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To cite this version:

Patrick Le Meur. On the Morita Reduced Versions of Skew Group Algebras of Path Algebras. Quarterly Journal of Mathematics, 2020, 71 (3), 10.1093/qmathj/haaa014 . hal-02361334v2

HAL Id: hal-02361334

https://hal.science/hal-02361334v2

Submitted on 16 Feb 2023

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ON THE MORITA REDUCED VERSIONS OF SKEW GROUP ALGEBRAS OF PATH ALGEBRAS

PATRICK LE MEUR

Abstract. Let R be the skew group algebra of a finite group acting on the path algebra of a quiver. This article develops both theoretical and practical methods to do computations in the Morita reduced algebra associated to R. Reiten and Riedtmann proved that there exists an idempotent e of R such that the algebra eRe is both Morita equivalent to R and isomorphic to the path algebra of some quiver which was described by Demonet. This article gives explicit formulas for the decomposition of any element of eRe as a linear combination of paths in the quiver described by Demonet. This is done by expressing appropriate compositions and pairings in a suitable monoidal category which takes into account the representation theory of the finite group.

Introduction

In [13], Reiten and Riedtmann initiated the investigation of the skew group algebras of Artin algebras from the viewpoint of homological dimensions and representation theory. Their article was followed up by many research works illustrating the following principle: Artin algebras share many properties, whether of homological or of representation theoretic nature, with their associated skew group algebras. For instance, see [12] for interactions with Koszul duality; [4, 11] for interactions with piecewise hereditary algebras; and [14, 3] for interactions with preprojective algebras and McKay correspondence.

Let $\Bbbk$ be an algebraically closed field, Q be a finite quiver that may have oriented cycles, and G be a finite group with order not divisible by $\mathrm{char}(\Bbbk)$ and acting on $\Bbbk Q$ by algebra automorphisms in such a way that both the set of (primitive idempotents of) vertices and the vector subspace generated by the arrows are stabilised by this action. Following [13], the skew group algebra $\Bbbk Q * G$ is hereditary.

While Reiten and Riedtmann described a quiver whose path algebra is Morita equivalent to $\Bbbk Q * G$ when G is cyclic, the description of such a quiver in general is recent. In [3], Demonet described an idempotent $\tilde{e}$ of $\Bbbk Q * G$ and a quiver Q_G such that $\tilde{e} \cdot (\Bbbk Q * G) \cdot \tilde{e}$ is Morita equivalent to $\Bbbk Q * G$ and isomorphic to $\Bbbk Q_G$. This, however, does not yield a completely described isomorphism, which might be source of trouble. Here is a situation taken from Calabi-Yau algebras and where such trouble may occur. Following [10], if W is a G-invariant potential on Q, then G acts on the Ginzburg dg algebra $\mathcal{A}(Q, W)$, defined in [6], and $\mathcal{A}(Q, W) * G$ is Morita equivalent to $\mathcal{A}(Q_G, W_G)$, where W_G is the potential on Q_G which, as a linear combination of oriented cycles, corresponds to $\tilde{e} \cdot W \cdot \tilde{e}$ under any chosen isomorphism $\Bbbk Q_G \to \tilde{e} \cdot (\Bbbk Q * G) \cdot \tilde{e}$. The problem here is that describing that linear combination explicitly is not easy. Up to now, the only fairly general explicit descriptions of W_G are due to Amiot and Plamondon [1] when G is of order 2 and to Giovannini and Pasquali [8] when G is cyclic, the stabiliser of each vertex is either trivial or the whole group, and every cycle appearing in W goes through fixed

Date: June 30, 2020.
2010 *Mathematics Subject Classification.* Primary 16S35; Secondary 16W22, 20C15, 18D10.
Key words and phrases. Path algebra; skew group algebra; monoidal category; intertwiner.

vertices only (or, through vertices with trivial stabilisers only, respectively) as soon as it goes through two of them; during the revision of the present article, Giovannini, Pasquali and Plamondon presented [9] an extension of the latter description to the case where G is finite and abelian.

This raises the question of decomposing explicitly the elements of $\tilde{e} \cdot (\Bbbk Q * G) \cdot \tilde{e}$ as linear combinations of paths in Q_G under an isomorphism $\Bbbk Q_G \to \tilde{e} \cdot (\Bbbk Q * G) \cdot \tilde{e}$.

Demonet's description of Q_G involves the representation theory of the stabilisers of the vertices of Q. It is hence expectable that answering the above-mentioned question should involve representation theory. Answers of this kind already exist in particular cases where Q has only one vertex. Here is an example due to Ginzburg [6]. He proved that, if G is a finite subgroup of $\mathrm{SL}_3(\mathbb{C})$, then $\mathbb{C}[x, y, z] * G$ is Calabi-Yau in dimension 3. This was done by taking Q to be the quiver with one vertex and three loops x, y, and z, by taking $W = xyz - xzy$, and by expressing W_G in terms of the monoidal category of finite dimensional representations of G. See [2] for a generalisation to $\mathbb{C}[x_1, \ldots, x_n] * G$ where G is a finite subgroup of $\mathrm{SL}_n(\mathbb{C})$, in which case Q is the quiver with one vertex and n loops. Note that this generalisation is (detailed and) successfully applied by Giovannini in [7] to find subgroups Γ of $\mathrm{SL}_{n+1}(\mathbb{C})$ such that $\mathbb{C}[x_1, \ldots, x_{n+1}] * \Gamma$ is Morita equivalent to the $(n+1)$-preprojective algebra of an n-representation finite algebra.

This article hence gives explicit formulas for the decomposition of the elements of $\tilde{e} \cdot (\Bbbk Q * G) \cdot \tilde{e}$ as linear combinations of paths in Q_G under the isomorphism $\Bbbk Q_G \to \tilde{e} \cdot (\Bbbk Q * G) \cdot \tilde{e}$. These formulas are obtained by translating explicitly the result of natural operations and pairings in the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$ of finite dimensional A-bimodules, where A is the direct product of the group algebras of the stabilisers of the vertices of Q.

On one hand, these formulas provide an interpretation of the coefficients of these linear combinations in terms of the representation theory of the stabilisers of the vertices of Q. On the other hand, they are explicit enough to be implemented by a computer or a human being able to manipulate the irreducible representations of the involved groups. In specific cases, for instance when the action of the group elements transforms arrows to scalar multiples of arrows and the stabilisers are cyclic, these formulas simplify in a combinatorial way. Finally, these formulas give a complete solution to the problem of computing a potential W_G on Q_G such that the Ginzburg dg algebra $\mathcal{A}(Q_G, W_G)$ is Morita equivalent to $\mathcal{A}(Q, W) * G$, for any given G-invariant potential W on Q.

1. Definitions and main results

Throughout the article, $\Bbbk$ denotes an algebraically closed field. For all finite dimensional algebras Λ, their categories of finite dimensional left modules are denoted by $\mathrm{mod}(\Lambda)$.

This section presents the main results of this text as well as the needed definitions and notation. These definitions are illustrated with a running example throughout the section. A full application of the main results is presented on an example in section 2. The remaining sections are devoted to the proofs of these results.

Setting 1.1. Let Q_0, S, G, and M be as follows.

- Q_0 is a finite set and S is the semi-simple $\Bbbk$-algebra $\Bbbk Q_0$.
- G is a finite group with order not divisible by $\mathrm{char}(\Bbbk)$ and acting on Q_0.
- M is a finite dimensional S-bimodule, considered as a collection of vector spaces $({}_iM_j)_{(i,j) \in Q_0^2}$, endowed with an action of G written exponentially and such that ${}^g({}_iM_j) = {}_{g \cdot i}M_{g \cdot j}$ for all $i, j \in Q_0$ and $g \in G$.

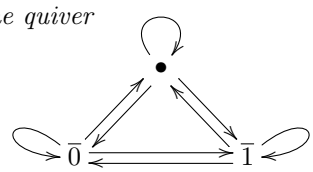
The choice of a basis of ${}_iM_j$, for all $i, j \in Q_0$ determines a quiver Q such that $T_S(M) \simeq \Bbbk Q$ as $\Bbbk$-algebras. All finite quivers arise in this way.

The actions of G on S and M induce an action on $T_S(M)$ by algebra automorphisms such that ${}^g(m_1 \otimes \cdots \otimes m_n) = {}^g m_1 \otimes \cdots \otimes {}^g m_n$ for all $g \in G$ and $m_1, \ldots, m_n \in M$. Recall that the skew group algebra $T_S(M) * G$ is the $\Bbbk$-algebra with underlying vector space $T_S(M) \otimes G$, where the tensor $u \otimes v$ is denoted by $u * v$ for all $u \in T_S(M)$ and $v \in \Bbbk G$, and with product such that $(u*g)\cdot(u'*g') = (u \cdot {}^g u') * gg'$ for all $u, u' \in T_S(M)$ and $g, g' \in G$. It is customary to consider $T_S(M)$ and $\Bbbk G$ as subalgebras of $T_S(M) * G$ in the canonical way.

Notation 1.2. The following data is used throughout the article.

- $[G\backslash Q_0]$ denotes a complete set of representatives of the orbits in Q_0.
- For all $i \in Q_0$,
  - G_i denotes the stabiliser of i and e_i denotes the corresponding primitive idempotent of S,
  - $[G/G_i]$ denotes a complete set of representatives of cosets modulo G_i,
  - $\mathrm{irr}(G_i)$ denotes a complete set of representatives of the isomorphism classes of the irreducible representations of G_i,
  - and ε_U denotes a primitive idempotent of $\Bbbk G_i$ such that $U \simeq \Bbbk G_i \cdot \varepsilon_U$ in $\mathrm{mod}(\Bbbk G_i)$, for all $U \in \mathrm{irr}(G_i)$.
- $\tilde{e}$ denotes the idempotent $\sum_{(i,U)} e_i * \varepsilon_U$ of $T_S(M) * G$, where i runs through $[G\backslash Q_0]$ and U runs through $\mathrm{irr}(G_i)$.

Example 1.3. *Let Q be the quiver*



where $\overline{0}$ and $\overline{1}$ denote the classes in $\mathbb{Z}/2\mathbb{Z}$. For all vertices i, j of Q, the arrow from i to j is denoted by $x_{i,j}$. Let G be the group $\langle \tau \mid \tau^2 \rangle$ of order 2. The actions of G on Q and on M are assumed to be the ones such that

$$\tau \cdot \bullet = \bullet, \quad \tau \cdot \overline{0} = \overline{1}\,,$$

and, for all vertices i, j of Q,

$${}^\tau x_{i,j} = -x_{\tau\cdot i, \tau\cdot j}\,.$$

- *Let $[G\backslash Q_0]$ be $\{\bullet, \overline{0}\}$. Note that $G_\bullet = \{\mathrm{Id}, \tau\}$ and $G_{\overline{0}} = \{\mathrm{Id}\}$.*
- *Let $[G/G_\bullet]$ and $[G/G_{\overline{0}}]$ be $\{\mathrm{Id}\}$ and $\{\mathrm{Id}, \tau\}$, respectively.*
- *For all $k \in \mathbb{Z}/2\mathbb{Z}$, denote by ε_k the idempotent $\frac{1}{2}(\mathrm{Id} + (-1)^k \tau)$; denote by ρ_k the irreducible representation $\Bbbk \cdot \varepsilon_k$ of G.*
- *Let $\mathrm{irr}(G_\bullet)$ and $\mathrm{irr}(G_{\overline{0}})$ be $\{\rho_k \mid k \in \mathbb{Z}/2\mathbb{Z}\}$ and $\{\Bbbk \cdot \mathrm{Id}\}$, respectively.*

In this situation, $\tilde{e} = e_\bullet * \varepsilon_{\overline{0}} + e_\bullet * \varepsilon_{\overline{1}} + e_{\overline{0}} * \mathrm{Id}$.

1.1. **Demonet's quiver of the Morita reduction.** Demonet proved in [3] that the skew group algebra $T_S(M) * G$ is Morita equivalent to the path algebra of a quiver as follows.

Setting 1.4. Let Q_G be a quiver with the following properties.

- The vertices are the couples (i, U), where $i \in [G\backslash Q_0]$ and $U \in \mathrm{irr}(G_i)$.
- For all vertices (i, U) and (j, V), the arrows $(i, U) \to (j, V)$ form a basis over $\Bbbk$ of $\mathrm{Hom}_{\Bbbk G_i}(U, M(i, j; V))$, here $M(i, j; V) = \oplus_{y \in [G/G_j]}\, {}_iM_{y\cdot j} \otimes_\Bbbk yV$, where yV stands for $y \otimes_\Bbbk V \subset \Bbbk G \otimes_\Bbbk V$, see Definition 1.8 for details.

Example 1.5. *This is the continuation of Example 1.3. In order to describe a quiver Q_G such as in Setting 1.4, Table 1 escribes the G_i-module $M(i,j;U)$ for all possible i,j,U.*

$M(i,j;U)$	*Basis over* $\Bbbk$	*Action of* G_i
$M(\bullet,\bullet;\rho_k)$	$x_{\bullet,\bullet}\otimes\varepsilon_k$	${}^{\tau}(x_{\bullet,\bullet}\otimes\varepsilon_k)=(-1)^{k+1}x_{\bullet,\bullet}\otimes\varepsilon_k$
$M(\bullet,\overline{0};\Bbbk\cdot\mathrm{Id})$	$(x_{\bullet,k}\otimes\tau^k)_{k\in\mathbb{Z}/2\mathbb{Z}}$	${}^{\tau}(x_{\bullet,k}\otimes\tau^k)=-x_{\bullet,k+\overline{1}}\otimes\tau^{k+\overline{1}}$
$M(\overline{0},\bullet;\rho_k)$	$x_{\overline{0},\bullet}\otimes\varepsilon_k$	*Trivial*
$M(\overline{0},\overline{0};\Bbbk\cdot\mathrm{Id})$	$(x_{\overline{0},k}\otimes\tau^k)_{k\in\mathbb{Z}/2\mathbb{Z}}$	*Trivial*

TABLE 1. The G_i-modules $M(i,j;U)$ ($k\in\mathbb{Z}/2\mathbb{Z}$).

For all $k\in\mathbb{Z}/2\mathbb{Z}$, denote by a_k and b_k the following respective morphisms of $G_\bullet$-modules

$$\begin{array}{rcl} \rho_{k+\overline{1}} & \longrightarrow & M(\bullet,\bullet;\rho_k) \\ \varepsilon_{k+\overline{1}} & \longmapsto & x_{\bullet,\bullet}\otimes\varepsilon_k \end{array} \qquad \begin{array}{rcl} \rho_k & \longrightarrow & M(\bullet,\overline{0};\Bbbk\cdot\mathrm{Id}) \\ \varepsilon_k & \longmapsto & \frac{1}{2}(x_{\bullet,\overline{0}}\otimes\mathrm{Id}+(-1)^{k+\overline{1}}x_{\bullet,\overline{1}}\otimes\tau) \end{array}$$

and denote by c_k and d_k the following respective morphisms of $G_{\overline{0}}$-modules

$$\begin{array}{rcl} \Bbbk\cdot\mathrm{Id} & \longrightarrow & M(\overline{0},\bullet;\rho_k) \\ \mathrm{Id} & \longmapsto & x_{\overline{0},\bullet}\otimes\varepsilon_k \end{array} \qquad \begin{array}{rcl} \Bbbk\cdot\mathrm{Id} & \longrightarrow & M(\overline{0},\overline{0};\Bbbk\cdot\mathrm{Id}) \\ \mathrm{Id} & \longmapsto & (-1)^k x_{\overline{0},k}\otimes\tau^k\,. \end{array}$$

Then, Q_G may be taken equal to

$$\begin{tikzcd} (\bullet,\rho_{\overline{0}}) \arrow[dd, bend right, "a_{\overline{1}}"'] \arrow[r, shift left, "b_{\overline{0}}"] & (0,\Bbbk\cdot\mathrm{Id}) \arrow[l, shift left, "c_{\overline{0}}"] \arrow[loop above, "d_{\overline{0}}"] \arrow[d, shift left, "c_{\overline{1}}"] \\ (\bullet,\rho_{\overline{1}}) \arrow[u, bend right, "a_{\overline{0}}"'] \arrow[r, shift right, "b_{\overline{1}}"'] & (0,\Bbbk\cdot\mathrm{Id}) \arrow[u, shift left, "b_{\overline{1}}"] \arrow[loop below, "d_{\overline{1}}"] \end{tikzcd}\,,$$

Demonet's result may be reformulated as follows, see Section 5 for more details.

Theorem 1.6 ([3])**.** *The $\Bbbk$-algebras $\Bbbk Q_G$ and $T_S(M)*G$ are Morita equivalent. Besides, assuming that $U=\Bbbk G_i\cdot\varepsilon_U$ for all vertices (i,U) of Q_G, then there exists an isomorphism of algebras*

$$\Bbbk Q_G \xrightarrow{\sim} \tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e} \tag{1.1.1}$$

which maps every arrow $f\colon(i,U)\to(j,V)$ of Q_G to $f(\varepsilon_U)$.

1.2. **Monoidal categories as a toolbox for intertwiners.** The present article introduces a $\Bbbk$-algebra, called the algebra of intertwiners relative to M and denoted by $\mathcal{I}\mathrm{ntw}$, and an isomorphism of $\Bbbk$-algebras $\Bbbk Q_G\to\mathcal{I}\mathrm{ntw}^{\mathrm{op}}$ together with explicit formulas decomposing any given element of $\mathcal{I}\mathrm{ntw}$ in the basis consisting of the images of the paths in Q_G under $\Bbbk Q_G\to\mathcal{I}\mathrm{ntw}^{\mathrm{op}}$. See Theorem 1.25 for details. Composing this isomorphism with the inverse of (1.1.1) results in a $\Bbbk$-algebra isomorphism $\tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}\to\mathcal{I}\mathrm{ntw}^{\mathrm{op}}$. A description of it is given in Theorem 1.27 hence providing a decomposition, under (1.1.1), of any element of $\tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}$ in the basis of $\Bbbk Q_G$ consisting of the paths in Q_G. These decompositions involve non-degenerate pairings between intertwiners relative to M and intertwiners relative to the dual vector space M^*. Both the product in $\mathcal{I}\mathrm{ntw}$ and the pairings are explicit reformulations of natural operations and pairings in a suitable monoidal category.

Remark 1.7 ([5])**.** Let $(\mathcal{C},\otimes)$ be a monoidal $\Bbbk$-linear category with finite dimensional morphism spaces.

(1) Given objects X, Y, U, V, and W of $\mathcal{C}$, and following the ideas presented in [6, Subsection 4.4] for the monoidal categories of finite subgroups of $\mathrm{SL}_3(\mathbb{C})$, there is a natural operation between morphisms $U \to X \otimes V$ and $V \to Y \otimes W$,

$$\begin{array}{ccc} \mathcal{C}(U, X \otimes V) \times \mathcal{C}(V, Y \otimes W) & \longrightarrow & \mathcal{C}(U, X \otimes Y \otimes W) \\ (f, f') & \longmapsto & (X \otimes f') \circ f\,. \end{array}$$

(2) Given an object X of $\mathcal{C}$, recall that a *left dual* of X is an object X' of $\mathcal{C}$ together with an adjunction $(X \otimes -) \vdash (X' \otimes -)$ of endofunctors of $\mathcal{C}$ ([5, Definition 2.10.1, Proposition 2.10.8]). If X has a left dual, then there exists a non-degenerate pairing, for all objects U and V of $\mathcal{C}$ such that $\mathcal{C}(U, U) = \Bbbk \cdot \mathrm{Id}_U$,

$$\begin{array}{cccc} \langle -|- \rangle\colon & \mathcal{C}(U, X \otimes V) \otimes \mathcal{C}(V, X' \otimes U) & \longrightarrow & \Bbbk \\ & f \otimes \phi & \longmapsto & \langle f|\phi \rangle \end{array}$$

such that $\langle f|\phi \rangle \cdot \mathrm{Id}_U = \overline{\phi} \circ f$, where $\overline{\phi} \in \mathcal{C}(X \otimes V, U)$ is adjoint to ϕ; because $\mathcal{C}(U, U) = \Bbbk \cdot \mathrm{Id}_U$, a scalar is associated to every endomorphism of U, and hence to $\overline{\phi} \circ f$, which makes it possible to define a pairing with values in $\Bbbk$. When $\mathcal{C}$ is a multitensor category this pairing is just a combination of the adjunction $(X \otimes -) \vdash (X' \otimes -)$ and the non-degenerate pairing $\mathcal{C}(U, X \otimes V) \otimes \mathcal{C}(X \otimes V, U) \to \Bbbk$ which takes into account U being simple in $\mathcal{C}$ (see [5, Example 7.9.14]).

The monoidal category $\mathcal{C}$ considered here is the category $\mathrm{mod}(A^e)$ of finite dimensional A-bimodules with tensor product being $\otimes_A$, where A is as follows,

$$A = \Bbbk \times \Pi_{i \in Q_0} \Bbbk G_i\,. \tag{1.2.1}$$

Note that A is a semi-simple $\Bbbk$-algebra; accordingly, any object X of $\mathcal{C}$ has a left dual given by the dual vector space X^*. The isolated factor $\Bbbk$ in the definition of A is included so that $\mathcal{C}$ contains $\mathrm{mod}(\Bbbk G_i)$ as a full additive subcategory, yet not as a monoidal subcategory, for all $i \in Q_0$. The reason for this inclusion is the following, at the same time it clarifies the role of $\mathcal{C}$. The text manipulates morphism spaces of the shape

$$\mathrm{Hom}_{\Bbbk G_{i_0}}(U,\, e_{i_0}(M \otimes_\Bbbk \Bbbk G) e_{i_1} \otimes_{\Bbbk G_{i_1}} \cdots \otimes_{\Bbbk G_{i_{n-1}}} e_{i_{n-1}}(M \otimes_\Bbbk \Bbbk G) e_{i_n} \otimes_{\Bbbk G_{i_n}} V) \tag{1.2.2}$$

where $i_0, \dots, i_n$ are vertices of Q whereas $U \in \mathrm{mod}(\Bbbk G_{i_0})$ and $V \in \mathrm{mod}(\Bbbk G_{i_n})$; note that each one of $e_{i_{t-1}}(M \otimes_\Bbbk \Bbbk G) e_{i_t}$ (for $1 \leqslant t \leqslant n$) is an object of $\mathcal{C}$, and so are U and V because of the isolated factor $\Bbbk$ in A; the above-mentioned morphism space is hence equal to

$$\mathcal{C}(U,\, e_{i_0}(M \otimes_\Bbbk \Bbbk G) e_{i_1} \otimes \cdots \otimes e_{i_{n-1}}(M \otimes_\Bbbk \Bbbk G) e_{i_n} \otimes V) \tag{1.2.3}$$

and it is manipulated in this text using the tools recalled in Remark 1.7.

1.3. **The interwiners.** Here are the base ingredients needed to define $\mathcal{I}$ntw.

Definition 1.8. Let n be a non-negative integer. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0. Let $V \in \mathrm{mod}(\Bbbk G_{i_n})$.

(1) For all sequences $\underline{y} = y_1, \dots, y_n$, where $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, let $M_{\underline{y}}(\underline{i}; V)$ be the following vector space,

$$M_{\underline{y}}(\underline{i}; V) = {}_{i_0}M_{y_1 \cdot i_1} \otimes_\Bbbk {}_{y_1 \cdot i_1}M_{y_1 y_2 \cdot i_2} \otimes_\Bbbk \cdots \otimes_\Bbbk {}_{y_1 \cdots y_{n-1} \cdot i_{n-1}}M_{y_1 \cdots y_n \cdot i_n} \otimes_\Bbbk y_1 \cdots y_n V\,,$$

where $y_1 \cdots y_n V$ stands for the vector subspace $y_1 \cdots y_n \otimes_\Bbbk V$ of $\Bbbk G \otimes_\Bbbk V$.

(2) Define the $\Bbbk G_{i_0}$-module $M(\underline{i};V)$ as follows. Its underlying vector space is

$$\bigoplus_{\underline{y}} M_{\underline{y}}(\underline{i};V)\,, \tag{1.3.1}$$

where $\underline{y}$ runs through all possible sequences such as in (1). For all $g \in G_{i_0}$, the action of g is written exponentially and defined as follows. For all $\underline{y}$, there exist a unique sequence $\underline{y}' = y'_1, \dots, y'_n$, where $y'_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, and a unique $h \in G_{i_n}$, such that

- $gy_1 \cdots y_t G_{i_t} = y'_1 \cdots y'_t G_{i_t}$ for all $t \in \{1, \dots, n\}$,
- and $gy_1 \cdots y_n = y'_1 \cdots y'_n h$;

then, the action of g transforms $M_{\underline{y}}(\underline{i};V)$ into $M_{\underline{y}'}(\underline{i};V)$ and, for all $m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n v \in M_{\underline{y}}(\underline{i};V)$,

$${}^g(m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n v) = {}^g m_1 \otimes \cdots \otimes {}^g m_n \otimes y'_1 \cdots y'_n {}^h v\,.$$

This does define a finite dimensional $\Bbbk G_{i_0}$-module. If $n = 0$, then $M(\underline{i};V) = V$. In the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$ considered previously, $M(\underline{i};V)$ is isomorphic to $M(\underline{i}) \otimes_A V$ where $M(\underline{i})$ is a $\Bbbk G_{i_0} - \Bbbk G_{i_n}$-bimodule defined by M and $\underline{i}$, see (4.1.1) for the definition of $M(\underline{i})$ and (4.1.7) for the isomorphism.

Note that, in the previous definition, if $U = \Bbbk G_{i_0} \cdot \varepsilon_U$ and $V = \Bbbk G_{i_n} \cdot \varepsilon_V$ for primitive idempotents $\varepsilon_U \in \Bbbk G_{i_0}$ and $\varepsilon_V \in \Bbbk G_{i_n}$, then $M(\underline{i};U)$ is a $\Bbbk G_{i_0}$-submodule of $T_S(M) * G$.

Example 1.9. *This is the continuation of Example 1.5. Let $k \in \mathbb{Z}/2\mathbb{Z}$. Then $M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_k)$ identifies with the $\Bbbk G_\bullet$-submodule of $T_S(M) * G$ whose underlying vector space is generated by $(x_{\bullet,a} x_{a,b} x_{b,\bullet} * \varepsilon_k)_{a,b \in \mathbb{Z}/2\mathbb{Z}}$. On this submodule, the action of τ is such that ${}^\tau(x_{\bullet,a} x_{a,b} x_{b,\bullet} * \varepsilon_k) = (-1)^{k+\overline{1}} x_{\bullet,a+\overline{1}} x_{a+\overline{1},b+\overline{1}} x_{b+\overline{1},\bullet} * \varepsilon_k$.*

The main results of this article are based on a description of the quiver $\Bbbk Q_G$ in terms of intertwiners.

Definition 1.10. An *intertwiner* relative to M is a morphism of $\Bbbk G_{i_0}$-modules $U \to M(\underline{i};V)$, for some sequence $\underline{i} = i_0, \dots, i_n$ in Q_0 and some modules $U \in \mathrm{mod}(\Bbbk G_{i_0})$ and $V \in \mathrm{mod}(\Bbbk G_{i_n})$.

Note that, in the previous definition, if $U = \Bbbk G_{i_0} \cdot \varepsilon_U$ and $V = \Bbbk G_{i_n} \cdot \varepsilon_V$ for primitive idempotents $\varepsilon_U \in \Bbbk G_{i_0}$ and $\varepsilon_V \in \Bbbk G_{i_n}$, then all intertwiners $U \to M(\underline{i};V)$ take their values in $T_S(M) * G$.

Example 1.11. *This is the continuation of Example 1.9. Let $s, t \in \mathbb{Z}/2\mathbb{Z}$. For all $a, b \in \mathbb{Z}/2\mathbb{Z}$, the following holds true in $M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_t)$,*

$$\varepsilon_s \cdot (x_{\bullet,a} x_{a,b} x_{b,\bullet} \otimes \varepsilon_t) = \frac{1}{2}(x_{\bullet,a} x_{a,b} x_{b,\bullet} \otimes \varepsilon_t + (-1)^{s+\overline{1}+t} x_{\bullet,a+\overline{1}} x_{a+\overline{1},b+\overline{1}} x_{b+\overline{1},\bullet} \otimes \varepsilon_t)\,.$$

Thus, $\mathrm{Hom}_{\Bbbk G_\bullet}(\rho_s, M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_t))$ is two-dimensional with basis $(F_a)_{a \in \mathbb{Z}/2\mathbb{Z}}$, where F_a denotes the following intertwiner relative to M, for all $a \in \mathbb{Z}/2\mathbb{Z}$,

$$\begin{array}{rcl} \rho_s & \longrightarrow & M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_t) \\ \varepsilon_s & \longmapsto & \frac{1}{2}(x_{\bullet,\overline{0}} x_{\overline{0},a} x_{a,\bullet} \otimes \varepsilon_t + (-1)^{s+\overline{1}+t} x_{\bullet,\overline{1}} x_{\overline{1},a+\overline{1}} x_{a+\overline{1},\bullet} \otimes \varepsilon_t)\,. \end{array}$$

The main purpose of this article is to provide explicit formulas in the main theorems. These formulas use the following notation.

Notation 1.12. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 where $n \geqslant 1$. Let $U \in \mathrm{mod}(\Bbbk G_{i_0})$ and $V \in \mathrm{mod}(\Bbbk G_{i_n})$. Let $f \in \mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(\underline{i};V))$. For all $u \in U$, write the decomposition of $f(u)$ along (1.3.1) as follows

$$f(u) = \sum_{\underline{y}} f_{\underline{y}}(u)\,;$$

and, for all sequences $\underline{y} = y_1, \dots, y_n$, where $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$, write symbolically the element $f_{\underline{y}}(u)$ of $M_{\underline{y}}(\underline{i}; V)$ as follows, with the sum sign omitted,

$$f_{\underline{y}}(u) = f_{\underline{y}}^{(1)}(u) \otimes \cdots \otimes f_{\underline{y}}^{(n)}(u) \otimes y_1 \cdots y_n f_{\underline{y}}^{(0)}(u)\,;$$

hence, $f_{\underline{y}}^{(t)}(u)$ is meant to lie in ${}_{y_1 \cdots y_{t-1} \cdot i_{t-1}} M_{y_1 \cdots y_t \cdot i_t}$, for all $t \in \{1, \dots, n\}$, and $f_{\underline{y}}^{(0)}(u)$ is meant to lie in V; as a whole,

$$f(u) = \sum_{\underline{y}} f_{\underline{y}}^{(1)}(u) \otimes \cdots \otimes f_{\underline{y}}^{(n)}(u) \otimes y_1 \cdots y_n f_{\underline{y}}^{(0)}(u)\,. \tag{1.3.2}$$

1.4. **The algebra of intertwiners.** Now here is the operation to be used as the multiplication in $\mathcal{I}$ntw.

Definition 1.13. Let $\underline{i}'' = i_0, \dots, i_{m+n}$ be a sequence in Q_0, where $m, n \geqslant 1$. Denote the sequences $i_0, \dots, i_m$ and $i_m, i_{m+1}, \dots, i_{m+n}$ by $\underline{i}$ and $\underline{i}'$, respectively. Let $U \in \mathrm{mod}(\Bbbk G_{i_0})$, $V \in \mathrm{mod}(\Bbbk G_{i_m})$, and $W \in \mathrm{mod}(\Bbbk G_{i_{m+n}})$. For all $f \in \mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(\underline{i}; V))$ and $f' \in \mathrm{Hom}_{\Bbbk G_{i_m}}(V, M(\underline{i}'; W))$, define a $\Bbbk$-linear mapping

$$f' \circledast f \colon U \to M(\underline{i}''; W)$$

by

$$f' \circledast f = \sum_{\underline{y''}} (f' \circledast f)_{\underline{y''}}\,, \tag{1.4.1}$$

where

- $\underline{y''}$ runs through all sequences $y_1, \dots, y_{m+n}$ with $y_t \in [G/G_{i_t}]$ for all t
- and $(f' \circledast f)_{\underline{y''}}$ is the mapping $U \to M_{\underline{y}''}(\underline{i}''; W)$ such that, for all $u \in U$,

$$\begin{aligned}(f' \circledast f)_{\underline{y''}}(u) \quad = \quad & f_{\underline{y}}^{(1)}(u) \otimes \cdots \otimes f_{\underline{y}}^{(m)}(u) \otimes \\ & {}_{y_1 \cdots y_m}(f_{\underline{y'}}^{\prime(1)}(f_{\underline{y}}^{(0)}(u))) \otimes \cdots \otimes {}_{y_1 \cdots y_m}(f_{\underline{y'}}^{\prime(n)}(f_{\underline{y}}^{(0)}(u))) \otimes \\ & y_1 \cdots y_{m+n} f_{\underline{y'}}^{\prime(0)}(f_{\underline{y}}^{(0)}(u))\,,\end{aligned}$$

  where $\underline{y} = y_1, \dots, y_m$ and $\underline{y'} = y_{m+1}, \dots, y_{m+n}$.

Note that, in the previous definition, if $U = \Bbbk G_{i_0} \cdot \varepsilon_U$, $V = \Bbbk G_{i_m} \cdot \varepsilon_V$, and $W = \Bbbk G_{i_{m+n}} \cdot \varepsilon_W$ for primitive idempotents $\varepsilon_U \in \Bbbk G_{i_0}$, $\varepsilon_V \in \Bbbk G_{i_m}$, and $\varepsilon_W \in \Bbbk G_{i_{m+n}}$, then

$$(f' \circledast f)(\varepsilon_U) = f(\varepsilon_U) \cdot f'(\varepsilon_v)\,, \tag{1.4.2}$$

where the product on the right-hand side is taken in $T_S(M) * G$.

Remark 1.14. The definition of $\circledast$ is technical. Actually, it is an explicit reformulation of a simple operation in the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$. More precisely, using the comment that follows Definition 1.8,

- first, $M(\underline{i}'') \simeq M(\underline{i}) \otimes_A M(\underline{i}')$, see Lemma 4.2;
- next, if $M(\underline{i}; V)$, $M(\underline{i}'; W)$, and $M(\underline{i}''; W)$ are identified with $M(\underline{i}) \otimes_A V$, $M(\underline{i}') \otimes_A W$, and $M(\underline{i}'') \otimes_A W$, respectively, then $\circledast$ is an explicit reformulation of the operation introduced in part (1) of Remark 1.7, see Lemma 4.4 for a precise statement and proof.

In view of the previous remark, $\circledast$ does yield intertwiners, see Lemma 4.4, and it is associative, see Proposition 4.5.

Definition 1.15. The *algebra of intertwiners* relative to M is denoted by $\mathcal{I}\mathrm{ntw}$ and defined by

$$\mathcal{I}\mathrm{ntw} = \left(\bigoplus_{n,\underline{i},U,V} \mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(\underline{i}; V)), \circledast \right) , \tag{1.4.3}$$

where n runs through all non negative integers, $\underline{i} = i_0, \dots, i_n$ runs through all sequences in $[G\backslash Q_0]$, and U and V run through $\mathrm{irr}(G_{i_0})$ and $\mathrm{irr}(G_{i_n})$, respectively.

It is hence possible to associate an element of $\mathcal{I}\mathrm{ntw}$ to every path in Q_G.

Notation 1.16. For all paths in Q_G

$$\gamma\colon (i_0, U_0) \xrightarrow{f_1} (i_1, U_1) \to \cdots \to (i_{n-1}, U_{n-1}) \xrightarrow{f_n} (i_n, U_n) ,$$

denote by f_γ the following element of $\mathcal{I}\mathrm{ntw}$ lying in $\mathrm{Hom}_{\Bbbk G_{i_0}}(U_0, M(i_0, \dots, i_n; U_n))$,

$$f_\gamma = f_n \circledast f_{n-1} \circledast \cdots \circledast f_1 . \tag{1.4.4}$$

1.5. **Dual intertwiners and their pairing with intertwiners.** As stated below, see Theorem 1.25, assigning f_γ to γ for every path γ in Q_G yields an isomorphism of algebras from $\Bbbk Q_G$ to $\mathcal{I}\mathrm{ntw}^{\mathrm{op}}$. The purpose of this article is to explain how to decompose any element of $\mathcal{I}\mathrm{ntw}$ as a linear combination of the f_γ's. This involves non-degenerate pairings between certain spaces of intertwiners relative to M and to M^*, respectively. Here, M^* denotes the dual vector space $\mathrm{Hom}_{\Bbbk}(M, \Bbbk)$, it has a structure of S-bimodule such that ${}_iM^*_j = ({}_jM_i)^*$ for all $i, j \in Q_0$, and G acts on M^* in such a way that, for all $\varphi \in M^*$ and $g \in G$,

$${}^g\varphi = \varphi({}^{g^{-1}}\bullet) .$$

This action is such that ${}^g({}_iM^*_j) = {}_{g\cdot i}M^*_{g\cdot j}$ for all $i, j \in Q_0$ and $g \in G$. Hence, the previous considerations may be applied to M^* instead of to M.

Example 1.17. *This is the continuation of Example 1.11. For all vertices i, j of Q, the vector space ${}_iM^*_j$ is one-dimensional; denote by $x'_{i,j}$ its base element, dual to the base element $x_{j,i}$ of ${}_jM_i$; Hence, ${}^\tau x'_{i,j} = -x'_{\tau(i),\tau(j)}$. After replacing M by M^*, the considerations of Example 1.11 yield that $\mathrm{Hom}_{\Bbbk G_\bullet}(\rho_t, M^*(\bullet, \overline{0}, \overline{0}, \bullet; \rho_s))$ is two-dimensional with basis $(\phi_a)_{a\in\mathbb{Z}/2\mathbb{Z}}$, where ϕ_a denotes the following intertwiner relative to M^*, for all $a \in \mathbb{Z}/2\mathbb{Z}$,*

$$\begin{array}{lcl} \rho_t & \longrightarrow & M^*(\bullet, \overline{0}, \overline{0}, \bullet; \rho_s) \\ \varepsilon_t & \longmapsto & x_{\bullet,\overline{0}} x_{\overline{0},a} x_{a,\bullet} \otimes \varepsilon_s + (-1)^{t+\overline{1}+s} x_{\bullet,\overline{1}} x_{\overline{1},a+\overline{1}} x_{a+\overline{1},\bullet} \otimes \varepsilon_s) . \end{array}$$

Notation 1.18. Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0, where $n \geqslant 0$. Let $U \in \mathrm{mod}(\Bbbk G_{i_0})$ and $V \in \mathrm{mod}(\Bbbk G_{i_n})$.

(1) Denote by $\underline{i}^o$ the sequence $i_n, i_{n-1}, \dots, i_0$.
(2) Let $\varphi \in \mathrm{Hom}_{\Bbbk G_{i_n}}(V, M^*(\underline{i}^o, U))$. Proceeding similarly as done in Notation 1.12,
   - denote by $\varphi^{\underline{x}}$ the composition of φ with the canonical projection $M^*(\underline{i}^o; U) \to M^*_{\underline{x}}(\underline{i}^o; U)$, for all sequences $\underline{x} = x_{n-1}, x_{n-2}, \dots, x_0$ where $x_t \in [G/G_{i_t}]$ for all $t \in \{0, \dots, n-1\}$,
   - and for all such $\underline{x}$ and all $v \in V$, write symbolically the element $\varphi^{\underline{x}}(v)$ of $M^*_{\underline{x}}(\underline{i}^o; U)$ as follows,

$$\varphi^{\underline{x}}(v) = \varphi^{\underline{x}}_{(n)}(v) \otimes \cdots \otimes \varphi^{\underline{x}}_{(1)}(v) \otimes x_{n-1} \cdots x_0 \varphi^{\underline{x}}_{(0)}(v) .$$

Hence, $\varphi^{\underline{x}}_{(t)}(v)$ is meant to lie in ${}_{(x_{n-1}x_{n-2}\cdots x_t)\cdot i_t}M^*_{(x_{n-1}x_{n-2}\cdots x_{t-1})\cdot i_{t-1}}$ for all $t \in \{1,\dots,n\}$ and $\varphi^{\underline{x}}_{(0)}(v)$ is meant to lie in U. As a whole,

$$\varphi(v) = \sum_{\underline{x}} \varphi^{\underline{x}}_{(n)}(v) \otimes \cdots \otimes \varphi^{\underline{x}}_{(1)}(v) \otimes x_{n-1}\cdots x_1 \varphi^{\underline{x}}_{(0)}(v)\,. \tag{1.5.1}$$

Here is an illustration of this notation.

Example 1.19. *This is the continuation of Example 1.17. Assume that $n = 2$ and $\underline{i} = \overline{0}, \overline{0}, \bullet$. Hence $\underline{i}^o = \bullet, \overline{0}, \overline{0}$. The underlying vector space of the $\Bbbk G_\bullet$-module $M^*(\underline{i}^o; \Bbbk \cdot \mathrm{Id})$ is freely generated by $(x'_{\bullet,a}x'_{a,b} \otimes \tau^b)_{a,b\in\mathbb{Z}/2\mathbb{Z}}$.*

Let $a, b, s \in \mathbb{Z}/2\mathbb{Z}$. Denote by φ the intertwiner $\rho_s \to M^*(\underline{i}^o; \Bbbk \cdot \mathrm{Id})$ such that $\varphi(\varepsilon_s) = \varepsilon_s \cdot (x'_{\bullet,a}x'_{a,b} \otimes \tau^b)$, that is $\varphi(\varepsilon_s) = \frac{1}{2}x'_{\bullet,a}x'_{a,b} \otimes \tau^b + \frac{(-1)^s}{2}x'_{\bullet,a+\overline{1}}x'_{a+\overline{1},b+\overline{1}} \otimes \tau^{b+\overline{1}}$. On one hand, $x'_{\bullet,a}x'_{a,b} = x'_{\bullet,\tau^a\cdot\overline{0}}x'_{\tau^a\cdot\overline{0},\tau^a\tau^{b-a}\cdot\overline{0}}$; Hence

$$\begin{cases} \varphi^{\tau^a,\tau^{b-a}}(\varepsilon_s) &= \frac{1}{2}x'_{\bullet,a}x'_{a,b} \otimes \tau^b \\ \varphi^{\tau^a,\tau^{b-a}}_{(2)}(\varepsilon_s) \otimes \varphi^{\tau^a,\tau^{b-a}}_{(1)}(\varepsilon_s) \otimes \varphi^{\tau^a,\tau^{b-a}}_{(0)}(\varepsilon_s) &= \frac{1}{2}x'_{\bullet,a} \otimes x'_{a,b} \otimes \mathrm{Id}\,. \end{cases}$$

On the other hand, $x'_{\bullet,a+\overline{1}}x'_{a+\overline{1},b+\overline{1}} = x'_{\bullet,\tau^{a+\overline{1}}\cdot\overline{0}}x'_{\tau^{a+\overline{1}}\cdot\overline{0},\tau^{a+\overline{1}}\tau^{b-a}\cdot\overline{0}}$; Hence

$$\begin{cases} \varphi^{\tau^{a+\overline{1}},\tau^{b-a}}(\varepsilon_s) &= \frac{(-1)^s}{2}x'_{\bullet,a+\overline{1}}x'_{a+\overline{1},b+\overline{1}} \otimes \tau^{b+\overline{1}} \\ \varphi^{\tau^{a+\overline{1}},\tau^{b-a}}_{(2)}(\varepsilon_s) \otimes \varphi^{\tau^{a+\overline{1}},\tau^{b-a}}_{(1)}(\varepsilon_s) \otimes \varphi^{\tau^{a+\overline{1}},\tau^{b-a}}_{(0)}(\varepsilon_s) &= \frac{(-1)^s}{2}x'_{\bullet,a+\overline{1}} \otimes x'_{a+\overline{1},b+\overline{1}} \otimes \mathrm{Id}\,. \end{cases}$$

In view of description of $\varphi(\varepsilon_s)$ made initially, all the remaining terms of the sum of (1.5.1) are zero.

The above mentioned pairings are described explicitly as follows.

Proposition 1.20. *Let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0, where $n \geqslant 0$. Let $U \in \mathrm{mod}(\Bbbk G_{i_0})$ and $V \in \mathrm{mod}(\Bbbk G_{i_n})$. Assume that U is simple. There exists a non-degenerate pairing*

$$(-|-)\colon \mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(\underline{i}; V)) \otimes_\Bbbk \mathrm{Hom}_{\Bbbk G_{i_n}}(V, M^*(\underline{i}^o; U)) \to \Bbbk$$

with the following property. For all $f \in \mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(\underline{i}; V))$ and for all $\varphi \in \mathrm{Hom}_{\Bbbk G_{i_n}}(V, M^*(\underline{i}^o; U))$, the scalar $(f|\varphi)$ is such that, for all $u \in U$, the element $(f|\varphi)\,u$ of U is equal to

$$\sum_{\underline{y}} \prod_{t=1}^{n} \varphi^{\underline{x}}_{(t)}(f^{(0)}_{\underline{y}}(u))({}^{(y_1\cdots y_n)^{-1}}f^{(t)}_{\underline{y}}(u))\ {}^{h_0}(\varphi^{\underline{x}}_{(0)}(f^{(0)}_{\underline{y}}(u)))\,, \tag{1.5.2}$$

where $\underline{y} = y_1, \dots, y_n$ runs through all sequences with $y_t \in [G/G_{i_t}]$ for all t and $\underline{x} = x_{n-1}, \dots, x_0$ and h_0 are uniquely determined by the following conditions:
(a) $x_t \in [G/G_{i_t}]$ for all t, and $h_0 \in G_{i_0}$.
(b) $(y_t \cdots y_n)^{-1} \in x_{n-1}x_{n-2}\cdots x_{t-1}G_{i_{t-1}}$ for all $t \in \{1, \dots, n\}$.
(c) $(y_1 \cdots y_n)^{-1} = x_{n-1}x_{n-2}\cdots x_0 h_0$.

In view of computations, here is a useful explanation regarding $\underline{x}$ and h_0 in the above proposition. For all sequences $\underline{y} = y_1, \dots, y_n$ in G such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$ and for all non zero tensors

$$m_1 \otimes \cdots \otimes m_n \in {}_{i_0}M_{y_1\cdot i_1} \otimes_\Bbbk \cdots \otimes_\Bbbk {}_{(y_1\cdots y_{n-1})\cdot i_{n-1}}M_{(y_1\cdots y_n)\cdot i_n}\,,$$

there exists a unique sequence $\underline{x} = x_{n-1}, x_{n-2}, \dots, x_0$ in G such that $x_t \in [G/G_{i_t}]$ for all $t \in \{0, \dots, n-1\}$ and

$${}^{(y_1\cdots y_n)^{-1}}(m_1 \otimes \cdots \otimes m_n) \in {}_{(x_{n-1}x_{n-2}\cdots x_0)\cdot i_0}M_{(x_{n-1}\cdots x_1)\cdot i_1} \otimes_\Bbbk \cdots \otimes_k {}_{x_{n-1}\cdot i_{n-1}}M_{i_n}\,;$$

besides, $(x_{n-1}x_{n-2}\cdots x_0)^{-1}(y_1 \cdots y_n)^{-1} \in G_{i_0}$. This sequence $\underline{x}$ is the one in the statement of Proposition 1.20 in which $h_0 = (x_{n-1}x_{n-2}\cdots x_0)^{-1}(y_1 \cdots y_n)^{-1}$.

When the action of G on M is such that arrows of Q are transformed into scalar multiples of arrows of Q, the expression (1.5.2) of $(-|-)$ simplifies to (2.1.3). The following example illustrates the computation of $(-|-)$ in this particular situation.

Example 1.21. *This is the continuation of Example 1.17. Let $s, t \in \mathbb{Z}/2\mathbb{Z}$. Let $a, b \in \mathbb{Z}/2\mathbb{Z}$. Consider $F_a \colon \rho_s \to M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_t)$ and $\phi_b \colon \rho_t \to M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_s)$ such as in Examples 1.11 and 1.17, respectively.*

In view of (2.1.3), $(F_a|\phi_b)$ is a sum of products of scalars, the sum is indexed by the paths in Q that appear in the expression of $F_a(\varepsilon_s)$. For all such paths γ, the scalars forming the corresponding product appear in Table 2 with bold face. For convenience, this table presents additional data, using the following notation, see Subsection 2.1 for more details. For a given path γ in Q:

- *α_γ denotes the coefficient of $\gamma \otimes \varepsilon_t$ in $F_a(\varepsilon_s)$;*
- *y_1, y_2, y_3 is the sequence of G such that $y_1, y_2 \in [G/G_{\overline{0}}]$, $y_3 \in [G/G_\bullet]$ and γ is a path of the shape $\bullet \to y_1 \cdot \overline{0} \to y_1 y_2 \cdot \overline{0} \to y_1 y_2 y_3 \cdot \bullet$;*
- *$(y_1 y_2 y_3)^{-1}(\gamma)$ denotes the path of Q and $\chi_{(y_1 y_2 y_3)^{-1}, \gamma}$ denotes the scalar such that ${}^{(y_1 y_2 y_3)^{-1}}\gamma = \chi_{(y_1 y_2 y_3)^{-1}, \gamma} \cdot (y_1 y_2 y_3)^{-1}(\gamma)$;*
- *$(y_1 y_2 y_3)^{-1}(\gamma)^*$ denotes the element of the basis of $T_S(M^*)$ dual to the basis of $T_S(M)$ consisting of the paths of Q, associated to $(y_1 y_2 y_3)^{-1}(\gamma)$;*
- *$\beta_{(y_1 y_2 y_3)^{-1}(\gamma)^*}$ denotes the coefficient of $(y_1 y_2 y_3)^{-1}(\gamma)^*$ in $\phi_b(\varepsilon_t)$;*
- *χ_{ρ_s} denotes the character of ρ_s.*

Thus,

$$(F_a|\phi_b) = \begin{cases} \frac{(-1)^{(s+\overline{1})a} \cdot (1+(-1)^t)}{2} & \text{if } a = b \\ 0 & \text{if not.} \end{cases} \tag{1.5.3}$$

γ	$x_{\bullet,\overline{0}} x_{\overline{0},a} x_{a,\bullet}$	$x_{\bullet,\overline{1}} x_{\overline{1},a+\overline{1}} x_{a+\overline{1},\bullet}$
α_γ	$\mathbf{\frac{1}{2}}$	$\mathbf{\frac{(-1)^{s+\overline{1}+t}}{2}}$
y_1, y_2, y_3	$\mathrm{Id}, \tau^a, \mathrm{Id}$	$\tau, \tau^a, \mathrm{Id}$
${}^{(y_1 y_2 y_3)^{-1}}\gamma$	$(-1)^a x_{\bullet,-a} x_{-a,\overline{0}} x_{\overline{0},\bullet}$	$(-1)^{a+\overline{1}} x_{\bullet,-a} x_{-a,\overline{0}} x_{\overline{0},\bullet}$
$\chi_{(\mathbf{y_1 y_2 y_3})^{-1},\gamma}$	$\mathbf{(-1)^a}$	$\mathbf{(-1)^{a+\overline{1}}}$
$(y_1 y_2 y_3)^{-1}(\gamma)^*$	$x'_{\bullet,\overline{0}} x'_{\overline{0},-a} x'_{-a,\bullet}$	$x'_{\bullet,\overline{0}} x'_{\overline{0},-a} x'_{-a,\bullet}$
$\beta_{(\mathbf{y_1 y_2 y_3})^{-1}(\gamma)^*}$	$\begin{cases} 1 & \text{if } b = -a \\ 0 & \text{if not} \end{cases}$	$\begin{cases} 1 & \text{if } b = -a \\ 0 & \text{if not} \end{cases}$
x_2, x_1, x_0, h_0	$\mathrm{Id}, \tau^{-a}, \mathrm{Id}, \tau^{-a}$	$\mathrm{Id}, \tau^{-a}, \mathrm{Id}, \tau^{-a-\overline{1}}$
$\chi_{\rho_s}(\mathbf{h_0})$	$\mathbf{(-1)^{sa}}$	$\mathbf{(-1)^{s(a+\overline{1})}}$
contribution of γ to $(F_a\|\phi_b)$	$\begin{cases} \frac{(-1)^{a(s+\overline{1})}}{2} & \text{if } b = -a \\ 0 & \text{if not} \end{cases}$	$\begin{cases} \frac{(-1)^{(s+\overline{1})a+t}}{2} & \text{if } b = -a \\ 0 & \text{if not} \end{cases}$

TABLE 2. The scalars appearing in the expression (2.1.3) of $(F_a|\phi_b)$

Remark 1.22. Like the operation $\circledast$, the pairing $(-|-)$ has a simple interpretation in the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$. Indeed,

- first, $M(\underline{i})$ admits $M^*(\underline{i}^o)$ as a left dual, see (4.1.5);
- next, if $M(\underline{i}; V)$ and $M^*(\underline{i}^o; U)$ are identified with $M(\underline{i}) \otimes_A V$ and $M^*(\underline{i}^o) \otimes_A U$, respectively, then $(-|-)$ is an explicit reformulation of $\langle -|- \rangle$ as introduced in part (2) of Remark 1.7, see (4.3.8).

Using $(-|-)$ yields intertwiners relative to M^* associated with the f_γ's for paths γ in Q_G.

Notation 1.23. Let Q_G be such as in Setting 1.4.

(1) Let $f \in (i,U) \to (j,V)$ be an arrow of Q_G. Using the basis of the vector space $\mathrm{Hom}_{\Bbbk G_i}(U, M(i,j;V))$ consisting of the arrows $(i,U) \to (j,V)$ of Q_G and using the associated $(-|-)$-dual basis of $\mathrm{Hom}_{\Bbbk G_j}(V, M^*(j,i;U))$, denote by $f^\vee$ the dual element associated with f. In other words, $f^\vee$ lies in $\mathrm{Hom}_{\Bbbk G_j}(V, M^*(j,i;U))$ and, for all arrows $f' \colon (i,U) \to (j,V)$ of Q_G,

$$(f'|f^\vee) = \begin{cases} 1 & \text{if } f' = f \\ 0 & \text{otherwise.} \end{cases}$$

(2) For all paths γ in Q_G

$$\gamma \colon (i_0, U_0) \xrightarrow{f_1} (i_1, U_1) \to \cdots \to (i_{n-1}, U_{n-1}) \xrightarrow{f_n} (i_n, U_n)\,,$$

denote by φ_γ the following intertwiner

$$\varphi_\gamma = f_1^\vee \circledast f_2^\vee \circledast \cdots \circledast f_n^\vee \in \mathrm{Hom}_{\Bbbk G_{i_n}}(U_n, M^*(i_n, i_{n-1}, \ldots, i_0; U_0))\,.$$

Example 1.24. *This is the continuation of Example 1.21, it illustrates Notation 1.23.*

Let $k \in \mathbb{Z}/2\mathbb{Z}$. Denote by a'_k and b'_k the following respective intertwiners relative to M^*

$$\begin{array}{lcl} \rho_k & \longrightarrow & M^*(\bullet, \bullet; \rho_{k+\overline{1}}) \\ \varepsilon_k & \longmapsto & x'_{\bullet,\bullet} \otimes \varepsilon_{k+\overline{1}} \end{array} \qquad \begin{array}{lcl} \Bbbk \cdot \mathrm{Id} & \longrightarrow & M^*(\overline{0}, \bullet; \rho_k) \\ \mathrm{Id} & \longmapsto & x'_{\overline{0},\bullet} \otimes \varepsilon_k\,. \end{array}$$

Denote by c'_k and d'_k the following respective intertwiners relative to M^*

$$\begin{array}{lcl} \rho_k & \longrightarrow & M^*(\bullet, \overline{0}; \Bbbk \cdot \mathrm{Id}) \\ \varepsilon_k & \longmapsto & (x'_{\bullet,\overline{0}} \otimes \mathrm{Id} + (-1)^{k+\overline{1}} x'_{\bullet,\overline{1}} \otimes \tau) \end{array} \qquad \begin{array}{lcl} \Bbbk \cdot \mathrm{Id} & \longrightarrow & M^*(\overline{0}, \overline{0}; \Bbbk \cdot \mathrm{Id}) \\ \mathrm{Id} & \longmapsto & x'_{\overline{0},k} \otimes \tau^k\,. \end{array}$$

Then $(a_k|a'_k) = 1$, $(b_k|b'_k) = 1$, $(c_k|c'_k) = 1$ and $(d_k|d'_\ell) = \delta_{k,\ell}$ for all $\ell \in \mathbb{Z}/2\mathbb{Z}$, where $\delta_{k,\ell}$ stands for the Kronecker symbol. Using (2.1.3), the intermediate steps of the computation of these pairings are given in Tables 3, 4, 5 and 6 (see Example 1.21). In view of the shape of Q_G, these equalilities entail that $a_k^\vee = a'_k$, $b_k^\vee = b'_k$, $c_k^\vee = c_k$ and $d_k^\vee = d'_k$.

γ	α_γ	y_1	${}^{y_1^{-1}}\gamma$	$\chi_{\mathbf{y_1^{-1}},\gamma}$	$y_1^{-1}(\gamma)^*$	$\beta_{\mathbf{y_1^{-1}}(\gamma)^*}$	x_0, h_0	$\chi_{\rho_{\mathbf{k+\overline{1}}}}(\mathbf{h_0})$	*contr. of* γ
$x_{\bullet,\bullet}$	$\mathbf{\frac{1}{2}}$	Id	$x_{\bullet,\bullet}$	$\mathbf{1}$	$x'_{\bullet,\bullet}$	$\mathbf{1}$	Id, Id	$\mathbf{1}$	$\mathbf{1}$

TABLE 3. $(a_k|a'_k) = 1$

γ	α_γ	y_1	${}^{y_1^{-1}}\gamma$	$\chi_{\mathbf{y_1^{-1}},\gamma}$	$y_1^{-1}(\gamma)^*$	$\beta_{\mathbf{y_1^{-1}}(\gamma)^*}$	x_0, h_0	$\chi_{\rho_{\mathbf{k}}}(\mathbf{h_0})$	*contr. of* γ
$x_{\bullet,\overline{0}}$	$\mathbf{\frac{1}{2}}$	Id	$x_{\bullet,\overline{0}}$	$\mathbf{1}$	$x'_{\overline{0},\bullet}$	$\mathbf{1}$	Id, Id	$\mathbf{1}$	$\mathbf{\frac{1}{2}}$
$x_{\bullet,\overline{1}}$	$\mathbf{\frac{(-1)^{k+\overline{1}}}{2}}$	τ	$-x_{\bullet,\overline{0}}$	$\mathbf{-1}$	$x'_{\overline{0},\bullet}$	$\mathbf{1}$	Id, τ	$\mathbf{(-1)^k}$	$\mathbf{\frac{1}{2}}$

TABLE 4. $(b_k|b'_k) = \frac{1}{2} + \frac{1}{2} = 1$

γ	α_γ	y_1	${}^{y_1^{-1}}\gamma$	$\chi_{\mathbf{y_1^{-1}},\gamma}$	$y_1^{-1}(\gamma)^*$	$\beta_{\mathbf{y_1^{-1}}(\gamma)^*}$	x_0, h_0	$\chi_{\rho_{\mathbf{k}}}(\mathbf{h_0})$	*contr. of* γ
$x_{\overline{0},\bullet}$	$\mathbf{1}$	Id	$x_{\overline{0},\bullet}$	$\mathbf{1}$	$x'_{\bullet,\overline{0}}$	$\mathbf{1}$	Id, Id	$\mathbf{1}$	$\mathbf{\frac{1}{2}}$

TABLE 5. $(c_k|c'_k) = 1$

Now it is possible to give examples of intertwiners relative to M^* and associated to paths in Q_G. Let $s, t, a \in \mathbb{Z}/2\mathbb{Z}$. Denote by γ the following path of Q_G.

$$(\bullet, \rho_s) \xrightarrow{b_s} (\overline{0}, \Bbbk \cdot \mathrm{Id}) \xrightarrow{d_a} (\overline{0}, \Bbbk \cdot \mathrm{Id}) \xrightarrow{c_t} (\bullet, \rho_t)\,.$$

γ	α_γ	y_1	${}^{y_1^{-1}}\gamma$	$\chi_{\mathbf{y_1^{-1}},\gamma}$	$y_1^{-1}(\gamma)^*$	$\beta_{\mathbf{y_1^{-1}}(\gamma)^*}$	x_0, h_0	$\chi_{\mathrm{k}\cdot\mathrm{Id}}(\mathbf{h_0})$	*contr. of* γ
$x_{\overline{0},k}$	$(-\mathbf{1})^{\mathbf{k}}$	τ^{k}	$(-1)^k x_{k,\overline{0}}$	$(-\mathbf{1})^{\mathbf{k}}$	$x'_{\overline{0},k}$	$\delta_{\mathbf{k},\ell}$	τ^{-k}, Id	$\mathbf{1}$	$\delta_{\mathbf{k},\ell}$

TABLE 6. $(d_k|d'_\ell) = \delta_{k,\ell}$

Then $\varphi_\gamma = b_s^\vee \circledast d_a^\vee \circledast c_t^\vee = b'_s \circledast d'_a \circledast c'_t$. *This is the intertwiner* $\rho_t \to M^*(\bullet, \overline{0}, \overline{0}, \bullet; \rho_s)$ *such that (see (1.4.2))*

$$\begin{aligned}\varphi_\gamma(\varepsilon_t) &= c'_t(\varepsilon_t)\cdot d'_a(\mathrm{Id})\cdot b'_s(\mathrm{Id})\\ &= (x'_{\bullet,\overline{0}}\otimes \mathrm{Id} + (-1)^{t+\overline{1}} x'_{\bullet,\overline{1}}\otimes\tau)\cdot(x'_{\overline{0},a}\otimes\tau^a)\cdot(x'_{\overline{0},\bullet}\otimes\varepsilon_s)\\ &= (-1)^{a(s+\overline{1})}(x'_{\bullet,\overline{0}}x'_{\overline{0},a}x'_{a,\bullet}\otimes\varepsilon_s + (-1)^{t+\overline{1}+s}x'_{\bullet,\overline{1}}x'_{\overline{1},a+\overline{1}}x'_{a+\overline{1},\bullet}\otimes\varepsilon_s)\,.\end{aligned}$$

Thus, using the notation introduced in Example 1.17, $\varphi_\gamma = (-1)^{a(s+\overline{1})}\phi_a$.

1.6. **The comparison of the algebra of intertwiners with the Morita reduction.** Now, here is the result comparing $\Bbbk Q_G$ and $\mathcal{I}\mathrm{ntw}$.

Theorem 1.25. *Keep Settings 1.1 and 1.4.*

(1) Assigning f_γ *to every path* γ *in* Q_G *yields an isomorphism of algebras*

$$\Bbbk Q_G \xrightarrow{\sim} \mathcal{I}\mathrm{ntw}^{\mathrm{op}}\,. \tag{1.6.1}$$

(2) For all $f \in \mathcal{I}\mathrm{ntw}$, *say lying in* $\mathrm{Hom}_{\Bbbk G_{i_0}}(U_0, M(i_0,\dots,i_n;U_n))$,

$$f = \sum_\gamma (f|\varphi_\gamma)\, f_\gamma\,, \tag{1.6.2}$$

where γ *runs through all paths of* Q_G *of the shape* $(i_0,U_0)\to(i_1,\bullet)\to\cdots\to(i_{n-1},\bullet)\to(i_n,U_n)$.

It is now possible to explain how to decompose any element of $\tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}$ along the paths in Q_G *via* the isomorphism $\Bbbk Q_G \xrightarrow{(1.1.1)} \tilde{e}(T_S(M)*G)\cdot\tilde{e}$. This is done with the following construction and assuming that $U = \Bbbk G_i\cdot\varepsilon_U$ for all vertices (i,U) of Q_G. Given a tensor $m_1\otimes\cdots\otimes m_n * g \in e_{i_0}\cdot(M^{\otimes_S n}*G)\cdot e_{i_n}$, where $n\geqslant 0$ and $i_0, i_n \in [G\backslash Q_0]$, then

$$m_1\otimes\cdots\otimes m_n * g \in \big({}_{i_0}M_{y_1\cdot i_1}\otimes_{\Bbbk}\cdots\otimes_{\Bbbk}{}_{(y_1\cdots y_{n-1})\cdot i_{n-1}}M_{(y_1\cdots y_n)\cdot i_n}\big) * y_1\cdots y_n \Bbbk G_{i_n}$$

for unique $i_1,\dots,i_{n-1}\in[G\backslash Q_0]$ and $y_t\in[G/G_{i_t}]$, for all $t\in\{1,\dots,n\}$; this defines an intertwiner, for all $U\in\mathrm{irr}(G_{i_0})$ and $V\in\mathrm{irr}(G_{i_n})$,

$$\begin{array}{ccc}\Bbbk G_{i_0}\cdot\varepsilon_U & \longrightarrow & M(i_0,\dots,i_n;\Bbbk G_{i_n}\cdot\varepsilon_V)\\ u\cdot\varepsilon_U & \longmapsto & (u*\varepsilon_U)\cdot(m_1\otimes\cdots\otimes m_n*g)\cdot(e_{i_n}*\varepsilon_V)\,,\end{array}$$

where the products on the right-hand side are meant in $T_S(M)*G$; summing these intertwiners over all $U\in\mathrm{irr}(G_{i_0})$ and $V\in\mathrm{irr}(G_{i_n})$ defines an element of $\mathcal{I}\mathrm{ntw}$. This construction may be extended linearly to a linear mapping

$$\hat{e}\cdot(T_S(M)*G)\cdot\hat{e}\longrightarrow\mathcal{I}\mathrm{ntw}^{\mathrm{op}}\,,$$

where $\hat{e}=\sum_{i\in[G\backslash Q_0]}e_i$. Denote by Ξ the restriction of this mapping to $\tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}$,

$$\Xi\colon \tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}\longrightarrow\mathcal{I}\mathrm{ntw}^{\mathrm{op}}\,. \tag{1.6.3}$$

Remark 1.26. Here is how $\hat{e}$ and $\tilde{e}$ are related. Given idempotents a and b of an algebra Λ, say that b is a *reduction* of a whenever Λb is a direct summand of Λa in $\mathrm{Mod}(\Lambda)$ and the algebras $\mathrm{End}_\Lambda(\Lambda b)$ and $\mathrm{End}_\Lambda(\Lambda a)$ are Morita equivalent. Now $\tilde{e}$ $(=\sum_{i\in[G\backslash Q_0]}e_i*(\sum_{U\in\mathrm{irr}(G_i)}\varepsilon_U))$ is a reduction of $\hat{e}*1_{\Bbbk G}$ $(=\sum_{i\in[G\backslash Q_0]}e_i*1_{\Bbbk G_i})$ in the algebra $T_S(M)*G$.

Theorem 1.27. *Let Q_0, S, G, and M be as in Setting 1.1. Let Q_G be as in Setting 1.4. Assume that $U = \Bbbk G_i \cdot \varepsilon_U$ for all vertices (i, U) of Q_G. Then, Ξ, as introduced in (1.6.3), is an isomorphism of $\Bbbk$-algebras such that the following diagram is commutative,*

$$\begin{tikzcd} & \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e} \arrow[d, "\Xi"] \\ \Bbbk Q_G \arrow[ur, "(1.1.1)"] \arrow[r, "(1.6.1)"'] & \mathcal{I}\mathrm{ntw}^{\mathrm{op}} \end{tikzcd} \tag{1.6.4}$$

In particular, if (1.1.1) is used as an identification, then, for all $\theta \in \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$,

$$\theta = \sum_{\gamma \text{ path in } Q_G} (\Xi(\theta) | \varphi_\gamma) \cdot \gamma \,. \tag{1.6.5}$$

Example 1.28. *This is the continuation of Example 1.24. Denote by w the path $x_{\bullet,\overline{0}} x_{\overline{0},\overline{1}} x_{\overline{1},\bullet}$ in Q, this is an oriented cycle of length three going clockwise in the picture of Q in Example 1.3. The present example describes the intertwiner $\Xi(\tilde{e} \cdot (w * \mathrm{Id}) \cdot \tilde{e})$ as well as its preimage under the isomorphism $\Bbbk Q_G \xrightarrow{(1.6.1)} \mathcal{I}\mathrm{ntw}^{\mathrm{op}}$.*

Because of the definition,

$$\Xi(\tilde{e} \cdot (w * \mathrm{Id}) \cdot \tilde{e}) = \sum_{s,t \in \mathbb{Z}/2\mathbb{Z}} \Xi(\tilde{e} \cdot (w * \mathrm{Id}) \cdot \tilde{e})_{s,t} \,,$$

where, for all $s, t \in \mathbb{Z}/2\mathbb{Z}$, the piece of notation $\Xi(\tilde{e} \cdot (w * \mathrm{Id}) \cdot \tilde{e})_{s,t}$ stands for the intertwiner $\rho_s \to M(\bullet, \overline{0}, \overline{0}, \bullet; \rho_t)$ which maps ε_s to $(e_\bullet \otimes \varepsilon_s) \cdot \tilde{e} \cdot (w \otimes \mathrm{Id}) \cdot \tilde{e} \cdot (e_\bullet \otimes \varepsilon_t)$. For all $s, t \in \mathbb{Z}/2\mathbb{Z}$, the latter expression is equal to $(1_{\Bbbk Q} \otimes \varepsilon_s) \cdot (w \otimes \varepsilon_t)$, which in turn is equal to $\frac{1}{2}(x_{\bullet,\overline{0}} x_{\overline{0},\overline{1}} x_{\overline{1},\bullet} \otimes \varepsilon_t + (-1)^{s+\overline{1}+t} x_{\bullet,\overline{1}} x_{\overline{1},\overline{0}} x_{\overline{0},\bullet} \otimes \varepsilon_t)$; therefore, using the notation of Example 1.11,

$$\Xi(\tilde{e} \cdot (w * \mathrm{Id}) \cdot \tilde{e})_{s,t} = F_{\overline{0}} \,.$$

Now, let $s, t, a \in \mathbb{Z}/2\mathbb{Z}$; using the computation of the pairings of the shape $(F_a | \phi_b)$ made in Example 1.21 as well as the description of the intertwiners relative to M^* associated to paths in Q_G made in Example 1.24, and denoting by γ the path $(\bullet, \rho_s) \xrightarrow{b_s} (\overline{0}, \Bbbk \cdot \mathrm{Id}) \xrightarrow{d_a} (\overline{0}, \Bbbk \cdot \mathrm{Id}) \xrightarrow{c_t} (\bullet, \rho_t)$ of Q_G,

$$(\Xi(\tilde{e}(w \otimes \mathrm{Id}) \cdot \tilde{e})_{s,t} | \varphi_\gamma) = \begin{cases} \frac{1+(-1)^t}{2} & \text{if } a = \overline{0} \\ 0 & \text{if not;} \end{cases}$$

Therefore $\Xi(\tilde{e} \cdot (w \otimes \mathrm{Id}) \cdot \tilde{e})_{s,t} = \frac{1+(-1)^t}{2} \cdot b_s d_{\overline{0}} c_t$ (see (1.6.5)). As a whole, $\Xi(\tilde{e} \cdot (w \otimes \mathrm{Id}) \cdot \tilde{e}) = b_{\overline{0}} d_{\overline{0}} c_{\overline{0}} + b_{\overline{1}} d_{\overline{0}} c_{\overline{0}}$.

2. Practical aspects

Let Q_0, S, G and M be as in Setting 1.1.

Applying Theorem 1.27 for computations requires to compute Q_G first. Actually, rather than computing explicitly the intertwiners relative to M which form the arrows of Q_G, it is simpler to

(1) compute a basis of $\mathrm{Hom}_{\Bbbk G_j}(\tau, M^*(j, i; \rho))$ for all pairs (i, ρ) and (j, τ) such that $i, j \in [G \backslash Q_0]$, $\rho \in \mathrm{irr}(G_i)$, and $\tau \in \mathrm{irr}(G_j)$.

Because of the pairing $(-|-)$, this yields a quiver Q_G such as in Setting 1.4 as well as the intertwiners φ_α for all arrows α of Q_G. This avoids computing explicitly the intertwiners corresponding to the arrows of Q_G, which are not used in the formulas of Theorem 1.27.

Given an element θ of $\tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}$, say homogeneous of degree n for the grading induced by the tensor powers of M, the decomposition into a linear combination of paths in Q_G of the inverse image of θ under the isomorphism $\Bbbk Q_G \xrightarrow{(1.1.1)} \tilde{e}\cdot(T_S(M)*G)\cdot\tilde{e}$ may be computed as follows using Theorem 1.27,

(2) compute $\Xi(\theta)$, which is a sum of intertwiners relative to M, see (1.6.3);
(3) compute the intertwiners φ_γ for all paths γ of length n in Q_G, see Notation 1.23;
(4) compute the pairing $(\Xi(\theta)|\varphi_\gamma)$ for all such paths γ, see Proposition 1.20.

The desired linear combination of paths is hence $\sum_\gamma(\Xi(\theta)|\varphi_\gamma)\cdot\gamma$ (see (1.6.5)).

These computations may be performed by a computer that is able to determine the irreducible representations of $\mathrm{irr}(G_i)$ for all $i\in[G\backslash Q_0]$. This section illustrates these computations.

2.1. **How to pair intertwiners?** In view of (1.4.2), the main difficulty in the computations listed in the introduction of Section 2 lies in computing pairings for $(-|-)$. This section explains how to compute (1.5.2) in combinatorial terms in the following specific setting.

Setting 2.1. Let Q_0, S, G and M be as in Setting 1.1. Let Q be a quiver with vertex set being Q_0 and such that M is equal to the vector space with basis elements being the arrows of Q. Let Q_G be as in Setting 1.4. Assume that

- the stabiliser G_i is abelian for all $i\in Q_0$,
- $U=\Bbbk G_i\cdot\varepsilon_U$ for all vertices (i,U) of Q_G, denote by χ_U the character $G_i\to\Bbbk^\times$ of U,
- and, for all $g\in G$ and all paths γ in Q, there exists a scalar $\chi_{g,\gamma}\in\Bbbk^\times$ and a path $g(\gamma)$ such that ${}^g\gamma=\chi_{g,\gamma}\,g(\gamma)$.

If G is abelian, then it is possible to find a quiver Q and to choose irreducible representations U of G_i, for all $i\in[G\backslash Q_0]$, fitting in this setting.

Notation 2.2. The following notation relative to M is useful.

- For all arrows $a\colon i\to j$ of Q, denote by a^* the element of M^* such that $a^*(a)=1$ and $a^*(b)=0$ for all arrows b of Q distinct from a.
- Denote by Q^* the quiver with vertex set being Q_0 and whose arrows are the linear forms a^* introduced just before. This quiver is isomorphic to the opposite quiver of Q.
- For all paths $\gamma\colon t_0\xrightarrow{a_1}t_1\to\cdots\to t_{n-1}\xrightarrow{a_n}t_n$ in Q, denote by γ^* the path $t_n\xrightarrow{a_n^*}t_{n-1}\to\cdots\to t_1\xrightarrow{a_1^*}t_0$ in Q^*.

Note that every path in Q starting in some vertex i_0 lying in $[G\backslash Q_0]$ is of the shape $i_0\to y_1\cdot i_1\to\cdots\to(y_1\cdots y_n)\cdot i_n$ for a unique sequence $i_1,\dots,i_n$ in $[G\backslash Q_0]$ and a unique sequence $y_1,\dots,y_n$ in G such that $y_t\in[G/G_{i_t}]$ for all $t\in\{1,\dots,n\}$.

Let $\underline{i}=i_0,\dots,i_n$ be a sequence in $[G\backslash Q_0]$, consider $U\in\mathrm{irr}(G_{i_0})$ and $V\in\mathrm{irr}(G_{i_n})$, and let $f\colon U\to M(\underline{i};V)$ and $\varphi\colon V\to M^*(\underline{i}^o;U)$ be intertwiners. Since the representation V is one dimensional, then,

$$f(\varepsilon_u)=\sum_\gamma\alpha_\gamma\gamma\otimes y_1y_2\cdots y_n\varepsilon_V\,, \tag{2.1.1}$$

where γ runs through all paths in Q of the shape

$$i_0\to y_1\cdot i_1\to y_1y_2\cdot i_2\to\cdots\to(y_1\cdots y_n)\cdot i_n$$

for some (unique) sequence $\underline{y}=y_1,\dots,y_n$ such that $y_t\in[G/G_{i_t}]$ for all $t\in\{1,\dots,n\}$, and $\alpha_\gamma\in\Bbbk$. Note that the considered paths are precisely the ones of the

shape $i_0 \to j_1 \to \cdots \to j_n$ such that j_t lies in the orbit of i_t for all $t \in \{1, \dots, n\}$. Similarly,

$$\varphi(\varepsilon_V) = \sum_{\gamma'} \beta_{\gamma'} \gamma' \otimes x_{n-1} \cdots x_0 \varepsilon_U \,, \tag{2.1.2}$$

where γ' runs through all paths in Q of the shape

$$i_n \to x_{n-1} \cdot i_{n-1} \to x_{n-1}x_{n-2} \cdot i_{n-2} \to \cdots \to (x_{n-1}x_{n-2} \cdots x_0) \cdot i_0$$

for some (unique) sequence $\underline{x} = x_{n-1}, \dots, x_0$ such that $x_t \in [G/G_{i_t}]$ for all $t \in \{0, \dots, n-1\}$, and $\beta_{\gamma'} \in \Bbbk$.

Lemma 2.3. *Using the decompositions* (2.1.1) *and* (2.1.2)*, then*

$$(f|\varphi) = \sum_{\gamma} \alpha_\gamma \cdot \beta_{(y_1 \cdots y_n)^{-1}(\gamma)^*} \cdot \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \cdot \chi_U(h_0) \,, \tag{2.1.3}$$

where γ runs through all paths in Q of the shape $i_0 \to j_1 \to \cdots \to j_n$ such that j_t lies in the orbit of i_t for all $t \in \{1, \dots, n\}$, where $\alpha_\gamma \in \Bbbk$, and where, for all such paths γ,

- *$\underline{y} = y_1, \dots, y_n$ denotes the sequence in G such that $y_t \in [G/G_{i_t}]$, for all $t \in \{1, \dots, n\}$ and γ is a path of the shape $i_0 \to y_1 \cdot i_1 \to \cdots \to y_1 \cdots y_n \cdot i_n$,*
- *h_0 is the element of G_{i_0} such as in (c) in Proposition 1.20,*
- *$\chi_{(y_1 \cdots y_n)^{-1}, \gamma}$ is the element of $\Bbbk^\times$ and $(y_1 \cdots y_n)^{-1}(\gamma)$ is the path in Q such that ${}^{(y_1 \cdots y_n)^{-1}}\gamma = \chi_{(y_1 \cdots y_n)^{-1}, \gamma}\, (y_1 \cdots y_n)^{-1}(\gamma)$,*
- *and $((y_1 \cdots y_n)^{-1}(\gamma))^*$ is the reverse path in Q^* associated to the path $(y_1 \cdots y_n)^{-1}(\gamma)$ in Q.*

Proof. For all sequences $\underline{y} = y_1, \dots, y_n$ and $\underline{x} = x_{n-1}, \dots, x_0$ such that $x_t \in [G/G_{i_t}]$ and $y_t \in [G/G_{i_t}]$ for all t, denote by $\mathcal{C}(\underline{y})$ and $\mathcal{C}'(\underline{x})$ the sets of paths in Q and in Q^* of the shape $i_0 \to y_1 \cdot i_1 \to \cdots \to y_1 \cdots y_n \cdot i_n$ and $i_n \to x_{n-1} \cdot i_{n-1} \to \cdots \to x_{n-1} \cdots x_0 \cdot i_0$, respectively. For all $\underline{y}$,

$$f_{\underline{y}}(\varepsilon_U) = \sum_{\gamma \in \mathcal{C}(\underline{y})} \alpha_\gamma \gamma \otimes y_1 \cdots y_n \varepsilon_U \,;$$

hence, the following term that serves in the definition of $(f|\varphi)$, see (1.5.2),

$${}^{(y_1 \cdots y_n)^{-1}} f^{(1)}_{\underline{y}}(\varepsilon_U) \otimes \cdots \otimes {}^{(y_1 \cdots y_n)^{-1}} f^{(n)}_{\underline{y}}(\varepsilon_U) \otimes f^{(0)}_{\underline{y}}(\varepsilon_U) \,,$$

is equal to

$$\sum_{\gamma \in \mathcal{C}(\underline{y})} \alpha_\gamma \cdot \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \cdot \underbrace{(y_1 \cdots y_n)^{-1}(\gamma)}_{\in M^{\otimes_S n}} \otimes \varepsilon_U \,.$$

Similarly, for all $\underline{x}$,

$$\varphi^{\underline{x}}(\varepsilon_V) = \sum_{\gamma' \in \mathcal{C}(\underline{x})} \beta_{\gamma'} \gamma' \otimes x_{n-1} \cdots x_0 \varepsilon_U \,.$$

Therefore, given $\underline{y}$, the "Π_t"-term in (1.5.2) is equal to

$$\sum_{\gamma \in \mathcal{C}(\underline{y}),\, \gamma' \in \mathcal{C}(\underline{x})} \alpha_\gamma \cdot \beta_{\gamma'}\, \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \cdot b_n^*(c_n) \cdots b_2^*(c_2) \cdot b_1^*(c_1) \cdot \chi_U(h_0) \cdot \varepsilon_U \,, \tag{2.1.4}$$

where

- $\underline{x} = x_{n-1}, \dots, x_0$ and h_0 are the elements of G determined by $\underline{y}$ and by (a), (b), and (c) in Proposition 1.20,
- $b_1^*, \dots, b_n^*$ are the arrows of Q^* such that γ' is $\cdot \xrightarrow{b_n^*} \cdot \xrightarrow{b_{n-1}^*} \cdots \xrightarrow{b_1^*} \cdot$,
- and $c_1, \dots, c_n$ are the arrows of Q such that $(y_1 \cdots y_n)^{-1}(\gamma)$ is $\cdot \xrightarrow{c_1} \cdot \xrightarrow{c_2} \cdots \xrightarrow{c_n} \cdot$.

By definition of the arrows $b_1^*, \dots, b_n^*$, the summand of (2.1.4) with index γ is non-zero only if the paths $\cdot \xrightarrow{b_1} \cdot \xrightarrow{b_2} \cdots \xrightarrow{b_n} \cdot$ and $(y_1 \cdots y_n)^{-1}(\gamma)$ in Q are equal, that is, only if $\gamma' = ((y_1 \cdots y_n)^{-1}(\gamma))^*$. Summing (2.1.4) over all possible $\underline{y}$ yields that $(f|\varphi)\varepsilon_U$ is equal to

$$\sum_{\gamma} \alpha_\gamma \beta_{((y_1 \cdots y_n)^{-1}(\gamma))^*} \chi_{(y_1 \cdots y_n)^{-1}, \gamma} \chi_U(h_0) \varepsilon_U \,.$$

This proves (2.1.3). □

2.2. **An example.** In this example, Q is the following quiver whose vertices are the elements of $\mathbb{Z}/4\mathbb{Z}$

(2.2.1)

$$\begin{tikzcd}
 & \overline{0} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\overline{4} \arrow[ur, shift left] \arrow[d, shift left] & & \overline{1} \arrow[ul, shift left] \arrow[d, shift left] \\
\overline{3} \arrow[u, shift left] \arrow[rr, shift left] & & \overline{2} \arrow[u, shift left] \arrow[ll, shift left]
\end{tikzcd}$$

and G is the dihedral group of order 10

$$G = \langle c, \tau \mid c^5, \tau^2, \tau c \tau c \rangle \,.$$

Denote by ε the group homomorphism $G \to \{-1, 1\}$ such that $\varepsilon(c) = 1$ and $\varepsilon(\tau) = -1$. Denote the arrows of Q by $x_{i,j} \colon i \to j$. For convenience, denote the arrows of Q^* by $x'_{i,j} \colon i \to j$, hence $(x_{j,i})^* = x'_{i,j}$. The actions of G on Q_0 and M are assumed to be the ones such that, for all $i \in Q_0$ and $g \in G$, and for all arrows $x_{i,j}$ of Q,

$$c \cdot i = i + \overline{1}, \;\; \tau \cdot i = -i, \;\; {}^g x_{i,j} = \varepsilon(g) x_{g \cdot i, g \cdot j} \,.$$

The purpose of this section is to make a full illustration of Theorem 1.27, that is, for all $w \in T_S(M) * G$,

- to describe the intertwiner $\Xi(\theta)$, where θ denotes $\tilde{e} \cdot w \cdot \tilde{e}$;
- and to describe the preimage of θ under the isomorphism $\Bbbk Q_G \xrightarrow{(1.1.1)} \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$.

Let $[G \backslash Q_0]$ be $\{\overline{0}\}$. Note that $G_{\overline{0}} = \{\mathrm{Id}, \tau\}$. Let $[G/G_{\overline{0}}]$ be $\{\mathrm{Id}, c, c^2, c^3, c^4\}$. For all $s \in \{0, 1\}$, denote by ε_s the following primitive idempotent of $\Bbbk G_{\overline{0}}$,

$$\varepsilon_s = \frac{1}{2}(\mathrm{Id} + (-1)^s \tau)$$

and denote $\varepsilon_s \cdot \Bbbk G_{\overline{0}}$ by ρ_s. Finally, let $\mathrm{irr}(G_{\overline{0}})$ be equal to $\{\rho_0, \rho_1\}$. In particular,

$$\tilde{e} = e_{\overline{0}} * \varepsilon_0 + e_{\overline{0}} * \varepsilon_1 \,.$$

The setting presented below is assumed until the end of the subsection. In view of the purpose stated previously, this is not a loss of generality in making this assumption because

- the elements of the shape $p * g$, where p is a path in Q and $g \in G$, form a basis of $T_S(M) * G$ as a vector space;
- every $g \in G$ is equal to $c^\ell \tau^r$ for a unique couple $(\ell, r) \in \{0, 1, 2, 3, 4\} \times \{0, 1\}$;
- for all paths p of Q and for all $(\ell, r) \in \{0, 1, 2, 3, 4\} \times \{0, 1\}$, the product $\tilde{e} \cdot (p * g) \cdot \tilde{e}$ vanishes as soon as p is not a path from $\overline{0}$ to $-\overline{\ell}$.

Setting 2.4. Assume that w is equal to $p * c^\ell \tau^r$, where $(\ell, r) \in \{0, 1, 2, 3, 4\} \times \{0, 1\}$ and p is a path in Q from $\overline{0}$ to $-\overline{\ell}$. Denote the length of p by n. Denote by $i_0, \dots, i_n$ the sequence of vertices of Q such that p has the shape $i_0 \to i_1 \to \cdots \to i_n$. Thus,

$p = x_{i_0,i_1}x_{i_1,i_2}\cdots x_{i_{n-1},i_n}$. In particular, $i_0 = \overline{0}$, $i_n = -\overline{\ell}$ and $i_k - i_{k-1} \in \{-\overline{1}, \overline{1}\}$ for all $k \in \{1, \dots, n\}$.

2.2.1. *Computation of Q_G.* By definition, for all $t \in \{0,1\}$,

$$M^*(\overline{0}, \overline{0}; \rho_t) = \mathrm{Span}(x'_{\overline{0},\overline{1}} \otimes c\varepsilon_t,\ x'_{\overline{0},\overline{4}} \otimes c^4\varepsilon_t)\,,$$

where the action of τ is given as follows

$$\begin{cases} x'_{\overline{0},\overline{1}} \otimes c\varepsilon_t & \longmapsto & (-1)^{1+t}x'_{\overline{0},\overline{4}} \otimes c^4\varepsilon_t \\ x'_{\overline{0},\overline{4}} \otimes c^4\varepsilon_t & \longmapsto & (-1)^{1+t}x'_{\overline{0},\overline{1}} \otimes c\varepsilon_t\,. \end{cases}$$

Hence, for all $s, t \in \{0,1\}$,

$$\mathrm{Hom}_{\Bbbk G_{\overline{0}}}(\rho_s, M^*(\overline{0}, \overline{0}; \rho_t)) = \mathrm{Span}(\varphi_{s,t})\,,$$

where $\varphi_{s,t}$ is the intertwiner such that $\varphi_{s,t}(\varepsilon_s) = \varepsilon_s \cdot (x'_{\overline{0},\overline{1}} \otimes c\varepsilon_t)$ $(= \varepsilon_s \cdot (x'_{\overline{0},\overline{1}} \otimes c) \cdot \varepsilon_t)$, that is,

$$\varphi_{s,t}(\varepsilon_s) = x'_{\overline{0},\overline{1}} \otimes c\varepsilon_t + (-1)^{s+t+1}x'_{\overline{0},\overline{4}} \otimes c^4\varepsilon_t\,.$$

Denote by $f_{s,t}$ the intertwiner $\rho_s \to M(\overline{0}, \overline{0}; \rho_t)$ such that $(f_{s,t}|\varphi_{t,s}) = 1$, for all $s, t \in \{0,1\}$. Then Q_G may be taken equal to the following quiver

$$\begin{tikzcd} (1,\rho_0) \arrow[loop above, "f_{0,0}"] \arrow[r, shift left, "f_{0,1}"] & (1,\rho_1) \arrow[loop above, "f_{1,1}"] \arrow[l, shift left, "f_{1,0}"] \end{tikzcd}\,.$$

2.2.2. *The intertwiner $\Xi(\theta)$.* By definition, $\Xi(\theta)$ decomposes as

$$\Xi(\theta) = \sum_{s,t\in\{0,1\}} \Xi(\theta)_{s,t}\,, \tag{2.2.2}$$

where, for all $s, t \in \{0,1\}$, the intertwiner $\Xi(\theta)_{s,t} \colon \rho_s \to M(\underbrace{0, \dots, 0}_{n+1}; \rho_t)$ is such that $\Xi(\theta)_{s,t}(\varepsilon_s) = \varepsilon_s \cdot \tilde{e} \cdot w \cdot \tilde{e} \cdot \varepsilon_t$, see (1.6.3). Note that

$$\begin{aligned} \Xi(\theta)_{s,t}(\varepsilon_s) &= \varepsilon_s \cdot \tilde{e} \cdot w \cdot \tilde{e} \cdot \varepsilon_t \\ &= \tfrac{1}{2}(\mathrm{Id} + (-1)^s\tau) \cdot (p \otimes c^\ell\tau^r) \cdot \varepsilon_t \\ &= \tfrac{(-1)^{rt}}{2}(\mathrm{Id} + (-1)^s\tau) \cdot (p \otimes c^\ell\varepsilon_t) \\ &= \tfrac{(-1)^{rt}}{2}\left(p \otimes c^\ell\varepsilon_t + (-1)^{s+t}({}^\tau p) \otimes c^{-\ell}\varepsilon_t\right)\,, \end{aligned} \tag{2.2.3}$$

where ${}^\tau p = (-1)^n x_{-i_0,-i_1}x_{-i_1,-i_2}\cdots x_{-i_{n-1},-i_n}$.

2.2.3. *Intertwiners relative to M^* associated to paths of Q_G.* Consider a path of length denoted by m in Q_G, say $q \colon (0, \rho_{s_0}) \to (0, \rho_{s_1}) \to \cdots \to (0, \rho_{s_m})$, where $s_0, \dots, s_m \in \{-1, 1\}$. Then φ_q is the following intertwiner $\rho_{s_m} \to M^*(\underbrace{0, \dots, 0}_{n+1}; \rho_{s_0})$,

$$\varphi_q = \varphi_{s_1,s_0} \circledast \varphi_{s_2,s_1} \circledast \cdots \circledast \varphi_{s_m,s_{m-1}}\,.$$

Consider $\varphi_q(\varepsilon_{s_m})$ as an element of $\Bbbk Q^* * G$; following (1.4.2) it equals the product $(x'_{\overline{0},\overline{1}} \otimes (-1)^{s_m+s_{m-1}+1}x_{\overline{0},\overline{4}} \otimes c^4)\cdots(x'_{\overline{0},\overline{1}} \otimes (-1)^{s_1+s_0+1}x_{\overline{0},\overline{4}} \otimes c^4)$. Hence

$$\varphi_q(\varepsilon_{s_m}) = \sum(-1)^{\sum s_k+s_{k-1}+1}x'_{\overline{0},j_{m-1}}x'_{j_{m-1}j_{m-2}}\cdots x'_{j_1,j_0} \otimes c^{j_0}\,, \tag{2.2.4}$$

where

- the first sum runs over all sequences $j_{m-1}, \dots, j_0 \in \mathbb{Z}/4\mathbb{Z}$ such that the expression $x'_{\overline{0},j_{m-1}}\cdots x'_{j_1,j_0}$ is a path in Q', which means that $j_t - j_{t-1} \in \{-\overline{1}, \overline{1}\}$ for all $t \in \{1, \dots, m\}$ (taking $j_m = \overline{0}$),

- and the second sum (in the exponent) runs over all indices $k \in \{1, \dots, m\}$ such that $x'_{j_k, j_{k-1}}$ of Q' goes anticlockwise when Q' is drawn like Q in (2.2.1), which means that $j_k - j_{k-1} = \overline{1}$.

2.2.4. *Computation of the pairing* $(\Xi(\theta)|\varphi_q)$. Let $q\colon (0, \rho_{s_0}) \to (0, \rho_{s_1}) \to \cdots \to (0, \rho_{s_m})$ be a path in Q_G like in 2.2.3. Because of the definition of the action of G on $\Bbbk Q$, the pairing $(\Xi(\theta)|\varphi_q)$ $(= (\Xi(\theta)_{s_0, s_m}|\varphi_q))$ is given by (2.1.3). In particular, because of (2.2.3) and (2.2.4),

- this pairing vanishes if $m \neq n$, that is, p and q have different lengths,
- if $m = n$ then, for all paths γ distinct from both $x_{i_0, i_1} x_{i_1, i_2} \cdots x_{i_{n-1}, i_n}$ and $x_{i_0, -i_1} x_{-i_1, -i_2} \cdots x_{-i_{n-1}, -i_n}$, the term of index γ in (2.2.3) vanishes.

Using (2.2.3) and (2.2.4), the scalars appearing in (2.1.3) may be computed directly for these two possibilities for γ, assuming that $m = n$. Table 7 describes these scalars, they appear in bold face. Recall that $i_0 = \overline{0}$.

γ	$x_{i_0, i_1} x_{i_1, i_2} \cdots x_{i_{n-1}, i_n}$	$x_{i_0, -i_1} x_{-i_1, -i_2} \cdots x_{-i_{n-1}, -i_n}$
α_γ	$\mathbf{\frac{(-1)^{rs_n}}{2}}$	$\mathbf{\frac{(-1)^{rs_n + s_0 + s_n + n}}{2}}$
$y_1, \dots, y_n$	$c^{i_1 - i_0}, c^{i_2 - i_1}, \dots, c^{i_n - i_{n-1}}$	$c^{-i_1 + i_0}, c^{-i_2 + i_1}, \dots, c^{-i_n + i_{n-1}}$
$y_1 \cdots y_n$	c^{i_n}	c^{-i_n}
$(y_1 \cdots y_n)^{-1}{}_\gamma$	$x_{-i_n, i_1 - i_n} x_{i_1 - i_n, i_2 - i_n} \cdots x_{i_{n-1} - i_n, \overline{0}}$	$x_{i_n, i_n - i_1} x_{i_n - i_1, i_n - i_2} \cdots x_{i_n - i_{n-1}, \overline{0}}$
$\chi_{(\mathbf{y_1} \cdots \mathbf{y_n})^{-1}, \gamma}$	$\mathbf{1}$	$\mathbf{1}$
$(y_1 \cdots y_n)^{-1}(\gamma)^*$	$x'_{\overline{0}, i_{n-1} - i_n} \cdots x'_{i_2 - i_n, i_1 - i_n} x'_{i_1 - i_n, -i_n}$	$x'_{\overline{0}, i_n - i_{n-1}} \cdots x'_{i_n - i_2, i_n - i_1} x'_{i_n - i_1, i_n}$
$\beta_{(\mathbf{y_1} \cdots \mathbf{y_n})^{-1}(\gamma)^*}$	$\mathbf{(-1)^x}$ where x is $\sum_{\substack{1 \leqslant k \leqslant n \\ \text{s.t. } i_k - i_{k-1} = \overline{1}}} s_k + s_{k-1} + 1$	$\mathbf{(-1)^x}$, where x is $\sum_{\substack{1 \leqslant k \leqslant n \\ \text{s.t. } i_k - i_{k-1} = -\overline{1}}} s_k + s_{k-1} + 1$
h_0	Id	Id
$\chi_{\rho_{\mathbf{s_0}}}(\mathbf{h_0})$	$\mathbf{1}$	$\mathbf{1}$

TABLE 7. The coefficients in the expression (2.1.3) of $(\Xi(\theta)_{s_0, s_m}|\varphi_q)$ according to γ

Thus

$$(\Xi(\theta)|\varphi_q) = \frac{(-1)^{rs_0}}{2} (-1)^{\sum_{i_k - i_{k-1} = \overline{1}} (s_k + s_{k-1} + 1)} + \frac{(-1)^{rs_0 + s_0 + s_n + n}}{2} (-1)^{\sum_{i_k - i_{k-1} = -\overline{1}} (s_k + s_{k-1} + 1)}.$$

As whole (after factoring and simplifying),

$$(\Xi(\theta)|\varphi_q) = \begin{cases} (-1)^{rs_0} (-1)^{\sum_{i_k - i_{k-1} = \overline{1}} (s_k + s_{k-1} + 1)} & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.5}$$

2.2.5. *Decomposition of θ into paths in Q_G.* Because of Theorem 1.27, the preimage of θ under the isomorphism $\Bbbk Q_G \xrightarrow{(1.1.1)} \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ is equal to

$$\sum_{q \text{ path in } Q_G} (\Xi(\theta)|\varphi_q) \cdot q\,.$$

In view of (2.2.5), this preimage is hence equal to

$$\sum_{s_0, \dots, s_n \in \{0,1\}} (-1)^{rs_0} (-1)^{\sum_{i_k - i_{k-1} = \overline{1}} s_k + s_{k-1} + 1} \cdot ((0, \rho_{s_0}) \to (0, \rho_{s_1}) \to \cdots \to (0, \rho_{s_n}))\,.$$

2.2.6. *Illustration.* Assume that $w = x_{\overline{0}, \overline{1}} x_{\overline{1}, \overline{2}} x_{\overline{2}, \overline{3}} x_{\overline{3}, \overline{4}} x_{\overline{4}, \overline{0}} * \mathrm{Id} - x_{\overline{0}, \overline{4}} x_{\overline{4}, \overline{3}} x_{\overline{3}, \overline{2}} x_{\overline{2}, \overline{1}} x_{\overline{1}, \overline{0}} * \mathrm{Id}$. Applying 2.2.5 yields that the preimage of $\tilde{e} \cdot w \cdot \tilde{e}$ under the isomorphism $\Bbbk Q_G \xrightarrow{(1.1.1)} \tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ is equal to $-2 \sum_\gamma \gamma$, where γ runs through all oriented cycles of length 5 in Q_G.

3. Details on the monoidal category of bimodules over groups

This section details some isomorphisms and properties in the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$ and related to the operation, say $\times$, and the pairing $\langle -|- \rangle$ considered in Remark 1.7.

- Section 3.1 describes the left dual of a tensor product of objects in $\mathrm{mod}(A^e)$.
- Section 3.2 details the the adjunction $(X \otimes_A -) \vdash (X^* \otimes_A -)$ for a given $X \in \mathrm{mod}(A^e)$.
- Section 3.3 details the compatibility of $\langle -|- \rangle$.

The existence of the isomorphisms and adjunctions discussed in these sections may be part of the folklore (see [5, Exercise 2.10.16]). However, the proof of Theorem 1.25 is based on their description, which is the reason for detailing these here. For the sake of simplicity, the presentation is made independently of the monoidal category $(\mathrm{mod}(A^e), \otimes_A)$ and of the framework of Theorem 1.25.

Here are some conventions. In any group, e denotes the neutral element; and $\mathrm{Hom}_{\Bbbk G}$ and $\otimes_{\Bbbk G}$ are denoted by Hom_G and $\otimes_G$, respectively, for all finite groups G.

Given finite groups G and H, by "*a module* ${}_G X_H$" is meant a $\Bbbk G - \Bbbk H$-bimodule, by "*a module* ${}_G X$" is meant a left $\Bbbk G$-module, and by "*by a module* X_H is meant a right $\Bbbk H$-module.

Given finite groups G and H and given a $\Bbbk G - \Bbbk H$-bimodule X, the dual vector space X^* is a $\Bbbk H - \Bbbk G$-bimodule in a natural way ($h \cdot \phi \cdot g = \phi(g \cdot \bullet \cdot h)$ for all $g \in G$, $h \in H$ and $\phi \in X^*$). In particular, taking H to be the trivial group, this defines a functor $X \mapsto X^*$ from left $\Bbbk G$-modules to right $\Bbbk G$-modules.

3.1. Tensor product of duals and dual of a tensor product.

Lemma 3.1. *Let H be a finite group with order not divisible by* $\mathrm{char}(\Bbbk)$. *For all finite dimensional $\Bbbk H$-modules X_H and ${}_H Y$, there is a functorial bijective mapping*

$$\begin{array}{rcl} Y^* \otimes_H X^* & \to & (X \otimes_H Y)^* \\ \phi_2 \otimes \phi_1 & \mapsto & (x \otimes y \mapsto \sum_{h \in H} \phi_1(x \cdot h)\phi_2(h^{-1} \cdot y)) \,. \end{array} \tag{3.1.1}$$

Proof. The mapping is well defined. When $X = \Bbbk H_H$ and $Y = {}_H \Bbbk H$, it identifies with the following one, where $\{\delta_h\}_{h \in H}$ denotes the canonical basis of $(\Bbbk H)^*$,

$$\begin{array}{rcl} (\Bbbk H)^* \otimes_H (\Bbbk H)^* & \to & (\Bbbk H)^* \\ \delta_{h_2} \otimes \delta_{h_1} & \mapsto & \delta_{h_1 h_2} \,; \end{array}$$

and the latter is bijective. Since X_H and ${}_H Y$ are finite dimensional and projective, then (3.1.1) is bijective. □

The previous lemma generalises as follows. Given an integer $n \geqslant 2$, a sequence of groups $G_0, \ldots, G_n$ and a finite dimensional $\Bbbk G_{i-1} - \Bbbk G_i$-bimodule U_i for every $i \in \{1, \ldots, n\}$, there is a functorial bijective mapping

$$X_n^* \otimes_{G_{n-1}} \cdots \otimes_{G_1} X_1^* \to (X_1 \otimes_{G_1} \cdots \otimes_{G_{n-1}} X_n)^* \,, \tag{3.1.2}$$

which maps any $\phi_n \otimes \cdots \otimes \phi_1$ to the following linear form on $X_1 \otimes_{G_1} \cdots \otimes_{G_{n-1}} X_n$,

$$x_1 \otimes \cdots \otimes x_n \mapsto \sum_{g_1, \ldots, g_{n-1}} \prod_{t=1}^{n} \phi_t(g_{t-1}^{-1} \cdot x_t \cdot g_t) \,,$$

where g_t runs through G_t $(1 \leqslant t \leqslant n-1)$ and $g_0 = g_n = e$.

3.2. **Adjunctions.**

Lemma 3.2. *Let G and H be finite groups with orders not divisible by* $\mathrm{char}(\Bbbk)$. *Let ${}_GX_H$, ${}_HV$ and ${}_GU$ be (bi)modules over $\Bbbk G$ and $\Bbbk H$ such that X and V are finite dimensional. Then, there exists a functorial isomorphism*

$$\Psi_X^{V,U}\colon \mathrm{Hom}_H(V, X^* \otimes_G U) \to \mathrm{Hom}_G(X \otimes_H V, U)\,,$$

such that, for all $\phi \in \mathrm{Hom}_H(V, X^* \otimes_G U)$, $x \in X$ and $v \in V$,

$$(\Psi_X^{V,U}(\phi))(x \otimes v) = \sum_{g\in G} (\phi^{(2)}(v))(g^{-1} \cdot x)g \cdot \phi^{(1)}(v)\,, \tag{3.2.1}$$

where the following notation is used with sum sign omitted, for all $v \in V$,

$$\phi(v) = \phi^{(2)}(v) \otimes \phi^{(1)}(v) \in X^* \otimes_G U\,.$$

In particular, the pair of functors $(X \otimes_H -, X^* \otimes_G -)$ between finite dimensional left $\Bbbk H$-modules and finite dimensional left $\Bbbk G$-modules is adjoint.

Proof. Using the definition of the structure of $H-G$-bimodule of X^*, simple changes of variable in the sum "$\sum_{g\in G}$" show that $\Psi_X^{V,U}$ is indeed well defined.

Since ${}_HV$ is finitely generated and projective, it suffices to prove the lemma assuming that $V = \Bbbk H$. In this case, $\Psi_X^{V,U}$ identifies with the mapping $\lambda_{X,U}$ defined (for all left $\Bbbk G$-modules X and U) by

$$\begin{array}{rcl} X^* \otimes_G U & \to & \mathrm{Hom}_G(X, U) \\ \phi \otimes u & \mapsto & (x \mapsto \sum_{g\in G} \phi(g^{-1} \cdot x)g \cdot u)\,. \end{array} \tag{3.2.2}$$

Now, $\lambda_{\Bbbk G,\Bbbk G}$ identifies with the following bijective mapping (recall that $|G| \in \Bbbk^\times$).

$$\begin{array}{rcl} \Bbbk G^* \otimes_G \Bbbk G & \to & \Bbbk G \\ \phi \otimes e & \mapsto & \sum_{g\in G} \phi(g^{-1})g\,. \end{array}$$

Since ${}_GX$ is finite dimensional and projective, and since ${}_GU$ is projective, it follows that $\lambda_{X,U}$ is bijective. Thus, $\Psi_X^{V,U}$ is bijective. □

Lemma 3.3. *Let G, H, K be finite groups with orders not divisible by* $\mathrm{char}(\Bbbk)$. *Let ${}_GX_H$, ${}_HY_K$, ${}_KW$ and ${}_GU$ be finite dimensional modules. Then, the following diagram commutes*

$$\begin{array}{ccc} \mathrm{Hom}_K(W, Y^* \otimes_H X^* \otimes_G U) & \longrightarrow & \mathrm{Hom}_K(W, (X \otimes_H Y)^* \otimes_G U) \\ \Big\downarrow{\scriptstyle \Psi_Y^{W,X^*\otimes_G U}} & & \Big\downarrow{\scriptstyle \Psi_{X\otimes_H Y}^{W,U}} \\ \mathrm{Hom}_H(Y \otimes_K W, X^* \otimes_G U) & \xrightarrow[\Psi_X^{Y\otimes_K W,U}]{} & \mathrm{Hom}_G(X \otimes_H Y \otimes_K W, U)\,, \end{array} \tag{3.2.3}$$

where the top horizontal arrow is obtained upon applying $\mathrm{Hom}_K(W, - \otimes_G U)$ *to (3.1.1).*

Proof. In view of the compatibility of the involved mappings with direct sum decompositions of the finite dimensional projective modules ${}_KW$ and ${}_GU$, it is sufficient to prove the lemma assuming that $W = \Bbbk K$ and $U = \Bbbk G$. In this case, the given diagram identifies with the following one

$$\begin{array}{ccc} Y^* \otimes_H X^* & \xrightarrow{(3.1.1)} & (X \otimes_H Y)^* \\ \Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \alpha} \\ \mathrm{Hom}_H(Y, X^*) & \xrightarrow[\gamma]{} & \mathrm{Hom}_G(X \otimes_H Y, \Bbbk G)\,, \end{array} \tag{3.2.4}$$

where, after using (3.2.2),

- α is given by $\phi \mapsto (x \otimes y \mapsto \sum_{g\in G} \phi(g^{-1} \cdot x \otimes y)g)$,
- β is given by $\phi_2 \otimes \phi_1 \mapsto (y \mapsto \sum_{h\in H} \phi_2(h^{-1} \cdot y)\underbrace{h \cdot \phi_1}_{\phi_1(\bullet h)})$ and
- γ is given by $f \mapsto (x \otimes y \mapsto \sum_{g\in G}(f(y))(g^{-1} \cdot x)g)$.

It is direct to check that (3.2.4) commutes, which proves the lemma. □

3.3. **Pairing of morphisms.**

Definition 3.4. Let G and H be finite groups whose orders are not divisible by $\mathrm{char}(\Bbbk)$. Let ${}_GX_H$, ${}_HV$, and ${}_GU$ be (bi)modules such that X and V are finite dimensional. Assume that U is simple. Define a pairing

$$\langle -|- \rangle \colon \quad \mathrm{Hom}_G(U, X \otimes_H V) \otimes_{\Bbbk} \mathrm{Hom}_H(V, X^* \otimes_G U) \quad \to \quad \Bbbk \tag{3.3.1}$$

as follows. For all $f \in \mathrm{Hom}_G(U, X \otimes_H V)$ and $\phi \in \mathrm{Hom}_H(V, X^* \otimes_G U)$,

$$\langle f|\phi \rangle \cdot \mathrm{Id}_U = \Psi_X^{V,U}(\phi) \circ f \tag{3.3.2}$$

(following Lemma 3.2, the pairing $\langle -|- \rangle$ is non-degenerate).

This pairing is compatible with the following operation on morphisms.

Definition 3.5. Let G, H and K be finite groups whose orders are not divisible by $\mathrm{char}(\Bbbk)$. Let ${}_GX_H$, ${}_HY_K$, ${}_GU$, ${}_HV$ and ${}_KW$ be (bi)modules. For all $f_1 \in \mathrm{Hom}_G(U, X \otimes_H V)$ and $f_2 \in \mathrm{Hom}_H(V, Y \otimes_K W)$, define $f_2 \times f_1 \in \mathrm{Hom}_G(U, X \otimes_H Y \otimes_K W)$ as the following composite morphism

$$f_2 \times f_1 \colon U \xrightarrow{f_1} X \otimes_H V \xrightarrow{(X \otimes_H f_2)} X \otimes_H Y \otimes_K W\,. \tag{3.3.3}$$

Here is the above-mentioned compatibility.

Lemma 3.6. *Let*

- *G, H, and K be finite groups with orders not divisible by* $\mathrm{char}(\Bbbk)$,
- *${}_GX_H$, ${}_HY_K$, ${}_GU$, ${}_HV$ and ${}_KW$ be (bi)modules such that X, Y, and W are finite dimensional and U and V are simple,*
- *$f_1 \in \mathrm{Hom}_G(U, X \otimes_H V)$ and $f_2 \in \mathrm{Hom}_H(V, Y \otimes_K W)$ and*
- *$\phi_1 \in \mathrm{Hom}_H(V, X^* \otimes_G U)$ and $\phi_2 \in \mathrm{Hom}_K(W, Y^* \otimes_H V)$.*

Denote by λ the following isomorphism obtained upon applying $- \otimes_G U$ to (3.1.1),

$$Y^* \otimes_H X^* \otimes_G U \xrightarrow{\sim} (X \otimes_H Y)^* \otimes_G U\,.$$

Then, $\langle f_2 \times f_1 | \lambda \circ (\phi_1 \times \phi_2) \rangle = \langle f_2|\phi_2 \rangle \cdot \langle f_1|\phi_1 \rangle$.

Proof. Using (3.2.3) yields that

$$\Psi^{W,U}_{X \otimes_H Y}(\lambda \circ (\phi_1 \times \phi_2)) = \Psi_X^{Y \otimes_K W,U} \circ \Psi_Y^{W,X^* \otimes_G U}(\phi_1 \times \phi_2)\,.$$

Recall that $\phi_1 \times \phi_2$ is the following composite morphism

$$W \xrightarrow{\phi_2} Y^* \otimes_H V \xrightarrow{Y^* \otimes_H \phi_1} Y^* \otimes_H X^* \otimes_G U\,.$$

Since $\Psi_X^{\bullet,\bullet}$ and $\Psi_Y^{\bullet,\bullet}$ are adjunction isomorphisms, then

$$\begin{aligned} \Psi_X^{Y \otimes_K W,U} \circ \Psi_Y^{W,X^* \otimes_G U}(\phi_1 \times \phi_2) &= \Psi_X^{Y \otimes_K W,U}(\phi_1 \circ \Psi_Y^{W,V}(\phi_2)) \\ &= \Psi_X^{V,U}(\phi_1) \circ (X \otimes_H \Psi_Y^{W,V}(\phi_2))\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \Psi^{W,U}_{X \otimes_H Y}(\lambda \circ (\phi_1 \times \phi_2)) \circ (f_2 \times f_1) &= \Psi_X^{V,U}(\phi_1) \circ (X \otimes_H \Psi_Y^{W,V}(\phi_2)) \circ \\ &\quad (X \otimes_H f_2) \circ f_1 \\ &= \Psi_X^{V,U}(\phi_1) \circ (X \otimes_H \langle f_2|\phi_2 \rangle \cdot \mathrm{Id}_V) \circ f_1 \\ &= \langle f_1|\phi_1 \rangle \cdot \langle f_2|\phi_2 \rangle \cdot \mathrm{Id}_U\,. \end{aligned}$$

The conclusion of the lemma then follows from the definition of $\langle -|- \rangle$. □

The previous lemma has the following generalisation. Consider

- an integer $n \geqslant 2$,
- finite groups $G_0, \dots, G_n$ with orders not divisible by $\mathrm{char}(\Bbbk)$,
- a finite dimensional left $\Bbbk G_i$-module U_i for every $i \in \{0, \dots, n\}$, such that $U_0, \dots, U_{n-1}$ are simple,
- a finite dimensional $\Bbbk G_{i-1} - \Bbbk G_i$-bimodule X_i for every $i \in \{1, \dots, n\}$,
- $f_i \in \mathrm{Hom}_{G_{i-1}}(U_{i-1}, X_i \otimes_{G_i} U_i)$ and $\phi_i \in \mathrm{Hom}_{G_i}(U_i, X_i^* \otimes_{G_i} U_{i-1})$ for every $i \in \{1, \dots, n\}$.

Denote by $\lambda^{(n)}$ the isomorphism obtained upon applying $- \otimes_{G_0} U_0$ to (3.1.2),

$$\lambda^{(n)} \colon X_n^* \otimes_{G_{n-1}} \cdots \otimes_{G_1} X_1^* \otimes_{G_0} U_0 \xrightarrow{\sim} (X_1 \otimes_{G_1} \cdots \otimes_{G_{n-1}} X_n)^* \otimes_{G_0} U_0 \,.$$

Then,

$$\langle f_n \times \cdots \times f_1 | \lambda^{(n)} \circ (\phi_1 \times \cdots \times \phi_n) \rangle = \prod_{t=1}^{n} \langle f_i | \phi_i \rangle \,. \tag{3.3.4}$$

4. Intertwiners

Let Q_0, S, G and M be as in Setting 1.1. This section establishes properties on intertwiners relative to M which are needed to prove Theorem 1.25. This includes the properties of $\circledast$ mentioned in Section 1 as well as Proposition 1.20 on the existence of the pairing $(-|-)$. This is done in several steps,

- first, by proving that, in Definition 1.8, the $\Bbbk G_{i_0}$-module $M(\underline{i}; V)$ is isomorphic to $M(\underline{i}) \otimes_{G_{i_n}} V$ for a suitable $\Bbbk G_{i_0} - \Bbbk G_{i_n}$-bimodule $M(\underline{i})$;
- next, by proving that, under this isomorphism, $\circledast$ corresponds to the operation $\times$ of Definition 3.5;
- and finally by defining $(-|-)$ so that, under this isomorphism, it corresponds to the pairing $\langle -|- \rangle$ of Definition 3.4.

These steps are proceeded in turn in Sections 4.1, 4.2, and 4.3, respectively. They use the bimodule MG defined as follows. Note that the action of G on S yields the skew group algebra $S * G$. Like in Section 3, for all groups H, the pieces of notation e, $\otimes_H$ and Hom_H stand for the neutral element of H, for $\otimes_{\Bbbk H}$, and for $\mathrm{Hom}_{\Bbbk H}$, respectively.

Definition 4.1. Define MG as the following $S * G - S * G$-bimodule.

- Its underlying vector space is $M \otimes_{\Bbbk} \Bbbk G$, a tensor $m \otimes g$ is denoted by $m * g$ for all $m \in M$ and $g \in G$.
- The actions of S and G on the left and on the right are such that, for all $m \in M$, $g, h, k \in G$ and $s, s' \in S$,

$$\begin{array}{rcl} s \cdot (m * g) \cdot s' & = & s m^{\,g} s' * g \\ h \cdot (m * g) \cdot k & = & {}^{h}m * hgk \,. \end{array}$$

The $S * G - S * G$-bimodule $M^* G$ is defined similarly after replacing M by M^*.

4.1. **The space $M(\underline{i})$.** Let n be a positive integer. Let $\underline{i} = i_0, \dots, i_n$ a sequence in Q_0. Denote by $M(\underline{i})$ the following $\Bbbk G_{i_0} - \Bbbk G_{i_n}$-bimodule,

$$M(\underline{i}) = e_{i_0} M G e_{i_1} \otimes_{G_{i_1}} \cdots \otimes_{G_{i_{n-1}}} e_{i_{n-1}} M G e_{i_n} \,. \tag{4.1.1}$$

Note that this yields the $\Bbbk G_{i_n} - \Bbbk G_{i_0}$-bimodule $M^*(\underline{i}^o)$. These bimodules are dual to each other. Here is an explicit isomorphism used later. Consider the isomorphism (3.1.2) taking $X_t = e_{i_{t-1}} M^* G e_{i_t}$ for all $t \in \{1, \dots, n\}$,

$$e_{i_n} (MG)^* e_{i_{n-1}} \otimes_{G_{i_{n-1}}} \cdots \otimes_{G_{i_1}} e_{i_1} (MG)^* e_{i_0} \xrightarrow{(3.1.2)} M(\underline{i})^* \,. \tag{4.1.2}$$

Since the following mapping is a non-degenerate pairing

$$\begin{array}{rccc} [-|-]: & MG \otimes_{\Bbbk} M^*G & \to & \Bbbk \\ & (m*g)\otimes(\varphi*h) & \mapsto & \varphi({}^{g^{-1}}m)\delta_{e,hg}\,, \end{array}$$

it induces an isomorphism (of $S*G-S*G$-bimodules)

$$\begin{array}{ccc} M^*G & \to & (MG)^* \\ \bullet & \mapsto & [-|\bullet] \end{array} \tag{4.1.3}$$

the n-th tensor power of which induces an isomorphism of $\Bbbk G_{i_n}-\Bbbk G_{i_0}$-bimodules,

$$M^*(\underline{i}^o) \xrightarrow{\sim} e_{i_n}(MG)^*e_{i_{n-1}} \otimes_{G_{i_{n-1}}} \cdots \otimes_{G_{i_1}} e_{i_1}(MG)^*e_{i_0}\,. \tag{4.1.4}$$

Composing (4.1.2) and (4.1.4) yields an isomorphism of $\Bbbk G_{i_n}-\Bbbk G_{i_0}$-bimodules

$$\Lambda_{\underline{i}}\colon M^*(\underline{i}^o) \xrightarrow{\sim} M(\underline{i})^*\,, \tag{4.1.5}$$

which is induced by the mapping from $(MG)^{\otimes_{\Bbbk} n}$ to $((MG)^{\otimes_{\Bbbk} n})^*$ which assigns to any tensor $(\varphi_n*g_n)\otimes\cdots\otimes(\varphi_1*g_1)$ the following linear form on $(MG)^{\otimes_{\Bbbk} n}$,

$$(m_1*g_1')\otimes\cdots\otimes(m_n*g_n') \mapsto \sum_{k_1,\ldots,k_{n-1}}\prod_{t=1}^{n}\varphi_t({}^{k_t^{-1}g_t'^{-1}}m_t)\delta_{e,g_tk_{t-1}^{-1}g_t'k_t}\,, \tag{4.1.6}$$

where δ denotes the Kronecker symbol, k_t runs through G_{i_t} and $k_n=k_0=e$.

The bimodules $M(\underline{i})$ and $M^*(\underline{i}^o)$ are hence objects of the monoidal category $(\mathrm{mod}(A^e),\otimes_A)$ in which they are left dual to each other. The following lemma describes tensor products of these objects in this monoidal category. The same statements are valid after replacing M by M^*.

Lemma 4.2. *Let m,n be positive integers. Let $\underline{i''}=i_0,\ldots,i_{m+n}$ be a sequence in Q_0. Denote the sequences $i_0,\ldots,i_m$ and $i_m,\ldots,i_{m+n}$ by $\underline{i}$ and $\underline{i'}$, respectively.*

(1) $M(\underline{i''})=M(\underline{i})\otimes_{G_{i_m}}M(\underline{i'})$.

(2) The following diagram is commutative

$$\begin{tikzcd} M^*(\underline{i'}^o)\otimes_{G_{i_m}} M^*(\underline{i}^o) \arrow[r, "\Lambda_{\underline{i'}}\otimes_{G_{i_m}}\Lambda_{\underline{i}}"] \arrow[d, equal] & M(\underline{i'})^*\otimes_{G_{i_m}} M(\underline{i})^* \arrow[d, "(3.1.1)"] \\ M^*(\underline{i''}^o) \arrow[r, "\Lambda_{\underline{i''}}"'] & M(\underline{i''})^*\,. \end{tikzcd}$$

Proof. (1) follows from the definition of the spaces $M(\underline{\bullet})$ and (2) follows from the definition of the morphisms $\Lambda_{\underline{\bullet}}$ in terms of (3.1.2) and (4.1.3). □

The following result is the link between the bimodules $M(\underline{\bullet})$ and the modules of Definition 1.8. This link makes it possible to translate the results of Section 3 in terms of intertwiners. From now on, the piece of notation $*$ used for tensors in MG is often dropped in order to avoid overloaded notation.

Lemma 4.3. *Let $\underline{i}=i_0,\ldots,i_n$ be a sequence in Q_0, where $n\geqslant 1$. Let $U_n\in\mathrm{mod}(\Bbbk G_{i_n})$. There is an isomorphism in $\mathrm{mod}(\Bbbk G_{i_0})$,*

$$\Theta^M_{\underline{i};U_n}\colon M(\underline{i};U_n)\xrightarrow{\sim} M(\underline{i})\otimes_{G_{i_n}}U_n \tag{4.1.7}$$

given by

$$m_1\otimes\cdots\otimes m_n\otimes y_1\cdots y_nu \mapsto m_1y_1\otimes {}^{y_1^{-1}}m_2y_2\otimes\cdots\otimes {}^{(y_1\cdots y_{n-1})^{-1}}m_ny_n\otimes u\,.$$

When there is no risk of confusion, $\Theta^M_{\underline{i};U_n}$ is denoted by $\Theta_{\underline{i};U_n}$.

Proof. This is straightforward to check. □

The following natural transformation of functors from $\mathrm{mod}(\Bbbk G_{i_0})$ to $\mathrm{mod}(\Bbbk)$, defined by $\Theta_{\underline{i};U_n}$, is hence an isomorphism of functors,

$$\begin{array}{ccc} \mathrm{Hom}_{G_{i_0}}(-, M(\underline{i};U_n)) & \xrightarrow{\sim} & \mathrm{Hom}_{G_{i_0}}(-, M(\underline{i}) \otimes_{G_{i_n}} U_n) \\ - & \longmapsto & \Theta_{\underline{i};U_n} \circ - \,. \end{array} \tag{4.1.8}$$

4.2. **Compositions of intertwiners.** It is now possible to compare the operation $\circledast$ on intertwiners, see Definition 1.13, and the operation $\times$ of Definition 3.5.

Lemma 4.4. *Let $\underline{i}'' = i_0, \dots, i_{m+n}$ be a sequence in Q_0, where $m, n \geqslant 1$. Denote the sequences $i_0, \dots, i_m$ and $i_m, i_{m+1}, \dots, i_{m+n}$ by $\underline{i}$ and $\underline{i}'$, respectively. Let $U_0 \in \mathrm{mod}(\Bbbk G_{i_0})$, $U_m \in \mathrm{mod}(\Bbbk G_{i_m})$ and $U_{m+n} \in \mathrm{mod}(\Bbbk G_{i_{m+n}})$. Finally, let $f \in \mathrm{Hom}_{\Bbbk G_{i_0}}(U_0, M(\underline{i};U_m))$ and $f' \in \mathrm{Hom}_{\Bbbk G_{i_m}}(U_m, M(\underline{i}';U_{m+n}))$. Then,*

$$(\Theta_{\underline{i}';U_{m+n}} \circ f') \times (\Theta_{\underline{i};U_m} \circ f) = \Theta_{\underline{i}'';U_{m+n}} \circ (f' \circledast f)\,. \tag{4.2.1}$$

As a consequence, $f' \circledast f \in \mathrm{Hom}_{G_{i_0}}(U_0, M(\underline{i}'';U_{m+n}))$.

Proof. Let $u \in U_0$. Since $\Theta_{\underline{i}';U_{m+n}} \circ f'$ lies in $\mathrm{Hom}_{G_{i_m}}(U_m, M(\underline{i}'') \otimes_{G_{i_{m+n}}} U_{m+n})$ and $\Theta_{\underline{i};U_m} \circ f$ lies in $\mathrm{Hom}_{G_{i_0}}(U_0, M(\underline{i}) \otimes_{G_{i_m}} U_m)$, then

$$((\Theta_{\underline{i}';U_{m+n}} \circ f') \times (\Theta_{\underline{i};U_m} \circ f))(u) \underset{(3.3.3)}{=} (M(\underline{i}) \otimes_{\Bbbk G_{i_m}} (\Theta_{\underline{i}';U_{m+n}} \circ f')) \circ (\Theta_{\underline{i};U_m} \circ f)(u)\,.$$

Using the definition of $\Theta_{\underline{i};U_m}$, see (4.1.7), the right-hand side is equal to

$$\begin{aligned} &(M(\underline{i}) \otimes_{\Bbbk G_{i_m}} (\Theta_{\underline{i}';U_{m+n}} \circ f'))(\textstyle\sum_{\underline{y}} f^{(1)}_{\underline{y}}(u) y_1 \otimes {}^{y_1^{-1}} f^{(2)}_{\underline{y}}(u) y_2 \otimes \cdots \\ &\cdots \otimes {}^{(y_1 \cdots y_{m-1})^{-1}} (f^{(m)}_{\underline{y}}(u)) y_m \otimes f^{(0)}_{\underline{y}}(u))\,, \end{aligned}$$

where $\underline{y}$ runs through all sequences $y_1, \dots, y_m$ such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, m\}$. Using the definition of $\Theta_{\underline{i}';U_{m+n}}$, see (4.1.7), this is equal to

$$\begin{aligned} &\textstyle\sum_{\underline{y},\underline{y}'} f^{(1)}_{\underline{y}}(u) y_1 \otimes \cdots \otimes {}^{(y_1 \cdots y_{m-1})^{-1}} (f^{(m)}_{\underline{y}}(u)) y_m \otimes \\ &f'^{(1)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u)) y_{m+1} \otimes {}^{y_{m+1}^{-1}} (f'^{(2)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u))) y_{m+2} \otimes \cdots \\ &\cdots \otimes {}^{(y_{m+1} \cdots y_{m+n})^{-1}} (f'^{(n)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u))) y_{m+n} \otimes f'^{(0)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u))\,, \end{aligned}$$

where $\underline{y}$ is as before and $\underline{y}'$ runs through all sequences $y_{m+1}, \dots, y_{m+n}$ such that $y_{m+t} \in [G/G_{i_{m+t}}]$ for all $t \in \{1, \dots, n\}$. Using the definition of $\Theta_{\underline{i}'';U_{m+n}}$, see (4.1.7), this is equal to

$$\begin{aligned} &\Theta_{\underline{i}'';U_{m+n}}(\textstyle\sum_{\underline{y}''} f^{(1)}_{\underline{y}}(u) \otimes \cdots \otimes f^{(m)}_{\underline{y}}(u) \otimes {}^{y_1 \cdots y_m} (f'^{(1)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u))) \otimes \cdots \\ &\cdots \otimes {}^{y_1 \cdots y_m} (f'^{(n)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u))) \otimes y_1 \cdots y_{m+n} f'^{(0)}_{\underline{y}'}(f^{(0)}_{\underline{y}}(u)))\,, \end{aligned}$$

where $\underline{y}''$ runs through all sequences $y_1, \dots, y_{m+n}$ such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, m+n\}$ and where $\underline{y}$ and $\underline{y}'$ stand for the sequences $y_1, \dots, y_m$ and $y_{m+1}, \dots, y_{m+n}$, respectively. Finally, using the definition of $\circledast$, see (1.4.1), this is equal to $(\Theta_{\underline{i}'';U_{m+n}} \circ (f' \circledast f))(u)$. Whence (4.2.1). Note that $f' \circledast f$ is a morphism of $\Bbbk G_{i_0}$-modules because

- $\Theta_{\underline{i}'';U_{m+n}}$ is an isomorphism of $\Bbbk G_{i_0}$-modules,
- f and $\Theta_{\underline{i};U_m}$ are morphisms of $\Bbbk G_{i_0}$-modules,
- f' and $\Theta_{\underline{i}';U_{m+n}}$ are morphisms of $\Bbbk G_{i_m}$-modules,
- and the operation "$\times$" yields morphisms of $\Bbbk G_{i_0}$-modules (see (3.3.3)).

□

Lemma 4.4 shows that Definition 1.15 yields a, possibly non-associative, algebra.

Proposition 4.5. $\mathcal{I}$ntw *is an associative and unital algebra.*

Proof. Note that $\mathcal{I}$ntw is unital. Indeed, for all $i \in [G\backslash Q_0]$ and $U \in \mathrm{irr}(G_i)$, denote by $\varepsilon_{i,U}$ the intertwiner $\mathrm{Id}_U \in \mathrm{Hom}_{\Bbbk G_i}(U, M(i;U))$, recall that $M(i;U) = U$. Then $\sum_{i,U} \varepsilon_{i,U}$ is a unity for $\circledast$.

In the framework of Definition 3.5, the equality $f_3 \times (f_2 \times f_1) = (f_3 \times f_2) \times f_1$ holds whenever both sides make sense because of the associativity of the tensor product. In view of Lemma 4.4, this entails that $\circledast$ is associative. □

4.3. **The pairing on intertwiners.** Let n be a positive integer, let $\underline{i} = i_0, \dots, i_n$ be a sequence in Q_0 and let U_0 and U_n be left modules over $\Bbbk G_{i_0}$ and $\Bbbk G_{i_n}$, respectively. To every $\varphi \in \mathrm{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0))$ is associated a $\Bbbk$-linear mapping

$$\Phi(\varphi)\colon M(\underline{i}; U_n) \to U_0$$

defined as follows. For every sequence of representatives $y_1, \dots, y_n$ ($y_t \in [G/G_{i_t}]$) there exist unique sequences

$$\underline{x} = x_{n-1}, \dots, x_0 \ (x_t \in [G/G_{i_t}]), \text{ and } h_{n-1}, \dots, h_0 \ (h_t \in G_{i_t})$$

such that, for all $t \in \{1, \dots, n\}$,

$$(y_t \cdots y_n)^{-1} = x_{n-1} \cdots x_{t-1} h_{t-1}\,; \tag{4.3.1}$$

these sequences are determined by a decreasing induction: $y_n^{-1} = x_{n-1}h_{n-1}$ and $h_t y_t^{-1} = x_{t-1}h_{t-1}$ for all $t \in \{1, \dots, n-1\}$; then, for every $m_1 \otimes \cdots \otimes m_t \otimes y_1 \cdots y_n u$ lying in $M(\underline{i}; U_n)$, define the element $\Phi(\varphi)(m_1 \otimes \cdots \otimes m_t \otimes y_1 \cdots y_n u)$ of U_0, using Notation 1.18, as

$$\prod_{t=1}^{n} \varphi^{\underline{x}}_{(t)}(u) \left({}^{(y_1 \cdots y_n)^{-1}} m_t \right)^{h_0^{-1}} (\varphi^{\underline{x}}_{(0)}(u))\,. \tag{4.3.2}$$

This does make sense, because $m_t \in {}_{y_1 \cdots y_{t-1} \cdot i_{t-1}} M_{y_1 \cdots y_t \cdot i_t}$, because $\varphi^{\underline{x}}_{(t)}$ is a linear form on ${}_{x_{n-1} \cdots x_{t-1} \cdot i_{t-1}} M_{x_{n-1} \cdots x_t \cdot i_t}$, and because of (4.3.1).

After appropriate identifications, Φ is related to the adjunction isomorphism $\Psi^{\bullet,\bullet}_{M(\underline{i})}$ of section 3.2 as explained in the following result. Recall that $\Psi^{U_n,U_0}_{M(\underline{i})}$, $\Theta^{M^*}_{\underline{i}^o;U_0}$ and $\Lambda_{\underline{i}}$ are defined in (3.2.1), (4.1.7), and (4.1.5), respectively.

Lemma 4.6. *Keep the previous setting. Denote by φ' the following composite morphism of left $\Bbbk G_{i_n}$-modules,*

$$U_n \xrightarrow{\varphi} M^*(\underline{i}^o; U_0) \xrightarrow{\Theta^{M^*}_{\underline{i}^o;U_0}} M^*(\underline{i}^o) \otimes_{G_{i_0}} U_0 \xrightarrow{\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0} M(\underline{i})^* \otimes_{G_{i_0}} U_0\,.$$

Then,

$$\Phi(\varphi) = \Psi^{U_n,U_0}_{M(\underline{i})}(\varphi') \circ \Theta^{M}_{\underline{i};U_n}\,. \tag{4.3.3}$$

In particular, $\Phi(\varphi) \in \mathrm{Hom}_{G_{i_0}}(M(\underline{i}; U_n), U_0)$.

Proof. The composition $\Theta^{M^*}_{\underline{i}^o;U_0} \circ \varphi$ is given by (see (4.1.7))

$$u \mapsto \sum_{\underline{x}} \varphi^{\underline{x}}_{(n)}(u) x_{n-1} \otimes {}^{x_{n-1}^{-1}} \varphi^{\underline{x}}_{(n-1)}(u) x_{n-2} \otimes \cdots \otimes {}^{(x_{n-1} \cdots x_1)^{-1}} \varphi^{\underline{x}}_{(1)}(u) x_0 \otimes \varphi^{\underline{x}}_{(0)}(u)\,,$$

where $\underline{x} = x_{n-1}, \dots, x_0$ runs through all sequences of representatives ($x_t \in [G/G_{i_t}]$ for all $0 \leqslant t \leqslant n-1$). Hence, following (4.1.6), the morphism φ' is given by

$$u \mapsto \sum_{\underline{x}} \phi^{\underline{x}}(u) \otimes \varphi^{\underline{x}}_{(0)}(u)$$

where $\phi^{\underline{x}}(u)$ is the linear form on $M(\underline{i})$, which is a quotient of $(MG)^{\otimes_{\Bbbk} n}$, induced by the linear form on $(MG)^{\otimes_{\Bbbk} n}$ mapping any $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n)$ to

$$\sum_{k_1,\ldots,k_{n-1}} \prod_{t=1}^{n} \Big({}^{(x_{n-1}x_{n-2}\cdots x_t)^{-1}}(\varphi^{\underline{x}}_{(t)}(u)) \Big) \Big({}^{k_t^{-1}g_t^{-1}} m_t \Big) \cdot \delta_{e, x_{t-1}k_{t-1}^{-1}g_t k_t}\,, \tag{4.3.4}$$

where δ denotes the Kronecker symbol, $k_0 = k_n = e$ and k_t runs through G_{i_t} for $1 \leqslant t \leqslant n-1$.

In order to simplify the expression of (4.3.4), consider a tensor $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n)$ in $(MG)^{\otimes_{\Bbbk} n}$; for all $k_1, \ldots, k_{n-1}$ in G, the following assertions are equivalent, keeping $k_0 = k_n = e$,

(i) $(\forall t \in \{1, \ldots, n-1\})\ e = x_{t-1}k_{t-1}^{-1}g_t k_t$,
(ii) $k_t = (x_{t-1} \cdots x_1 x_0 g_1 g_2 \cdots g_t)^{-1}$ for all $t \in \{1, \ldots, n\}$, and $g_1 \cdots g_n$ is the inverse of $x_{n-1} \cdots x_1 x_0$;

given that each ${}^{(x_{n-1}x_{n-2}\cdots x_t)^{-1}}(\varphi^{\underline{x}}_{(t)}(u))$ in (4.3.4) is a linear form on ${}_{x_{t-1}i_{t-1}}M_{i_t}$ and each k_t is constrained to lie in G_{i_t}, then (4.3.4) is equal to

$$\prod_{t=1}^{n} (\varphi^{\underline{x}}_{(t)}(u))\, ({}^{x_{n-1}\cdots x_1 x_0 g_1 g_2 \cdots g_{t-1}} m_t)\, \delta_{e, g_1 g_2 \cdots g_n x_{n-1} \cdots x_1 x_0}\,,$$

which simplifies to

$$\prod_{t=1}^{n} (\varphi^{\underline{x}}_{(t)}(u)) \Big({}^{(g_t g_{t+1} \cdots g_n)^{-1}} m_t \Big)\, \delta_{(g_1 g_2 \cdots g_n)^{-1}, x_{n-1} \cdots x_1 x_0}\,. \tag{4.3.5}$$

Thus, the morphism φ' is given by $u \mapsto \sum_{\underline{x}} \phi^{\underline{x}}(u) \otimes \varphi^{\underline{x}}_{(0)}(u)$, where $\phi^{\underline{x}}(u)$ is the linear form on $M(\underline{i})$ induced by the mapping assigning (4.3.5) to any element $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n)$ of $(MG)^{\otimes_{\Bbbk} n}$.

Hence (see Lemma 3.2), $\Psi^{U_n, U_0}_{M(\underline{i})}(\varphi')$ is the morphism from $M(\underline{i}) \otimes_{G_{i_n}} U_n$ to U_0 induced by the mapping from $(MG)^{\otimes_{\Bbbk} n} \otimes_{\Bbbk} U_n$ to U_0 which maps any $(m_1 * g_1) \otimes \cdots \otimes (m_n * g_n) \otimes u$ to

$$\begin{aligned} & \textstyle\sum_{\underline{x}} \sum_{g \in G_{i_0}} \phi^{\underline{x}}(u)(g^{-1} m_1 g_1 \otimes m_2 g_2 \otimes \cdots \otimes m_n g_n)\ {}^{g}(\varphi^{\underline{x}}_{(0)}(u)) \\ = \ & \textstyle\sum_{g\,,\,\underline{x}} \prod_{t=1}^{n} \varphi^{\underline{x}}_{(t)}(u)({}^{(g_t \cdots g_n)^{-1}} m_t) \delta_{(g_1 \cdots g_n)^{-1} g, x_{n-1} \cdots x_0}\ {}^{g}(\varphi^{\underline{x}}_{(0)}(u))\,. \end{aligned}$$

Now, let $m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n u \in M(\underline{i}; U_n)$. Recall that its image under $\Theta^{M}_{\underline{i}; U_n}$ is $m_1 y_1 \otimes {}^{y_1^{-1}} m_2 y_2 \otimes \cdots \otimes {}^{(y_1 \cdots y_{n-1})^{-1}} m_n y_n \otimes u$ (see (4.1.7)). Hence, its image under $\Psi^{U_n, U_0}_{M(\underline{i})}(\varphi') \circ \Theta^{M}_{\underline{i}; U_n}$ is

$$\sum_{g, \underline{x}} \prod_{t=1}^{n} \varphi^{\underline{x}}_{(t)}(u)({}^{(y_1 \cdots y_n)^{-1}} m_t) \delta_{(y_1 \cdots y_n)^{-1} g, x_{n-1} \cdots x_0}\ {}^{g}(\varphi^{\underline{x}}_{(0)}(u))\,. \tag{4.3.6}$$

Note that, given $g \in G_{i_0}$ and $\underline{x}$, then, for all $t \in \{1, \ldots, n\}$,

- $\varphi^{\underline{x}}_{(t)}(u) \in ({}_{x_{n-1} \cdots x_{t-1} \cdot i_{t-1}} M_{x_{n-1} \cdots x_t \cdot i_t})^*$ and
- ${}^{(y_1 \cdots y_n)^{-1}} m_t \in {}_{(y_t \cdots y_n)^{-1} \cdot i_{t-1}} M_{(y_{t+1} \cdots y_n)^{-1} \cdot i_t}$;

in particular, if for a given index $(g, \underline{x})$ in (4.3.6) the product "$\prod_{t=1}^{n}$" is non-zero then this product is the unique non-zero term of the sum (4.3.6). More precisely, let $\{x_t, h_t\}_{0 \leqslant t \leqslant n-1}$ be the unique collection of elements in G such that, for all $t \in \{0, \ldots, n-1\}$,

- $x_t \in [G/G_{i_t}]$,
- $h_t \in G_{i_t}$ and
- $(y_{t+1} \cdots y_n)^{-1} = x_{n-1} \cdots x_t h_t$;

then, the only possibly non-zero term of the sum (4.3.6) is the one with index $(h_0^{-1}, \underline{x})$. Therefore, (4.3.6) equals

$$\prod_{t=1}^{n} \varphi^{\underline{x}}_{(t)}(u)({}^{(y_1\cdots y_n)^{-1}}m_t)\ {}^{h_0^{-1}}(\varphi^{\underline{x}}_{(0)}(u))\,,$$

which is equal to $\Phi(\varphi)(m_1 \otimes \cdots \otimes m_n \otimes y_1 \cdots y_n u)$ (see (4.3.2)). Thus, $\Phi(\varphi) = \Psi^{U_n,U_0}_{M(\underline{i})}(\varphi') \circ \Theta^{M}_{\underline{i};U_n}$. This proves (4.3.3). The second assertion of the lemma is a direct consequence of (4.3.3). □

It follows from Lemma 4.6 that the following mapping is bijective,

$$\Phi\colon \operatorname{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0)) \to \operatorname{Hom}_{G_{i_0}}(M(\underline{i}; U_n), U_0)\,.$$

This yields a non-degenerate pairing defined as follows.

Definition 4.7. Let $\underline{i} = i_0, \ldots, i_n$ be a sequence in Q_0, where $n \geqslant 1$. Let $U_0 \in \operatorname{mod}(\Bbbk G_{i_0})$ be simple. Let $U_n \in \operatorname{mod}(\Bbbk G_{i_n})$. Define $(-|-)$ to be the non-degenerate pairing

$$(-|-)\colon \operatorname{Hom}_{G_{i_0}}(U_0, M(\underline{i}; U_n)) \otimes_{\Bbbk} \operatorname{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o; U_0)) \to \Bbbk\,,$$

such that, for all $f \in \operatorname{Hom}_{G_{i_0}}(U_0, M(\underline{i}; U_n))$ and $\varphi \in \operatorname{Hom}_{G_{i_n}}(U_n, M^*(\underline{i}^o, U_0))$,

$$(f|\varphi) \cdot \operatorname{Id}_{U_0} = \Phi(\varphi) \circ f\,; \tag{4.3.7}$$

With this definition, it follows from (3.3.2) and (4.3.3) that

$$(f|\varphi) = \langle \Theta^{M}_{\underline{i};U_n} \circ f | (\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Theta^{M^*}_{\underline{i}^o;U_0} \circ \varphi \rangle\,. \tag{4.3.8}$$

It is now possible to prove Proposition 1.20.

Proof of Proposition 1.20. Note that (1.5.2) is equal to $(\Phi(\varphi))(f(u))$, see (4.3.2). Hence, the pairing $(-|-)$ claimed by the proposition is the one of Definition 4.7. □

The pairing $(-|-)$ is compatible with the composition of (dual) intertwiners in the following sense.

Lemma 4.8. *Let m, n be positive integers, let $\underline{i''} = i_0, \ldots, i_{m+n}$ be a sequence in Q_0, let U_0 and U_m be simple left modules over $\Bbbk G_{i_0}$ and $\Bbbk G_{i_m}$, respectively, and let U_{m+n} be a left $\Bbbk G_{i_{m+n}}$-module. Then, for all intertwiners*

$$\begin{aligned} f_1 &\colon U_0 \to M(\underline{i}; U_m)\,, & \varphi_1 &\colon U_m \to M^*(\underline{i}^o; U_0)\,,\\ f_2 &\colon U_m \to M(\underline{i'}; U_{m+n})\,, & \varphi_2 &\colon U_{m+n} \to M^*(\underline{i'}^o; U_m)\,, \end{aligned}$$

where $\underline{i} = i_0, \ldots, i_m$ and $\underline{i'} = i_m, \ldots, i_{m+n}$, the following holds

$$(f_2 \circledast f_1 | \varphi_1 \circledast \varphi_2) = (f_2|\varphi_2) \cdot (f_1|\varphi_1)\,. \tag{4.3.9}$$

Proof. In order to avoid overloaded expressions, the following shorthand notation is used in this proof: $\Theta_{\underline{i};U_0}$ for $\Theta^{M}_{\underline{i};U_0}$ and $\Theta'_{\underline{i};U_0}$ for $\Theta^{M^*}_{\underline{i}^o;U_0}$. Define $f'_1, f'_2, \varphi'_1, \varphi'_2$ as follows

$$\begin{aligned} f'_1 &= \Theta_{\underline{i};U_m} \circ f_1\,, & \varphi'_1 &= (\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Theta'_{\underline{i};U_0} \circ \varphi_1\,,\\ f'_2 &= \Theta_{\underline{i'};U_{m+n}} \circ f_2\,, & \varphi'_2 &= (\Lambda_{\underline{i'}} \otimes_{G_{i_m}} U_m) \circ \Theta'_{\underline{i'};U_m} \circ \varphi_2\,. \end{aligned}$$

It follows from (4.3.8) that $(f_2 \circledast f_1 | \varphi_1 \circledast \varphi_2)$ is equal to

$$\langle \Theta_{\underline{i''},U_{m+n}} \circ (f_2 \circledast f_1) | (\Lambda_{\underline{i''}} \otimes_{G_{i_0}} U_0) \circ \Theta'_{\underline{i''},U_0} \circ (\varphi_1 \circledast \varphi_2) \rangle\,;$$

applying Lemma 4.4 twice, first for M and next for M^*, yields that the previous expression is equal to

$$\langle \underbrace{(\Theta_{\underline{i'},U_{m+n}} \circ f_2)}_{f'_1} \times \underbrace{(\Theta_{\underline{i},U_m} \circ f_1)}_{f'_2} | (\Lambda_{\underline{i''}} \otimes_{G_{i_0}} U_0) \circ \Big[(\Theta'_{\underline{i},U_0} \circ \varphi_1) \times (\Theta'_{\underline{i'},U_m} \circ \varphi_2) \Big] \rangle\,;$$

according to Lemma 4.2 this is equal to

$$\langle f_1' \times f_2' | \lambda \circ (\Lambda_{\underline{i'}} \otimes_{G_{i_m}} \Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Big[(\Theta'_{\underline{i},U_0} \circ \varphi_1) \times (\Theta'_{\underline{i'},U_m} \circ \varphi_2) \Big] \rangle \,, \tag{4.3.10}$$

where λ is as in Lemma 3.6. Now, since the following composite morphisms are equal due to the associativity of the tensor product

$$M^*(\underline{i'}^o) \underset{G_{i_m}}{\otimes} U_m \xrightarrow{1 \otimes (\Theta'_{\underline{i};U_0} \circ \varphi_1)} M^*(\underline{i'}^o) \underset{G_{i_m}}{\otimes} M^*(\underline{i}^o) \underset{G_{i_0}}{\otimes} U_0 \xrightarrow{\Lambda_{\underline{i'}} \otimes \Lambda_{\underline{i}} \otimes 1} M(\underline{i'})^* \underset{G_{i_m}}{\otimes} M(\underline{i})^* \underset{G_{i_0}}{\otimes} U_0$$

and

$$M^*(\underline{i'}^o) \underset{G_{i_m}}{\otimes} U_m \xrightarrow{\Lambda_{\underline{i'}} \otimes 1} M(\underline{i'})^* \underset{G_{i_m}}{\otimes} U_m \xrightarrow{1 \otimes (\Theta'_{\underline{i};U_0} \circ \varphi_1)} M(\underline{i'})^* \underset{G_{i_m}}{\otimes} M^*(\underline{i}^o) \underset{G_{i_0}}{\otimes} U_0 \xrightarrow{1 \otimes \Lambda_{\underline{i}} \otimes 1} M(\underline{i'})^* \underset{G_{i_m}}{\otimes} M(\underline{i})^* \underset{G_{i_0}}{\otimes} U_0 \,,$$

then $(\Lambda_{\underline{i'}} \otimes_{G_{i_m}} \Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Big[(\Theta'_{\underline{i},U_0} \circ \varphi_1) \times (\Theta'_{\underline{i'},U_m} \circ \varphi_2) \Big]$ is equal to

$$\underbrace{[(\Lambda_{\underline{i}} \otimes_{G_{i_0}} U_0) \circ \Theta'_{\underline{i};U_0} \circ \varphi_1]}_{\varphi_1'} \times \underbrace{[(\Lambda_{\underline{i'}} \otimes_{G_{i_m}} U_m) \circ \Theta'_{\underline{i'};U_m} \circ \varphi_1]}_{\varphi_2'}.$$

Therefore,

$$\begin{array}{rcl} (4.3.10) & = & \langle f_2' \times f_1' | \lambda \circ (\varphi_1' \times \varphi_2') \rangle \\ & \underset{\text{Lemma 3.6}}{=} & \langle f_2' | \varphi_2' \rangle \cdot \langle f_1' | \varphi_1' \rangle \\ & \underset{(4.3.8)}{=} & (f_2 | \varphi_2) \cdot (f_1 | \varphi_1) \,. \end{array}$$

Thus, $(f_2 \circledast f_1 | \varphi_1 \circledast \varphi_2) = (f_1|\varphi_1) \cdot (f_2|\varphi_2)$. □

5. Proof of the main results

This section proves Theorems 1.25 and 1.27. Keeping Q_0, S, G and M as in Setting 1.1 and Q_G as in Setting 1.4, the proofs use the following notation:

- denote by $\hat{e}$ the idempotent $\sum_{i \in [G \backslash Q_0]} e_i$ of S, this idempotent is not to be confused with $\tilde{e}$ which is equal to $\sum_{(i,U)} e_i * \varepsilon_U$, where (i,U) runs through the vertices of Q_G;
- denote by S_1 the $\Bbbk$-algebra $\Pi_{i \in [G \backslash Q_0]} \Bbbk G_i$.

First, it is necessary to check that the quiver Q_G corresponds to the one given by Demonet in [3].

Proof of Theorem 1.6. Following [3], see p. 1057, the algebra $T_S(M) * G$ is Morita equivalent to $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, which is isomorphic to the path algebra of a quiver having the same vertices as Q_G and such that the arrows from a vertex (i,U) to a vertex (j,V) form a basis of the vector space $(e_i * \varepsilon_U) \cdot (MG) \cdot (e_j * \varepsilon_V)$. Now, considering e_i, e_j, ε_U, and ε_V as elements of $S * G$,

$$\begin{array}{rcl} (e_i * \varepsilon_U) \cdot (MG) \cdot (e_j * \varepsilon_V) & = & \varepsilon_U \cdot (e_i \cdot (MG) \cdot e_j) \cdot \varepsilon_V \\ & \underset{(4.1.1)}{=} & \varepsilon_U \cdot M(i,j) \cdot \varepsilon_V \\ & \simeq & \mathrm{Hom}_{\Bbbk G_i}(U, M(i,j) \otimes_{G_j} V) \\ & \underset{(4.1.7)}{\simeq} & \mathrm{Hom}_{\Bbbk G_i}(U, M(i,j;V)) \,. \end{array}$$

Whence the first statement of the theorem. Now, if $U = \Bbbk G_i \cdot \varepsilon_U$ for all $(i,U) \in Q_G$, then $f(\varepsilon_U)$ does belong to $(e_i * \varepsilon_U) \cdot (MG) \cdot (e_j * \varepsilon_V)$, which is contained in $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, for all arrows $f \colon (i,U) \to (j,V)$ of Q_G. Whence the isomorphism (1.1.1). □

The proof of Theorem 1.25 uses Morita equivalences in the graded sense. Two ($\mathbb{N}$-)graded finite dimensional $\Bbbk$-algebras Λ_1 and Λ_2 are Morita equivalent in the graded sense if $\Lambda_2 \simeq \mathrm{End}_{\Lambda_1}(P)^{\mathrm{op}}$ for some graded projective Λ_1-module P which is a generator of the category of graded Λ_1-modules, and hence of $\mathrm{mod}(\Lambda_1)$. Both

Λ_1 and Λ_2 are isomorphic as graded algebras to basic graded finite dimensional algebras; moreover Λ_1 and Λ_2 are Morita equivalent in the graded sense if and only if their basic versions are isomorphic as graded algebras.

Proof of Theorem 1.25. Denote by Φ the algebra homomorphism $\Bbbk Q_G \to \mathcal{I}\mathrm{ntw}^{\mathrm{op}}$ given in the statement of the theorem. Hence, $\Phi(\gamma) = f_\gamma$ for all paths γ in Q_G.

(1) Let γ and γ' be parallel paths in Q_G. It follows from the definition of $\varphi_{\gamma'}$, see Notation 1.23, and from (4.3.9) that

$$(f_\gamma|\varphi_{\gamma'}) = \begin{cases} 1 & \text{if } \gamma = \gamma', \\ 0 & \text{otherwise.} \end{cases} \tag{5.0.1}$$

Since Φ maps non parallel paths in Q_G into distinct components of $\mathcal{I}\mathrm{ntw}$ in the decomposition (1.4.3), then Φ is injective. In order to prove that Φ is an isomorphism, it is hence sufficient to prove that, for all integers n, the homogeneous components of degree n of the graded algebras $\Bbbk Q_G$ and $\mathcal{I}\mathrm{ntw}$ have the same dimensions. Note that $\Bbbk Q_G$ is graded by the length of the paths in Q_G and $\mathcal{I}\mathrm{ntw}$ is graded so that every intertwiner lying in a space of the shape $\mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(i_0, \dots, i_n; V))$ is homogeneous of degree n.

First, following [3], see Theorem 1.6, the algebras $\Bbbk Q_G$ and $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$ are isomorphic $\Bbbk$-algebras both Morita equivalent to $T_S(M) * G$. Actually, the proof given in [3] works in the graded setting.

Next, $\hat{e}$ may be considered as an idempotent of $T_S(M) * G$. In this sense,

$$\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e} = \tilde{e} \cdot \hat{e}(T_S(M) * G) \cdot \hat{e} \cdot \tilde{e}\,.$$

Following [3], see p. 1057, the algebra $\hat{e} \cdot (T_S(M) * G) \cdot \hat{e}$ is Morita equivalent to $T_{S_1}(\hat{e} \cdot (MG) \cdot \hat{e})$. Again, this is still true in the graded sense. Hence, $\tilde{e} \cdot (T_S(M) * G) \cdot \tilde{e}$, which is basic, is isomorphic to $\tilde{e} \cdot T_{S_1}(\hat{e} \cdot (MG) \cdot \hat{e}) \cdot \tilde{e}$ as a graded algebra.

Finally, by definition of $\tilde{e}$ and $\hat{e}$, the homogeneous component of degree n of $\tilde{e} \cdot T_{S_1}(\hat{e} \cdot (MG) \cdot \hat{e}) \cdot \tilde{e}$ is equal to

$$\bigoplus_{i_0, i_n \in [G \backslash Q_0],\, U_0 \in \mathrm{irr}(G_{i_0}),\, U_n \in \mathrm{irr}(G_{i_n})} (e_{i_0} * \varepsilon_{U_0}) \cdot (MG)^{\otimes_{S_1} n} \cdot (e_{i_n} * \varepsilon_{U_n})\,.$$

This is equal to

$$\bigoplus_{i_0, \dots, i_n \in [G \backslash Q_0],\, U_0 \in \mathrm{irr}(G_{i_0}),\, U_n \in \mathrm{irr}(G_{i_n})} \varepsilon_{U_0} \cdot M(i_0, \dots, i_n) \cdot \varepsilon_{U_n}\,.$$

This is isomorphic to

$$\bigoplus_{i_0, \dots, i_n, U_0, U_n} \mathrm{Hom}_{\Bbbk G_{i_0}}(U_0, \underbrace{M(i_0, \dots, i_n) \otimes_{G_{i_n}} V}_{\simeq M(i_0, \dots, i_n; V)})\,.$$

These considerations prove that the homogeneous components of degree n of $\Bbbk Q_G$ and $\mathcal{I}\mathrm{ntw}$ are isomorphic. Thus, Φ is an isomorphism.

(2) follows from (5.0.1) and from the fact that Φ is surjective. □

Proof of Theorem 1.27. First, Ξ is bijective. Indeed, on one hand, there is an equality of vector subspaces of $T_S(M) * G$,

$$\hat{e} \cdot (T_S(M) * G) \cdot \hat{e} = \bigoplus_{\underline{i}, \underline{y}} \left({}_{i_0}M_{y_1 \cdot i_1} \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} {}_{y_1 \cdots y_{n-1} \cdot i_{n-1}}M_{y_1 \cdots y_n \cdot i_n} \right) * y_1 \cdots y_n \Bbbk G\,,$$

where $\underline{i}$ runs through all sequences $i_0, \dots, i_n$ of $[G \backslash Q_0]$ and $\underline{y}$ runs through all sequences $y_1, \dots, y_n$ such that $y_t \in [G/G_{i_t}]$ for all $t \in \{1, \dots, n\}$. On the other

hand, for all vertices (i, U) and (j, V) of Q_G and for all $\Bbbk G_i - \Bbbk G_j$-bimodules N, the following mapping is bijective,

$$\begin{array}{ccc} \mathrm{Hom}_{\Bbbk G_i}(U, N \cdot \varepsilon_V) & \longrightarrow & \varepsilon_U \cdot N \cdot \varepsilon_V \\ f & \mapsto & f(\varepsilon_U)\,. \end{array}$$

Hence, Ξ is bijective and Ξ^{-1} is given by $\Xi^{-1}(f) = f(\varepsilon_U)$ for all $f \in \mathcal{I}\mathrm{ntw}$ with domain denoted by U.

Next, Ξ is a morphism of $\Bbbk$-algebras. Indeed, for all $f \in \mathrm{Hom}_{\Bbbk G_{i_0}}(U, M(\underline{i}; V))$ lying in $\mathcal{I}\mathrm{ntw}$, then $\Xi^{-1}(f) = f(\varepsilon_U)$. Therefore, with the setting of Definition 1.13, then $(f' \circledast f)(\varepsilon_U)$ is equal to the product $f(\varepsilon_V) \cdot f'(\varepsilon_U)$ in $T_S(M) * G$, see (1.4.1). Accordingly, Ξ^{-1}, and hence Ξ, is a $\Bbbk$-algebra homomorphism.

Finally, the diagram (1.6.4) is commutative. Indeed, all its arrows are $\Bbbk$-algebra homomorphisms. Hence, it suffices to prove the commutativity on the arrows of Q_G, which follows from the definitions of Ξ, (1.1.1), and (1.6.1). Note that (1.6.5) follows from (1.6.2). □

6. Acknowledgements

I thank Edson Ribeiro Alvares for helpful comments on previous versions of this article. I also thank the referee for several suggestions that improved the presentation of the article.

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