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ON THE MORITA REDUCED VERSIONS OF SKEW GROUP ALGEBRAS OF PATH ALGEBRAS

PATRICK LE MEUR

ABSTRACT. Let R be the skew group algebra of a finite group acting on the path algebra of a quiver. This article develops both theoretical and practical methods to do computations in the Morita reduced algebra associated to R . Reiten and Riedtmann proved that there exists an idempotent e of R such that the algebra eRe is both Morita equivalent to R and isomorphic to the path algebra of some quiver which was described by Demonet. This article gives explicit formulas for the decomposition of any element of eRe as a linear combination of paths in the quiver described by Demonet. This is done by expressing appropriate compositions and pairings in a suitable monoidal category which takes into account the representation theory of the finite group.

INTRODUCTION

In [12], Reiten and Riedtmann initiated the investigation of the skew group algebras of Artin algebras from the viewpoint of homological dimensions and representation theory. Their article was followed-up by many research works illustrating the following principle: Artin algebras share many properties, whether of homological or of representation theoretic nature, with their associated skew group algebras. For instance, see [11] for interactions with Koszul duality; [4, 10] for interactions with piecewise hereditary algebras; and [13, 3] for interactions with preprojective algebras and McKay correspondence.

Let \mathbb{k} be an algebraically closed field, Q be a finite quiver which may have oriented cycles, and G be a finite group with order not divisible by $\text{char}(\mathbb{k})$ and acting on $\mathbb{k}Q$ by algebra automorphisms in such a way that both the set of (primitive idempotents of) vertices and the vector subspace generated by the arrows are stabilised by this action. Following [12], the skew group algebra $\mathbb{k}Q * G$ is hereditary.

While Reiten and Riedtmann described a quiver whose path algebra is Morita equivalent to $\mathbb{k}Q * G$ when G is cyclic, the description of such a quiver in general is recent. In [3], Demonet described an idempotent \tilde{e} of $\mathbb{k}Q * G$ and a quiver Q_G such that $\tilde{e} \cdot (\mathbb{k}Q * G) \cdot \tilde{e}$ is Morita equivalent to $\mathbb{k}Q * G$ and isomorphic to $\mathbb{k}Q_G$. This, however, does not yield a completely described isomorphism, which might be source of trouble. Here is a situation taken from Calabi-Yau algebras and where such trouble may occur. Following [9], if W is a G -invariant potential on Q , then G acts on the Ginzburg dg algebra $\mathcal{A}(Q, W)$, defined in [6], and $\mathcal{A}(Q, W) * G$ is Morita equivalent to $\mathcal{A}(Q_G, W_G)$, where W_G is the potential on Q_G which, as a linear combination of oriented cycles, corresponds to $\tilde{e} \cdot W \cdot \tilde{e}$ under any chosen isomorphism $\mathbb{k}Q_G \rightarrow \tilde{e} \cdot (\mathbb{k}Q * G) \cdot \tilde{e}$. The problem here is that describing that linear combination explicitly is not easy. Up to now, the only fairly general explicit descriptions of W_G are due to Amiot and Plamondon ([1]) when G is of order 2 and to Giovannini and Pasquali ([8]) when G is cyclic, the stabiliser of each vertex is either trivial or the whole group, and every cycle appearing in W goes through

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fixed vertices only (or, through vertices with trivial stabilisers only, respectively) as soon as it goes through two of them.

This raises the question of decomposing explicitly the elements of $\tilde{e} \cdot (\mathbb{k}Q * G) \cdot \tilde{e}$ as linear combinations of paths in Q_G under an isomorphism $\mathbb{k}Q_G \rightarrow \tilde{e} \cdot (\mathbb{k}Q * G) \cdot \tilde{e}$.

Demonet's description of Q_G involves the representation theory of the stabilisers of the vertices of Q . It is hence expectable that answering the above mentioned question should involve representation theory. Answers of this kind already exist in particular cases where Q has only one vertex. Here is an example due to Ginzburg ([6]). He proved that, if G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, then $\mathbb{C}[x, y, z] * G$ is Calabi-Yau in dimension 3. This was done by taking Q to be the quiver with one vertex and three loops x, y , and z , by taking $W = xyz - xzy$, and by expressing W_G in terms of the monoidal category of finite dimensional representations of G . See [2] for a generalisation to $\mathbb{C}[x_1, \dots, x_n] * G$ where G is a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$, in which case Q is the quiver with one vertex and n loops. Note that this generalisation is (detailed and) successfully applied by Giovannini in [7] to find subgroups Γ of $\mathrm{SL}_{n+1}(\mathbb{C})$ such that $\mathbb{C}[x_1, \dots, x_{n+1}] * \Gamma$ is Morita equivalent to the $(n+1)$ -preprojective algebra of an n -representation finite algebra.

This article hence gives explicit formulas for the decomposition of the elements of $\tilde{e} \cdot (\mathbb{k}Q * G) \cdot \tilde{e}$ as linear combinations of paths in Q_G under the isomorphism $\mathbb{k}Q_G \rightarrow \tilde{e} \cdot (\mathbb{k}Q * G) \cdot \tilde{e}$. These formulas are obtained by translating explicitly the result of natural operations and pairings in the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$ of finite dimensional A -bimodules, where A is the direct product of the group algebras of the stabilisers of the vertices of Q .

On one hand, these formulas provide an interpretation of the coefficients of these linear combinations in terms of the representation theory of the stabilisers of the vertices of Q . On the other hand, they are explicit enough to be implemented by a computer or a human being able to manipulate the irreducible representations of the involved groups. In specific cases, for instance when the action of the group elements transform arrows to scalar multiples of arrows and the stabilisers are cyclic, these formulas simplify in a combinatorial way. Finally, these formulas give a complete solution to the problem of computing a potential W_G on Q_G such that the Ginzburg dg algebra $\mathcal{A}(Q_G, W_G)$ is Morita equivalent to $\mathcal{A}(Q, W) * G$, for any given G -invariant potential W on Q .

1. DEFINITIONS AND MAIN RESULTS

Throughout the article, \mathbb{k} denotes an algebraically closed field. For all finite dimensional algebras Λ , their categories of finite dimensional left modules are denoted by $\mathrm{mod}(\Lambda)$.

This section presents the main results of this text as well as the needed definitions and notation. These results are illustrated in section 2. The remaining sections are devoted to the proofs of these results.

Setting 1.1. Let Q_0, S, G , and M be as follows.

- Q_0 is a finite set and S is the semi-simple \mathbb{k} -algebra $\mathbb{k}Q_0$.
- G is a finite group with order not divisible by $\mathrm{char}(\mathbb{k})$ and acting on Q_0 .
- M is a finite dimensional S -bimodule, considered as a collection of vector spaces $({}_i M_j)_{(i,j) \in Q_0^2}$, endowed with an action of G written exponentially and such that ${}^g({}_i M_j) = {}_{g \cdot i} M_{g \cdot j}$ for all $i, j \in Q_0$ and $g \in G$.

The choice of a basis of ${}_i M_j$, for all $i, j \in Q_0$ determines a quiver Q such that $T_S(M) \simeq \mathbb{k}Q$ as \mathbb{k} -algebras. All finite quivers arise in this way.

The actions of G on S and M induce an action on $T_S(M)$ by algebra automorphisms such that ${}^g(m_1 \otimes \dots \otimes m_n) = {}^g m_1 \otimes \dots \otimes {}^g m_n$ for all $g \in G$ and

$m_1, \dots, m_n \in M$. Recall that the skew group algebra $T_S(M) * G$ is the \mathbb{k} -algebra with underlying vector space $T_S(M) \otimes G$, where the tensor $u \otimes v$ is denoted by $u * v$ for all $u \in T_S(M)$ and $v \in \mathbb{k}G$, and with product such that $(u * g) \cdot (u' * g') = (u \cdot {}^g u') * gg'$ for all $u, u' \in T_S(M)$ and $g, g' \in G$. It is customary to consider $T_S(M)$ and $\mathbb{k}G$ as subalgebras of $T_S(M) * G$ in the canonical way.

Notation 1.2. The following data is used throughout the article.

- $[G \setminus Q_0]$ denotes a complete set of representatives of the orbits in Q_0 .
- For all $i \in Q_0$,
 - G_i denotes the stabiliser of i and e_i denotes the corresponding primitive idempotent of S ,
 - $[G/G_i]$ denotes a complete set of representatives of cosets modulo G_i ,
 - $\text{irr}(G_i)$ denotes a complete set of representatives of the isomorphism classes of the irreducible representations of G_i ,
 - and ε_U denotes a primitive idempotent of $\mathbb{k}G_i$ such that $U \simeq \mathbb{k}G_i \cdot \varepsilon_U$ in $\text{mod}(\mathbb{k}G_i)$, for all $U \in \text{irr}(G_i)$.
- \tilde{e} denotes the idempotent $\sum_{(i,U)} e_i * \varepsilon_U$ of $T_S(M) * G$, where i runs through $[G \setminus Q_0]$ and U runs through $\text{irr}(G_i)$.

Demonet proved in [3] that the skew group algebra $T_S(M) * G$ is Morita equivalent to the path algebra of a quiver as follows.

Setting 1.3. Let Q_G be a quiver with the following properties.

- The vertices are the couples (i, U) , where $i \in [G \setminus Q_0]$ and $U \in \text{irr}(G_i)$.
- For all vertices (i, U) and (j, V) , the arrows $(i, U) \rightarrow (j, V)$ form a basis over \mathbb{k} of $\text{Hom}_{\mathbb{k}G_i}(U, M(i, j; V))$, here $M(i, j; V) = \bigoplus_{y \in [G/G_j]} i M_{y \cdot j} \otimes_{\mathbb{k}} yV$, where yV stands for $y \otimes_{\mathbb{k}} V \subset \mathbb{k}G \otimes_{\mathbb{k}} V$, see Definition 1.6 for details.

Demonet's result may be reformulated as follows, see Section 5 for more details.

Theorem 1.4 ([3]). *The \mathbb{k} -algebras $\mathbb{k}Q_G$ and $T_S(M) * G$ are Morita equivalent. Besides, assuming that $U = \mathbb{k}G_i \cdot \varepsilon_U$ for all vertices (i, U) of Q_G , then there exists an isomorphism of algebras*

$$(1.0.1) \quad \mathbb{k}Q_G \xrightarrow{\sim} \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$$

which maps every arrow $f: (i, U) \rightarrow (j, V)$ of Q_G to $f(\varepsilon_U)$.

The present article introduces a \mathbb{k} -algebra, called the algebra of intertwiners relative to M and denoted by $\mathcal{I}ntw$, and an isomorphism of \mathbb{k} -algebras $\mathbb{k}Q_G \rightarrow \mathcal{I}ntw^{\text{op}}$ together with explicit formulas decomposing any given element of $\mathcal{I}ntw$ in the basis consisting of the images of the paths in Q_G under $\mathbb{k}Q_G \rightarrow \mathcal{I}ntw^{\text{op}}$. See Theorem 1.15 for details. Composing this isomorphism with the inverse of (1.0.1) results in a \mathbb{k} -algebra isomorphism $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e} \rightarrow \mathcal{I}ntw^{\text{op}}$. A description of it is given in Theorem 1.16 hence providing a decomposition, under (1.0.1), of any element of $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ in the basis of $\mathbb{k}Q_G$ consisting of the paths in Q_G . These decompositions involve non degenerate pairings between intertwiners relative to M and intertwiners relative to the dual vector space M^* . Both the product in $\mathcal{I}ntw$ and the pairings are explicit reformulations of natural operations and pairings in a suitable monoidal category.

Remark 1.5 ([5]). Let (\mathcal{C}, \otimes) be a monoidal \mathbb{k} -linear category with finite dimensional morphism spaces.

- (1) Given objects X, Y, U, V , and W of \mathcal{C} , there is a natural operation between morphisms $U \rightarrow X \otimes V$ and $V \rightarrow Y \otimes W$,

$$\begin{aligned} \mathcal{C}(U, X \otimes V) \times \mathcal{C}(V, Y \otimes W) &\longrightarrow \mathcal{C}(U, X \otimes Y \otimes W) \\ (f, f') &\longmapsto (X \otimes f') \circ f. \end{aligned}$$

- (2) Given an object X of \mathcal{C} , recall that a *left dual* of X is an object X' of \mathcal{C} together with an adjunction $(X \otimes -) \vdash (X' \otimes -)$ of endofunctors of \mathcal{C} . If X has a left dual, then there exists a non degenerate pairing, for all objects U and V of \mathcal{C} such that $\mathcal{C}(U, U) = \mathbb{k} \cdot \text{Id}_U$,

$$\begin{aligned} \langle - | - \rangle: \mathcal{C}(U, X \otimes V) \otimes \mathcal{C}(V, X' \otimes U) &\longrightarrow \mathbb{k} \\ f \otimes \phi &\longmapsto \langle f | \phi \rangle \end{aligned}$$

such that $\langle f | \phi \rangle \cdot \text{Id}_U = \bar{\phi} \circ f$, where $\bar{\phi} \in \mathcal{C}(X \otimes V, U)$ is adjoint to ϕ .

The monoidal category \mathcal{C} considered here is the category $\text{mod}(A^e)$ of finite dimensional A -bimodules with tensor product being \otimes_A , where A is as follows,

$$(1.0.2) \quad A = \mathbb{k} \times \prod_{i \in Q_0} \mathbb{k}G_i.$$

Note that A is a semi-simple \mathbb{k} -algebra; accordingly, any object X of \mathcal{C} has a left dual given by the dual vector space X^* . The isolated factor \mathbb{k} in the definition of A is included so that \mathcal{C} contains $\text{mod}(\mathbb{k}G_i)$ as a full additive subcategory, yet not as a monoidal subcategory, for all $i \in Q_0$. Note that a smaller algebra might be used instead of A ; however, the reason for introducing \mathcal{C} is to prove the needed properties of the product in $\mathcal{I}ntw$ and of the pairing; besides (1.0.2) has the advantage of being easy to describe.

Here are the base ingredients needed to define $\mathcal{I}ntw$.

Definition 1.6. Let n be a non negative integer. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 . Let $V \in \text{mod}(\mathbb{k}G_{i_n})$.

- (1) For all sequences $\underline{y} = y_1, \dots, y_n$, where $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, let $M_{\underline{y}}(\underline{i}; V)$ be the following vector space,

$$M_{\underline{y}}(\underline{i}; V) = {}_{i_0}M_{y_1 \cdot i_1} \otimes_{\mathbb{k}} {}_{y_1 \cdot i_1}M_{y_1 y_2 \cdot i_2} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} {}_{y_1 \cdots y_{n-1} \cdot i_{n-1}}M_{y_1 \cdots y_n \cdot i_n} \otimes_{\mathbb{k}} y_1 \cdots y_n V,$$

where $y_1 \cdots y_n V$ stands for the vector subspace $y_1 \cdots y_n \otimes_{\mathbb{k}} V$ of $\mathbb{k}G \otimes_{\mathbb{k}} V$.

- (2) Define the $\mathbb{k}G_{i_0}$ -module $M(\underline{i}; V)$ as follows. Its underlying vector space is

$$(1.0.3) \quad \bigoplus_{\underline{y}} M_{\underline{y}}(\underline{i}; V),$$

where \underline{y} runs through all possible sequences such as in (1). For all $g \in G_{i_0}$, the action of g is written exponentially and defined as follows. For all \underline{y} , there exist a unique sequence $\underline{y}' = y'_1, \dots, y'_n$, where $y'_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, and a unique $h \in G_{i_n}$, such that

- $gy_1 \cdots y_t G_{i_t} = y'_1 \cdots y'_t G_{i_t}$ for all $t \in \{1, \dots, n\}$,
- and $gy_1 \cdots y_n = y'_1 \cdots y'_n h$;

then, the action of g transforms $M_{\underline{y}}(\underline{i}; V)$ into $M_{\underline{y}'}(\underline{i}; V)$ and, for all $m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n v \in M_{\underline{y}}(\underline{i}; V)$,

$${}^g(m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n v) = {}^g m_1 \otimes \cdots \otimes {}^g m_n \otimes y'_1 \cdots y'_n h v.$$

This does define a finite dimensional $\mathbb{k}G_{i_0}$ -module. If $n = 0$, then $M(\underline{i}; V) = V$. In the monoidal category $(\text{mod}(A^e), \otimes_A)$ considered previously, $M(\underline{i}; V)$ is isomorphic to $M(\underline{i}) \otimes_A V$ where $M(\underline{i})$ is a $\mathbb{k}G_{i_0} - \mathbb{k}G_{i_n}$ -bimodule defined by M and \underline{i} , see (4.1.1) for the definition of $M(\underline{i})$ and (4.1.7) for the isomorphism.

Note that, in the previous definition, if $U = \mathbb{k}G_{i_0} \cdot \varepsilon_U$ and $V = \mathbb{k}G_{i_n} \cdot \varepsilon_V$ for primitive idempotents $\varepsilon_U \in \mathbb{k}G_{i_0}$ and $\varepsilon_V \in \mathbb{k}G_{i_n}$, then $M(\underline{i}; U)$ is a $\mathbb{k}G_{i_0}$ -submodule of $T_S(M) * G$.

The main results of this article are based on a description of the quiver $\mathbb{k}Q_G$ in terms of intertwiners.

Definition 1.7. An *intertwiner* relative to M is a morphism of $\mathbb{k}G_{i_0}$ -modules $U \rightarrow M(\underline{i}; V)$, for some sequence $\underline{i} = i_0, \dots, i_n$ in Q_0 and some modules $U \in \text{mod}(\mathbb{k}G_{i_0})$ and $V \in \text{mod}(\mathbb{k}G_{i_n})$.

Note that, in the previous definition, if $U = \mathbb{k}G_{i_0} \cdot \varepsilon_U$ and $V = \mathbb{k}G_{i_n} \cdot \varepsilon_V$ for primitive idempotents $\varepsilon_U \in \mathbb{k}G_{i_0}$ and $\varepsilon_V \in \mathbb{k}G_{i_n}$, then all intertwiners $U \rightarrow M(\underline{i}; V)$ take their values in $T_S(M) * G$.

The main purpose of this article is to provide explicit formulas in the main theorems. These formulas use the following notation.

Notation 1.8. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 where $n \geq 1$. Let $U \in \text{mod}(\mathbb{k}G_{i_0})$ and $V \in \text{mod}(\mathbb{k}G_{i_n})$. Let $f \in \text{Hom}_{\mathbb{k}G_{i_0}}(U, M(\underline{i}; V))$. For all $u \in U$, write the decomposition of $f(u)$ along (1.0.3) as follows

$$f(u) = \sum_{\underline{y}} f_{\underline{y}}(u);$$

and, for all sequences $\underline{y} = y_1, \dots, y_n$, where $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, write symbolically the element $f_{\underline{y}}(u)$ of $M_{\underline{y}}(\underline{i}; V)$ as follows, with the sum sign omitted,

$$f_{\underline{y}}(u) = f_{\underline{y}}^{(1)}(u) \otimes \dots \otimes f_{\underline{y}}^{(n)}(u) \otimes y_1 \cdots y_n f_{\underline{y}}^{(0)}(u);$$

hence, $f_{\underline{y}}^{(t)}(u)$ is meant to lie in ${}_{y_1 \cdots y_{t-1} \cdot i_{t-1}} M_{y_1 \cdots y_t \cdot i_t}$, for all $t \in \{1, \dots, n\}$, and $f_{\underline{y}}^{(0)}(u)$ is meant to lie in V ; as a whole,

$$(1.0.4) \quad f(u) = \sum_{\underline{y}} f_{\underline{y}}^{(1)}(u) \otimes \dots \otimes f_{\underline{y}}^{(n)}(u) \otimes y_1 \cdots y_n f_{\underline{y}}^{(0)}(u).$$

Now here is the operation to be used as the multiplication in $\mathcal{I}ntw$.

Definition 1.9. Let $\underline{i}'' = i_0, \dots, i_{m+n}$ be a sequence in Q_0 , where $m, n \geq 1$. Denote the sequences i_0, \dots, i_m and $i_m, i_{m+1}, \dots, i_{m+n}$ by \underline{i} and \underline{i}' , respectively. Let $U \in \text{mod}(\mathbb{k}G_{i_0})$, $V \in \text{mod}(\mathbb{k}G_{i_m})$, and $W \in \text{mod}(\mathbb{k}G_{i_{m+n}})$. For all $f \in \text{Hom}_{\mathbb{k}G_{i_0}}(U, M(\underline{i}; V))$ and $f' \in \text{Hom}_{\mathbb{k}G_{i_m}}(V, M(\underline{i}'; W))$, define a \mathbb{k} -linear mapping

$$f' \otimes f: U \rightarrow M(\underline{i}''; W)$$

by

$$(1.0.5) \quad f' \otimes f = \sum_{\underline{y}''} (f' \otimes f)_{\underline{y}''},$$

where

- \underline{y}'' runs through all sequences y_1, \dots, y_{m+n} with $y_t \in [G/G_{i_t}]$ for all t
- and $(f' \otimes f)_{\underline{y}''}$ is the mapping $U \rightarrow M_{\underline{y}''}(\underline{i}''; W)$ such that, for all $u \in U$,

$$\begin{aligned} (f' \otimes f)_{\underline{y}''}(u) &= f_{\underline{y}}^{(1)}(u) \otimes \dots \otimes f_{\underline{y}}^{(m)}(u) \otimes \\ &\quad y_1 \cdots y_m (f'_{\underline{y}'}^{(1)}(f_{\underline{y}}^{(0)}(u))) \otimes \dots \otimes y_1 \cdots y_m (f'_{\underline{y}'}^{(n)}(f_{\underline{y}}^{(0)}(u))) \otimes \\ &\quad y_1 \cdots y_{m+n} f_{\underline{y}'}^{(0)}(f_{\underline{y}}^{(0)}(u)), \end{aligned}$$

where $\underline{y} = y_1, \dots, y_m$ and $\underline{y}' = y_{m+1}, \dots, y_{m+n}$.

Note that, in the previous definition, if $U = \mathbb{k}G_{i_0} \cdot \varepsilon_U$, $V = \mathbb{k}G_{i_m} \cdot \varepsilon_V$, and $W = \mathbb{k}G_{i_{m+n}} \cdot \varepsilon_W$ for primitive idempotents $\varepsilon_U \in \mathbb{k}G_{i_0}$, $\varepsilon_V \in \mathbb{k}G_{i_m}$, and $\varepsilon_W \in \mathbb{k}G_{i_{m+n}}$, then

$$(1.0.6) \quad (f' \otimes f)(\varepsilon_U) = f(\varepsilon_U) \cdot f'(\varepsilon_V),$$

where the product on the right-hand side is taken in $T_S(M) * G$.

The definition of \otimes is technical. Actually, it is an explicit reformulation of a simple operation in the monoidal category $(\text{mod}(A^e), \otimes_A)$. More precisely, using the comment which follows Definition 1.6,

- first, $M(\underline{i}'') \simeq M(\underline{i}') \otimes_A M(\underline{i}')$, see Lemma 4.2;
- next, if $M(\underline{i}; V)$, $M(\underline{i}'; W)$, and $M(\underline{i}''; W)$ are identified with $M(\underline{i}) \otimes_A V$, $M(\underline{i}') \otimes_A W$, and $M(\underline{i}'') \otimes_A W$, respectively, then \otimes is an explicit reformulation of the operation introduced in part (1) of Remark 1.5, see Lemma 4.4 for a precise statement and proof.

Accordingly, \otimes does yield intertwiners, see Lemma 4.4, and it is associative, see Proposition 4.5.

Definition 1.10. The algebra of intertwiners relative to M is denoted by $\mathcal{I}ntw$ and defined by

$$(1.0.7) \quad \mathcal{I}ntw = \left(\bigoplus_{n, \underline{i}, U, V} \text{Hom}_{\mathbb{k}G_{i_0}}(U, M(\underline{i}; V)), \otimes \right),$$

where n runs through all non negative integers, $\underline{i} = i_0, \dots, i_n$ runs through all sequences in $[G \setminus Q_0]$, and U and V run through $\text{irr}(G_{i_0})$ and $\text{irr}(G_{i_n})$, respectively.

It is hence possible to associate an element of $\mathcal{I}ntw$ to every path in Q_G .

Notation 1.11. For all paths in Q_G

$$\gamma: (i_0, U_0) \xrightarrow{f_1} (i_1, U_1) \rightarrow \dots \rightarrow (i_{n-1}, U_{n-1}) \xrightarrow{f_n} (i_n, U_n),$$

denote by f_γ the following element of $\mathcal{I}ntw$ lying in $\text{Hom}_{\mathbb{k}G_{i_0}}(U_0, M(i_0, \dots, i_n; U_n))$,

$$(1.0.8) \quad f_\gamma = f_n \otimes f_{n-1} \otimes \dots \otimes f_1.$$

As stated below, see Theorem 1.15, assigning f_γ to γ for every path γ in Q_G yields an isomorphism of algebras from $\mathbb{k}Q_G$ to $\mathcal{I}ntw^{\text{op}}$. The purpose of this article is to explain how to decompose any element of $\mathcal{I}ntw$ as a linear combination of the f_γ 's. This involves non degenerate pairings between certain spaces of intertwiners relative to M and to M^* , respectively. Here, M^* denotes the dual vector space $\text{Hom}_{\mathbb{k}}(M, \mathbb{k})$, it has a structure of S -bimodule such that ${}_i M_j^* = ({}_j M_i)^*$ for all $i, j \in Q_0$, and G acts on M^* in such a way that, for all $\varphi \in M^*$ and $g \in G$,

$${}^g \varphi = \varphi(g^{-1} \bullet).$$

This action is such that ${}^g ({}_i M_j^*) = {}_{g \cdot i} M_{g \cdot j}^*$ for all $i, j \in Q_0$ and $g \in G$. Hence, the previous considerations may be applied to M^* instead of to M .

Notation 1.12. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 , where $n \geq 0$. Let $U \in \text{mod}(\mathbb{k}G_{i_0})$ and $V \in \text{mod}(\mathbb{k}G_{i_n})$.

- (1) Denote by \underline{i}^o the sequence i_n, i_{n-1}, \dots, i_0 .
- (2) Let $\varphi \in \text{Hom}_{\mathbb{k}G_{i_n}}(V, M^*(\underline{i}^o, U))$. Proceeding similarly as done in Notation 1.8,
 - denote by $\varphi^{\underline{x}}$ the composition of φ with the canonical projection $M^*(\underline{i}^o; U) \rightarrow M_{\underline{x}}^*(\underline{i}^o; U)$, for all sequences $\underline{x} = x_{n-1}, x_{n-2}, \dots, x_0$ where $x_t \in [G/G_{i_t}]$ for all $t \in \{0, \dots, n-1\}$,
 - and for all such \underline{x} and all $v \in V$, write symbolically the element $\varphi^{\underline{x}}(v)$ of $M_{\underline{x}}^*(\underline{i}^o; U)$ as follows,

$$\varphi^{\underline{x}}(v) = \varphi_{(n)}^{\underline{x}}(v) \otimes \dots \otimes \varphi_{(1)}^{\underline{x}}(v) \otimes x_{n-1} \cdots x_0 \varphi_{(0)}^{\underline{x}}(v).$$

Hence, $\varphi_{(t)}^{\underline{x}}(v)$ is meant to lie in ${}_{(x_{n-1}x_{n-2}\cdots x_t)\cdot i_t}M_{(x_{n-1}x_{n-2}\cdots x_{t-1})\cdot i_{t-1}}^*$ for all $t \in \{1, \dots, n\}$ and $\varphi_{(0)}^{\underline{x}}(v)$ is meant to lie in U . As a whole,

$$(1.0.9) \quad \varphi(v) = \sum_{\underline{x}} \varphi_{(n)}^{\underline{x}}(v) \otimes \cdots \otimes \varphi_{(1)}^{\underline{x}}(v) \otimes x_{n-1} \cdots x_1 \varphi_{(0)}^{\underline{x}}(v).$$

The above mentioned pairings are described explicitly as follows.

Proposition 1.13. *Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 , where $n \geq 0$. Let $U \in \text{mod}(\mathbb{k}G_{i_0})$ and $V \in \text{mod}(\mathbb{k}G_{i_n})$. Assume that U is simple. There exists a non degenerate pairing*

$$(-|-): \text{Hom}_{\mathbb{k}G_{i_0}}(U, M(\underline{i}; V)) \otimes_{\mathbb{k}} \text{Hom}_{\mathbb{k}G_{i_n}}(V, M^*(\underline{i}^o; U)) \rightarrow \mathbb{k}$$

with the following property. For all $f \in \text{Hom}_{\mathbb{k}G_{i_0}}(U, M(\underline{i}; V))$ and for all $\varphi \in \text{Hom}_{\mathbb{k}G_{i_n}}(V, M^*(\underline{i}^o; U))$, the scalar $(f|\varphi)$ is such that, for all $u \in U$, the element $(f|\varphi)u$ of U is equal to

$$(1.0.10) \quad \sum_{\underline{y}} \prod_{t=1}^n \varphi_{(t)}^{\underline{x}}(f_{\underline{y}}^{(0)}(u)) ({}^{(y_1 \cdots y_n)^{-1}} f_{\underline{y}}^{(t)}(u)) {}^{h_0}(\varphi_{(0)}^{\underline{x}}(f_{\underline{y}}^{(0)}(u))),$$

where $\underline{y} = y_1, \dots, y_n$ runs through all sequences with $y_t \in [G/G_{i_t}]$ for all t and $\underline{x} = x_{n-1}, \dots, x_0$ and h_0 are uniquely determined by the following conditions.

- (a) $x_t \in [G/G_{i_t}]$ for all t , and $h_0 \in G_{i_0}$.
- (b) $(y_t \cdots y_n)^{-1} \in x_{n-1}x_{n-2} \cdots x_{t-1}G_{i_{t-1}}$ for all $t \in \{0, \dots, n-1\}$.
- (c) $(y_1 \cdots y_n)^{-1} = x_{n-1}x_{n-2} \cdots x_0 h_0$.

In view of computations, here is a useful explanation regarding \underline{x} and h_0 in the above proposition. For all sequences $\underline{y} = y_1, \dots, y_n$ in G such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$ and for all non zero tensors

$$m_1 \otimes \cdots \otimes m_n \in {}_{i_0}M_{y_1 \cdot i_1} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} {}_{(y_1 \cdots y_{n-1}) \cdot i_{n-1}}M_{(y_1 \cdots y_n) \cdot i_n},$$

there exists a unique sequence $\underline{x} = x_{n-1}, x_{n-2}, \dots, x_0$ in G such that $x_t \in [G/G_{i_t}]$ for all $t \in \{0, \dots, n-1\}$ and

$${}^{(y_1 \cdots y_n)^{-1}}(m_1 \otimes \cdots \otimes m_n) \in {}_{(x_{n-1}x_{n-2} \cdots x_0) \cdot i_0}M_{(x_{n-1} \cdots x_1) \cdot i_1} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} {}_{x_{n-1} \cdot i_{n-1}}M_{i_n};$$

besides, $(x_{n-1}x_{n-2} \cdots x_0)^{-1}(y_1 \cdots y_n)^{-1} \in G_{i_0}$. This sequence \underline{x} is the one in the statement of Proposition 1.13 in which $h_0 = (x_{n-1}x_{n-2} \cdots x_0)^{-1}(y_1 \cdots y_n)^{-1}$.

Like the operation \circledast , the pairing $(-|-)$ has a simple interpretation in the monoidal category $(\text{mod}(A^e), \otimes_A)$. Indeed,

- first, $M(\underline{i})$ admits $M^*(\underline{i}^o)$ as a left dual, see (4.1.5);
- next, if $M(\underline{i}; V)$ and $M^*(\underline{i}^o; U)$ are identified with $M(\underline{i}) \otimes_A V$ and $M^*(\underline{i}^o) \otimes_A U$, respectively, then $(-|-)$ is an explicit reformulation of $\langle -|- \rangle$ as introduced in part (2) of Remark 1.5, see (4.3.8).

Using $(-|-)$ yields intertwiners relative to M^* associated with the f_γ 's for paths γ in Q_G .

Notation 1.14. Let Q_G be such as in Setting 1.3.

- (1) Let $f \in (i, U) \rightarrow (j, V)$ be an arrow of Q_G . Using the basis of the vector space $\text{Hom}_{\mathbb{k}G_i}(U, M(i, j; V))$ consisting of the arrows $(i, U) \rightarrow (j, V)$ of Q_G and using the associated $(-|-)$ -dual basis of $\text{Hom}_{\mathbb{k}G_j}(V, M^*(j, i; U))$, denote by f^\vee the dual element associated with f . In other words, f^\vee lies in $\text{Hom}_{\mathbb{k}G_j}(V, M^*(j, i; U))$ and, for all arrows $f' : (i, U) \rightarrow (j, V)$ of Q_G ,

$$(f'|f^\vee) = \begin{cases} 1 & \text{if } f' = f \\ 0 & \text{otherwise.} \end{cases}$$

(2) For all paths γ in Q_G

$$\gamma: (i_0, U_0) \xrightarrow{f_1} (i_1, U_1) \rightarrow \cdots \rightarrow (i_{n-1}, U_{n-1}) \xrightarrow{f_n} (i_n, U_n),$$

denote by φ_γ the following intertwiner

$$\varphi_\gamma = f_1^\vee \otimes f_2^\vee \otimes \cdots \otimes f_n^\vee \in \text{Hom}_{\mathbb{k}G_{i_n}}(U_n, M^*(i_n, i_{n-1}, \dots, i_0; U_0)).$$

Now, here is the result comparing $\mathbb{k}Q_G$ and $\mathcal{I}ntw$.

Theorem 1.15. *Keep Settings 1.1 and 1.3.*

(1) Assigning f_γ to every path γ in Q_G yields an isomorphism of algebras

$$(1.0.11) \quad \mathbb{k}Q_G \xrightarrow{\sim} \mathcal{I}ntw^{\text{op}}.$$

(2) For all $f \in \mathcal{I}ntw$, say lying in $\text{Hom}_{\mathbb{k}G_{i_0}}(U_0, M(i_0, \dots, i_n; U_n))$,

$$(1.0.12) \quad f = \sum_{\gamma} (f|\varphi_\gamma) f_\gamma,$$

where γ runs through all paths of Q_G of the shape $(i_0, U_0) \rightarrow (i_1, \bullet) \rightarrow \cdots \rightarrow (i_{n-1}, \bullet) \rightarrow (i_n, U_n)$.

It is now possible to explain how to decompose any element of $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ along the paths in Q_G via the isomorphism $\mathbb{k}Q_G \xrightarrow{(1.0.1)} \tilde{e}(T_S(M) * G) \cdot \tilde{e}$. This is done with the following construction and assuming that $U = \mathbb{k}G_i \cdot \varepsilon_U$ for all vertices (i, U) of Q_G . Given a tensor $m_1 \otimes \cdots \otimes m_n * g \in e_{i_0} \cdot (M^{\otimes_S n} * G) \cdot e_{i_n}$, where $n \geq 0$ and $i_0, i_n \in [G \setminus Q_0]$, then

$m_1 \otimes \cdots \otimes m_n * g \in ({}_{i_0}M_{y_1 \cdot i_1} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} ({}_{(y_1 \cdots y_{n-1}) \cdot i_{n-1}}M_{(y_1 \cdots y_n) \cdot i_n}) * y_1 \cdots y_n \mathbb{k}G_{i_n}$ for unique $i_1, \dots, i_{n-1} \in [G \setminus Q_0]$ and $y_t \in [G/G_{i_t}]$, for all $t \in \{1, \dots, n\}$; this defines an intertwiner, for all $U \in \text{irr}(G_{i_0})$ and $V \in \text{irr}(G_{i_n})$,

$$\begin{aligned} \mathbb{k}G_{i_0} \cdot \varepsilon_U &\longrightarrow M(i_0, \dots, i_n; \mathbb{k}G_{i_n} \cdot \varepsilon_V) \\ u \cdot \varepsilon_U &\longmapsto (u \cdot \varepsilon_U) \cdot (m_1 \otimes \cdots \otimes m_n * g) \cdot (e_{i_n} * \varepsilon_V), \end{aligned}$$

where the products on the right-hand side are meant in $T_S(M) * G$; summing these intertwiners over all $U \in \text{irr}(G_{i_0})$ and $V \in \text{irr}(G_{i_n})$ defines an element of $\mathcal{I}ntw$. This construction may be extended linearly to a linear mapping

$$\hat{e} \cdot (T_S(M) * G) \cdot \hat{e} \longrightarrow \mathcal{I}ntw^{\text{op}},$$

where $\hat{e} = \sum_{i \in [G \setminus Q_0]} e_i$. Denote by Ξ the restriction of this mapping to $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$,

$$(1.0.13) \quad \Xi: \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e} \longrightarrow \mathcal{I}ntw^{\text{op}}.$$

Theorem 1.16. *Let Q_0, S, G , and M be as in Setting 1.1. Let Q_G be as in Setting 1.3. Assume that $U = \mathbb{k}G_i \cdot \varepsilon_U$ for all vertices (i, U) of Q_G . Then, Ξ , as introduced in (1.0.13), is an isomorphism of \mathbb{k} -algebras such that the following diagram is commutative,*

$$(1.0.14) \quad \begin{array}{ccc} & & \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e} \\ & \xrightarrow{(1.0.1)} & \downarrow \Xi \\ \mathbb{k}Q_G & & \mathcal{I}ntw^{\text{op}} \\ & \xrightarrow{(1.0.11)} & \end{array}$$

In particular, if (1.0.1) is used as an identification, then, for all $\theta \in \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$,

$$(1.0.15) \quad \theta = \sum_{\gamma \text{ path in } Q_G} (\Xi(\theta)|\varphi_\gamma) \cdot \gamma.$$

2. PRACTICAL ASPECTS

Let Q_0 , S , G , and M be as in Setting 1.1.

Applying Theorem 1.16 for computations requires to compute Q_G first. Actually, rather than computing explicitly the intertwiners relative to M which form the arrows of Q_G , it is simpler to

- (1) compute a basis of $\text{Hom}_{G_j}(\tau, M^*(j, i; \rho))$ for all pairs (i, ρ) and (j, τ) such that $i, j \in [G \setminus Q_0]$, $\rho \in \text{irr}(G_i)$, and $\tau \in \text{irr}(G_j)$.

Because of the pairing $(-|-)$, this yields a quiver Q_G such as in Setting 1.3 as well as the intertwiners φ_α for all arrows α of Q_G . This avoids computing explicitly the intertwiners corresponding to the arrows of Q_G , which are not used in the formulas of Theorem 1.16.

Given an element θ of $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, say homogeneous of degree n for the grading induced by the tensor powers of M , the decomposition into a linear combination of paths in Q_G of the inverse image of θ under the isomorphism $\mathbb{k}Q_G \xrightarrow{(1.0.1)} \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ may be computed as follows using Theorem 1.16,

- (2) compute $\Xi(\theta)$, which is a sum of intertwiners relative to M , see (1.0.13);
- (3) compute the intertwiners φ_γ for all paths γ of length n in Q_G , see Notation 1.14;
- (4) compute the pairing $(\Xi(\theta)|\varphi_\gamma)$ for all such paths γ , see Proposition 1.13.

The desired linear combination of paths is hence $\sum_\gamma (\Xi(\theta)|\varphi_\gamma) \cdot \gamma$ (see (1.0.15)).

These computations may be performed by a computer which is able to determine the irreducible representations of $\text{irr}(G_i)$ for all $i \in [G \setminus Q_0]$. This section illustrates these computations.

2.1. How to pair intertwiners? In view of (1.0.6), the main difficulty in the computations listed in the introduction of Section 2 lies in computing pairings for $(-|-)$. This section explains how to compute (1.0.10) in combinatorial terms in the following specific setting.

Setting 2.1. Let Q_0 , S , G , and M be as in Setting 1.1. Let Q be a quiver with vertex set being Q_0 and such that M is equal to the vector space with basis elements being the arrows of Q . Let Q_G be as in Setting 1.3. Assume that

- the stabiliser G_i is abelian for all $i \in Q_0$,
- $U = \mathbb{k}G_i \cdot \varepsilon_U$ for all vertices (i, U) of Q_G , denote by χ_U the character $G_i \rightarrow \mathbb{k}^\times$ of U ,
- and, for all $g \in G$ and all paths γ in Q , there exists a scalar $\chi_{g, \gamma} \in \mathbb{k}^\times$ and a path $g(\gamma)$ such that ${}^g\gamma = \chi_{g, \gamma} g(\gamma)$.

If G is abelian, then it is possible to find a quiver Q and to choose irreducible representations U of G_i , for all $i \in [G \setminus Q_0]$, fitting in this setting.

Notation 2.2. The following notation relative to M is useful.

- For all arrows $a: i \rightarrow j$ of Q , denote by a^* the element of M^* such that $a^*(a) = 1$ and $a^*(b) = 0$ for all arrows b of Q distinct from a .
- Denote by Q^* the quiver with vertex set being Q_0 and whose arrows are the linear forms a^* introduced just before. This quiver is isomorphic to the opposite quiver of Q .
- For all paths $\gamma: t_0 \xrightarrow{a_1} t_1 \rightarrow \cdots \rightarrow t_{n-1} \xrightarrow{a_n} t_n$ in Q , denote by γ^* the path $t_n \xrightarrow{a_n^*} t_{n-1} \rightarrow \cdots \rightarrow t_1 \xrightarrow{a_1^*} t_0$ in Q^* .

Note that every path in Q starting in some vertex i_0 lying in $[G \setminus Q_0]$ is of the shape $i_0 \rightarrow y_1 \cdot i_1 \rightarrow \cdots \rightarrow (y_1 \cdots y_n) \cdot i_n$ for a unique sequence i_1, \dots, i_n in $[G \setminus Q_0]$ and a unique sequence y_1, \dots, y_n in G such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$.

Let $\underline{i} = i_0, \dots, i_n$ be a sequence in $[G \backslash Q_0]$, consider $U \in \text{irr}(G_{i_0})$ and $V \in \text{irr}(G_{i_n})$, and let $f: U \rightarrow M(\underline{i}; V)$ and $\varphi: V \rightarrow M^*(\underline{i}^o; U)$ be intertwiners. Since the representation V is one dimensional, then,

$$(2.1.1) \quad f(\varepsilon_U) = \sum_{\gamma} \alpha_{\gamma} \gamma \otimes y_1 y_2 \cdots y_n \varepsilon_V,$$

where γ runs through all paths in Q of the shape

$$i_0 \rightarrow y_1 \cdot i_1 \rightarrow y_1 y_2 \cdot i_2 \rightarrow \cdots \rightarrow (y_1 \cdots y_n) \cdot i_n$$

for some (unique) sequence $\underline{y} = y_1, \dots, y_n$ such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, and $\alpha_{\gamma} \in \mathbb{k}$. Note that the considered paths are precisely the ones of the shape $i_0 \rightarrow j_1 \rightarrow \cdots \rightarrow j_n$ such that j_t lies in the orbit of i_t for all $t \in \{1, \dots, n\}$. Similarly,

$$(2.1.2) \quad \varphi(\varepsilon_V) = \sum_{\gamma'} \beta_{\gamma'} \gamma' \otimes x_{n-1} \cdots x_0 \varepsilon_U,$$

where γ' runs through all paths in Q of the shape

$$i_n \rightarrow x_{n-1} \cdot i_{n-1} \rightarrow x_{n-1} x_{n-2} \cdot i_{n-2} \rightarrow \cdots \rightarrow (x_{n-1} x_{n-2} \cdots x_0) \cdot i_0$$

for some (unique) sequence $\underline{x} = x_{n-1}, \dots, x_0$ such that $x_t \in [G/G_{i_t}]$ for all $t \in \{0, \dots, n-1\}$, and $\beta_{\gamma'} \in \mathbb{k}$.

Lemma 2.3. *Using the decompositions (2.1.1) and (2.1.2), then*

$$(2.1.3) \quad (f|\varphi) = \sum_{\gamma} \alpha_{\gamma} \cdot \beta_{(y_1 \cdots y_n)^{-1}(\gamma)^*} \cdot \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \cdot \chi_U(h_0),$$

where γ runs through all paths in Q of the shape $i_0 \rightarrow j_1 \rightarrow \cdots \rightarrow j_n$ such that j_t lies in the orbit of i_t for all $t \in \{1, \dots, n\}$, where $\alpha_{\gamma} \in \mathbb{k}$, and where, for all such paths γ ,

- $\underline{y} = y_1, \dots, y_n$ denotes the sequence in G such that $y_t \in [G/G_{i_t}]$, for all $t \in \{1, \dots, n\}$ and γ is a path of the shape $i_0 \rightarrow y_1 \cdot i_1 \rightarrow \cdots \rightarrow y_1 \cdots y_n \cdot i_n$,
- h_0 is the element of G_{i_0} such as in (c) in Proposition 1.13,
- $\chi_{(y_1 \cdots y_n)^{-1}, \gamma}$ is the element of \mathbb{k}^{\times} and $(y_1 \cdots y_n)^{-1}(\gamma)$ is the path in Q such that $(y_1 \cdots y_n)^{-1} \gamma = \chi_{(y_1 \cdots y_n)^{-1}, \gamma} (y_1 \cdots y_n)^{-1}(\gamma)$,
- and $((y_1 \cdots y_n)^{-1}(\gamma))^*$ is the reverse path in Q^* associated to the path $(y_1 \cdots y_n)^{-1}(\gamma)$ in Q .

Proof. For all sequences $\underline{y} = y_1, \dots, y_n$ and $\underline{x} = x_{n-1}, \dots, x_0$ such that $x_t \in [G/G_{i_t}]$ and $y_t \in [G/G_{i_t}]$ for all t , denote by $\mathcal{C}(\underline{y})$ and $\mathcal{C}'(\underline{x})$ the sets of paths in Q and in Q^* of the shape $i_0 \rightarrow y_1 \cdot i_1 \rightarrow \cdots \rightarrow y_1 \cdots y_n \cdot i_n$ and $i_n \rightarrow x_{n-1} \cdot i_{n-1} \rightarrow \cdots \rightarrow x_{n-1} \cdots x_0 \cdot i_0$, respectively. For all \underline{y} ,

$$f_{\underline{y}}(\varepsilon_U) = \sum_{\gamma \in \mathcal{C}(\underline{y})} \alpha_{\gamma} \gamma \otimes y_1 \cdots y_n \varepsilon_U;$$

hence, the following term which serves in the definition of $(f|\varphi)$, see (1.0.10),

$$(y_1 \cdots y_n)^{-1} f_{\underline{y}}^{(1)}(\varepsilon_U) \otimes \cdots \otimes (y_1 \cdots y_n)^{-1} f_{\underline{y}}^{(n)}(\varepsilon_U) \otimes f_{\underline{y}}^{(0)}(\varepsilon_U),$$

is equal to

$$\sum_{\gamma \in \mathcal{C}(\underline{y})} \alpha_{\gamma} \cdot \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \cdot \underbrace{(y_1 \cdots y_n)^{-1}(\gamma)}_{\in M^{\otimes n}} \otimes \varepsilon_U.$$

Similarly, for all \underline{x} ,

$$\varphi^{\underline{x}}(\varepsilon_V) = \sum_{\gamma' \in \mathcal{C}(\underline{x})} \beta_{\gamma'} \gamma' \otimes x_{n-1} \cdots x_0 \varepsilon_U.$$

Therefore, given \underline{y} , the “ Π_t ”-term in (1.0.10) is equal to

$$(2.1.4) \quad \sum_{\gamma \in \mathcal{C}(\underline{y}), \gamma' \in \mathcal{C}(\underline{x})} \alpha_\gamma \cdot \beta_{\gamma'} \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \cdot b_n^*(c_n) \cdots b_2^*(c_2) \cdot b_1^*(c_1) \cdot \chi_U(h_0) \cdot \varepsilon_U,$$

where

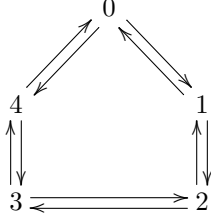
- $\underline{x} = x_{n-1}, \dots, x_0$ and h_0 are the elements of G determined by \underline{y} and by (a), (b), and (c) in Proposition 1.13,
- b_1^*, \dots, b_n^* are the arrows of Q^* such that γ' is $\cdot \xrightarrow{b_n^*} \cdot \xrightarrow{b_{n-1}^*} \cdots \xrightarrow{b_1^*} \cdot$,
- and c_1, \dots, c_n are the arrows of Q such that $(y_1 \cdots y_n)^{-1}(\gamma)$ is $\cdot \xrightarrow{c_1} \cdot \xrightarrow{c_2} \cdots \xrightarrow{c_n} \cdot$.

By definition of the arrows b_1^*, \dots, b_n^* , the summand of (2.1.4) with index γ is non zero only if the paths $\cdot \xrightarrow{b_1} \cdot \xrightarrow{b_2} \cdots \xrightarrow{b_n} \cdot$ and $(y_1 \cdots y_n)^{-1}(\gamma)$ in Q are equal, that is, only if $\gamma' = ((y_1 \cdots y_n)^{-1}(\gamma))^*$. Summing (2.1.4) over all possible \underline{y} yields that $(f|\varphi)\varepsilon_U$ is equal to

$$\sum_{\gamma} \alpha_\gamma \beta_{((y_1 \cdots y_n)^{-1}(\gamma))^*} \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \chi_U(h_0) \varepsilon_U.$$

This proves (2.1.3). \square

2.2. An example. In this example, Q is the following quiver



and G is the dihedral group of order 10

$$G = \langle c, \tau \mid c^5, \tau^2, \tau c \tau c \rangle.$$

Denote by ε the group homomorphism $G \rightarrow \{-1, 1\}$ such that $\varepsilon(c) = 1$ and $\varepsilon(\tau) = -1$. Denote the arrows of Q by $x_{i,j}: i \rightarrow j$. For convenience, denote the arrows of Q^* by $x'_{i,j}: i \rightarrow j$, hence $(x_{j,i})^* = x'_{i,j}$. The actions of G on Q_0 and M are assumed to be the ones such that, for all $i \in Q_0$ and $g \in G$, and for all arrows $x_{i,j}$ of Q ,

$$c \cdot i = i + 1 \pmod{5}, \quad \tau \cdot i = 5 - i, \quad {}^g x_{i,j} = \varepsilon(g) x_{g \cdot i, g \cdot j}.$$

Let $[G \backslash Q_0]$ be $\{0\}$. Note that $G_0 = \{\text{Id}, \tau\}$. Let $[G/G_0]$ be $\{\text{Id}, c, c^2, c^3, c^4\}$. For all $s \in \{0, 1\}$, denote by ε_s the following primitive idempotent of $\mathbb{k}G_0$,

$$\varepsilon_s = \frac{1}{2}(\text{Id} + (-1)^s \tau)$$

and denote $\varepsilon_s \cdot \mathbb{k}G_0$ by ρ_s . Finally, let $\text{irr}(G_0)$ be equal to $\{\rho_0, \rho_1\}$. In particular,

$$\tilde{e} = e_0 * \varepsilon_0 + e_0 * \varepsilon_1.$$

Denote by W the following element of $\mathbb{k}Q$,

$$W = x_{0,1} x_{1,2} x_{2,3} x_{3,4} x_{4,0} - x_{0,4} x_{4,3} x_{3,2} x_{2,1} x_{1,0}.$$

Denote $\tilde{e} \cdot W \cdot \tilde{e}$ by θ .

2.2.1. *Computation of Q_G .* By definition, for all $t \in \{0, 1\}$,

$$M^*(0, 0; \rho_t) = \text{Span}(x'_{0,1} \otimes c\varepsilon_t, x'_{0,4} \otimes c^4\varepsilon_t),$$

where the action of τ is given as follows

$$\begin{cases} x'_{0,1} \otimes c\varepsilon_t & \mapsto (-1)^{1+t}x'_{0,4} \otimes c^4\varepsilon_t \\ x'_{0,4} \otimes c^4\varepsilon_t & \mapsto (-1)^{1+t}x'_{0,1} \otimes c\varepsilon_t. \end{cases}$$

Hence, for all $s, t \in \{0, 1\}$,

$$\text{Hom}_{\mathbb{k}G_0}(\rho_s, M^*(0, 0; \rho_t)) = \text{Span}(\varphi_{s,t}),$$

where $\varphi_{s,t}$ is the intertwiner such that $\varphi_{s,t}(\varepsilon_s) = \varepsilon_s \cdot (x'_{0,1} \otimes c\varepsilon_t) (= \varepsilon_s \cdot (x'_{0,1} \otimes c) \cdot \varepsilon_t)$, that is,

$$\varphi_{s,t}(\varepsilon_s) = x'_{0,1} \otimes c\varepsilon_t + (-1)^{s+t+1}x'_{0,4} \otimes c^4\varepsilon_t.$$

Denote by $f_{s,t}$ the intertwiner $\rho_s \rightarrow M(0, 0; \rho_t)$ such that $(f_{s,t}|\varphi_{t,s}) = 1$, for all $s, t \in \{0, 1\}$. Then Q_G may be taken equal to the following quiver

$$\begin{array}{ccc} & \begin{array}{c} \text{\scriptsize } f_{0,0} \\ \curvearrowright \\ (1, \rho_0) \end{array} & \begin{array}{c} \text{\scriptsize } f_{1,1} \\ \curvearrowright \\ (1, \rho_1) \end{array} \\ & \begin{array}{c} \xrightarrow{\text{\scriptsize } f_{0,1}} \\ \xleftarrow{\text{\scriptsize } f_{1,0}} \end{array} & \end{array}$$

2.3. **The intertwiner $\Xi(\theta)$.** By definition, $\Xi(\theta)$ decomposes as

$$(2.3.1) \quad \Xi(\theta) = \sum_{r,w \in \{0,1\}} \Xi(\theta)_{r,w},$$

where, for all $r, w \in \{0, 1\}$, the intertwiner $\Xi(\theta)_{r,w}: \rho_r \rightarrow M(0, 0, 0, 0, 0, 0; \rho_w)$ is such that $\Xi(\theta)_{r,w}(\varepsilon_r) = \varepsilon_r \cdot \tilde{e} \cdot W \cdot \tilde{e} \cdot \varepsilon_w$, see (1.0.13) and note that $\varepsilon_r \cdot \tilde{e} = e_0 * \varepsilon_r$. Hence,

$$(2.3.2) \quad \begin{aligned} \Xi(\theta)_{r,w}(\varepsilon_r) &= \frac{1}{2}(1 + (-1)^{r+w})x_{0,1}x_{1,2}x_{2,3}x_{3,4}x_{4,0} \otimes \varepsilon_w \\ &\quad - \frac{1}{2}(1 + (-1)^{r+w})x_{0,4}x_{4,3}x_{3,2}x_{2,1}x_{1,0} \otimes \varepsilon_w. \end{aligned}$$

2.4. **Intertwiners relative to M^* associated to paths of Q_G .** Consider a path $\gamma: (0, \rho_r) \rightarrow (0, \rho_s) \rightarrow (0, \rho_t) \rightarrow (0, \rho_u) \rightarrow (0, \rho_v) \rightarrow (0, \rho_w)$ of length 5 in Q_G . Then φ_γ is the following intertwiner $\rho_w \rightarrow M^*(0, 0, 0, 0, 0, 0; \rho_r)$,

$$\varphi_\gamma = \varphi_{s,r} \otimes \varphi_{t,s} \otimes \varphi_{u,t} \otimes \varphi_{v,u} \otimes \varphi_{w,v}.$$

Consider $\varphi_\gamma(\varepsilon_w)$ as an element of $\mathbb{k}Q^* * G$, see (1.0.6), then it equals

$$(2.4.1) \quad \begin{aligned} &(x'_{0,1} \otimes c + (-1)^{v+w+1}x'_{0,4} \otimes c^4) \cdot (x'_{0,1} \otimes c + (-1)^{u+v+1}x'_{0,4} \otimes c^4) \cdot \\ &(x'_{0,1} \otimes c + (-1)^{t+u+1}x'_{0,4} \otimes c^4) \cdot (x'_{0,1} \otimes c + (-1)^{s+t+1}x'_{0,4} \otimes c^4) \cdot \\ &(x'_{0,1} \otimes c + (-1)^{r+s+1}x'_{0,4} \otimes c^4) \cdot \varepsilon_t. \end{aligned}$$

2.5. **Decomposition of θ into paths in Q_G .** For all paths $(0, \rho_r) \rightarrow (0, \rho_s) \rightarrow (0, \rho_t) \rightarrow (0, \rho_u) \rightarrow (0, \rho_v) \rightarrow (0, \rho_w)$ of length 5 in Q_G , Table 1 describes the terms appearing in (2.1.3) when f is taken equal to $\Xi(\theta)_{r,w}$ (see (2.3.2)) and φ is taken equal to $\varphi_{s,r} \otimes \varphi_{t,s} \otimes \varphi_{u,t} \otimes \varphi_{v,u} \otimes \varphi_{w,v}$ (see (2.4.1)). Thus,

γ	$x_{0,1}x_{1,2}x_{2,3}x_{3,4}x_{4,0}$	$x_{0,4}x_{4,3}x_{3,2}x_{2,1}x_{1,0}$
α_γ	$\frac{1}{2}(1 + (-1)^{r+w})$	$-\frac{1}{2}(1 + (-1)^{r+w})$
y_1, \dots, y_5	c, c, c, c, c	$c^{-1}, c^{-1}, c^{-1}, c^{-1}, c^{-1}$
$\chi_{(y_1 \cdots y_5)^{-1}, \gamma}$	1	1
$((y_1 \cdots y_5)^{-1}(\gamma))^*$	$x'_{0,4}x'_{4,3}x'_{3,2}x'_{2,1}x'_{1,0}$	$x'_{0,1}x'_{1,2}x'_{2,3}x'_{3,4}x'_{4,5}$
$\beta_{(y_1 \cdots y_5)^{-1}(\gamma)^*}$	$(-1)^{1+r+w}$	1
h_0	1	1
$\chi_{\rho_r}(h_0)$	1	1

TABLE 1. Computation of $(\Xi(\theta)_{r,w}|\varphi_{s,r} \otimes \varphi_{t,s} \otimes \varphi_{u,t} \otimes \varphi_{v,u} \otimes \varphi_{w,v})$

$$(\Xi(\theta)_{r,s}|\varphi_{s,r} \otimes \varphi_{t,s} \otimes \varphi_{u,t} \otimes \varphi_{v,u} \otimes \varphi_{w,v}) = -(1 + (-1)^{r+w}).$$

Applying (1.0.15) yields that the inverse image of $\tilde{e} \cdot W \cdot \tilde{e}$ under the isomorphism $\mathbb{k}Q_G \xrightarrow{(1.0.1)} \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ is equal to $-2 \sum_{\gamma} \gamma$, where γ runs through all oriented cycles of length 5 in Q_G .

3. DETAILS ON THE MONOIDAL CATEGORY OF BIMODULES OVER GROUPS

This section details some isomorphisms and properties in the monoidal category $(\text{mod}(A^e), \otimes_A)$ and related to the operation, say \times , and the pairing $\langle - | - \rangle$ considered in Remark 1.5.

- Section 3.1 describes the left dual of a tensor product of objects in $\text{mod}(A^e)$.
- Section 3.2 details the the adjunction $(X \otimes_A -) \vdash (X^* \otimes_A -)$ for a given $X \in \text{mod}(A^e)$.
- Section 3.3 details the compatibility of $\langle - | - \rangle$.

The existence of the isomorphisms and adjunctions discussed in these sections may be part of the folklore (see [5, Exercise 2.10.16]). However, the proof of Theorem 1.15 is based on their description, which is the reason for detailing these here. For the sake of simplicity, the presentation is made independently of the monoidal category $(\text{mod}(A^e), \otimes_A)$ and of the framework of Theorem 1.15.

Here are some conventions. In any group, e denotes the neutral element; and $\text{Hom}_{\mathbb{k}G}$ and $\otimes_{\mathbb{k}G}$ are denoted by Hom_G and \otimes_G , respectively, for all finite groups G .

Given finite groups G and H , by “a module ${}_G X_H$ ” is meant a $\mathbb{k}G - \mathbb{k}H$ -bimodule, by “a module ${}_G X$ ” is meant a left $\mathbb{k}G$ -module, and by “by a module X_H ” is meant a right $\mathbb{k}H$ -module.

Given finite groups G and H and given a $\mathbb{k}G - \mathbb{k}H$ -bimodule X , the dual vector space X^* is a $\mathbb{k}H - \mathbb{k}G$ -bimodule in a natural way ($h \cdot \phi \cdot g = \phi(g \cdot \bullet \cdot h)$ for all $g \in G$, $h \in H$ and $\phi \in X^*$). In particular, taking H to be the trivial group, this defines a functor $X \mapsto X^*$ from left $\mathbb{k}G$ -modules to right $\mathbb{k}G$ -modules.

3.1. Tensor product of duals and dual of a tensor product.

Lemma 3.1. *Let H be a finite group with order not divisible by $\text{char}(\mathbb{k})$. For all finite dimensional $\mathbb{k}H$ -modules X_H and ${}_H Y$, there is a functorial bijective mapping*

$$(3.1.1) \quad \begin{aligned} Y^* \otimes_H X^* &\rightarrow (X \otimes_H Y)^* \\ \phi_2 \otimes \phi_1 &\mapsto (x \otimes y \mapsto \sum_{h \in H} \phi_1(x \cdot h) \phi_2(h^{-1} \cdot y)). \end{aligned}$$

Proof. The mapping is well defined. When $X = \mathbb{k}H_H$ and $Y = {}_H \mathbb{k}H$, it identifies with the following one, where $\{\delta_h\}_{h \in H}$ denotes the canonical basis of $(\mathbb{k}H)^*$,

$$\begin{aligned} (\mathbb{k}H)^* \otimes_H (\mathbb{k}H)^* &\rightarrow (\mathbb{k}H)^* \\ \delta_{h_2} \otimes \delta_{h_1} &\mapsto \delta_{h_1 h_2}; \end{aligned}$$

and the latter is bijective. Since X_H and ${}_H Y$ are finite dimensional and projective, then (3.1.1) is bijective. \square

The previous lemma generalises as follows. Given an integer $n \geq 2$, a sequence of groups G_0, \dots, G_n and a finite dimensional $\mathbb{k}G_{i-1} - \mathbb{k}G_i$ -bimodule U_i for every $i \in \{1, \dots, n\}$, there is a functorial bijective mapping

$$(3.1.2) \quad X_n^* \otimes_{G_{n-1}} \cdots \otimes_{G_1} X_1^* \rightarrow (X_1 \otimes_{G_1} \cdots \otimes_{G_{n-1}} X_n)^*,$$

which maps any $\phi_n \otimes \cdots \otimes \phi_1$ to the following linear form on $X_1 \otimes_{G_1} \cdots \otimes_{G_{n-1}} X_n$,

$$x_1 \otimes \cdots \otimes x_n \mapsto \sum_{g_1, \dots, g_{n-1}} \prod_{t=1}^n \phi_t(g_{t-1}^{-1} \cdot x_t \cdot g_t),$$

where g_t runs through G_t ($1 \leq t \leq n-1$) and $g_0 = g_n = e$.

3.2. Adjunctions.

Lemma 3.2. *Let G and H be finite groups with orders not divisible by $\text{char}(\mathbb{k})$. Let ${}_G X_H$, ${}_H V$ and ${}_G U$ be (bi)modules over $\mathbb{k}G$ and $\mathbb{k}H$ such that X and V are finite dimensional. Then, there exists a functorial isomorphism*

$$\Psi_X^{V,U} : \text{Hom}_H(V, X^* \otimes_G U) \rightarrow \text{Hom}_G(X \otimes_H V, U),$$

such that, for all $\phi \in \text{Hom}_H(V, X^* \otimes_G U)$, $x \in X$ and $v \in V$,

$$(3.2.1) \quad (\Psi_X^{V,U}(\phi))(x \otimes v) = \sum_{g \in G} (\phi^{(2)}(v))(g^{-1} \cdot x) g \cdot \phi^{(1)}(v),$$

where the following notation is used with sum sign omitted, for all $v \in V$,

$$\phi(v) = \phi^{(2)}(v) \otimes \phi^{(1)}(v) \in X^* \otimes_G U.$$

In particular, the pair of functors $(X \otimes_H -, X^* \otimes_G -)$ between finite dimensional left $\mathbb{k}H$ -modules and finite dimensional left $\mathbb{k}G$ -modules is adjoint.

Proof. Using the definition of the structure of H - G -bimodule of X^* , simple changes of variable in the sum " $\sum_{g \in G}$ " show that $\Psi_X^{V,U}$ is indeed well-defined.

Since ${}_H V$ is finitely generated and projective, it suffices to prove the lemma assuming that $V = \mathbb{k}H$. In this case, $\Psi_X^{V,U}$ identifies with the mapping $\lambda_{X,U}$ defined (for all left $\mathbb{k}G$ -modules X and U) by

$$(3.2.2) \quad \begin{array}{ccc} X^* \otimes_G U & \rightarrow & \text{Hom}_G(X, U) \\ \phi \otimes u & \mapsto & (x \mapsto \sum_{g \in G} \phi(g^{-1} \cdot x) g \cdot u). \end{array}$$

Now, $\lambda_{\mathbb{k}G, \mathbb{k}G}$ identifies with the following bijective mapping (recall that $|G| \in \mathbb{k}^\times$).

$$\begin{array}{ccc} \mathbb{k}G^* \otimes_G \mathbb{k}G & \rightarrow & \mathbb{k}G \\ \phi \otimes e & \mapsto & \sum_{g \in G} \phi(g^{-1}) g. \end{array}$$

Since ${}_G X$ is finite dimensional and projective, and since ${}_G U$ is projective, it follows that $\lambda_{X,U}$ is bijective. Thus, $\Psi_X^{V,U}$ is bijective. \square

Lemma 3.3. *Let G, H, K be finite groups with orders not divisible by $\text{char}(\mathbb{k})$. Let ${}_G X_H$, ${}_H Y_K$, ${}_K W$ and ${}_G U$ be finite dimensional modules. Then, the following diagram commutes*

$$(3.2.3) \quad \begin{array}{ccc} \text{Hom}_K(W, Y^* \otimes_H X^* \otimes_G U) & \xrightarrow{\quad\quad\quad} & \text{Hom}_K(W, (X \otimes_H Y)^* \otimes_G U) \\ \Psi_Y^{W, X^* \otimes_G U} \downarrow & & \downarrow \Psi_{X \otimes_H Y}^{W, U} \\ \text{Hom}_H(Y \otimes_K W, X^* \otimes_G U) & \xrightarrow{\Psi_X^{Y \otimes_K W, U}} & \text{Hom}_G(X \otimes_H Y \otimes_K W, U), \end{array}$$

where the top horizontal arrow is obtained upon applying $\text{Hom}_K(W, - \otimes_G U)$ to (3.1.1).

Proof. In view of the compatibility of the involved mappings with direct sum decompositions of the finite dimensional projective modules ${}_K W$ and ${}_G U$, it is sufficient to prove the lemma assuming that $W = \mathbb{k}K$ and $U = \mathbb{k}G$. In this case, the given diagram identifies with the following one

$$(3.2.4) \quad \begin{array}{ccc} Y^* \otimes_H X^* & \xrightarrow{(3.1.1)} & (X \otimes_H Y)^* \\ \beta \downarrow & & \downarrow \alpha \\ \text{Hom}_H(Y, X^*) & \xrightarrow{\gamma} & \text{Hom}_G(X \otimes_H Y, \mathbb{k}G), \end{array}$$

where, after using (3.2.2),

- α is given by $\phi \mapsto (x \otimes y \mapsto \sum_{g \in G} \phi(g^{-1} \cdot x \otimes y)g)$,
- β is given by $\phi_2 \otimes \phi_1 \mapsto (y \mapsto \sum_{h \in H} \phi_2(h^{-1} \cdot y) \underbrace{h \cdot \phi_1}_{\phi_1(\bullet h)})$ and
- γ is given by $f \mapsto (x \otimes y \mapsto \sum_{g \in G} (f(y))(g^{-1} \cdot x)g)$.

It is direct to check that (3.2.4) commutes, which proves the lemma. \square

3.3. Pairing of morphisms.

Definition 3.4. Let G and H be finite groups whose orders are not divisible by $\text{char}(\mathbb{k})$. Let ${}_G X_H$, ${}_H V$, and ${}_G U$ be (bi)modules such that X and V are finite dimensional. Assume that U is simple. Define a pairing

$$(3.3.1) \quad \langle - | - \rangle: \text{Hom}_G(U, X \otimes_H V) \otimes_{\mathbb{k}} \text{Hom}_H(V, X^* \otimes_G U) \rightarrow \mathbb{k}$$

as follows. For all $f \in \text{Hom}_G(U, X \otimes_H V)$ and $\phi \in \text{Hom}_H(V, X^* \otimes_G U)$,

$$(3.3.2) \quad \langle f | \phi \rangle \cdot \text{Id}_U = \Psi_X^{V,U}(\phi) \circ f$$

(following Lemma 3.2, the pairing $\langle - | - \rangle$ is non degenerate).

This pairing is compatible with the following operation on morphisms.

Definition 3.5. Let G , H , and K be finite groups whose orders are not divisible by $\text{char}(\mathbb{k})$. Let ${}_G X_H$, ${}_H Y_K$, ${}_G U$, ${}_H V$, and ${}_K W$ be (bi)modules. For all $f_1 \in \text{Hom}_G(U, X \otimes_H V)$ and $f_2 \in \text{Hom}_H(V, Y \otimes_K W)$, define $f_2 \times f_1 \in \text{Hom}_G(U, X \otimes_H Y \otimes_K W)$ as the following composite morphism

$$(3.3.3) \quad f_2 \times f_1: U \xrightarrow{f_1} X \otimes_H V \xrightarrow{(X \otimes_H f_2)} X \otimes_H Y \otimes_K W.$$

Here is the above mentioned compatibility.

Lemma 3.6. *Let*

- G , H , and K be finite groups with orders not divisible by $\text{char}(\mathbb{k})$,
- ${}_G X_H$, ${}_H Y_K$, ${}_G U$, ${}_H V$ and ${}_K W$ be (bi)modules such that X , Y , and W are finite dimensional and U and V are simple,
- $f_1 \in \text{Hom}_G(U, X \otimes_H V)$ and $f_2 \in \text{Hom}_H(V, Y \otimes_K W)$ and
- $\phi_1 \in \text{Hom}_H(V, X^* \otimes_G U)$ and $\phi_2 \in \text{Hom}_K(W, Y^* \otimes_H V)$.

Denote by λ the following isomorphism obtained upon applying $- \otimes_G U$ to (3.1.1),

$$Y^* \otimes_H X^* \otimes_G U \xrightarrow{\sim} (X \otimes_H Y)^* \otimes_G U.$$

Then, $\langle f_2 \times f_1 | \lambda \circ (\phi_1 \times \phi_2) \rangle = \langle f_2 | \phi_2 \rangle \cdot \langle f_1 | \phi_1 \rangle$.

Proof. Using (3.2.3) yields that

$$\Psi_{X \otimes_H Y}^{W,U}(\lambda \circ (\phi_1 \times \phi_2)) = \Psi_X^{Y \otimes_K W, U} \circ \Psi_Y^{W, X^* \otimes_G U}(\phi_1 \times \phi_2).$$

Recall that $\phi_1 \times \phi_2$ is the following composite morphism

$$W \xrightarrow{\phi_2} Y^* \otimes_H V \xrightarrow{Y^* \otimes_H \phi_1} Y^* \otimes_H X^* \otimes_G U.$$

Since $\Psi_X^{\bullet, \bullet}$ and $\Psi_Y^{\bullet, \bullet}$ are adjunction isomorphisms, then

$$\begin{aligned} \Psi_X^{Y \otimes_K W, U} \circ \Psi_Y^{W, X^* \otimes_G U}(\phi_1 \times \phi_2) &= \Psi_X^{Y \otimes_K W, U}(\phi_1 \circ \Psi_Y^{W, V}(\phi_2)) \\ &= \Psi_X^{V, U}(\phi_1) \circ (X \otimes_H \Psi_Y^{W, V}(\phi_2)). \end{aligned}$$

Therefore,

$$\begin{aligned} \Psi_{X \otimes_H Y}^{W,U}(\lambda \circ (\phi_1 \times \phi_2)) \circ (f_2 \times f_1) &= \Psi_X^{V,U}(\phi_1) \circ (X \otimes_H \Psi_Y^{W,V}(\phi_2)) \circ \\ &\quad (X \otimes_H f_2) \circ f_1 \\ &= \Psi_X^{V,U}(\phi_1) \circ (X \otimes_H \langle f_2 | \phi_2 \rangle \cdot \text{Id}_V) \circ f_1 \\ &= \langle f_1 | \phi_1 \rangle \cdot \langle f_2 | \phi_2 \rangle \cdot \text{Id}_U. \end{aligned}$$

The conclusion of the lemma then follows from the definition of $\langle - | - \rangle$. \square

The previous lemma has the following generalisation. Consider

- an integer $n \geq 2$,
- finite groups G_0, \dots, G_n with orders not divisible by $\text{char}(\mathbb{k})$,
- a finite dimensional left $\mathbb{k}G_i$ -module U_i for every $i \in \{0, \dots, n\}$, such that U_0, \dots, U_{n-1} are simple,
- a finite dimensional $\mathbb{k}G_{i-1} - \mathbb{k}G_i$ -bimodule X_i for every $i \in \{1, \dots, n\}$,
- $f_i \in \text{Hom}_{G_{i-1}}(U_{i-1}, X_i \otimes_{G_i} U_i)$ and $\phi_i \in \text{Hom}_{G_i}(U_i, X_i^* \otimes_{G_i} U_{i-1})$ for every $i \in \{1, \dots, n\}$.

Denote by $\lambda^{(n)}$ the isomorphism obtained upon applying $- \otimes_{G_0} U_0$ to (3.1.2),

$$\lambda^{(n)}: X_n^* \otimes_{G_{n-1}} \cdots \otimes_{G_1} X_1^* \otimes_{G_0} U_0 \xrightarrow{\sim} (X_1 \otimes_{G_1} \cdots \otimes_{G_{n-1}} X_n)^* \otimes_{G_0} U_0.$$

Then,

$$(3.3.4) \quad \langle f_n \times \cdots \times f_1 | \lambda^{(n)} \circ (\phi_1 \times \cdots \times \phi_n) \rangle = \prod_{t=1}^n \langle f_t | \phi_t \rangle.$$

4. INTERTWINERS

Let Q_0, S, G , and M be as in Setting 1.1. This section establishes properties on intertwiners relative to M which are needed to prove Theorem 1.15. This includes the properties of \otimes mentioned in Section 1 as well as Proposition 1.13 on the existence of the pairing $\langle - | - \rangle$. This is done in several steps,

- first, by proving that, in Definition 1.6, the $\mathbb{k}G_{i_0}$ -module $M(\underline{i}; V)$ is isomorphic to $M(\underline{i}) \otimes_{G_{i_n}} V$ for a suitable $\mathbb{k}G_{i_0} - \mathbb{k}G_{i_n}$ -bimodule $M(\underline{i})$;
- next, by proving that, under this isomorphism, \otimes corresponds to the operation \times of Definition 3.5;
- and finally by defining $\langle - | - \rangle$ so that, under this isomorphism, it corresponds to the pairing $\langle - | - \rangle$ of Definition 3.4.

These steps are proceeded in turn in Sections 4.1, 4.2, and 4.3, respectively. They use the bimodule MG defined as follows. Note that, the action of G on S yields the skew group algebra $S * G$. Like in Section 3, for all groups H , the pieces of notation e , \otimes_H , and Hom_H stand for the neutral element of H , for $\otimes_{\mathbb{k}H}$, and for $\text{Hom}_{\mathbb{k}H}$, respectively.

Definition 4.1. Define MG as the following $S * G - S * G$ -bimodule.

- Its underlying vector space is $M \otimes_{\mathbb{k}} \mathbb{k}G$, a tensor $m \otimes g$ is denoted by $m * g$ for all $m \in M$ and $g \in G$.
- The actions of S and G on the left and on the right are such that, for all $m \in M$, $g, h, k \in G$ and $s, s' \in S$,

$$\begin{aligned} s \cdot (m * g) \cdot s' &= sm^g s' * g \\ h \cdot (m * g) \cdot k &= {}^h m * h g k. \end{aligned}$$

The $S * G - S * G$ -bimodule M^*G is defined similarly after replacing M by M^* .

4.1. The space $M(\underline{i})$. Let n be a positive integer. Let $\underline{i} = i_0, \dots, i_n$ a sequence in Q_0 . Denote by $M(\underline{i})$ the following $\mathbb{k}G_{i_0} - \mathbb{k}G_{i_n}$ -bimodule,

$$(4.1.1) \quad M(\underline{i}) = e_{i_0} M G e_{i_1} \otimes_{G_{i_1}} \cdots \otimes_{G_{i_{n-1}}} e_{i_{n-1}} M G e_{i_n}.$$

Note that this yields the $\mathbb{k}G_{i_n} - \mathbb{k}G_{i_0}$ -bimodule $M^*(\underline{i}^\circ)$. These bimodules are dual to each other. Here is an explicit isomorphism used later. Consider the isomorphism (3.1.2) taking $X_t = e_{i_{t-1}} M^* G e_{i_t}$ for all $t \in \{1, \dots, n\}$,

$$(4.1.2) \quad e_{i_n} (MG)^* e_{i_{n-1}} \otimes_{G_{i_{n-1}}} \cdots \otimes_{G_{i_1}} e_{i_1} (MG)^* e_{i_0} \xrightarrow{(3.1.2)} M(\underline{i})^*.$$

Since the following mapping is a non degenerate pairing

$$\begin{aligned} [-|-]: \quad MG \otimes_{\mathbb{k}} M^*G &\rightarrow \mathbb{k} \\ (m * g) \otimes (\varphi * h) &\mapsto \varphi(g^{-1}m)\delta_{e,hg}, \end{aligned}$$

it induces an isomorphism (of $S * G - S * G$ -bimodules)

$$(4.1.3) \quad \begin{aligned} M^*G &\rightarrow (MG)^* \\ \bullet &\mapsto [-|\bullet] \end{aligned}$$

the n -th tensor power of which induces an isomorphism of $\mathbb{k}G_{i_n} - \mathbb{k}G_{i_0}$ -bimodules,

$$(4.1.4) \quad M^*(\underline{i}^o) \xrightarrow{\sim} e_{i_n}(MG)^* e_{i_{n-1}} \otimes_{G_{i_{n-1}}} \cdots \otimes_{G_{i_1}} e_{i_1}(MG)^* e_{i_0}.$$

Composing (4.1.2) and (4.1.4) yields an isomorphism of $\mathbb{k}G_{i_n} - \mathbb{k}G_{i_0}$ -bimodules

$$(4.1.5) \quad \Lambda_{\underline{i}}: M^*(\underline{i}^o) \xrightarrow{\sim} M(\underline{i})^*,$$

which is induced by the mapping from $(MG)^{\otimes_{\mathbb{k}} n}$ to $((MG)^{\otimes_{\mathbb{k}} n})^*$ which assigns to any tensor $(\varphi_n * g_n) \otimes \cdots \otimes (\varphi_1 * g_1)$ the following linear form on $(MG)^{\otimes_{\mathbb{k}} n}$,

$$(4.1.6) \quad (m_1 * g'_1) \otimes \cdots \otimes (m_n * g'_n) \mapsto \sum_{k_1, \dots, k_{n-1}} \prod_{t=1}^n \varphi_t(k_t^{-1} g_t'^{-1} m_t) \delta_{e, g_t k_{t-1}^{-1} g_t' k_t},$$

where δ denotes the Kronecker symbol, k_t runs through G_{i_t} and $k_n = k_0 = e$.

The bimodules $M(\underline{i})$ and $M^*(\underline{i}^o)$ are hence objects of the monoidal category $(\text{mod}(A^e), \otimes_A)$ in which they are left dual to each other. The following lemma describes tensor products of these objects in this monoidal category. The same statements are valid after replacing M by M^* .

Lemma 4.2. *Let m, n be positive integers. Let $\underline{i}'' = i_0, \dots, i_{m+n}$ be a sequence in Q_0 . Denote the sequences i_0, \dots, i_m and i_m, \dots, i_{m+n} by \underline{i} and \underline{i}' , respectively.*

- (1) $M(\underline{i}'') = M(\underline{i}) \otimes_{G_{i_m}} M(\underline{i}')$.
- (2) *The following diagram is commutative*

$$\begin{array}{ccc} M^*(\underline{i}'^o) \otimes_{G_{i_m}} M^*(\underline{i}^o) & \xrightarrow{\Lambda_{\underline{i}'} \otimes_{G_{i_m}} \Lambda_{\underline{i}}} & M(\underline{i}')^* \otimes_{G_{i_m}} M(\underline{i})^* \\ \parallel & & \downarrow (3.1.1) \\ M^*(\underline{i}''^o) & \xrightarrow{\Lambda_{\underline{i}''}} & M(\underline{i}'')^*. \end{array}$$

Proof. (1) follows from the definition of the spaces $M(\bullet)$ and (2) follows from the definition of the morphisms Λ_{\bullet} in terms of (3.1.2) and (4.1.3). \square

The following result is the link between the bimodules $M(\bullet)$ and the modules of Definition 1.6. This link makes it possible to translate the results of Section 3 in terms of intertwiners. From now on, the piece of notation $*$ used for tensors in MG is often dropped in order to avoid overloaded notation.

Lemma 4.3. *Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 , where $n \geq 1$. Let $U_n \in \text{mod}(\mathbb{k}G_{i_n})$. There is an isomorphism in $\text{mod}(\mathbb{k}G_{i_0})$,*

$$(4.1.7) \quad \Theta_{\underline{i}; U_n}^M: M(\underline{i}; U_n) \xrightarrow{\sim} M(\underline{i}) \otimes_{G_{i_n}} U_n$$

given by

$$m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n u \mapsto m_1 y_1 \otimes y_1^{-1} m_2 y_2 \otimes \cdots \otimes (y_1 \cdots y_{n-1})^{-1} m_n y_n \otimes u.$$

When there is no risk of confusion, $\Theta_{\underline{i}; U_n}^M$ is denoted by $\Theta_{\underline{i}; U_n}$.

Proof. This is straightforward to check. \square

Note that $\Theta_{\underline{i}; U_n}$ induces a functorial bijection, for all $U_0 \in \text{mod}(\mathbb{k}G_{i_0})$,

$$(4.1.8) \quad \begin{array}{ccc} \text{Hom}_{G_{i_0}}(U_0, M(\underline{i}; U_n)) & \xrightarrow{\sim} & \text{Hom}_{G_{i_0}}(U_0, M(\underline{i}) \otimes_{G_{i_n}} U_n) \\ f & \mapsto & \Theta_{\underline{i}; U_n} \circ f. \end{array}$$

4.2. Compositions of intertwiners. It is now possible to compare the operation \circledast on intertwiners, see Definition 1.9, and the operation \times of Definition 3.5.

Lemma 4.4. *Let $\underline{i}'' = i_0, \dots, i_{m+n}$ be a sequence in Q_0 , where $m, n \geq 1$. Denote the sequences i_0, \dots, i_m and $i_m, i_{m+1}, \dots, i_{m+n}$ by \underline{i} and \underline{i}' , respectively. Let $U_0 \in \text{mod}(\mathbb{k}G_{i_0})$, $U_m \in \text{mod}(\mathbb{k}G_{i_m})$, and $U_{m+n} \in \text{mod}(\mathbb{k}G_{i_{m+n}})$. Finally, let $f \in \text{Hom}_{\mathbb{k}G_{i_0}}(U_0, M(\underline{i}; U_m))$ and $f' \in \text{Hom}_{\mathbb{k}G_{i_m}}(U_m, M(\underline{i}'; U_{m+n}))$. Then,*

$$(4.2.1) \quad (\Theta_{\underline{i}'; U_{m+n}} \circ f') \times (\Theta_{\underline{i}; U_m} \circ f) = \Theta_{\underline{i}''; U_{m+n}} \circ (f' \circledast f).$$

As a consequence, $f' \circledast f \in \text{Hom}_{G_{i_0}}(U_0, M(\underline{i}''; U_{m+n}))$.

Proof. Let $u \in U_0$. Since $\Theta_{\underline{i}'; U_{m+n}} \circ f'$ lies in $\text{Hom}_{G_{i_m}}(U_m, M(\underline{i}''; U_{m+n}) \otimes_{G_{i_{m+n}}} U_{m+n})$ and $\Theta_{\underline{i}; U_m} \circ f$ lies in $\text{Hom}_{G_{i_0}}(U_0, M(\underline{i}) \otimes_{G_{i_m}} U_m)$, then

$$((\Theta_{\underline{i}'; U_{m+n}} \circ f') \times (\Theta_{\underline{i}; U_m} \circ f))(u) \stackrel{(3.3.3)}{=} (M(\underline{i}) \otimes_{\mathbb{k}G_{i_m}} (\Theta_{\underline{i}'; U_{m+n}} \circ f')) \circ (\Theta_{\underline{i}; U_m} \circ f)(u).$$

Using the definition of $\Theta_{\underline{i}; U_m}$, see (4.1.7), the right-hand side is equal to

$$\begin{aligned} & (M(\underline{i}) \otimes_{\mathbb{k}G_{i_m}} (\Theta_{\underline{i}'; U_{m+n}} \circ f')) (\sum_{\underline{y}} f_{\underline{y}}^{(1)}(u) y_1 \otimes y_1^{-1} f_{\underline{y}}^{(2)}(u) y_2 \otimes \dots \\ & \dots \otimes (y_1 \dots y_{m-1})^{-1} (f_{\underline{y}}^{(m)}(u)) y_m \otimes f_{\underline{y}}^{(0)}(u)), \end{aligned}$$

where \underline{y} runs through all sequences y_1, \dots, y_m such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, m\}$. Using the definition of $\Theta_{\underline{i}'; U_{m+n}}$, see (4.1.7), this is equal to

$$\begin{aligned} & \sum_{\underline{y}, \underline{y}'} f_{\underline{y}}^{(1)}(u) y_1 \otimes \dots \otimes (y_1 \dots y_{m-1})^{-1} (f_{\underline{y}}^{(m)}(u)) y_m \otimes \\ & f_{\underline{y}'}^{(1)}(f_{\underline{y}}^{(0)}(u)) y_{m+1} \otimes y_{m+1}^{-1} (f_{\underline{y}'}^{(2)}(f_{\underline{y}}^{(0)}(u))) y_{m+2} \otimes \dots \\ & \dots \otimes (y_{m+1} \dots y_{m+n})^{-1} (f_{\underline{y}'}^{(n)}(f_{\underline{y}}^{(0)}(u))) y_{m+n} \otimes f_{\underline{y}'}^{(0)}(f_{\underline{y}}^{(0)}(u)), \end{aligned}$$

where \underline{y} is as before and \underline{y}' runs through all sequences y_{m+1}, \dots, y_{m+n} such that $y_{m+t} \in [G/G_{i_{m+t}}]$ for all $t \in \{1, \dots, n\}$. Using the definition of $\Theta_{\underline{i}''; U_{m+n}}$, see (4.1.7), this is equal to

$$\begin{aligned} & \Theta_{\underline{i}''; U_{m+n}} (\sum_{\underline{y}''} f_{\underline{y}''}^{(1)}(u) \otimes \dots \otimes f_{\underline{y}''}^{(m)}(u) \otimes y_1 \dots y_m (f_{\underline{y}''}^{(1)}(f_{\underline{y}''}^{(0)}(u))) \otimes \dots \\ & \dots \otimes y_1 \dots y_m (f_{\underline{y}''}^{(n)}(f_{\underline{y}''}^{(0)}(u))) \otimes y_1 \dots y_{m+n} f_{\underline{y}''}^{(0)}(f_{\underline{y}''}^{(0)}(u)), \end{aligned}$$

where \underline{y}'' runs through all sequences y_1, \dots, y_{m+n} such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, m+n\}$ and where \underline{y} and \underline{y}' stand for the sequences y_1, \dots, y_m and y_{m+1}, \dots, y_{m+n} , respectively. Finally, using the definition of \circledast , see (1.0.5), this is equal to $(\Theta_{\underline{i}''; U_{m+n}} \circ (f' \circledast f))(u)$. Whence (4.2.1). Note that $f' \circledast f$ is a morphism of $\mathbb{k}G_{i_0}$ -modules because

- $\Theta_{\underline{i}''; U_{m+n}}$ is an isomorphism of $\mathbb{k}G_{i_0}$ -modules,
- f and $\Theta_{\underline{i}; U_m}$ are morphisms of $\mathbb{k}G_{i_0}$ -modules,
- f' and $\Theta_{\underline{i}'; U_{m+n}}$ are morphisms of $\mathbb{k}G_{i_m}$ -modules,
- and the operation “ \times ” yields morphisms of $\mathbb{k}G_{i_0}$ -modules (see (3.3.3)).

□

Lemma 4.4 shows that Definition 1.10 yields a, possibly non associative, algebra.

Proposition 4.5. *Intw is an associative and unital algebra.*

Proof. Note that $\mathcal{I}ntw$ is unital. Indeed, for all $i \in [G \setminus Q_0]$ and $U \in \text{irr}(G_i)$, denote by $\varepsilon_{i,U}$ the intertwiner $\text{Id}_U \in \text{Hom}_{\mathbb{k}G_i}(U, M(i; U))$, recall that $M(i; U) = U$. Then $\sum_{i,U} \varepsilon_{i,U}$ is a unity for \otimes .

In the framework of Definition 3.5, the equality $f_3 \times (f_2 \times f_1) = (f_3 \times f_2) \times f_1$ holds whenever both sides make sense because of the associativity of the tensor product. In view of Lemma 4.4, this entails that \otimes is associative. \square

4.3. The pairing on intertwiners. Let n be a positive integer, let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 and let U_0 and U_n be left modules over $\mathbb{k}G_{i_0}$ and $\mathbb{k}G_{i_n}$, respectively. To every $\varphi \in \text{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0))$ is associated a \mathbb{k} -linear mapping

$$\Phi(\varphi): M(\underline{i}; U_n) \rightarrow U_0$$

defined as follows. For every sequence of representatives y_1, \dots, y_n ($y_t \in [G/G_{i_t}]$) there exist unique sequences

$$\underline{x} = x_{n-1}, \dots, x_0 \quad (x_t \in [G/G_{i_t}]), \quad \text{and} \quad h_{n-1}, \dots, h_0 \quad (h_t \in G_{i_t})$$

such that, for all $t \in \{1, \dots, n\}$,

$$(4.3.1) \quad (y_t \cdots y_n)^{-1} = x_{n-1} \cdots x_{t-1} h_{t-1};$$

these sequences are determined by a decreasing induction: $y_n^{-1} = x_{n-1} h_{n-1}$ and $h_t y_t^{-1} = x_{t-1} h_{t-1}$ for all $t \in \{1, \dots, n-1\}$; then, for every $m_1 \otimes \cdots \otimes m_t \otimes y_1 \cdots y_n u$ lying in $M(\underline{i}; U_n)$, define the element $\Phi(\varphi)(m_1 \otimes \cdots \otimes m_t \otimes y_1 \cdots y_n u)$ of U_0 , using Notation 1.12, as

$$(4.3.2) \quad \prod_{t=1}^n \varphi_{(t)}^{\underline{x}}(u) \left((y_1 \cdots y_n)^{-1} m_t \right) h_0^{-1} (\varphi_{(0)}^{\underline{x}}(u)).$$

This does make sense, because $m_t \in {}_{y_1 \cdots y_{t-1} \cdot i_{t-1}} M_{y_1 \cdots y_t \cdot i_t}$, because $\varphi_{(t)}^{\underline{x}}$ is a linear form on ${}_{x_{n-1} \cdots x_{t-1} \cdot i_{t-1}} M_{x_{n-1} \cdots x_t \cdot i_t}$, and because of (4.3.1).

After appropriate identifications, Φ is related to the adjunction isomorphism $\Psi_{M(\underline{i})}^{\bullet, \bullet}$ of section 3.2 as explained in the following result. Recall that $\Psi_{M(\underline{i})}^{U_n, U_0}$, $\Theta_{\underline{i}^o; U_0}^{M^*}$ and $\Lambda_{\underline{i}}$ are defined in (3.2.1), (4.1.7), and (4.1.5), respectively.

Lemma 4.6. *Keep the previous setting. Denote by φ' the following composite morphism of left $\mathbb{k}G_{i_n}$ -modules,*

$$U_n \xrightarrow{\varphi} M^*(\underline{i}^o; U_0) \xrightarrow{\Theta_{\underline{i}^o; U_0}^{M^*}} M^*(\underline{i}^o) \otimes_{G_{i_0}} U_0 \xrightarrow{\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0} M(\underline{i})^* \otimes_{G_{i_0}} U_0.$$

Then,

$$(4.3.3) \quad \Phi(\varphi) = \Psi_{M(\underline{i})}^{U_n, U_0}(\varphi') \circ \Theta_{\underline{i}; U_n}^M.$$

In particular, $\Phi(\varphi) \in \text{Hom}_{G_{i_0}}(M(\underline{i}; U_n), U_0)$.

Proof. The composition $\Theta_{\underline{i}^o; U_0}^{M^*} \circ \varphi$ is given by (see (4.1.7))

$$u \mapsto \sum_{\underline{x}} \varphi_{(n)}^{\underline{x}}(u) x_{n-1} \otimes x_{n-1}^{-1} \varphi_{(n-1)}^{\underline{x}}(u) x_{n-2} \otimes \cdots \otimes (x_{n-1} \cdots x_1)^{-1} \varphi_{(1)}^{\underline{x}}(u) x_0 \otimes \varphi_{(0)}^{\underline{x}}(u),$$

where $\underline{x} = x_{n-1}, \dots, x_0$ runs through all sequences of representatives ($x_t \in [G/G_{i_t}]$) for all $0 \leq t \leq n-1$). Hence, following (4.1.6), the morphism φ' is given by

$$u \mapsto \sum_{\underline{x}} \phi^{\underline{x}}(u) \otimes \varphi_{(0)}^{\underline{x}}(u)$$

where $\phi^{\underline{x}}(u)$ is the linear form on $M(\underline{i})$, which is a quotient of $(MG)^{\otimes_{\mathbb{k}} n}$, induced by the linear form on $(MG)^{\otimes_{\mathbb{k}} n}$ mapping any $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n)$ to

$$(4.3.4) \quad \sum_{k_1, \dots, k_{n-1}} \prod_{t=1}^n \left((x_{n-1} x_{n-2} \cdots x_t)^{-1} (\varphi_{(t)}^{\underline{x}}(u)) \right) \left(k_t^{-1} g_t^{-1} m_t \right) \cdot \delta_{e, x_{t-1} k_{t-1}^{-1} g_t k_t},$$

where δ denotes the Kronecker symbol, $k_0 = k_n = e$ and k_t runs through G_{i_t} for $1 \leq t \leq n-1$.

In order to simplify the expression of (4.3.4), consider a tensor $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n)$ in $(MG)^{\otimes_{\mathbb{k}} n}$; for all k_1, \dots, k_{n-1} in G , the following assertions are equivalent, keeping $k_0 = k_n = e$,

- (i) $(\forall t \in \{1, \dots, n-1\}) e = x_{t-1} k_{t-1}^{-1} g_t k_t$,
- (ii) $k_t = (x_{t-1} \cdots x_1 x_0 g_1 g_2 \cdots g_t)^{-1}$ for all $t \in \{1, \dots, n\}$, and $g_1 \cdots g_n$ is the inverse of $x_{n-1} \cdots x_1 x_0$;

given that each $(x_{n-1} x_{n-2} \cdots x_t)^{-1} (\varphi_{(t)}^{\underline{x}}(u))$ in (4.3.4) is a linear form on $x_{t-1} i_{t-1} M_{i_t}$ and each k_t is constrained to lie in G_{i_t} , then (4.3.4) is equal to

$$\prod_{t=1}^n (\varphi_{(t)}^{\underline{x}}(u)) (x_{n-1} \cdots x_1 x_0 g_1 g_2 \cdots g_{t-1} m_t) \delta_{e, g_1 g_2 \cdots g_n x_{n-1} \cdots x_1 x_0},$$

which simplifies to

$$(4.3.5) \quad \prod_{t=1}^n (\varphi_{(t)}^{\underline{x}}(u)) \left((g_t g_{t+1} \cdots g_n)^{-1} m_t \right) \delta_{(g_1 g_2 \cdots g_n)^{-1}, x_{n-1} \cdots x_1 x_0}.$$

Thus, the morphism φ' is given by $u \mapsto \sum_{\underline{x}} \phi^{\underline{x}}(u) \otimes \varphi_{(0)}^{\underline{x}}(u)$, where $\phi^{\underline{x}}(u)$ is the linear form on $M(\underline{i})$ induced by the mapping assigning (4.3.5) to any element $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n)$ of $(MG)^{\otimes_{\mathbb{k}} n}$.

Hence (see Lemma 3.2), $\Psi_{M(\underline{i})}^{U_n, U_0}(\varphi')$ is the morphism from $M(\underline{i}) \otimes_{G_{i_n}} U_n$ to U_0 induced by the mapping from $(MG)^{\otimes_{\mathbb{k}} n} \otimes_{\mathbb{k}} U_n$ to U_0 which maps any $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n) \otimes u$ to

$$\begin{aligned} & \sum_{\underline{x}} \sum_{g \in G_{i_0}} \phi^{\underline{x}}(u) (g^{-1} m_1 g_1 \otimes m_2 g_2 \otimes \cdots \otimes m_n g_n) {}^g (\varphi_{(0)}^{\underline{x}}(u)) \\ &= \sum_{g, \underline{x}} \prod_{t=1}^n \varphi_{(t)}^{\underline{x}}(u) ((g_t \cdots g_n)^{-1} m_t) \delta_{(g_1 \cdots g_n)^{-1}, x_{n-1} \cdots x_0} {}^g (\varphi_{(0)}^{\underline{x}}(u)). \end{aligned}$$

Now, let $m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n u \in M(\underline{i}; U_n)$. Recall that its image under $\Theta_{\underline{i}; U_n}^M$ is $m_1 y_1 \otimes y_1^{-1} m_2 y_2 \otimes \cdots \otimes (y_1 \cdots y_{n-1})^{-1} m_n y_n \otimes u$ (see (4.1.7)). Hence, its image under $\Psi_{M(\underline{i})}^{U_n, U_0}(\varphi') \circ \Theta_{\underline{i}; U_n}^M$ is

$$(4.3.6) \quad \sum_{g, \underline{x}} \prod_{t=1}^n \varphi_{(t)}^{\underline{x}}(u) ((y_1 \cdots y_n)^{-1} m_t) \delta_{(y_1 \cdots y_n)^{-1}, x_{n-1} \cdots x_0} {}^g (\varphi_{(0)}^{\underline{x}}(u)).$$

Note that, given $g \in G_{i_0}$ and \underline{x} , then, for all $t \in \{1, \dots, n\}$,

- $\varphi_{(t)}^{\underline{x}}(u) \in (x_{n-1} \cdots x_{t-1} i_{t-1} M_{x_{n-1} \cdots x_t i_t})^*$ and
- $(y_1 \cdots y_n)^{-1} m_t \in (y_t \cdots y_n)^{-1} i_{t-1} M_{(y_{t+1} \cdots y_n)^{-1} i_t}$;

in particular, if for a given index (g, \underline{x}) in (4.3.6) the product “ $\prod_{t=1}^n$ ” is nonzero then this product is the unique nonzero term of the sum (4.3.6). More precisely, let $\{x_t, h_t\}_{0 \leq t \leq n-1}$ be the unique collection of elements in G such that, for all $t \in \{0, \dots, n-1\}$,

- $x_t \in [G/G_{i_t}]$,
- $h_t \in G_{i_t}$ and
- $(y_{t+1} \cdots y_n)^{-1} = x_{n-1} \cdots x_t h_t$;

then, the only possibly nonzero term of the sum (4.3.6) is the one with index $(h_0^{-1}, \underline{x})$. Therefore, (4.3.6) equals

$$\prod_{t=1}^n \varphi_{(t)}^{\underline{x}}(u) (y_1 \cdots y_n)^{-1} m_t h_0^{-1} (\varphi_{(0)}^{\underline{x}}(u)),$$

which is equal to $\Phi(\varphi)(m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n u)$ (see (4.3.2)). Thus, $\Phi(\varphi) = \Psi_{M(\underline{i})}^{U_n, U_0}(\varphi') \circ \Theta_{\underline{i}; U_n}^M$. This proves (4.3.3). The second assertion of the lemma is a direct consequence of (4.3.3). \square

It follows from Lemma 4.6 that the following mapping is bijective,

$$\Phi: \text{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0)) \rightarrow \text{Hom}_{G_{i_0}}(M(\underline{i}; U_n), U_0).$$

This yields a non degenerate pairing defined as follows.

Definition 4.7. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 , where $n \geq 1$. Let $U_0 \in \text{mod}(\mathbb{k}G_{i_0})$ be simple. Let $U_n \in \text{mod}(\mathbb{k}G_{i_n})$. Define $(-|-)$ to be the non degenerate pairing

$$(-|-): \text{Hom}_{G_{i_0}}(U_0, M(\underline{i}; U_n)) \otimes_{\mathbb{k}} \text{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0)) \rightarrow \mathbb{k},$$

such that, for all $f \in \text{Hom}_{G_{i_0}}(U_0, M(\underline{i}; U_n))$ and $\varphi \in \text{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0))$,

$$(4.3.7) \quad (f|\varphi) \cdot \text{Id}_{U_0} = \Phi(\varphi) \circ f;$$

With this definition, it follows from (3.3.2) and (4.3.3) that

$$(4.3.8) \quad (f|\varphi) = \langle \Theta_{\underline{i}; U_n}^M \circ f | (\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Theta_{\underline{i}^o; U_0}^{M^*} \circ \varphi \rangle.$$

It is now possible to prove Proposition 1.13.

Proof of Proposition 1.13. The expression (1.0.10) is equal to $(\Phi(\varphi))(f(u))$, see (4.3.2). Hence, the pairing $(-|-)$ claimed by the proposition is the one of Definition 4.7. \square

The pairing $(-|-)$ is compatible with the composition of (dual) intertwiners in the following sense.

Lemma 4.8. Let m, n be positive integers, let $\underline{i}'' = i_0, \dots, i_{m+n}$ be a sequence in Q_0 , let U_0 and U_m be simple left modules over $\mathbb{k}G_{i_0}$ and $\mathbb{k}G_{i_m}$, respectively, and let U_{m+n} be a left $\mathbb{k}G_{i_{m+n}}$ -module. Then, for all intertwiners

$$\begin{aligned} f_1: U_0 &\rightarrow M(\underline{i}; U_m), & \varphi_1: U_m &\rightarrow M^*(\underline{i}^o; U_0), \\ f_2: U_m &\rightarrow M(\underline{i}'; U_{m+n}), & \varphi_2: U_{m+n} &\rightarrow M^*(\underline{i}'^o; U_m), \end{aligned}$$

where $\underline{i} = i_0, \dots, i_m$ and $\underline{i}' = i_m, \dots, i_{m+n}$, the following holds

$$(4.3.9) \quad (f_2 \otimes f_1 | \varphi_1 \otimes \varphi_2) = (f_2 | \varphi_2) \cdot (f_1 | \varphi_1).$$

Proof. In order to avoid overloaded expressions, the following shorthand notation is used in this proof: $\Theta_{\underline{i}; U_0}$ for $\Theta_{\underline{i}; U_0}^M$ and $\Theta'_{\underline{i}; U_0}$ for $\Theta_{\underline{i}^o; U_0}^{M^*}$. Define $f'_1, f'_2, \varphi'_1, \varphi'_2$ as follows

$$\begin{aligned} f'_1 &= \Theta_{\underline{i}; U_m} \circ f_1, & \varphi'_1 &= (\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Theta'_{\underline{i}; U_0} \circ \varphi_1, \\ f'_2 &= \Theta_{\underline{i}'; U_{m+n}} \circ f_2, & \varphi'_2 &= (\Lambda_{\underline{i}'} \otimes_{G_{i_m}} U_m) \circ \Theta'_{\underline{i}'; U_m} \circ \varphi_2. \end{aligned}$$

It follows from (4.3.8) that $(f_2 \otimes f_1 | \varphi_1 \otimes \varphi_2)$ is equal to

$$\langle (\Theta_{\underline{i}'; U_{m+n}} \circ f_2) | (\Lambda_{\underline{i}'} \otimes_{G_{i_0}} U_0) \circ \Theta'_{\underline{i}'; U_0} \circ (\varphi_1 \otimes \varphi_2) \rangle;$$

applying Lemma 4.4 twice, first for M and next for M^* , yields that the previous expression is equal to

$$\langle \underbrace{(\Theta_{\underline{i}'; U_{m+n}} \circ f_2)}_{f'_2} \times \underbrace{(\Theta_{\underline{i}; U_m} \circ f_1)}_{f'_1} | (\Lambda_{\underline{i}''} \otimes_{G_{i_0}} U_0) \circ \left[(\Theta'_{\underline{i}; U_0} \circ \varphi_1) \times (\Theta'_{\underline{i}'; U_m} \circ \varphi_2) \right] \rangle;$$

according to Lemma 4.2 this is equal to

$$(4.3.10) \quad \langle f'_1 \times f'_2 | \lambda \circ (\Lambda_{\underline{i}'} \otimes_{G_{i_m}} \Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ [(\Theta'_{\underline{i}', U_0} \circ \varphi_1) \times (\Theta'_{\underline{i}', U_m} \circ \varphi_2)] \rangle,$$

where λ is as in Lemma 3.6. Now, since the following composite morphisms are equal due to the associativity of the tensor product

$$M^*(\underline{i}'^o) \otimes_{G_{i_m}} U_m \xrightarrow{1 \otimes (\Theta'_{\underline{i}', U_0} \circ \varphi_1)} M^*(\underline{i}'^o) \otimes_{G_{i_m}} M^*(\underline{i}^o) \otimes_{G_{i_0}} U_0 \xrightarrow{\Lambda_{\underline{i}'} \otimes \Lambda_{\underline{i}} \otimes 1} M(\underline{i}')^* \otimes_{G_{i_m}} M(\underline{i})^* \otimes_{G_{i_0}} U_0$$

and

$$M^*(\underline{i}'^o) \otimes_{G_{i_m}} U_m \xrightarrow{\Lambda_{\underline{i}'} \otimes 1} M(\underline{i}')^* \otimes_{G_{i_m}} U_m \xrightarrow{1 \otimes (\Theta'_{\underline{i}', U_0} \circ \varphi_1)} M(\underline{i}')^* \otimes_{G_{i_m}} M^*(\underline{i}^o) \otimes_{G_{i_0}} U_0 \xrightarrow{1 \otimes \Lambda_{\underline{i}} \otimes 1} M(\underline{i}')^* \otimes_{G_{i_m}} M(\underline{i})^* \otimes_{G_{i_0}} U_0,$$

then $(\Lambda_{\underline{i}'} \otimes_{G_{i_m}} \Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ [(\Theta'_{\underline{i}', U_0} \circ \varphi_1) \times (\Theta'_{\underline{i}', U_m} \circ \varphi_2)]$ is equal to

$$\underbrace{[(\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Theta'_{\underline{i}', U_0} \circ \varphi_1]}_{\varphi'_1} \times \underbrace{[(\Lambda_{\underline{i}'} \otimes_{G_{i_m}} U_m) \circ \Theta'_{\underline{i}', U_m} \circ \varphi_1]}_{\varphi'_2}.$$

Therefore,

$$(4.3.10) \quad \begin{aligned} &= \langle f'_2 \times f'_1 | \lambda \circ (\varphi'_1 \times \varphi'_2) \rangle \\ &\stackrel{\text{Lemma 3.6}}{=} \langle f'_2 | \varphi'_2 \rangle \cdot \langle f'_1 | \varphi'_1 \rangle \\ &\stackrel{(4.3.8)}{=} (f_2 | \varphi_2) \cdot (f_1 | \varphi_1). \end{aligned}$$

Thus, $(f_2 \otimes f_1 | \varphi_1 \otimes \varphi_2) = (f_1 | \varphi_1) \cdot (f_2 | \varphi_2)$. \square

5. PROOF OF THE MAIN RESULTS

This section proves Theorems 1.15 and 1.16. Keeping Q_0 , S , G , and M as in Setting 1.1 and Q_G as in Setting 1.3, the proofs use the following notation:

- denote by \hat{e} the idempotent $\sum_{i \in [G \setminus Q_0]} e_i$ of S , this idempotent is not to be confused with \tilde{e} which is equal to $\sum_{(i, U)} e_i * \varepsilon_U$, where (i, U) runs through the vertices of Q_G ;
- denote by S_1 the \mathbb{k} -algebra $\prod_{i \in [G \setminus Q_0]} \mathbb{k}G_i$.

First, it is necessary to check that the quiver Q_G corresponds to the one given by Demonet in [3].

Proof of Theorem 1.4. Following [3], see p. 1057, the algebra $T_S(M) * G$ is Morita equivalent to $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, which is isomorphic to the path algebra of a quiver having the same vertices as Q_G and such that the arrows from a vertex (i, U) to a vertex (j, V) form a basis of the vector space $(e_i * \varepsilon_U) \cdot (MG) \cdot (e_j * \varepsilon_V)$. Now, considering e_i , e_j , ε_U , and ε_V as elements of $S * G$,

$$\begin{aligned} (e_i * \varepsilon_U) \cdot (MG) \cdot (e_j * \varepsilon_V) &= \varepsilon_U \cdot (e_i \cdot (MG) \cdot e_j) \cdot \varepsilon_V \\ &= \varepsilon_U \cdot M(i, j) \cdot \varepsilon_V \\ (4.1.1) \quad &\simeq \text{Hom}_{\mathbb{k}G_i}(U, M(i, j) \otimes_{G_j} V) \\ &\simeq \text{Hom}_{\mathbb{k}G_i}(U, M(i, j; V)). \\ (4.1.7) \quad & \end{aligned}$$

Whence the first statement of the theorem. Now, if $U = \mathbb{k}G_i \cdot \varepsilon_U$ for all $(i, U) \in Q_G$, then $f(\varepsilon_U)$ does belong to $(e_i * \varepsilon_U) \cdot (MG) \cdot (e_j * \varepsilon_V)$, which is contained in $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, for all arrows $f: (i, U) \rightarrow (j, V)$ of Q_G . Whence the isomorphism (1.0.1). \square

The proof of Theorem 1.15 uses Morita equivalences in the graded sense. Two $(\mathbb{N}$ -)graded finite dimensional \mathbb{k} -algebras Λ_1 and Λ_2 are Morita equivalent in the graded sense if $\Lambda_2 \simeq \text{End}_{\Lambda_1}(P)^{\text{op}}$ for some graded projective Λ_1 -module P which is a generator of the category of graded Λ_1 -modules, and hence of $\text{mod}(\Lambda_1)$. Both

Λ_1 and Λ_2 are isomorphic as graded algebras to basic graded finite dimensional algebras; moreover Λ_1 and Λ_2 are Morita equivalent in the graded sense if and only if their basic versions are isomorphic as graded algebras.

Proof of Theorem 1.15. Denote by Φ the algebra homomorphism $\mathbb{k}Q_G \rightarrow \mathcal{I}ntw^{\text{op}}$ given in the statement of the theorem. Hence, $\Phi(\gamma) = f_\gamma$ for all paths γ in Q_G .

(1) Let γ and γ' be parallel paths in Q_G . It follows from the definition of $\varphi_{\gamma'}$, see Notation 1.14, and from (4.3.9) that

$$(5.0.11) \quad (f_\gamma | \varphi_{\gamma'}) = \begin{cases} 1 & \text{if } \gamma = \gamma', \\ 0 & \text{otherwise.} \end{cases}$$

Since Φ maps non parallel paths in Q_G into distinct components of $\mathcal{I}ntw$ in the decomposition (1.0.7), then Φ is injective. In order to prove that Φ is an isomorphism, it is hence sufficient to prove that, for all integers n , the homogeneous components of degree n of the graded algebras $\mathbb{k}Q_G$ and $\mathcal{I}ntw$ have the same dimensions. Note that $\mathbb{k}Q_G$ is graded by the length of the paths in Q_G and $\mathcal{I}ntw$ is graded so that every intertwiner lying in a space of the shape $\text{Hom}_{\mathbb{k}G_{i_0}}(U, M(i_0, \dots, i_n; V))$ is homogeneous of degree n .

First, following [3], see Theorem 1.4, the algebras $\mathbb{k}Q_G$ and $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ are isomorphic \mathbb{k} -algebras both Morita equivalent to $T_S(M) * G$. Actually, the proof given in [3] works in the graded setting.

Next, \hat{e} may be considered as an idempotent of $T_S(M) * G$. In this sense,

$$\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e} = \tilde{e} \cdot \hat{e}(T_S(M) * G) \cdot \hat{e} \cdot \tilde{e}.$$

Following [3], see p. 1057, the algebra $\hat{e} \cdot (T_S(M) * G) \cdot \hat{e}$ is Morita equivalent to $T_{S_1}(\hat{e} \cdot (MG) \cdot \hat{e})$. Again, this is still true in the graded sense. Hence, $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, which is basic, is isomorphic to $\tilde{e} \cdot T_{S_1}(\hat{e} \cdot (MG) \cdot \hat{e}) \cdot \tilde{e}$ as a graded algebra.

Finally, by definition of \tilde{e} and \hat{e} , the homogeneous component of degree n of $\tilde{e} \cdot T_{S_1}(\hat{e} \cdot (MG) \cdot \hat{e}) \cdot \tilde{e}$ is equal to

$$\bigoplus_{i_0, i_n \in [G \setminus Q_0], U_0 \in \text{irr}(G_{i_0}), U_n \in \text{irr}(G_{i_n})} (e_{i_0} * \varepsilon_{U_0}) \cdot (MG)^{\otimes_{S_1} n} \cdot (e_{i_n} * \varepsilon_{U_n}).$$

This is equal to

$$\bigoplus_{i_0, \dots, i_n \in [G \setminus Q_0], U_0 \in \text{irr}(G_{i_0}), U_n \in \text{irr}(G_{i_n})} \varepsilon_{U_0} \cdot M(i_0, \dots, i_n) \cdot \varepsilon_{U_n}.$$

This is isomorphic to

$$\bigoplus_{i_0, \dots, i_n, U_0, U_n} \text{Hom}_{\mathbb{k}G_{i_0}}(U_0, \underbrace{M(i_0, \dots, i_n) \otimes_{G_{i_n}} V}_{\simeq M(i_0, \dots, i_n; V)}).$$

These considerations prove that the homogeneous components of degree n of $\mathbb{k}Q_G$ and $\mathcal{I}ntw$ are isomorphic. Thus, Φ is an isomorphism.

(2) follows from (5.0.11) and from the fact that Φ is surjective. \square

Proof of Theorem 1.16. First, Ξ is bijective. Indeed, on one hand, there is an equality of vector subspaces of $T_S(M) * G$,

$$\hat{e} \cdot (T_S(M) * G) \cdot \hat{e} = \bigoplus_{\underline{i}, \underline{y}} ({}_{i_0} M_{y_1 \cdot i_1} \otimes_{\mathbb{k}} \cdots \otimes_{\mathbb{k}} {}_{y_1 \cdots y_{n-1} \cdot i_{n-1}} M_{y_1 \cdots y_n \cdot i_n}) * y_1 \cdots y_n \mathbb{k}G,$$

where \underline{i} runs through all sequences i_0, \dots, i_n of $[G \setminus Q_0]$ and \underline{y} runs through all sequences y_1, \dots, y_n such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$. On the other

hand, for all vertices (i, U) and (j, V) of Q_G and for all $\mathbb{k}G_i - \mathbb{k}G_j$ -bimodules N , the following mapping is bijective,

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbb{k}G_i}(U, N \cdot \varepsilon_V) & \longrightarrow & \varepsilon_U \cdot N \cdot \varepsilon_V \\ f & \mapsto & f(\varepsilon_U). \end{array}$$

Hence, Ξ is bijective and Ξ^{-1} is given by $\Xi^{-1}(f) = f(\varepsilon_U)$ for all $f \in \mathcal{I}ntw$ with domain denoted by U .

Next, Ξ is a morphism of \mathbb{k} -algebras. Indeed, for all $f \in \mathrm{Hom}_{\mathbb{k}G_{i_0}}(U, M(\underline{i}; V))$ lying in $\mathcal{I}ntw$, then $\Xi^{-1}(f) = f(\varepsilon_U)$. Therefore, with the setting of Definition 1.9, then $(f' \otimes f)(\varepsilon_U)$ is equal to the product $f(\varepsilon_V) \cdot f'(\varepsilon_U)$ in $T_S(M) * G$, see (1.0.5). Accordingly, Ξ^{-1} , and hence Ξ , is a \mathbb{k} -algebra homomorphism.

Finally, the diagram (1.0.14) is commutative. Indeed, all its arrows are \mathbb{k} -algebra homomorphisms. Hence, it suffices to prove the commutativity on the arrows of Q_G , which follows from the definitions of Ξ , (1.0.1), and (1.0.11). Note that (1.0.15) follows from (1.0.12). \square

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