \alpha_2 - \beta_2 - \gamma_2 = 0 \right\}.$$

Taking the exponential map, any $(g_1, g_2, g_3) \in H_0$ is then of the form

$$\left(\begin{bmatrix} e^{\alpha_1} \cos \alpha_2 & -e^{\alpha_1} \sin \alpha_2 \\ e^{\alpha_1} \sin \alpha_2 & e^{\alpha_1} \cos \alpha_2 \end{bmatrix}, \begin{bmatrix} e^{\beta_1} \cos \beta_2 & -e^{\beta_1} \sin \beta_2 \\ e^{\beta_1} \sin \beta_2 & e^{\beta_1} \cos \beta_2 \end{bmatrix}, \begin{bmatrix} e^{\gamma_1} \cos \gamma_2 & -e^{\gamma_1} \sin \gamma_2 \\ e^{\gamma_1} \sin \gamma_2 & e^{\gamma_1} \cos \gamma_2 \end{bmatrix} \right),$$

where $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 - \beta_2 - \gamma_2 = 0$. An argument similar to [23, Lemma 2.1] shows that H is contained in $N_G(H_0)$, the normalizer of H_0 . In fact any $(g_1, g_2, g_3) \in N_G(H_0)$ is of the form

$$\left(\begin{bmatrix} \pm\eta_1 & 0 \\ 0 & \eta_1 \end{bmatrix} h_1, \begin{bmatrix} \pm\eta_2 & 0 \\ 0 & \eta_2 \end{bmatrix} h_2, \begin{bmatrix} \pm\eta_3 & 0 \\ 0 & \eta_3 \end{bmatrix} h_3 \right),$$

where $(h_1, h_2, h_3) \in H_0$, and $\eta_1 \eta_2 \eta_3 \neq 0$. If $(g_1, g_2, g_3) \in H$, then $\eta_1 \eta_2 \eta_3 = \pm 1$. Thus any $(g_1, g_2, g_3) \in H$ takes one of the following eight forms:

$$\begin{aligned} & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} h_1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} h_2, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} h_3 \right), \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} h_1, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} h_2, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} h_3 \right), \\ & \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} h_1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} h_2, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} h_3 \right), \left(\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} h_1, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} h_2, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} h_3 \right), \\ & \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} h_1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} h_2, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} h_3 \right), \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} h_1, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} h_2, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} h_3 \right), \\ & \left(\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} h_1, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} h_2, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} h_3 \right), \left(\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} h_1, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} h_2, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} h_3 \right), \end{aligned}$$

where $(h_1, h_2, h_3) \in H_0$. For any $(h_1, h_2, h_3) \in H_0$, we have $\det(h_i) > 0$ for $i = 1, 2, 3$, and so for any $(g_1, g_2, g_3) \in H$, we have either $\det(g_i) > 0$ or $\det(g_i) < 0$ for all $i = 1, 2, 3$.

Therefore $S = G/H$ has the following four connected components:

$$\begin{aligned} & \{(g_1, g_2, g_3) \cdot B : \det(g_1) \det(g_2) > 0, \det(g_1) \det(g_3) > 0, \det(g_2) \det(g_3) > 0\}, \\ & \{(g_1, g_2, g_3) \cdot B : \det(g_1) \det(g_2) > 0, \det(g_1) \det(g_3) < 0, \det(g_2) \det(g_3) < 0\}, \\ & \{(g_1, g_2, g_3) \cdot B : \det(g_1) \det(g_2) < 0, \det(g_1) \det(g_3) > 0, \det(g_2) \det(g_3) < 0\}, \\ & \{(g_1, g_2, g_3) \cdot B : \det(g_1) \det(g_2) < 0, \det(g_1) \det(g_3) < 0, \det(g_2) \det(g_3) > 0\}. \end{aligned} \quad \square$$

5. HIGHER-ORDER CONNECTEDNESS OF X -RANK

In general it is difficult to compute the fundamental and higher homotopy groups of $s_r(X)$, the set of X -rank- r points. We will instead compute it for an open dense subset of identifiable points, defined as follows.

Definition 5.1. Let W be a finite-dimensional vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and $X \subsetneq \mathbb{P}W$ be an irreducible nondegenerate nonsingular projective variety. Here a X -rank- r point is called *identifiable* if it has a unique X -rank- r decomposition. We say that X is *r -identifiable* if a general point of $s_r(X)$ has a unique X -rank- r decomposition.

More precisely, if X is r -identifiable, then $s_r(X)$ contains an open dense subset $\mathcal{U} \subseteq s_r(X)$ such that any point in \mathcal{U} has a unique X -rank- r decomposition, i.e., a general point is simply a point in \mathcal{U} . This also makes precise the intuitive interpretation of “ X is r -identifiable” as “almost all points in $s_r(X)$ admits a unique X -rank- r decomposition.”

We will first need to define the set of points to be excluded from consideration. Let

$$(5.1) \quad D_r := \{x \in s_r(X) : \text{rank}(x) < r \text{ or } x \text{ has non-unique rank-}r \text{ decompositions}\}.$$

The next result gives the fundamental and higher homotopy groups of $s_r(X) \setminus D_r$ under some mild conditions.

Proposition 5.2. Let X be r -identifiable over \mathbb{F} and

$$(5.2) \quad c := \text{codim}_{\mathbb{R}}(s_r^{-1}(D_r), (\widehat{X} \setminus \{0\})^r) > 2,$$

Then

$$\pi_k(s_r(X) \setminus D_r) \cong \begin{cases} \pi_1(\widehat{X} \setminus \{0\})^r \rtimes \mathfrak{S}_r & \text{if } k = 1, \\ \pi_k(\widehat{X} \setminus \{0\})^r & \text{if } c \geq 4 \text{ and } 2 \leq k \leq c - 2. \end{cases}$$

Here the semidirect product \rtimes is given by the action of the symmetric group \mathfrak{S}_r on $\pi_1(\widehat{X} \setminus \{0\})^r$ as permutations.

Proof. Recall that s_r also denotes the map in (2.2). Slightly abusing notation, we will also use s_r to denote the restriction of s_r on $(\widehat{X} \setminus \{0\})^r \setminus s_r^{-1}(D_r)$.

Since \mathfrak{S}_r acts on $(\widehat{X} \setminus \{0\})^r$ as Deck transformations and

$$s_r : (\widehat{X} \setminus \{0\})^r \setminus s_r^{-1}(D_r) \rightarrow s_r(X) \setminus D_r$$

gives an $r!$ -fold normal covering space of $s_r(X) \setminus D_r$. Therefore the quotient group equals

$$(5.3) \quad \pi_1(s_r(X) \setminus D_r) / \pi_1((\widehat{X} \setminus \{0\})^r \setminus s_r^{-1}(D_r)) = \mathfrak{S}_r.$$

If X is r -identifiable and the codimension condition is satisfied, then by Theorem 3.5,

$$\pi_1((\widehat{X} \setminus \{0\})^r \setminus s_r^{-1}(D_r)) \cong \pi_1((\widehat{X} \setminus \{0\})^r) \cong \pi_1(\widehat{X} \setminus \{0\})^r,$$

and by (5.3),

$$\pi_1(s_r(X) \setminus D_r) \cong \pi_1(\widehat{X} \setminus \{0\})^r \rtimes \mathfrak{S}_r,$$

the semidirect product of $\pi_1(\widehat{X} \setminus \{0\})^r$ and \mathfrak{S}_r .

If $c \geq 4$ and $2 \leq k \leq c - 2$, the isomorphism between $\pi_k(s_r(X) \setminus D_r)$ and $\pi_k(\widehat{X} \setminus \{0\})^r$ follows from Theorem 3.5 and the fact that the k -sphere \mathbb{S}^k is simply connected when $k \geq 2$, which implies that every map from \mathbb{S}^k to $s_r(X) \setminus D_r$ can be lifted to $(\widehat{X} \setminus \{0\})^r \setminus s_r^{-1}(D_r)$, by the lifting property of covering spaces. \square

Proposition 5.2 and, as we will soon see, most of the results in Section 6, will depend on identifiability. For the case of Segre variety, i.e., $X = \text{Seg}(\mathbb{P}W_1 \times \cdots \times \mathbb{P}W_d)$, the best known result for identifiability is the Kruskal uniqueness theorem [32]. Here we will present another result of this nature that holds for more general X . Under generic identifiability condition, it tells us that there is a close relation between the nonsingularity of a point and the uniqueness of its X -rank- r decomposition. When applied to tensors, this will allow us to use the same line of arguments in [18] to find more identifiable points than those given by the Kruskal uniqueness theorem.

Proposition 5.3. Let X be r -identifiable over $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . If $x = x_1 + \cdots + x_r \in s_r(X)$ is a nonsingular point of $\widehat{\sigma}_r(X)$ and $\dim_{\mathbb{F}}(\mathbb{T}_{x_1}\widehat{X} + \cdots + \mathbb{T}_{x_r}\widehat{X}) = r \dim_{\mathbb{F}}\widehat{X}$, then x has a unique X -rank- r decomposition.

Proof. Since X is r -identifiable, $\dim_{\mathbb{F}}\widehat{\sigma}_r(X) = r \dim_{\mathbb{F}}\widehat{X}$. Since x is nonsingular, by [3, Corollary 1.8], $x = x_1 + \cdots + x_r$ has X -rank r . Since $\dim_{\mathbb{F}}(\mathbb{T}_{x_1}\widehat{X} + \cdots + \mathbb{T}_{x_r}\widehat{X}) = r \dim_{\mathbb{F}}\widehat{X} = \dim_{\mathbb{F}}\widehat{\sigma}_r(X)$, we have

$$s_{r*}(\mathbb{T}_{x_1}\widehat{X} \oplus \cdots \oplus \mathbb{T}_{x_r}\widehat{X}) = \mathbb{T}_{x_1}\widehat{X} + \cdots + \mathbb{T}_{x_r}\widehat{X} = \mathbb{T}_x\widehat{\sigma}_r(X),$$

and so the linear map s_{r*} has full rank at (x_1, \dots, x_r) . Therefore, for each $x_i \in \widehat{X}$, there is an open ball $B(x_i, \varepsilon_i) \subseteq \widehat{X}$ such that the restricted map

$$s_r|_{B(x_1, \varepsilon_1) \times \cdots \times B(x_r, \varepsilon_r)}$$

is a local diffeomorphism. Suppose $x_1 + \cdots + x_r = y_1 + \cdots + y_r$ for some $y_1, \dots, y_r \in \widehat{X}$, and $\{x_1, \dots, x_r\} \neq \{y_1, \dots, y_r\}$. By [41, Lemma 1], for each y_i , there is an open ball $B(y_i, \delta_i)$ such that the image $s_r(B(y_1, \delta_1) \times \cdots \times B(y_r, \delta_r))$ contains a nonempty open subset $\mathcal{U} \subseteq \widehat{\sigma}_r(X)$. By the curve selection lemma [39, Chapter 3], x is an accumulation point of \mathcal{U} . Since $s_r(B(x_1, \varepsilon_1) \times \cdots \times B(x_r, \varepsilon_r))$ is open and contains x as an interior point,

$$s_r(B(x_1, \varepsilon_1) \times \cdots \times B(x_r, \varepsilon_r)) \cap s_r(B(y_1, \delta_1) \times \cdots \times B(y_r, \delta_r))$$

contains a nonempty open subset $\mathcal{V} \subseteq \widehat{\sigma}_r(X)$ such that x is an accumulation point of \mathcal{V} . As $\{x_1, \dots, x_r\} \neq \{y_1, \dots, y_r\}$, there is at least one open ball $B(x_i, \varepsilon_i)$ satisfying

$$B(x_i, \varepsilon_i) \cap B(y_j, \delta_j) = \emptyset$$

for all $j = 1, \dots, r$. This shows that a general point in \mathcal{U} has at least two X -rank- r decompositions, contradicting the r -identifiability of X . \square

6. HIGHER-ORDER CONNECTEDNESS OF TENSOR RANK

Our calculations of the fundamental groups and higher homotopy groups of fixed-rank tensors will rely heavily on geometric information, notably knowledge of the singular loci of the secant varieties. As such our discussion will be limited to rank- r tensors where $r = 1, 2, 3$. The main difficulty in extending these calculations to rank- r tensors for $r \geq 4$ is that the singular loci of the r th secant varieties of the Segre variety are still unknown for $r \geq 4$. The same difficulty will limit our calculations in Section 7 for the homotopy groups of symmetric tensors to those of symmetric ranks ≤ 3 .

Parts of our results in Propositions 6.2, 6.3, 6.5, and 6.6 will be stated in terms of higher homotopy groups of spheres $\pi_k(\mathbb{S}^n)$. So in cases⁶ where these are known, we may determine the explicit

⁶See [46] for an extensive list of known $\pi_k(\mathbb{S}^n)$ for many values of (k, n) .

homotopy group for the set of low-rank tensors in question. This is a consequence of our relating higher homotopy groups of low-rank identifiable tensors to higher homotopy groups of spheres via (6.3) and (6.5). For instance, the vanishing of higher homotopy groups in Propositions 6.2 and 6.3 are directly obtained from these. In principle, we could derive many more explicit results easily using the list in [46], but we omit these calculations to avoid a tedious case-by-case discussion.

6.1. Fundamental and higher homotopy groups of complex rank- r tensors. To deduce the fundamental group of the set of rank- r tensors for small values of r , we apply the results in Section 6.1 to the case where X is the Segre variety. To be precise, let W_1, \dots, W_d be finite dimensional vector spaces over $\mathbb{F} = \mathbb{C}$ or \mathbb{R} . As usual, we will assume that all complex vector spaces are of (complex) dimensions at least two throughout this section. Let $d \geq 3$ and $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$ be the Segre variety. When $r = 2$, by [38], the singular locus of $\sigma_2(X)$ takes the form

$$Y := \bigcup_{1 \leq i < j \leq d} \mathbb{P}W_1 \times \dots \times \mathbb{P}W_{i-1} \times \mathbb{P}W_{i+1} \times \dots \times \mathbb{P}W_{j-1} \times \mathbb{P}W_{j+1} \times \dots \times \mathbb{P}W_d \times \sigma_2(\mathbb{P}W_i \times \mathbb{P}W_j).$$

Note that although the set

$$\mathbb{P}W_1 \times \dots \times \mathbb{P}W_{i-1} \times \mathbb{P}W_{i+1} \times \dots \times \mathbb{P}W_{j-1} \times \mathbb{P}W_{j+1} \times \dots \times \mathbb{P}W_d \times \sigma_2(\mathbb{P}W_i \times \mathbb{P}W_j)$$

lies in a different tensor space $\mathbb{P}((\bigotimes_{k \neq i, j} W_k) \otimes W_i \otimes W_j)$ for different i and j , we adopt the convention⁷ of identifying images under the isomorphism induced by permuting factors

$$\left(\bigotimes_{k \neq i, j} W_k \right) \otimes W_i \otimes W_j \cong \bigotimes_{k=1}^d W_k.$$

With this convention, we may safely write $Y \subseteq \sigma_2(X)$.

If $x \in \widehat{Y} \cap s_2(X)$, then $\text{rank}(x) < 2$ or x does not have a unique rank-2 decomposition. On the other hand, by [15, Proposition 1.1],

$$s_2(X) \setminus \widehat{Y} = \{a_1 \otimes \dots \otimes a_d + b_1 \otimes \dots \otimes b_d \in W_1 \otimes \dots \otimes W_d : \\ a_i, b_i \text{ linearly independent for at least three } i \in \{1, \dots, d\}\}.$$

By Proposition 5.3, every point in $s_2(X) \setminus \widehat{Y}$ has a unique decomposition, i.e., $s_2(X) \setminus D_2 = s_2(X) \setminus \widehat{Y}$, where D_2 is as defined in (5.1). This explicit description allows us to deduce the fundamental group of $s_2(X) \setminus D_2$.

Theorem 6.1 (Fundamental group of complex tensor rank). *Let $d \geq 3$ and W_1, \dots, W_d be complex vector spaces of dimensions n_1, \dots, n_d .*

(i) *The set of rank-one complex tensors has fundamental group*

$$\pi_1(\{A \in W_1 \otimes \dots \otimes W_d : \text{rank}(A) = 1\}) = 0.$$

(ii) *Let $n_1 \leq \dots \leq n_d$ and $(n_1 - 1) + \dots + (n_{d-2} - 1) > 1$. Then set of the rank-two identifiable complex tensors has fundamental group*

$$\pi_1(\{A \in W_1 \otimes \dots \otimes W_d : \text{rank}(A) = 2, A \text{ is identifiable}\}) = \mathbb{Z}/2\mathbb{Z}.$$

Proof. (i) Let $\mathcal{O}_X^\circ(-1)$ be the bundle in (3.1) with $S = X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$. The projection $p_2: \mathcal{O}_X^\circ(-1) \rightarrow W_1 \otimes \dots \otimes W_d$ is a homeomorphism between $\mathcal{O}_X^\circ(-1)$ and the set of rank-one tensors. So the fundamental group of the set of rank-one tensors is the same as

⁷For the benefit of readers unaccustomed with the coordinate-free approach towards tensors, we note that this is one of its benefits — the standard isomorphisms $U \otimes V \cong V \otimes U$ and $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ for any vector spaces U, V, W allows one to ignore the ordering of the vector spaces appearing in a tensor product.

that of $\mathcal{O}_X^\circ(-1)$. If we fix a choice of Hermitian metrics on W_1, \dots, W_d , we have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & \mathbb{S}^{2n_i-1} & \longrightarrow & \mathbb{P}W_i \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C} \setminus \{0\} & \longrightarrow & \mathcal{O}_{\mathbb{P}W_i}^\circ(-1) & \longrightarrow & \mathbb{P}W_i \end{array}$$

where \mathbb{S}^{2n_i-1} is regarded as the unit sphere in W_i and \mathbb{S}^1 as that in \mathbb{C} . Thus $\mathcal{O}_{\mathbb{P}W_i}^\circ(-1)$ has the same homotopy type as \mathbb{S}^{2n_i-1} . Consider the sequence

$$\mathbb{Z} \xrightarrow{j_*} \pi_1(\mathcal{O}_X^\circ(-1)) \rightarrow 0$$

induced by $\mathbb{C} \setminus \{0\} \xrightarrow{j} \mathcal{O}_X^\circ(-1) \rightarrow X$. For any $[v_1 \otimes \dots \otimes v_d] \in X$, we may assume that $\|v_1\| = \dots = \|v_d\| = 1$. Thus a generator of $\pi_1(\mathbb{C} \setminus \{0\}) = \mathbb{Z}$ can be realized as the unit circle in the complex line spanned by $v_1 \otimes \dots \otimes v_d$, i.e., $\lambda \cdot v_1 \otimes \dots \otimes v_d$, where $\lambda \in \mathbb{C}$ has $|\lambda| = 1$. Since

$$\lambda \cdot v_1 \otimes \dots \otimes v_d = (\lambda v_1) \otimes v_2 \otimes \dots \otimes v_d,$$

this unit circle can be realized as the unit circle in the complex line spanned by $v_1 \in W_1$, i.e., a generator of $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ in the sequence $\pi_1(\mathbb{S}^1) \rightarrow \pi_1(\mathbb{S}^{2n_1-1}) \rightarrow \pi_1(\mathbb{P}W_1)$. Since $\pi_1(\mathbb{S}^{2n_1-1}) = 0$ for $n_1 \geq 2$, we get $j_*(\mathbb{Z}) = 0$, and therefore $\pi_1(\mathcal{O}_X^\circ(-1)) = 0$.

(ii) Let $x = a_1 \otimes \dots \otimes a_{d-2} \otimes a_{d-1} \otimes a_d + a_1 \otimes \dots \otimes a_{d-2} \otimes b_{d-1} \otimes b_d \in D_2$. Then

$$\begin{aligned} s_2^{-1}(x) &= \{(a_1 \otimes \dots \otimes a_{d-2} \otimes u_{d-1} \otimes u_d, a_1 \otimes \dots \otimes a_{d-2} \otimes v_{d-1} \otimes v_d) \in \widehat{X}^2: \\ &\quad u_{d-1} \otimes u_d + v_{d-1} \otimes v_d = a_{d-1} \otimes a_d + b_{d-1} \otimes b_d\}, \end{aligned}$$

which implies that

$$\text{codim}_{\mathbb{C}}(s_2^{-1}(D_r), (\widehat{X} \setminus \{0\})^2) = (n_1 - 1) + \dots + (n_{d-2} - 1) > 1.$$

Let W be a complex vector space, and $N \subsetneq M \subseteq W$ be two subsets in W . Recall that for two complex manifolds $N \subsetneq M$,

$$(6.1) \quad \text{codim}_{\mathbb{R}}(N, M) = 2 \text{codim}_{\mathbb{C}}(N, M),$$

and that this extends to the case where M and N are each a union of finitely many disjoint complex manifolds (where dimension is defined as the maximum dimension of the constituent manifolds). Therefore we have

$$\text{codim}_{\mathbb{R}}(s_2^{-1}(D_r), (\widehat{X} \setminus \{0\})^2) > 2.$$

Given that $\pi_1(\widehat{X} \setminus \{0\}) = 0$ by part (i), it follows from Proposition 5.2 that the set of complex rank-two identifiable d -tensors has fundamental group $\mathbb{Z}/2\mathbb{Z}$. \square

We will move on to the higher homotopy groups. Again $X = \text{Seg}(\mathbb{P}W_1 \times \dots \times \mathbb{P}W_d)$ will denote the Segre variety in the proofs below. Note that there is no loss of generality in assuming that W_1, \dots, W_d are arranged in nondecreasing order of dimension — otherwise, we just need to replace n_1 with $\min\{n_1, \dots, n_d\}$ in the statements of the next two results.

Theorem 6.2 (Higher homotopy groups of complex rank-one tensors). *Let $d \geq 3$ and W_1, \dots, W_d be complex vector spaces of dimensions $n_1 \leq \dots \leq n_d$. Then*

$$\pi_2(\{A \in W_1 \otimes \dots \otimes W_d : \text{rank}(A) = 1\}) = \mathbb{Z}^d,$$

and

$$\pi_k(\{A \in W_1 \otimes \dots \otimes W_d : \text{rank}(A) = 1\}) \cong \prod_{j=1}^d \pi_k(\mathbb{S}^{2n_j-1}) \quad \text{for all } k \geq 3.$$

In particular, if $3 \leq k \leq 2n_1 - 2$, then

$$\pi_k(\{A \in W_1 \otimes \cdots \otimes W_d : \text{rank}(A) = 1\}) = 0.$$

Proof. By Theorem 3.6, the fiber bundle $\mathbb{C} \setminus \{0\} \rightarrow \mathcal{O}_X^\circ(-1) \rightarrow X$ yields the long exact sequence

$$\cdots \rightarrow \pi_k(\mathbb{C} \setminus \{0\}) \rightarrow \pi_k(\mathcal{O}_X^\circ(-1)) \rightarrow \pi_k(X) \rightarrow \pi_{k-1}(\mathbb{C} \setminus \{0\}) \rightarrow \cdots$$

As $\pi_k(\mathbb{C} \setminus \{0\}) = 0$ for all $k \geq 2$, and $\pi_1(\mathbb{C} \setminus \{0\})$ is isomorphic to $\pi_1(\mathcal{O}_X^\circ(-1))$, we get

$$(6.2) \quad \pi_k(\{A \in W_1 \otimes \cdots \otimes W_d : \text{rank}(A) = 1\}) \cong \pi_k(X) \cong \prod_{j=1}^d \pi_k(\mathbb{P}W_j)$$

for all $k \geq 2$, as required. From the fiber bundle $\mathbb{S}^1 \rightarrow \mathbb{S}^{2n+1} \rightarrow \mathbb{C}\mathbb{P}^n$ we obtain⁸

$$(6.3) \quad \pi_k(\mathbb{C}\mathbb{P}^n) \cong \begin{cases} 0 & \text{if } k = 1 \text{ or } 3 \leq k \leq 2n, \\ \mathbb{Z} & \text{if } k = 2 \text{ or } 2n + 1, \\ \pi_k(\mathbb{S}^{2n+1}) & \text{if } k \geq 2n + 2. \end{cases}$$

Combined with (6.2), we obtain the required higher homotopy groups for the set of complex rank-one tensors. \square

Theorem 6.3 (Higher homotopy groups of identifiable complex rank-two tensors). *Let $d \geq 3$ and W_1, \dots, W_d be complex vector spaces of dimensions $n_1 \leq \cdots \leq n_d$ with*

$$n_1 + \cdots + n_{d-2} \geq d.$$

We have

$$\pi_2(\{A \in W_1 \otimes \cdots \otimes W_d : \text{rank}(A) = 2, A \text{ identifiable}\}) = \mathbb{Z}^{2d}.$$

Let k be such that

$$1 < k/2 \leq \left(\sum_{j=1}^{d-2} n_j \right) - d + 1.$$

Then

$$\pi_k(\{A \in W_1 \otimes \cdots \otimes W_d : \text{rank}(A) = 2, A \text{ identifiable}\}) \cong \prod_{j=1}^d \pi_k(\mathbb{S}^{2n_j-1})^2.$$

In particular, if $d \geq 4$ and $3 \leq k \leq 2(n_1 - 1)$ or $d = 3$, $n_1 \geq 3$, and $3 \leq k \leq 2(n_1 - 2)$, then

$$\pi_k(\{A \in W_1 \otimes \cdots \otimes W_d : \text{rank}(A) = 2, A \text{ identifiable}\}) = 0.$$

Proof. Let

$$c := \text{codim}_{\mathbb{R}}(s_2^{-1}(D_2), (\widehat{X} \setminus \{0\})^2) = 2 \left(\sum_{j=1}^{d-2} n_j \right) - (d - 2).$$

By Proposition 5.2, if $c \geq 4$ and $2 \leq k \leq c - 2$, then

$$\pi_k(s_2(X) \setminus D_2) \cong \pi_k(\widehat{X} \setminus \{0\})^2,$$

and since $\widehat{X} \setminus \{0\}$ is exactly the set of complex rank-one tensors, by (6.2),

$$(6.4) \quad \pi_k(s_2(X) \setminus D_2) \cong \prod_{j=1}^d \pi_k(\mathbb{P}W_j)^2.$$

By (6.3) and (6.4), we obtain the k th homotopy group of the set of identifiable complex rank-two tensors for $2 \leq k \leq c - 2$, assuming that $c \geq 4$. \square

⁸When $n = 1$, we may identify $\mathbb{C}\mathbb{P}^1$ with \mathbb{S}^2 topologically, implying that $\pi_k(\mathbb{C}\mathbb{P}^1) = \pi_k(\mathbb{S}^2)$ for all $k = 1, 2, \dots$. For example, we have $\pi_1(\mathbb{C}\mathbb{P}^1) = 0$ and $\pi_2(\mathbb{C}\mathbb{P}^1) \cong \pi_3(\mathbb{C}\mathbb{P}^1) \cong \mathbb{Z}$.

6.2. Fundamental and higher homotopy groups of real rank- r tensors. We now turn our attention to the real case, using ideas similar to those used in the complex case: We will consider a fiber bundle and a double covering for real rank-one tensors and identifiable real rank-two tensors respectively. From these geometric constructions, we will calculate the homotopy groups of these real low rank tensors: Theorems 6.4, 6.5, and 6.6 are respectively the real analogues of Theorems 6.1, 6.2, and 6.3. As usual, throughout this section, we will assume that all real vector spaces have (real) dimensions at least two.

Theorem 6.4 (Fundamental groups of real tensor rank). *Let $d \geq 3$ and V_1, \dots, V_d be real vector spaces of real dimensions n_1, \dots, n_d . Let $m := \#\{i : \dim_{\mathbb{R}}(V_i) = 2\}$.*

(i) *The set of rank-one real tensors has fundamental group*

$$\pi_1(\{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) = 1\}) = \begin{cases} \mathbb{Z}^d & \text{if } m = d, \\ \mathbb{Z}^m \times (\mathbb{Z}/2\mathbb{Z})^{d-m-1} & \text{if } 0 \leq m < d. \end{cases}$$

(ii) *Let $n_1 \leq \cdots \leq n_d$ and $(n_1 - 1) + \cdots + (n_{d-2} - 1) > 2$. Then the set of rank-two identifiable real tensors has fundamental group*

$$\begin{aligned} \pi_1(\{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) = 2, A \text{ is identifiable}\}) \\ = \begin{cases} \mathbb{Z}^{2d} \rtimes \mathbb{Z}/2\mathbb{Z} & \text{if } m = d, \\ (\mathbb{Z}^{2m} \times (\mathbb{Z}/2\mathbb{Z})^{2d-2m-2}) \rtimes \mathbb{Z}/2\mathbb{Z} & \text{if } 0 \leq m < d. \end{cases} \end{aligned}$$

Proof. Let $X = \text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d)$ and let $\mathcal{O}_X^\circ(-1)$ be the bundle in (3.1) with $S = X$.

(i) As in the proof of the complex case in Theorem 6.1, the projection $p_2: \mathcal{O}_X^\circ(-1) \rightarrow V_1 \otimes \cdots \otimes V_d$ is a homeomorphism and it suffices to determine the fundamental group of $\mathcal{O}_X^\circ(-1)$. The fiber bundle

$$\mathbb{R} \setminus \{0\} \rightarrow \mathcal{O}_X^\circ(-1) \rightarrow X$$

induces the long exact sequence

$$0 \rightarrow \pi_1(\mathcal{O}_X^\circ(-1)) \rightarrow \pi_1(X) \rightarrow \pi_0(\mathbb{R} \setminus \{0\}) \rightarrow 0.$$

Since $\pi_1(X) = \mathbb{Z}^m \times (\mathbb{Z}/2\mathbb{Z})^{d-m}$ and $\pi_0(\mathbb{R} \setminus \{0\}) = \mathbb{Z}/2\mathbb{Z}$, we get

$$\pi_1(\mathcal{O}_X^\circ(-1)) = \begin{cases} \mathbb{Z}^d & \text{if } m = d, \\ \mathbb{Z}^m \times (\mathbb{Z}/2\mathbb{Z})^{d-m-1} & \text{if } 0 \leq m < d. \end{cases}$$

(ii) Since

$$\text{codim}_{\mathbb{R}}(s_2^{-1}(D_2), (\widehat{X} \setminus \{0\})^2) = (n_1 - 1) + \cdots + (n_{d-2} - 1) > 2,$$

applying Proposition 5.2 with the fundamental group obtained in part (i) gives us the required result. \square

Theorem 6.5 (Higher homotopy groups of real rank-one tensors). *Let $d \geq 3$ and V_1, \dots, V_d be real vector spaces of real dimensions $n_1 \leq \cdots \leq n_d$. For any $k \geq 2$, we have*

$$\pi_k(\{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) = 1\}) \cong \prod_{j=1}^d \pi_k(\mathbb{S}^{n_j-1}).$$

In particular, if $2 \leq k \leq n_1 - 1$, then

$$\pi_k(\{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) = 1\}) = 0.$$

Proof. Let $X = \text{Seg}(\mathbb{P}V_1 \times \cdots \times \mathbb{P}V_d)$. The fiber bundle $\mathbb{R} \setminus \{0\} \rightarrow \mathcal{O}_X^\circ(-1) \rightarrow X$ induces an isomorphism

$$\pi_k(\{A \in V_1 \otimes \cdots \otimes V_d : \text{rank}(A) = 1\}) \cong \pi_k(\mathcal{O}_X^\circ(-1)) \cong \pi_k(X) \cong \prod_{j=1}^d \pi_k(\mathbb{P}V_j)$$

for all $k \geq 2$ as $\pi_k(\mathbb{R} \setminus \{0\}) = 0$. Recall that homotopy groups of real projective spaces are isomorphic to those of spheres, i.e., the double cover $\mathbb{S}^n \rightarrow \mathbb{R}\mathbb{P}^n$ gives isomorphism $\pi_k(\mathbb{R}\mathbb{P}^n) \cong \pi_k(\mathbb{S}^n)$ for all $k \geq 2$. For easy reference, a list⁹ of homotopy groups of real projective n -spaces for $n \geq 2$ is as follows:

$$(6.5) \quad \pi_k(\mathbb{R}\mathbb{P}^n) \cong \begin{cases} \mathbb{Z}_2 & \text{if } k = 1, \\ 0 & \text{if } 2 \leq k \leq n-1, \\ \mathbb{Z} & \text{if } k = n, \\ \pi_k(\mathbb{S}^n) & \text{if } n+1 \leq k. \end{cases} \quad \square$$

The homotopy groups of identifiable real rank-two tensors follows directly from Proposition 5.2 with $r = 2$.

Theorem 6.6 (Higher homotopy groups of identifiable real rank-two tensors). *Let $d \geq 3$ and V_1, \dots, V_d be real vector spaces of real dimensions $n_1 \leq \dots \leq n_d$ with*

$$n_1 + \dots + n_{d-2} \geq d + 2.$$

Let k be such that

$$2 \leq k \leq \left(\sum_{j=1}^{d-2} n_j \right) - d.$$

Then

$$\pi_k(\{A \in V_1 \otimes \dots \otimes V_d : \text{rank}(A) = 2, A \text{ is identifiable}\}) \cong \prod_{j=1}^d \pi_k(\mathbb{S}^{n_j-1})^2.$$

In particular, if

$$2 \leq k \leq \min\left\{n_1 - 1, \left(\sum_{j=1}^{d-2} n_j\right) - d\right\},$$

then

$$\pi_k(\{A \in V_1 \otimes \dots \otimes V_d : \text{rank}(A) = 2, A \text{ is identifiable}\}) = 0.$$

7. HIGHER-ORDER CONNECTEDNESS OF SYMMETRIC TENSOR RANK

The remark that we made at the beginning of Section 6 also applies to symmetric tensor rank. Here we will again limit ourselves to symmetric rank- r symmetric tensors where $r = 1, 2$, or 3 . The difficulty in extending these results to $r \geq 4$ is that the singular loci of the r th secant varieties of the Veronese variety are still unknown for $r \geq 4$. Also, as in the previous section, two of our results, Propositions 7.2 and 7.5, will be stated in the terms of homotopy groups of spheres.

7.1. Fundamental and higher homotopy groups of complex symmetric rank- r tensors.

To deduce the fundamental group of the set of symmetric rank- r symmetric tensors for small values of r , we apply the results in Section 6.1 to the case where $X = \nu_d(\mathbb{P}W)$ is the Veronese variety, with W a finite-dimensional vector space over $\mathbb{F} = \mathbb{C}$ or \mathbb{R} of dimension at least two.

Theorem 7.1 (Fundamental groups of complex symmetric tensor rank). *Let $d \geq 3$ and W be a complex vector space.*

(i) *The set of symmetric rank-one complex symmetric tensors has fundamental group*

$$\pi_1(\{A \in \mathbb{S}^d(W) : \text{rank}_{\mathbb{S}}(A) = 1\}) = 0.$$

(ii) *Let $d \geq 3$ and $n > 2$. The set of symmetric rank-two complex symmetric tensors has fundamental group*

$$\pi_1(\{A \in \mathbb{S}^d(W) : \text{rank}_{\mathbb{S}}(A) = 2\}) = \mathbb{Z}/2\mathbb{Z}.$$

(iii) *Let $d \geq 5$ and $n > 2$. The set of symmetric rank-three complex symmetric tensors has fundamental group*

$$\pi_1(\{A \in \mathbb{S}^d(W) : \text{rank}_{\mathbb{S}}(A) = 3\}) = \mathfrak{S}_3.$$

⁹When $n = 1$, $\pi_1(\mathbb{R}\mathbb{P}^1) \cong \mathbb{Z}$ and all higher homotopy groups vanish.

Proof. (i) Let $\mathcal{O}_X^\circ(-1)$ be the bundle in (3.1) with $S = X = \nu_d(\mathbb{P}W)$. The projection $p_2: \mathcal{O}_X^\circ(-1) \rightarrow \mathbb{S}^d(W)$ defines a homeomorphism between $\mathcal{O}_X^\circ(-1)$ and the set of symmetric rank-one complex tensors. We have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{S}^1 & \longrightarrow & \mathbb{S}^{2n-1} & \longrightarrow & \mathbb{P}W \\ \downarrow \nu_d & & \downarrow & & \downarrow \nu_d \\ \mathbb{C} \setminus \{0\} & \longrightarrow & \mathcal{O}_X^\circ(-1) & \longrightarrow & X \end{array}$$

where \mathbb{S}^1 is the unit circle in \mathbb{C} and \mathbb{S}^{2n-1} is the unit sphere in W after fixing an Hermitian metric on W . Thus $\mathcal{O}_X^\circ(-1)$ and \mathbb{S}^{2n-1} have the same homotopy type, which implies that $\pi_1(\mathcal{O}_X^\circ(-1)) = 0$.

- (ii) When $r = 2$, the singular locus of $\sigma_2(X)$ is X by [30, Theorem 3.3]. When $d \geq 3$, since any $x \in \widehat{\sigma}_2(X)$ with $\text{rank}_\mathbb{S}(x) = 2$ must take the form $u^{\otimes d} + v^{\otimes d}$ for some u, v linearly independent, x is necessarily a nonsingular point. By Proposition 5.3, D_2 as defined in (5.1) equals \widehat{X} . It follows from (6.1) that

$$\text{codim}_\mathbb{R}(s_2^{-1}(D_2), (\widehat{X} \setminus \{0\})^2) = 2 \text{codim}_\mathbb{C}(s_2^{-1}(\widehat{X}), (\widehat{X} \setminus \{0\})^2) = 2(n-1) > 2.$$

By Proposition 5.2, the required fundamental group is $\mathbb{Z}/2\mathbb{Z}$.

- (iii) When $r = 3$, the singular locus of $\sigma_3(X)$ is $\sigma_2(X)$ by [24]. As $d \geq 5$, by [16, 13], for any $x \in \widehat{\sigma}_2(X)$, we must have $\text{rank}_\mathbb{S}(x) \neq 3$, which implies that $\widehat{\sigma}_2(X) \cap s_3(X) = s_2(X)$. By [34, Theorem 1.2] any $x \in s_3(X) \setminus s_2(X)$, which is a nonsingular point of $\widehat{\sigma}_3(X)$, has the form $x = u^{\otimes d} + v^{\otimes d} + w^{\otimes d}$, where $[u], [v], [w]$ are distinct points in the projective space $\mathbb{P}W$. By Proposition 5.3, this decomposition of x is unique. Hence D_3 as defined in (5.1) equals $s_2(X)$. Since

$$\text{codim}_\mathbb{R}(s_3^{-1}(D_3), (\widehat{X} \setminus \{0\})^3) = 2 \text{codim}_\mathbb{C}(s_3^{-1}(s_2(X)), (\widehat{X} \setminus \{0\})^3) = 2(n-1) > 2,$$

it follows from Proposition 5.2 that $\pi_1(s_3(X) \setminus s_2(X)) = \mathfrak{S}_3$. □

For the higher homotopy groups, we combine Proposition 5.2 with the long exact sequence of the fiber bundle $\mathbb{C} \setminus \{0\} \rightarrow \mathcal{O}_X^\circ(-1) \rightarrow X$ obtained from Theorem 3.6 and employ the same argument as in the proofs of Theorems 6.2, 6.3, 6.5, and 6.6. This gives us our next two results.

Theorem 7.2 (Higher homotopy groups of complex symmetric rank-one tensors). *Let $d \geq 3$ and W be a complex vector space. Then*

$$\pi_2(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 1\}) = \mathbb{Z}.$$

Let $k \geq 3$. Then

$$\pi_k(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 1\}) \cong \pi_k(\mathbb{S}^{2n-1}).$$

In particular, if $3 \leq k \leq 2(n-1)$, then

$$\pi_k(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 1\}) = 0.$$

Theorem 7.3 (Higher homotopy groups of complex symmetric rank-two and three tensors). *Let W be a complex vector space. Then*

$$\pi_2(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 2\}) = \mathbb{Z}^2 \quad \text{if } d \geq 3,$$

$$\pi_2(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 3\}) = \mathbb{Z}^3 \quad \text{if } d \geq 5.$$

Let $3 \leq k \leq 2(n-2)$. Then

$$\pi_k(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 2\}) = 0 \quad \text{if } d \geq 3,$$

$$\pi_k(\{A \in \mathbb{S}^d(W) : \text{rank}_\mathbb{S}(A) = 3\}) = 0 \quad \text{if } d \geq 5.$$

7.2. Fundamental and higher homotopy groups of real symmetric rank- r tensors. We next move on to the real case. The next three theorems are the real analogues of Theorems 7.1, 7.2, and 7.3.

Theorem 7.4 (Fundamental groups of real symmetric tensor rank). *Let V be a real vector space of dimension n .*

(i) *The set of symmetric rank-one real symmetric tensors has fundamental group*

$$\pi_1(\{A \in \mathcal{S}^d(V) : \text{rank}_{\mathcal{S}}(A) = 1\}) = \begin{cases} \mathbb{Z} & \text{if } n = 2 \text{ and } d \text{ is odd,} \\ 0 & \text{if } n > 2 \text{ and } d \text{ is odd,} \\ \mathbb{Z} & \text{if } n = 2 \text{ and } d \text{ is even,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n > 2 \text{ and } d \text{ is even.} \end{cases}$$

(ii) *Let $n > 3$ and $d \geq 3$. Then the set of real symmetric rank-two tensors has fundamental group*

$$\pi_1(\{A \in \mathcal{S}^d(V) : \text{rank}_{\mathcal{S}}(A) = 2\}) = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is even.} \end{cases}$$

(iii) *Let $n > 3$ and $d \geq 5$. Then the set of real symmetric rank-three tensors has fundamental group*

$$\pi_1(\{A \in \mathcal{S}^d(V) : \text{rank}_{\mathcal{S}}(A) = 3\}) = \begin{cases} \mathfrak{S}_3 & \text{if } d \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3 & \text{if } d \text{ is even.} \end{cases}$$

Proof. (i) Let $\mathcal{O}_X^\circ(-1)$ be the bundle in (3.1) with $S = X = \nu_d(\mathbb{P}V)$. As in the complex case, the projection $p_2: \mathcal{O}_X^\circ(-1) \rightarrow \mathcal{S}^d(V)$ defines a homeomorphism between $\mathcal{O}_X^\circ(-1)$ and the set of symmetric rank-one real tensors. The fiber bundle

$$\mathbb{R} \setminus \{0\} \rightarrow \mathcal{O}_X^\circ(-1) \rightarrow X$$

induces a long exact sequence

$$0 \rightarrow \pi_1(\mathcal{O}_X^\circ(-1)) \rightarrow \pi_1(X) \rightarrow \pi_0(\mathbb{R} \setminus \{0\}) \rightarrow \pi_0(\mathcal{O}_X^\circ(-1)) \rightarrow 0.$$

Since $\pi_0(\mathbb{R} \setminus \{0\}) = \mathbb{Z}/2\mathbb{Z}$,

$$\pi_0(\mathcal{O}_X^\circ(-1)) = \begin{cases} 0 & \text{if } d \text{ is odd,} \\ \mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is even,} \end{cases} \quad \text{and} \quad \pi_1(X) = \begin{cases} \mathbb{Z} & \text{if } n = 2, \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n > 2, \end{cases}$$

we obtain the required $\pi_1(\mathcal{O}_X^\circ(-1))$.

(ii) As in the complex case, D_2 as defined in (5.1) equals \widehat{X} . It follows from (6.1) that

$$\text{codim}_{\mathbb{R}}(s_2^{-1}(D_2), (\widehat{X} \setminus \{0\})^2) = \text{codim}_{\mathbb{R}}(s_2^{-1}(\widehat{X}), (\widehat{X} \setminus \{0\})^2) = (n-1) > 2.$$

By Proposition 5.2,

$$\pi_1(s_2(X) \setminus \widehat{X}) = \pi_1(\widehat{X} \setminus \{0\})^2 \rtimes \mathfrak{S}_2 = \begin{cases} \mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^2 \rtimes \mathbb{Z}/2\mathbb{Z} & \text{if } d \text{ is even.} \end{cases}$$

(iii) As in the complex case, D_3 as defined in (5.1) equals $s_2(X)$. Since

$$\text{codim}_{\mathbb{R}}(s_3^{-1}(D_3), (\widehat{X} \setminus \{0\})^3) = \text{codim}_{\mathbb{R}}(s_3^{-1}(s_2(X)), (\widehat{X} \setminus \{0\})^3) = (n-1) > 2,$$

it follows from Proposition 5.2 that

$$\pi_1(s_3(X) \setminus s_2(X)) = \pi_1(\mathcal{O}_X^\circ(-1))^3 \rtimes \mathfrak{S}_3 = \begin{cases} \mathfrak{S}_3 & \text{if } d \text{ is odd,} \\ (\mathbb{Z}/2\mathbb{Z})^3 \rtimes \mathfrak{S}_3 & \text{if } d \text{ is even.} \end{cases} \quad \square$$

Again, from (6.5) and the long exact sequence induced by the fiber bundle $\mathbb{R} \setminus \{0\} \rightarrow \mathcal{O}_X(-1) \rightarrow X$, we deduce the higher-homotopy groups in the real case.

Theorem 7.5 (Higher homotopy groups of real symmetric rank-one tensors). *Let $d \geq 3$ and V be a real vector space. Let $k \geq 2$. Then*

$$\pi_k(\{A \in \mathbf{S}^d(V) : \text{rank}_{\mathbf{S}}(A) = 1\}) \cong \pi_k(\mathbb{S}^{n-1}).$$

In particular, if $n \geq 4$ and $2 \leq k \leq n - 2$, then

$$\pi_k(\{A \in \mathbf{S}^d(V) : \text{rank}_{\mathbf{S}}(A) = 1\}) = 0.$$

Theorem 7.6 (Higher homotopy groups of real symmetric rank-two and three tensors). *Let V be a real vector space. If $2 \leq k \leq n - 3$, then*

$$\begin{aligned} \pi_k(\{A \in \mathbf{S}^d(V) : \text{rank}_{\mathbf{S}}(A) = 2\}) &= 0 && \text{if } d \geq 3, \\ \pi_k(\{A \in \mathbf{S}^d(V) : \text{rank}_{\mathbf{S}}(A) = 3\}) &= 0 && \text{if } d \geq 5. \end{aligned}$$

8. TOPOLOGY OF MULTILINEAR RANK

We will address the connectedness and calculate the homotopy groups of the set of tensors of a fixed multilinear rank. We start by recalling the notion.

Definition 8.1. Let V_1, \dots, V_d be vector spaces over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} of dimensions n_1, \dots, n_d respectively. Let $r_i \leq n_i$ be a positive integer $i = 1, \dots, d$. The *subspace variety* is the set

$$\begin{aligned} \text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d) \\ := \{A \in V_1 \otimes \dots \otimes V_d : A \in U_1 \otimes \dots \otimes U_d, U_i \subseteq V_i, \dim(U_i) = r_i, i = 1, \dots, d\}. \end{aligned}$$

We say that $A \in V_1 \otimes \dots \otimes V_d$ has *multilinear rank* (r_1, \dots, r_d) , or, in notation,

$$\mu\text{rank}(A) = (r_1, \dots, r_d),$$

if whenever $A \in \text{Sub}_{s_1, \dots, s_d}(V_1, \dots, V_d)$ for $s_i \leq r_i, i = 1, \dots, d$, we must have $r_i = s_i$ for all $i = 1, \dots, d$. In other words $\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d)$ is the smallest subspace variety that contains A .

Clearly, the definition implies that

$$\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d) = \{A \in V_1 \otimes \dots \otimes V_d : \mu\text{rank}(A) \leq (r_1, \dots, r_d)\}.$$

The subspace variety is very well studied [33] but in this article we are interested in the set of all tensors of multilinear rank *exactly* (r_1, \dots, r_d) , which we will denote by

$$(8.1) \quad X_{r_1, \dots, r_d}(V_1, \dots, V_d) := \{A \in V_1 \otimes \dots \otimes V_d : \mu\text{rank}(A) = (r_1, \dots, r_d)\}.$$

Every d -tensor may be regarded as a 2-tensor via *flattening* [33, 37]. The flattening map

$$(8.2) \quad b_i : V_1 \otimes \dots \otimes V_d \rightarrow V_i \otimes \left(\bigotimes_{j \neq i} V_j \right), \quad i = 1, \dots, d,$$

takes a d -tensor and sends it to a 2-tensor by ‘forgetting’ the tensor product structure in $\bigotimes_{j \neq i} V_j$. One may also characterize multilinear rank as

$$\mu\text{rank}(A) = (\text{rank}(b_1(A)), \dots, \text{rank}(b_d(A))),$$

where rank here denotes usual matrix rank, which, being coordinate independent, is defined on $V_i \otimes \left(\bigotimes_{j \neq i} V_j \right)$.

Note that if (r_1, \dots, r_d) is the multilinear rank of some tensor, then we must have

$$(8.3) \quad r_i \leq \prod_{j \neq i} r_j, \quad i = 1, \dots, d,$$

as it follows from (8.2) that $\text{rank}(b_i(A)) \leq \min\{\dim_{\mathbb{F}}(U_i), \dim_{\mathbb{F}}(\bigotimes_{j \neq i} U_j)\}$.

8.1. Path-connectedness of multilinear rank. While the subspace variety, being irreducible, is connected (in fact, contractible since it is a union of infinitely many linear subspaces of the ambient tensor space), it is less clear for the set of tensors of a fixed multilinear rank. For example, over $\mathbb{F} = \mathbb{R}$, when $d = 2$ and $r_1 = r_2 = n_1 = n_2 = n$, $X_{n,n}(V_1, V_2)$ is the set of $n \times n$ invertible real matrices, which is disconnected.

As one can surmise from the case $d = 2$, the situation over \mathbb{R} is more subtle and we will start with this first, leaving the complex case to the end.

For a finite-dimensional real vector space V , we write $\text{Gr}(r, V)$ for the *Grassmannian* of r -dimensional linear subspaces of V and $\mathcal{T}_{\text{Gr}(r, V)}$ for its *tautological vector bundle*, i.e., whose fiber over $U \in \text{Gr}(r, V)$ is U .

Let V_1, \dots, V_d be vector spaces of dimensions n_1, \dots, n_d respectively and r_1, \dots, r_d be positive integers such that $r_i \leq n_i$, $i = 1, \dots, d$. We write

$$G_{r_1, \dots, r_d} = \text{Gr}(r_1, V_1) \times \cdots \times \text{Gr}(r_d, V_d)$$

and $q_j : G_{r_1, \dots, r_d} \rightarrow \text{Gr}(r_j, V_j)$ for the j th projection. We write

$$\mathcal{T}_{r_1, \dots, r_d} = q_1^*(\mathcal{T}_{\text{Gr}(r_1, V_1)}) \otimes \cdots \otimes q_d^*(\mathcal{T}_{\text{Gr}(r_d, V_d)})$$

for the tensor product of the pullbacks of the tautological vector bundles, i.e., whose fiber over $(U_1, \dots, U_d) \in \text{Gr}(r_1, V_1) \times \cdots \times \text{Gr}(r_d, V_d)$ is $U_1 \otimes \cdots \otimes U_d$.

Let $p : \mathcal{T}_{r_1, \dots, r_d} \rightarrow G_{r_1, \dots, r_d}$ be the projection of the vector bundle $\mathcal{T}_{r_1, \dots, r_d}$ onto its base space G_{r_1, \dots, r_d} . We define the map

$$\rho_{r_1, \dots, r_d} : \mathcal{T}_{r_1, \dots, r_d} \rightarrow V_1 \otimes \cdots \otimes V_d, \quad (U_1, \dots, U_d, A) \mapsto A,$$

where $(U_1, \dots, U_d) \in G_{r_1, \dots, r_d}$ and $A \in U_1 \otimes \cdots \otimes U_d$. The image of ρ_{r_1, \dots, r_d} is $\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d)$ and ρ_{r_1, \dots, r_d} gives a *Kempf–Weyman desingularization* [49, 33] of $\text{Sub}_{r_1, \dots, r_d}(V_1, \dots, V_d)$.

Theorem 8.2 (Connectedness of multilinear rank over \mathbb{R}). *Let V_1, \dots, V_d be real vector spaces.*

(i) *The set of multilinear rank- (r_1, \dots, r_d) real tensors*

$$\{A \in V_1 \otimes \cdots \otimes V_d : \mu\text{rank}(A) = (r_1, \dots, r_d)\}$$

is connected if

$$r_i < \prod_{j \neq i} r_j \quad \text{for all } i = 1, \dots, d,$$

or if

$$r_i = \prod_{j \neq i} r_j < n_i \quad \text{for some } i = 1, \dots, d.$$

(ii) *The set of multilinear rank- (r_1, \dots, r_d) real tensors*

$$\{A \in V_1 \otimes \cdots \otimes V_d : \mu\text{rank}(A) = (r_1, \dots, r_d)\}$$

has two connected components if

$$r_i = \prod_{j \neq i} r_j = n_i \quad \text{for some } i = 1, \dots, d.$$

Proof. For brevity, we will write $X_{r_1, \dots, r_d} = X_{r_1, \dots, r_d}(V_1, \dots, V_d)$ for the set of multilinear rank- (r_1, \dots, r_d) tensors in this proof. Let $C \in V_1 \otimes \cdots \otimes V_d$ and $b_i(C) \in V_i \otimes (\bigotimes_{j \neq i} V_j)$ be the i th flattening of C as defined in (8.2). Let

$$\mathcal{X}_{r_1, \dots, r_d} := \{(U_1, \dots, U_d, C) \in \mathcal{T}_{r_1, \dots, r_d} : \text{rank}(b_i(C)) = r_i \text{ for } i = 1, \dots, d\}.$$

Then $\rho_{r_1, \dots, r_d} : \mathcal{X}_{r_1, \dots, r_d} \rightarrow X_{r_1, \dots, r_d}$ is an isomorphism. For each $i = 1, \dots, d$, let

$$\mathcal{S}_i := \{(U_1, \dots, U_d, C) \in \mathcal{T}_{r_1, \dots, r_d} : \text{rank}(b_i(C)) \leq r_i - 1\}.$$

Then

$$\mathcal{X}_{r_1, \dots, r_d} = \mathcal{T}_{r_1, \dots, r_d} \setminus \bigcup_{i=1}^d \mathcal{S}_i.$$

We observe that

$$\dim_{\mathbb{R}}(\mathcal{T}_{r_1, \dots, r_d}) = \sum_{i=1}^d r_i(n_i - r_i) + \prod_{i=1}^d r_i$$

and

$$\begin{aligned} \dim_{\mathbb{R}}(\mathcal{S}_i) &= \sum_{i=1}^d r_i(n_i - r_i) + (r_i - 1) + (r_i - 1) \prod_{j \neq i} r_j \\ (8.4) \quad &= \dim_{\mathbb{R}}(\mathcal{T}_{r_1, \dots, r_d}) - \left(\prod_{j \neq i} r_j - r_i + 1 \right). \end{aligned}$$

If $r_i < \prod_{j \neq i} r_j$, then (8.4) implies that \mathcal{S}_i has real codimension at least two in $\mathcal{T}_{r_1, \dots, r_d}$. By Theorem 3.5, we see that X_{r_1, \dots, r_d} is connected.

We next consider the case when $r_i = \prod_{j \neq i} r_j < n_i$ for some $i = 1, \dots, d$. Without loss of generality, we may assume that

$$r_1 = \prod_{i=2}^d r_i < n_1.$$

We want to prove that any two points (U_1, \dots, U_d, A) and (U'_1, \dots, U'_d, B) in X_{r_1, \dots, r_d} can be connected by a curve contained in X_{r_1, \dots, r_d} . We will first prove that since the base space $\text{Gr}(r_1, V_1) \times \dots \times \text{Gr}(r_d, V_d)$ of the bundle $\mathcal{T}_{r_1, \dots, r_d}$ is connected, there is a curve in X_{r_1, \dots, r_d} connecting (U'_1, \dots, U'_d, B) and (U_1, \dots, U_d, A') for some $A' \in U'_1 \otimes \dots \otimes U'_d$. We will then prove that (U_1, \dots, U_d, A') and (U_1, \dots, U_d, A) can be connected by a curve contained in X_{r_1, \dots, r_d} .

For each $i = 1, \dots, d$, let $\gamma_i : [0, 1] \rightarrow \text{Gr}(r_i, V_i)$ be a curve connecting $U'_i = \gamma_i(0) \in \text{Gr}(r_i, V_i)$ and $U_i = \gamma_i(1) \in \text{Gr}(r_i, V_i)$. Since $B \in U'_1 \otimes \dots \otimes U'_d$, we may write

$$B = \sum_{i_1, \dots, i_d=1}^{r_1, \dots, r_d} \lambda_{i_1 \dots i_d} u_{1, i_1} \otimes \dots \otimes u_{d, i_d},$$

where $u_{i,1}, \dots, u_{i,r_i}$ form a basis of U'_i , $i = 1, \dots, d$. Consider the curve $B(\cdot) : [0, 1] \rightarrow X_{r_1, \dots, r_d}$ defined by

$$B(t) = \sum_{i_1, \dots, i_d=1}^{r_1, \dots, r_d} \lambda_{i_1 \dots i_d} u_{1, i_1}(t) \otimes \dots \otimes u_{d, i_d}(t),$$

where $u_{i,1}(t), \dots, u_{i,r_i}(t)$ form a basis of $\gamma_i(t)$ for any $t \in [0, 1]$, with

$$u_{i,1}(0) = u_{i,1}, \dots, u_{i,r_i}(0) = u_{i,r_i}.$$

The curve $B(t)$ connects the point $B = B(0)$ with some $B(1) \in U'_1 \otimes \dots \otimes U'_d$. Moreover, $(\gamma_1(t), \dots, \gamma_d(t), B(t))$ defines a curve in X_{r_1, \dots, r_d} connecting (U'_1, \dots, U'_d, B) and $(U_1, \dots, U_d, B(1))$. If $(U_1, \dots, U_d, B(1))$ and (U_1, \dots, U_d, A) can also be connected by a curve in X_{r_1, \dots, r_d} , then so can (U_1, \dots, U_d, A) and (U'_1, \dots, U'_d, B) .

It remains to show that any two points (U_1, \dots, U_d, A) and (U_1, \dots, U_d, B) in X_{r_1, \dots, r_d} can be connected by a curve contained in X_{r_1, \dots, r_d} . Extend the basis $u_{1,1}, \dots, u_{1,r_1}$ of the subspace U_1 chosen earlier to a basis $u_{1,1}, \dots, u_{1,n_1}$ of V_1 . With respect to this basis, the first flattening of A and B have representation as matrices

$$b_1(A) = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{\prod_{i=2}^d n_i}, \quad b_1(B) = \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{\prod_{i=2}^d n_i},$$

where $I \in \mathbb{R}^{r_1 \times r_1}$ is the identity matrix and for some $M \in \mathbb{R}^{r_1 \times r_1}$.

We consider the map $\Phi : \mathbb{R}^{r_1} \rightarrow \text{Gr}(r_1, V_1)$ defined by

$$\Phi(t_1, \dots, t_{r_1}) = \text{span}\{u_{1,1} + t_1 u_{1,r_1+1}, \dots, u_{1,r_1} + t_{r_1} u_{1,r_1+1}\},$$

which is well-defined as $u_{1,1}, \dots, u_{1,r_1}$ are linearly independent. The image $\Phi(\mathbb{R}^{r_1}) \subseteq \text{Gr}(r_1, V_1)$ is a smooth submanifold — to see this, we determine the rank of the differential

$$d\Phi_{(t_1, \dots, t_{r_1})} : \mathbb{T}_{(t_1, \dots, t_{r_1})} \mathbb{R}^{r_1} \rightarrow \mathbb{T}_{\Phi(t_1, \dots, t_{r_1})} \text{Gr}(r_1, V_1).$$

Since every point $U \in \text{Gr}(r_1, V_1)$ may be written as $[u_1 \wedge \cdots \wedge u_{r_1}] \in \mathbb{P}\mathbb{R}^{\binom{n_1}{r_1}}$ by the Plücker embedding, where u_1, \dots, u_{r_1} form a basis of U , we obtain

$$d\Phi_{(t_1, \dots, t_{r_1})}(s_1, \dots, s_{r_1}) = ([u_{r_1+1}^1 \wedge u_2^1 \wedge \cdots \wedge u_{r_1}^1], \dots, [u_1^1 \wedge \cdots \wedge u_{r_1-1}^1 \wedge u_{r_1+1}^1])^\top,$$

which has full rank r_1 for all $(t_1, \dots, t_{r_1}) \in \mathbb{R}^{r_1}$. Note that Φ is not necessarily an embedding, but its image $\Phi(\mathbb{R}^{r_1})$ must be smooth and hence it is an immersed submanifold of $\text{Gr}(r_1, V_1)$.

Recall the notations in the two paragraphs preceding Theorem 8.2. Let $(U_1, \dots, U_d) \in G_{r_1, \dots, r_d}$ and consider the preimage

$$\mathcal{U} := p^{-1}(\Phi(\mathbb{R}^{r_1}) \times \{U_1\} \times \cdots \times \{U_d\}) \subseteq \mathcal{T}_{r_1, \dots, r_d}.$$

Since $\Phi(\mathbb{R}^{r_1})$ is a smooth submanifold of $\text{Gr}(r_1, V_1)$ and p is the projection map, \mathcal{U} is a smooth submanifold of $\mathcal{T}_{r_1, \dots, r_d}$. By its definition \mathcal{U} contains both (U_1, \dots, U_d, A) and (U_1, \dots, U_d, B) . Let $(U_1, \dots, U_d, C) \in \mathcal{U}$. Then its first flattening takes the form

$$b_1(C) = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n_1 \times \prod_{i=2}^d n_i},$$

for some $L \in \mathbb{R}^{(r_1+1) \times r_1}$. Set

$$\mathcal{R}_i := \{(U_1, \dots, U_d, C) \in \mathcal{U} : \text{rank}(b_i(C)) \leq r_i - 1\}, \quad i = 1, \dots, d.$$

We will show that $\mathcal{U} \setminus \bigcup_{i=1}^d \mathcal{R}_i$ is connected by comparing dimensions. Clearly,

$$\dim_{\mathbb{R}}(\mathcal{U}) = r_1 + \prod_{i=1}^d r_i$$

since $\Phi(\mathbb{R}^{r_1})$ has dimension r_1 and the fiber of p has dimension $\prod_{i=1}^d r_i$. The codimension of \mathcal{R}_1 in \mathcal{U} is at least two: \mathcal{R}_1 is the intersection of \mathcal{U} with the set $\mathcal{V} = \{(U_1, \dots, U_d, C) \in \mathcal{T}_{r_1, \dots, r_d} : \text{rank}(b_1(C)) \leq r_1 - 1\}$; as all $r_1 \times r_1$ minors of $b_1(C) = \begin{bmatrix} L & 0 \\ 0 & 0 \end{bmatrix}$ vanishes and L is an $(r_1 + 1) \times r_1$ matrix, $\mathcal{R}_1 = \mathcal{U} \cap \mathcal{V}$ must be of at least codimension two in \mathcal{U} . The same is true for $i = 2, \dots, d$, where

$$\dim_{\mathbb{R}}(\mathcal{R}_i) \leq r_1 + (r_i - 1) \prod_{j \neq i}^d r_j + (r_i - 1) = r_1 + \prod_{i=1}^d r_i - \left(\prod_{j \neq i}^d r_j - r_i + 1 \right);$$

by assumption, $\prod_{j \neq i}^d r_j > r_i$ for $i = 2, \dots, d$, and so we have $\dim_{\mathbb{R}}(\mathcal{R}_i) \leq r_1 + \prod_{i=1}^d r_i - 2$. Hence $\mathcal{U} \setminus \bigcup_{i=1}^d \mathcal{R}_i$ is connected by Theorem 3.5. In particular, there is a curve in $\mathcal{U} \setminus \bigcup_{i=1}^d \mathcal{R}_i \subseteq X_{r_1, \dots, r_d}$ connecting (U_1, \dots, U_d, A) and (U_1, \dots, U_d, B) , completing the proof in this case.

Finally, if $r_1 = \prod_{i=2}^d r_i = n_1$, we consider the map

$$f : X_{r_1, \dots, r_d} \rightarrow \mathbb{R}, \quad f(A) = \det(b_1(A)).$$

We see that X_{r_1, \dots, r_d} is a disjoint union of the preimages $f^{-1}(0, \infty)$ and $f^{-1}(-\infty, 0)$. It is straightforward — by an argument similar to the case $r_1 = \prod_{i=2}^d r_i < n_1$ — to show that both $f^{-1}(0, \infty)$ and $f^{-1}(-\infty, 0)$ are connected. Hence X_{r_1, \dots, r_d} has two connected components in this case. \square

As multilinear rank must necessarily satisfy (8.3), the three cases in Theorem 8.2 cover all possibilities.

For the case $\mathbb{F} = \mathbb{C}$, it follows from (8.3) that the *real* codimension in (8.4) is always at least two, and we easily obtain the following for complex tensors.

Theorem 8.3 (Connectedness of multilinear rank over \mathbb{C}). *Let W_1, \dots, W_d be complex vector spaces. The set of multilinear rank- (r_1, \dots, r_d) complex tensors*

$$\{A \in W_1 \otimes \cdots \otimes W_d : \mu\text{rank}(A) = (r_1, \dots, r_d)\}$$

is always connected.

8.2. Higher homotopy groups of multilinear rank. Let V be a real vector space of dimension n and let $r \leq n$. Theorem 3.6 allows one to determine $\pi_k(\text{Gr}(r, V))$ from the fiber bundle

$$\text{O}(r) \rightarrow \text{St}(r, V) \rightarrow \text{Gr}(r, V),$$

where $\text{O}(r)$ is the orthogonal group and $\text{St}(r, V)$ is the *Stiefel manifold* of r -frames in V . Since $\text{St}(r, V)$ is $(n - r - 1)$ -connected [27], $\pi_k(\text{St}(r, V)) = 0$ and thus

$$(8.5) \quad \pi_k(\text{Gr}(r, V)) \cong \pi_{k-1}(\text{O}(r))$$

for all $k \leq n - r - 1$.

We will study the homotopy groups of $X_{r_1, \dots, r_d}(V_1, \dots, V_d)$ for real vector spaces V_1, \dots, V_d . For nondegenerate results, we will assume that each $r_i \geq 2$. By (8.3), we must have

$$r_0 := \min_{i=1, \dots, d} \left[\left(\prod_{j \neq i} r_j \right) - r_i \right] \geq 0.$$

We will impose a slight restriction that $r_0 \geq 1$. Then it follows from (8.4) that

$$\text{codim}_{\mathbb{R}} \left(\bigcup_{i=1}^d \mathcal{S}_i, \mathcal{T}_{r_1, \dots, r_d} \right) = r_0 + 1 \geq 2.$$

So by Theorem 3.5 and Theorem 3.6, for $k < r_0$,

$$\begin{aligned} \pi_k(X_{r_1, \dots, r_d}(V_1, \dots, V_d)) &\cong \pi_k(\mathcal{T}_{r_1, \dots, r_d}) \cong \pi_k(\text{Gr}(r_1, V_1) \times \dots \times \text{Gr}(r_d, V_d)) \\ &\cong \pi_k(\text{Gr}(r_1, V_1)) \times \dots \times \pi_k(\text{Gr}(r_d, V_d)), \end{aligned}$$

which implies that when $n_i = \dim_{\mathbb{R}}(V_i)$ is large enough, the homotopy groups $\pi_k(X_{r_1, \dots, r_d}(V_1, \dots, V_d))$ do not depend on V_1, \dots, V_d , a consequence of (8.5). Hence when $k \leq \min\{r_0 - 1, n_1 - r_1 - 1, \dots, n_d - r_d - 1\}$, it follows from (8.5) that

$$\pi_k(X_{r_1, \dots, r_d}(V_1, \dots, V_d)) \cong \pi_{k-1}(\text{O}(r_1)) \times \dots \times \pi_{k-1}(\text{O}(r_d)).$$

The required homotopy groups then follows from the Bott Periodicity Theorem [10, 11]. We will state these formally below.

We will introduce a further abbreviation for the set of multilinear rank- (r_1, \dots, r_d) real tensors in (8.1). We write

$$X_{r_1, \dots, r_d}(n_1, \dots, n_d) := X_{r_1, \dots, r_d}(V_1, \dots, V_d)$$

if V_1, \dots, V_d are real vector spaces of dimensions n_1, \dots, n_d . The colimit of the sequence

$$X_{r_1, \dots, r_d}(n_1, \dots, n_d) \subseteq X_{r_1, \dots, r_d}(n_1 + 1, \dots, n_d + 1) \subseteq X_{r_1, \dots, r_d}(n_1 + 2, \dots, n_d + 2) \subseteq \dots$$

will be denoted by $X_{r_1, \dots, r_d}(\infty)$. Note that the homotopy groups $\pi_k(X_{r_1, \dots, r_d}(\infty))$ also repeat periodically for small k by Bott periodicity.

Theorem 8.4 (Higher homotopy groups of multilinear rank over \mathbb{R}). (i) For large enough $r_i < n_i$, when $0 < k \leq \min\{r_0 - 1, n_1 - r_1 - 1, \dots, n_d - r_d - 1\}$,

$k \bmod 8$	0	1	2	3	4	5	6	7
$\pi_k(X_{r_1, \dots, r_d}(n_1, \dots, n_d))$	\mathbb{Z}^d	$(\mathbb{Z}/2\mathbb{Z})^d$	$(\mathbb{Z}/2\mathbb{Z})^d$	0	\mathbb{Z}^d	0	0	0

(ii) For large enough r_i , when $0 < k < r_0$,

$k \bmod 8$	0	1	2	3	4	5	6	7
$\pi_k(X_{r_1, \dots, r_d}(\infty))$	\mathbb{Z}^d	$(\mathbb{Z}/2\mathbb{Z})^d$	$(\mathbb{Z}/2\mathbb{Z})^d$	0	\mathbb{Z}^d	0	0	0

The same argument applies to complex tensors of multilinear rank (r_1, \dots, r_d) with the unitary group $\text{U}(r)$ in place of $\text{O}(r)$. More precisely, let W_1, \dots, W_d be complex vector spaces of complex dimensions n_1, \dots, n_d respectively. We write

$$X_{r_1, \dots, r_d}^{\mathbb{C}}(n_1, \dots, n_d) := X_{r_1, \dots, r_d}(W_1, \dots, W_d),$$

for the set of multilinear rank- (r_1, \dots, r_d) complex tensors. In addition, let $X_{r_1, \dots, r_d}^{\mathbb{C}}(\infty)$ denote the colimit of the sequence

$$X_{r_1, \dots, r_d}^{\mathbb{C}}(n_1, \dots, n_d) \subseteq X_{r_1, \dots, r_d}^{\mathbb{C}}(n_1 + 1, \dots, n_d + 1) \subseteq X_{r_1, \dots, r_d}^{\mathbb{C}}(n_1 + 2, \dots, n_d + 2) \subseteq \dots$$

Then when $k \leq \min\{r_0 - 1, 2n_1 - 2r_1, \dots, 2n_d - 2r_d\}$,

$$\pi_k(X_{r_1, \dots, r_d}^{\mathbb{C}}(n_1, \dots, n_d)) \cong \pi_{k-1}(U(r_1)) \times \dots \times \pi_{k-1}(U(r_d)).$$

Theorem 8.5 (Higher homotopy groups of multilinear rank over \mathbb{C}). (i) For large enough $r_i < n_i$, when $0 < k \leq \min\{r_0 - 1, 2n_1 - 2r_1, \dots, 2n_d - 2r_d\}$,

$k \bmod 8$	0	1	2	3	4	5	6	7
$\pi_k(X_{r_1, \dots, r_d}^{\mathbb{C}}(n_1, \dots, n_d))$	\mathbb{Z}^d	0	\mathbb{Z}^d	0	\mathbb{Z}^d	0	\mathbb{Z}^d	0

(ii) For large enough r_i , when $0 < k < r_0$,

$k \bmod 8$	0	1	2	3	4	5	6	7
$\pi_k(X_{r_1, \dots, r_d}^{\mathbb{C}}(\infty))$	\mathbb{Z}^d	0	\mathbb{Z}^d	0	\mathbb{Z}^d	0	\mathbb{Z}^d	0

9. TOPOLOGY OF SYMMETRIC MULTILINEAR RANK

It is easy to see that for a symmetric tensor $A \in \mathcal{S}^d(V) \subseteq V^{\otimes d}$, its multilinear rank (r_1, \dots, r_d) must satisfy $r_1 = \dots = r_d$. We may therefore define a corresponding notion of symmetric subspace variety and symmetric multilinear rank.

Definition 9.1. Let V be a vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} of dimension n . Let $r \leq n$ be a positive integer. The *symmetric subspace variety* is the set

$$\text{Sub}_r(V) := \{A \in \mathcal{S}^d(V) : A \in \mathcal{S}^d(U), U \subseteq V, \dim(U) = r\}.$$

We say that $A \in \mathcal{S}^d(V)$ has *symmetric multilinear rank* r , or, in notation,

$$\mu\text{rank}_{\mathcal{S}}(A) = r,$$

if whenever $A \in \text{Sub}_s(V)$, we must have $r = s$. In other words $\text{Sub}_r(V)$ is the smallest symmetric subspace variety that contains A .

Clearly, the definition implies that

$$\text{Sub}_r(V) = \{A \in \mathcal{S}^d(V) : \mu\text{rank}_{\mathcal{S}}(A) \leq r\}.$$

We are also interested in the set of tensors of multilinear rank *exactly* r , which we will denote by

$$(9.1) \quad Y_r(V) := \{A \in \mathcal{S}^d(V) : \mu\text{rank}_{\mathcal{S}}(A) = r\}.$$

9.1. Path-connectedness of symmetric multilinear rank. We study the connectedness of the set of symmetric tensors of symmetric multilinear rank r , i.e., $Y_r(V)$ as defined in (9.1). Here V is an n -dimensional vector space over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and $r = 1, \dots, n$.

Our approach in this section mirrors the one we used in Section 8.1 but is somewhat simpler this time. Let $\mathbb{F} = \mathbb{R}$. We consider the vector bundle \mathcal{Q}_r over $\text{Gr}(r, V)$ defined by

$$(9.2) \quad \mathcal{Q}_r := \{(U, A) \in \text{Gr}(r, V) \times \mathcal{S}^d(V) : A \in \mathcal{S}^d(U)\}$$

and the map

$$\rho_r : \mathcal{Q}_r \rightarrow \mathcal{S}^d(V), \quad (U, A) \mapsto A.$$

The image of ρ_r is precisely $\text{Sub}_r(V)$, the symmetric subspace variety as defined in Definition 9.1.

Theorem 9.2 (Connectedness of symmetric multilinear rank over \mathbb{R}). *Let V be a real vector space of dimension n .*

(i) When $r = 1$ and d is odd, the set of symmetric multilinear rank-one real tensors

$$\{A \in S^d(V) : \mu\text{rank}_S(A) = 1\}$$

is a connected set.

(ii) When $r = 1$ and d is even, the set of symmetric multilinear rank-one real tensors

$$\{A \in S^d(V) : \mu\text{rank}_S(A) = 1\}$$

has two connected components.

(iii) When $d = 2$, the set of symmetric multilinear rank- r real tensors

$$\{A \in S^d(V) : \mu\text{rank}_S(A) = r\}$$

has $r + 1$ connected components.

(iv) When $r \geq 2$ and $d \geq 3$, the set of symmetric multilinear rank- r real tensors

$$\{A \in S^d(V) : \mu\text{rank}_S(A) = r\}$$

is a connected set.

Proof. Note that when $r = 1$ or when $d = 2$, symmetric multilinear rank and symmetric rank coincide. Since the connectedness of the latter has been addressed in Proposition 4.2 and Theorem 4.4, we will focus on the last case where $r \geq 2$ and $d \geq 3$. Let

$$(9.3) \quad \mathcal{Y}_r := \{(U, A) \in \mathcal{Q}_r : \mu\text{rank}_S(A) = r\} \quad \text{and} \quad \mathcal{L}_r := \{(U, A) \in \mathcal{Q}_r : \mu\text{rank}_S(A) < r\}.$$

Then $\rho_r : \mathcal{Y}_r \rightarrow Y_r(V)$ is a homeomorphism and $\mathcal{Y}_r = \mathcal{Q}_r \setminus \mathcal{L}_r$. Observe that

$$\dim_{\mathbb{R}}(\mathcal{Q}_r) = r(n - r) + \binom{r + d - 1}{d},$$

and

$$(9.4) \quad \dim_{\mathbb{R}}(\mathcal{L}_r) = r(n - r) + (r - 1) + \binom{r + d - 2}{d} = \dim_{\mathbb{R}}(\mathcal{Q}_r) - \left[\binom{r + d - 2}{d - 1} - r + 1 \right].$$

If $r \geq 2$ and $d \geq 3$, then by (9.4), \mathcal{L}_r has real codimension at least two in \mathcal{Q}_r . Hence, by Theorem 3.5, Y_r is connected. \square

For the case $\mathbb{F} = \mathbb{C}$, when $d \geq 3$ and $r \geq 2$, the *real* codimension in (9.4) is always at least two. So the connectedness in the complex case follows easily from Theorem 3.7.

Theorem 9.3 (Connectedness of symmetric multilinear rank over \mathbb{C}). *Let W be a complex vector space. The set of symmetric multilinear rank- r complex tensors*

$$\{A \in S^d(W) : \mu\text{rank}_S(A) = r\}$$

is always connected.

9.2. Higher homotopy groups of symmetric multilinear rank. Let V be a vector space of dimension n over $\mathbb{F} = \mathbb{R}$ or \mathbb{C} . We will study the homotopy groups of the set $Y_r(V)$ of symmetric multilinear rank- r tensors. We will focus on the interesting case that $d \geq 3$, $r \geq 2$, and $n \geq 2$. In this case,

$$s_0 := \binom{r + d - 2}{d - 1} - r \geq 1,$$

and it follows from (9.4) that

$$\text{codim}_{\mathbb{R}}(\mathcal{L}_r, \mathcal{Q}_r) = s_0 + 1 \geq 2,$$

where \mathcal{L}_r and \mathcal{Q}_r are as defined in (9.3) and (9.2). So by Theorem 3.5 and Theorem 3.6, for $k < s_0$,

$$(9.5) \quad \pi_k(Y_r(V)) \cong \pi_k(\mathcal{Q}_r) \cong \pi_k(\text{Gr}(r, V)),$$

implying that when $\dim_{\mathbb{F}}(V)$ is large enough, the homotopy group $\pi_k(Y_r(V))$ does not depend on V . As in Section 8.2, we will write

$$Y_r(V) = \begin{cases} Y_r(n) & \text{if } V \text{ is a real vector space of real dimension } n, \\ Y_r^{\mathbb{C}}(n) & \text{if } V \text{ is a complex vector space of complex dimension } n, \end{cases}$$

The colimits of the sequences

$$Y_r(n) \subseteq Y_r(n+1) \subseteq Y_r(n+2) \subseteq \dots \quad \text{and} \quad Y_r^{\mathbb{C}}(n) \subseteq Y_r^{\mathbb{C}}(n+1) \subseteq Y_r^{\mathbb{C}}(n+2) \subseteq \dots$$

will be denoted by $Y_r(\infty)$ and $Y_r^{\mathbb{C}}(\infty)$ respectively. As in Section 8.2, we obtain the following results from (9.5) and Bott periodicity.

Theorem 9.4 (Higher homotopy groups of symmetric multilinear rank over \mathbb{R}).

(i) For large enough $r < n$, when $0 < k \leq \min\{s_0 - 1, n - r - 1\}$,

$$\begin{array}{c|cccccccc} k \bmod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \pi_k(Y_r(n)) & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & 0 \end{array}$$

(ii) For large enough r , when $0 < k < s_0$,

$$\begin{array}{c|cccccccc} k \bmod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \pi_k(Y_r(\infty)) & \mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & \mathbb{Z}/2\mathbb{Z} & 0 & \mathbb{Z} & 0 & 0 & 0 \end{array}$$

Theorem 9.5 (Higher homotopy groups of symmetric multilinear rank over \mathbb{C}).

(i) For large enough $r < n$, when $0 < k \leq \min\{s_0 - 1, 2n - 2r\}$,

$$\begin{array}{c|cccccccc} k \bmod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \pi_k(Y_r^{\mathbb{C}}(n)) & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \end{array}$$

(ii) For large enough r_i , when $0 < k < r_0$,

$$\begin{array}{c|cccccccc} k \bmod 8 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \pi_k(Y_r^{\mathbb{C}}(\infty)) & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \end{array}$$

10. CONCLUSION

We view our work in this article as a first step towards unraveling the topology of the set of fixed-rank tensors for various common notions of rank. There are still many unanswered questions, notably the higher homotopy groups of rank- r tensors and symmetric rank- r symmetric tensors when $r \geq 4$. However, from an applications point-of-view, the results in this article about path-connectedness and fundamental groups are relatively complete and provide full answers to questions about the feasibility of Riemannian optimization methods and homotopy continuation methods in low-rank approximations and rank decompositions of tensors. Two other aspects we left unexplored are: (i) possible connections with the very substantial body of work¹⁰ on the topology of algebraic varieties, and (ii) more general relations between singular loci and fundamental groups, leaving room for further future work.

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¹⁰See for instance <https://www.math.ias.edu/sp/topalgvar>.

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