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DUALITY FOR DIFFERENTIAL OPERATORS OF LIE-RINEHART ALGEBRAS

THIERRY LAMBRE AND PATRICK LE MEUR

Abstract. Let \((S, L)\) be a Lie-Rinehart algebra over a commutative ring \(R\). This article proves that, if \(S\) is flat as an \(R\)-module and has Van den Bergh duality in dimension \(n\), and if \(L\) is finitely generated and projective with constant rank \(d\) as an \(S\)-module, then the enveloping algebra of \((S, L)\) has Van den Bergh duality in dimension \(n + d\). When, moreover, \(S\) is Calabi-Yau and the \(d\)-th exterior power of \(L\) is free over \(S\), the article proves that the enveloping algebra is skew Calabi-Yau, and it describes a Nakayama automorphism of it. These considerations are specialised to Poisson enveloping algebras. They are also illustrated on Poisson structures over two and three dimensional polynomial algebras and on Nambu-Poisson structures on certain two dimensional hypersurfaces.

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Introduction

Rinehart introduced in [Rin63] the concept of Lie-Rinehart algebra \((S, L)\) over a commutative ring \(R\) and defined its enveloping algebra \(U\). This generalises both constructions of universal enveloping algebras of \(R\)-Lie algebras and algebras of differential operators of commutative \(R\)-algebras. In [Hue99], Huebschmann investigated Poincaré duality on the (co)homology groups of \((S, L)\). This duality is defined by the existence of a right \(U\)-module \(C\), called the dualising module of \((S, L)\) such that, for all left \(U\)-modules \(M\) and \(k \in \mathbb{N}\),

\[
\text{Ext}_{U}^{k}(S, M) \cong \text{Tor}_{d-k}^{U}(C, M).
\]

Chemla proved in [Che99] that for Lie-Rinehart algebras arising from affine complex Lie algebroids, the algebra \(U\) has a rigid dualising complex, which she determined, and has Van den Bergh duality [vdB98]. Having Van den Bergh duality in dimension \(n\) for an \(R\)-algebra \(A\) means that

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A is homologically smooth, that is A lies in the perfect derived category per(A¹) of the algebra A¹ := A ⊗_R A^{op}; and
• Ext^i_{A¹}(A, A¹) is zero for • ≠ 0 and invertible as an A-bimodule if • = n.

When this occurs, there is a functorial isomorphism, for all A-bimodules M and integers i (see [vdB98]),
\[ \text{Ext}^i_{A¹}(A, M) \cong \text{Tor}^A_{n-i}(A, \text{Ext}^n_{A¹}(A, A¹) \otimes_A M) ; \]
and Ext^i_{A¹}(A, A¹) is called the inverse dualising bimodule of A. Two classes of algebras with Van den Bergh duality are of particular interest, namely
• Calabi-Yau algebras, for which Ext^i_{A¹}(A, A¹) is required to be isomorphic to A as an A-bimodule (see [Gin06]); and
• skew Calabi-Yau algebras, for which there exists an automorphism
\[ \nu \in \text{Aut}_{R\text{-alg}}(A) \]
such that Ext^i_{A¹}(A, A¹) ≃ A^{\nu} as A-bimodules (see [RRZ14]); here A^{\nu} denotes the A-bimodule obtained from A by twisting the action of A on the right by \( \nu \).

The relevance of these algebras comes from their role in the noncommutative geometry initiated in [ASS7] and in the investigation of Calabi-Yau categories, and also from the specificities of their Hochschild cohomology when \( R \) is a field. For instance, it is proved in [Gin06, Lam10] that the Gerstenhaber bracket of the Hochschild cohomology of Calabi-Yau algebras have a BV generator.

This article investigates when the enveloping algebra \( U \) of a general Lie-Rinehart algebra \((S, L)\) over a commutative ring \( R \) has Van den Bergh duality.

It considers Lie-Rinehart algebras \((S, L)\) such that \( S \) has Van den Bergh duality and is flat as an \( R \)-module, and \( L \) is finitely generated and projective with constant rank \( d \) as an \( S \)-module. Under these conditions, it is proved that \( U \) has Van den Bergh duality. Note that, when \( R \) is a perfect field, the former condition amounts to saying that \( S \) is a smooth affine \( R \)-algebra [Kra07]. Note also that, under the latter condition, it is proved in [Hue99, Theorem 2.10] that \((S, L)\) has duality in the sense of [0.1]. Under the additional assumption that \( S \) is Calabi-Yau and \( L \) is free as an \( S \)-module, it appears as a corollary that \( U \) is skew Calabi-Yau, and a Nakayama automorphism may be described explicitly. These considerations are specialised to the situation where the Lie-Rinehart algebra \((S, L)\) arises from a Poisson structure on \( S \). Also they are illustrated by detailed examples in the following cases:
• For Poisson brackets on polynomial algebras in two or three variables;
• For Nambu-Poisson structures on two dimensional hypersurfaces of the shape \( 1 + T(x, y, z) = 0 \), where \( T \) is a weight homogeneous polynomial.

Throughout the article, \( R \) denotes a commutative ring, \((S, L)\) denotes a Lie-Rinehart algebra over \( R \) and \( U \) denotes its enveloping algebra. Given an \( R \)-Lie algebra \( \mathfrak{g} \), its universal enveloping algebra is denoted by \( U_R(\mathfrak{g}) \). For an \( R \)-algebra \( A \), the category of left \( A \)-modules is denoted by Mod(\( A \)) and Mod(\( A^{op} \)) is identified with the category of right \( A \)-modules. For simplicity, the piece of notation \( \otimes \) is used for \( \otimes_R \). All complexes have differential of degree +1.

### 1. Main results

A Lie-Rinehart algebra over a commutative ring \( R \) is a pair \((S, L)\) where \( S \) is a commutative \( R \)-algebra and \( L \) is a Lie \( R \)-algebra which is also a left \( S \)-module, endowed with a homomorphism of \( R \)-Lie algebras
\[
\begin{align*}
L & \to \text{Der}_R(S) \\
\alpha & \mapsto \partial_\alpha := \alpha(-)
\end{align*}
\]
such that, for all \( \alpha, \beta \in L \) and \( s \in S \),
\[
[a, s\beta] = s[a, \beta] + a(s)\beta.
\]

Following [Hue99], the enveloping algebra \( U \) of \((S, L)\) is identified with the algebra
\[
(S \rtimes L)/I,
\]
where \( S \rtimes L \) is the smash-product algebra of \( S \) by the action of \( L \) on \( S \) by derivations and \( I \) is the two-sided ideal of \( S \rtimes L \) generated by
\[
\{s \otimes \alpha - 1 \otimes s\alpha \mid s \in S, \alpha \in L\}
\]
(see Lemma 3.0.1); it is proved in [Hue99] that this set generates \( I \) as a right ideal.

As mentioned in the introduction, when \( L \) is a finitely generated \( S \)-module with constant rank \( d \), the Lie-Rinehart algebra \((S, L)\) has duality in the sense of \((0.1)\) with \( C = \Lambda^d_S L^\vee \). Here \(-\vee\) is the duality \( \text{Hom}_S(-, S) \) and \( \Lambda^d_S L^\vee \) is considered as a right \( U \)-module using the Lie derivative \( \lambda_\alpha \), for \( \alpha \in L \) (see [Hue99] Section 2),
\[
\lambda_\alpha: \Lambda^*_S L^\vee \to \Lambda^*_S L^\vee;
\]
this is the derivation of \( \Lambda^*_S L^\vee \) such that, for all \( s \in S, \varphi \in L^\vee \) and \( \beta \in L \),
\[
\lambda_\alpha(s) = \alpha(s) \quad \text{and} \quad \lambda_\alpha(\varphi)(\beta) = \alpha(\varphi(\beta)) - \varphi(\alpha, \beta).
\]
The right \( U \)-module structure of \( \Lambda^*_S L^\vee \) is such that, for all \( \varphi \in \Lambda^*_S L^\vee \) and \( \alpha \in L \),
\[
\varphi \cdot \alpha = -\lambda_\alpha(\varphi) .
\]

The first main result of the article gives sufficient conditions for \( U \) to have Van den Bergh duality. It also describes the inverse dualising bimodule. Here are some explanations on this description. On one hand, \( R \)-linear derivations \( \partial \in \text{Der}_R(S) \) act on \( \text{Ext}^n_S(S, S^n) \), \( n \in \mathbb{N} \), by Lie derivatives (see Section 3.3),
\[
\mathcal{L}_\partial: \text{Ext}^n_S(S, S^n) \to \text{Ext}^n_S(S, S^n).
\]
Combining with the action of \( L \) on \( S \) yields an action \( \alpha \otimes e \mapsto \alpha \cdot e \) of \( L \) on \( \text{Ext}^n_S(S, S^n) \) such that, for all \( \alpha \in L \) and \( e \in \text{Ext}^n_S(S, S^n) \),
\[
\alpha \cdot e = \mathcal{L}_{\lambda_\alpha}(e),
\]
Although this is not a \( U \)-module structure on \( \text{Ext}^n_S(S, S^n) \), it defines a left \( U \)-module structure on \( \Lambda^d_S L^\vee \otimes_S \text{Ext}^n_S(S, S^n), \) \( d \in \mathbb{N} \), such that, for all \( \alpha \in L, \varphi \in \Lambda^*_S L^\vee \) and \( e \in \text{Ext}^n_S(S, S^n) \),
\[
\alpha \cdot (\varphi \otimes e) = -\varphi \cdot \alpha \otimes e + \varphi \otimes \alpha \cdot e .
\]

On the other hand, consider the functor
\[
F: \text{Mod}(U) \to \text{Mod}(U^\circ)
\]
(see Section 3.3) such that, if \( N \in \text{Mod}(U) \), then \( F(N) \) equals \( U \otimes_S N \) in \( \text{Mod}(U) \) and has a right \( U \)-module structure defined by the following formula, for all \( \alpha \in L, u \in U \) and \( n \in N \),
\[
(u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n .
\]
This functor takes left \( U \)-modules which are invertible as \( S \)-modules to invertible \( U \)-bimodules (see Section 3.6). The main result of this article is the following.

**Theorem 1.** Let \( R \) be a commutative ring. Let \((S, L)\) be a Lie-Rinehart algebra over \( R \). Denote by \( U \) the enveloping algebra of \((S, L)\). Assume that
- \( S \) is flat as an \( R \)-module,
- \( S \) has Van den Bergh duality in dimension \( n \),
- \( L \) is finitely generated and projective with constant rank \( d \) as an \( S \)-module.
Then, $U$ has Van den Bergh duality in dimension $n+d$ and there is an isomorphism of $U$-bimodules

$$\text{Ext}^{n+d}_U(U, U^e) \simeq F(\Lambda^d_S L^\vee \otimes_S \text{Ext}^n_S(S, S^e)).$$

Note that when $R$ is Noetherian and $S$ is finitely generated as an $R$-algebra and projective as an $R$-module, then there is an isomorphism of $S$-(bi)modules

$$\text{Ext}^n_S(S, S^e) \simeq \Lambda^d_S \text{Der}_R(S);$$

this isomorphism is compatible with the actions by Lie derivatives (see Section 4.5). The above theorem was proved in [Che99, Theorem 4.4.1] when $R = \mathbb{C}$ and $S$ is finitely generated as a $\mathbb{C}$-algebra.

The preceding theorem specialises to the situation where the involved invertible $S$-modules are free. On one hand, when $(\Lambda^d_S L)^\vee$ is free as an $S$-module with free generator $\varphi_L$, there is an associated trace mapping

$$\lambda_L : L \to S$$

such that, for all $\alpha \in L$,

$$\varphi_L \cdot \alpha = \lambda_L(\alpha) \cdot \varphi_L,$$

where the action on the left-hand side is given by (1.2) and that on the right-hand side is just given by the $S$-module structure. On the other hand, when $S$ is Calabi-Yau in dimension $n$, each generator of the free of rank one $S$-module $\text{Ext}^n_S(S, S^e)$ determines a volume form $\omega_S \in \Lambda^n_S \Omega_S/R$, and the divergence

$$\text{div} : \text{Der}_R(S) \to S$$

associated with $\omega_S$ is defined by the following equality, for all $\partial \in \text{Der}_R(S)$

$$L_\partial(\omega_S) = \text{div}(\partial) \omega_S.$$

(see Section 4.5 for details). The second main result of the article then reads as follows.

**Theorem 2.** Let $R$ be a commutative ring. Let $(S, L)$ be a Lie-Rinehart algebra over $R$. Denote by $U$ the enveloping algebra of $(S, L)$. Assume that

- $S$ is flat as an $R$-module,
- $S$ is Calabi-Yau in dimension $n$,
- $L$ is finitely generated and projective with constant rank $d$ and $\Lambda^d_S L$ is free as $S$-modules.

Then, $U$ is skew Calabi-Yau with a Nakayama automorphism $\nu \in \text{Aut}_R(U)$ such that, for all $s \in S$ and $\alpha \in L$,

$$\nu(s) = s$$

$$\nu(\alpha) = \alpha + \lambda_L(\alpha) + \text{div}(\partial_\alpha),$$

where $\lambda_L$ is any trace mapping on $\Lambda^d_S L^\vee$ and $\text{div}$ is any divergence.

Among all Lie-Rinehart algebras, those arising from Poisson structures on $S$ play a special role because of the connection to Poisson (co)homology. Recall that any $R$-bilinear Poisson bracket $\{-,-\}$ on $S$ defines a Lie-Rinehart algebra structure on $(S, L) = (S, \Omega_{S/R})$ such that, for all $s, t \in S$,

- $\partial_{ds} = \{s, -\}$;
- $[ds, dt] = d\{s, t\}.$

In this case, the formulations of Theorems 1 and 2 simplify because, when $\Omega_{S/R}$ is projective with constant rank $n$ as an $S$-module, the right $U$-module structure of $\Lambda^n_S \Omega_{S/R}$ (see (1.2)) is given by classical Lie derivatives, that is, for all $s \in S$,

$$(1.3) \quad \lambda_{ds}(\varphi) = L_{(s, -)}(\varphi).$$

More precisely, these theorems specialise as follows.
Corollary 1. Let $R$ be a Noetherian ring. Let $(S, \{-,-\})$ be a finitely generated Poisson algebra over $R$. Denote by $U$ the enveloping algebra of the associated Lie Rinehart algebra $(S, \Omega_{S/R})$. Assume that

- $S$ is projective in $\text{Mod}(R)$;
- $S \in \text{per}(S^e)$;
- $\Omega_{S/R}$, which is then projective in $\text{Mod}(S)$, has constant rank $n$.

Then, $U$ has Van den Bergh duality in dimension $2n$ and there is an isomorphism of $U$-bimodules

$$\text{Ext}^n_{S^e}(U, U^e) \simeq U \otimes_S \Lambda^n_S \text{Der}_R(S) \otimes_S \Lambda^n_S \text{Der}_R(S),$$

where the right-hand side term is a left $U$-module in a natural way and a right $U$-module such that, for all $u \in U$, $\varphi, \varphi' \in \Lambda^n_S \text{Der}_R(S)$ and $s \in S$,

$$(u \otimes \varphi \otimes \varphi') \cdot ds = u ds \otimes \varphi \otimes \varphi' - u \otimes (\mathcal{L}_{\{s,-\}}(\varphi) \otimes \varphi' + \varphi \otimes \mathcal{L}_{\{s,-\}}(\varphi')).$$

In particular, if $S$ has a volume form, then $U$ is skew Calabi-Yau with a Nakayama automorphism $\nu: U \to U$ such that, for all $s \in S$,

$$\begin{cases}
\nu(s) = s \\
\nu(ds) = ds + 2 \text{div}([s, -]),
\end{cases}$$

where $\text{div}$ is the divergence of the chosen volume form.

For the case where $R = \mathbb{C}$ and $S$ is finitely generated as a $\mathbb{C}$-algebra, the above corollary is announced in [LWZ17] Theorem 0.7, Corollary 0.8 using the main results of [Che99].

This article is structured as follows. Section 2 presents useful information on the case where $S$ has Van den Bergh duality. Section 3 is devoted to technical lemmas on $U$-(bi)modules. In particular, it presents the above mentioned functor $F$ and its right adjoint $G$, which play an essential role in the proof of the main results. Section 4 introduces the action of $L$ on $\text{Ext}^*_S(S^e)$ by Lie derivatives. This structure is used in Section 5 in order to describe $\text{Ext}^*_S(U, U^e)$ and prove Theorem 1 and Corollary 1. Finally, Section 6 applies this corollary to a class of examples of Nambu-Poisson surfaces.

2. Poincaré duality for $S$

As proved in vdH98 when $R$ is field, if $S$ has Van den Bergh duality in dimension $n$, then there is a functorial isomorphism, for all $S$-bimodules $N$,

$$\text{Ext}^n_{S^e}(S, N) \simeq \text{Tor}^n_{S^e}(S, \text{Ext}^n_S(S, S^e) \otimes_S N).$$

It is direct to check that this is still the case without assuming that $R$ is a field. In view of the proof of the main results of the article, 2.1 relates the above mentioned isomorphism to the fundamental class of $S$, following [Lam10], and 2.2 relates Van den Bergh duality to the regularity of commutative algebras, following [Kra07].

2.1. Fundamental class and contraction. Consider a projective resolution $P^\bullet$ in $\text{Mod}(S^e)$,

$$\cdots \to P^{-2} \to P^{-1} \to P^0 \to S,$$

and let $p^0 \in P^0$ be such that $\epsilon(p^0) = 1_S$. For all $M, N \in \text{Mod}(S^e)$ and $n \in \mathbb{N}$, define the contraction

$$\text{Tor}^n_{S^e}(S, M) \times \text{Ext}^n_{S^e}(S, N) \to \text{Tor}^n_{S^e}(S, M \otimes_S N),$$

$$((\omega, \epsilon)) \mapsto \iota_\epsilon(\omega)$$

as the mapping induced by the following one

$$M \otimes_S P^{-n} \to \text{Hom}_R(\text{Hom}_{S^e}(P^{-n}, N), (M \otimes_S N) \otimes_S P^0)$$

$$x \otimes p \mapsto (\varphi \mapsto (x \otimes \varphi(p)) \otimes p^0).$$
This makes sense because $P^\bullet$ is concentrated in nonpositive degrees. The construction depends neither on the choice of $p^0$ nor on that of $P^\bullet$.

Following the proof of [Lam10, Proposition 3.3], when $S \in \per(S^\circ)$ and $n$ is taken equal to $\pd_{S^\circ}(S)$, the contraction induces an isomorphism for all $N \in \Mod(S^\circ)$,
\[
\Tor_S^n(S, \Ext_{S^\circ}^m(S, S^\circ)) \to \Hom_{S^\circ}(\Ext_{S^\circ}^m(S, N), \Tor_S^n(S, \Ext_{S^\circ}^m(S, S^\circ) \otimes_S N))
\]
\[\omega \mapsto \iota_\omega(n)\]
In the particular case $N = S^c$, the fundamental class of $S$ is the element $c_S \in \Tor_S^n(S, \Ext_{S^\circ}^m(S, S^\circ))$ such that
\[
(\iota_\omega(c_S))|_{\Ext_{S^\circ}^m(S, S^c)} = \Id_{\Ext_{S^\circ}^m(S, S^c)}.
\]

Following the arguments in the proof of [Lam10, Théorème 4.2], when $S$ has Van den Bergh duality in dimension $n$, which gives that $n = \pd_{S^\circ}(S)$, the contraction with $c_S$ induces an isomorphism, for all $N \in \Mod(S^\circ)$,
\begin{equation}
(2.1)
\iota_\omega(c_S) : \Ext_{S^\circ}^n(S, N) \to \Tor_S^n(S, \Ext_{S^\circ}^m(S, S^\circ) \otimes_S N).
\end{equation}

When $S$ is projective in $\Mod(R)$, the Hochschild complex $S^\otimes_{R^\oplus} + 2$ is a resolution of $S$ in $\Mod(S^\circ)$ and the contraction
\[
\Tor_S^n(S, M) \times \Ext_{S^\circ}^m(S, N) \to \Tor_{n-m}(S, M \otimes_S N)
\]
may be defined for all $M, N \in \Mod(S^\circ)$ and $m, n \in \mathbb{N}$, as the mapping induced at the level of Hochschild (co)chains by
\[
M \otimes S^\otimes_{R^\oplus} \times \Hom_R(S^\otimes_{R^\oplus}^m, N) \to (M \otimes S^\circ) \otimes (S^\otimes_{R^\oplus}^{m-n})
\]
\[
((x|s_1| \cdots|s_n), \psi) \mapsto (x \otimes \psi(s_1| \cdots|s_m)s_{m+1}| \cdots|s_n).
\]
When, in addition, $S$ has Van den Bergh duality in dimension $n$, then [Lam10, Théorème 4.2] asserts that the following mapping given by contraction with $c_S$ is an isomorphism, for all $N \in \Mod(S^\circ)$ and $m \in \mathbb{N}$,
\[
\iota_\omega(c_S) : \Ext_{S^\circ}^m(S, N) \to \Tor_S^n(S, \Ext_{S^\circ}^m(S, S^\circ) \otimes_S N).
\]

2.2. Relationship to regularity. The main results of this article assume that $S$ has Van den Bergh duality. For commutative algebras, this property is related to smoothness and regularity. The relationship is detailed in [Krâ07] for the case where $R$ is a perfect field and is summarised below in the present setting.

**Proposition 2.2.1 ([Krâ07]).** Let $R$ be a Noetherian commutative ring. Let $S$ be a finitely generated commutative $R$-algebra and projective as an $R$-module. Let $n \in \mathbb{N}$. The following properties are equivalent.

(i) $S$ has Van den Bergh duality in dimension $n$.

(ii) $\text{gl.dim}(S^\circ) < \infty$ and $\Omega_{S/R}$, which is then projective in $\Mod(S)$, has constant rank $n$.

When these properties are true, $\text{gl.dim}(S) < \infty$ and $\Ext_{S^\circ}^m(S, S^\circ) \simeq \Lambda_S^m \Der_R(S)$ as $S$-modules.

**Proof.** See [Krâ07] for full details. Since $S$ is projective over $R$, then $\pd_{(S^\circ)^\per}(S^\circ) \leq 2 \pd_{(S^\circ)^\per}(S)$ ([CE56, Chap. IX, Proposition 7.4]); besides, using the Hochschild resolution of $S$ in $\Mod(S^\circ)$ yields that
\[
\text{gl.dim}(S) \leq \pd_{(S^\circ)^\per}(S) \leq \text{gl.dim}(S^\circ);
\]
thus
\[
S \in \per(S^\circ) \quad \Leftrightarrow \quad \text{gl.dim}(S^\circ) < \infty \Rightarrow \text{gl.dim}(S) < \infty.
\]

(2.2)
Note also that, following [HKR62, Theorem 3.1],
\[ \text{gl.dim}(S^e) < \infty \Rightarrow \Omega_{S/R} \text{ is projective in Mod}(S). \] (2.3)

Denote by \( \mu \) the multiplication mapping \( S \otimes S \to S \). Assume \( \text{gl.dim}(S^e) < \infty \), let \( \mathfrak{p} \in \text{Spec}(S) \subseteq \text{Spec}(S^e) \) and denote by \( d \) the rank of \((\Omega_{S/R})_{\mathfrak{p}}\). Since \( \Omega_{S/R} \cong \text{Ker}(\mu)/\text{Ker}(\mu)^2 \) as modules over \( S \cong S^e/\text{Ker}(\mu) \) and \( \text{gl.dim}(S^e) < \infty \), the \((S^e)_{\mathfrak{p}}\)-module \( \text{Ker}(\mu)_{\mathfrak{p}} \) is generated by a regular sequence having \( d \) elements. There results a Koszul resolution of \( S_{\mathfrak{p}} \) in \( \text{Mod}((S^e)_{\mathfrak{p}}) \). Using this resolution and the isomorphism \( \text{Ext}^n_{S^e}(S,S^e)_{\mathfrak{p}} \cong \text{Ext}^n_{(S^e)_{\mathfrak{p}}}(S_{\mathfrak{p}},(S^e)_{\mathfrak{p}}) \) in \( \text{Mod}((S^e)_{\mathfrak{p}}) \) yields isomorphisms of \((S^e)_{\mathfrak{p}}\)-modules
\[ \text{Ext}^n_{S^e}(S,S^e)_{\mathfrak{p}} \cong \begin{cases} 0 & \text{if } \bullet \neq d \\ S_{\mathfrak{p}} & \text{if } \bullet = d. \end{cases} \] (2.4)

Now assume (i). Then, \( \text{gl.dim}(S^e) < \infty \) (see (2.2)), \( \Omega_{S/R} \) is projective (see (2.3)) and has constant rank \( n \) (see (2.4)). Conversely, assume that \( \text{gl.dim}(S^e) < \infty \) and \( \Omega_{S/R} \) has constant rank \( n \). Then, \( S \in \text{per}(S^e) \) (see (2.2)) and the \( S \)-module (equivalently, the symmetric \( S \)-bimodule) \( \text{Ext}^n_{S^e}(S,S^e) \) is zero if \( \bullet \neq n \) and is invertible if \( \bullet = n \) (see (2.4)). Thus,
\[(i) \iff (ii) .\]

Finally, assume that both (i) and (ii) are true. Then, \( \text{gl.dim}(S) < \infty \) (see (2.2)). Moreover, Van den Bergh duality [vdB98, Theorem 1] does apply here and provides an isomorphism of \( S \)-modules
\[ \text{Ext}^i_{S^e}(S,\text{Ext}^i_{S^e}(S,S^e)^{-1}) \cong \text{Tor}^S_n(S,S) , \]
whereas [HKR62, Theorem 3.1] yields an isomorphism of \( S \)-modules
\[ \text{Tor}^S_n(S,S) \cong \Lambda^S_0 \Omega_{S/R} . \]
Thus, \( \text{Ext}^n_{S^e}(S,S^e) \cong \Lambda^S_0 \text{Der}_R(S) \) in \( \text{Mod}(S) \). \( \square \)

3. Material on \( U \)-(bi)modules

The purpose of this section is to introduce an adjoint pair of functors \((F,G)\) between \( \text{Mod}(U) \) and \( \text{Mod}(U^*) \). In the proof of Theorem [4] the \( U \)-bimodule \( \text{Ext}^*_{S^e}(U,U^e) \) is described as the image under \( F \) of a certain left \( U \)-module which is invertible as an \( S \)-module. This section develops the needed properties of \( F \). Hence, [3.1] recalls the basic constructions of \( U \)-modules; [3.2] and [3.3] introduce the functors \( G \) and \( F \), respectively; [3.4] proves that \((F,G)\) is adjoint; and [3.5] introduces and collects basic properties of compatible left \( S \times L \)-modules, these are applied in Section [4] to the action of \( L \) on \( \text{Ext}^*_{S^e}(S,S^e) \) by Lie derivatives; and [3.6] proves that the functor \( F \) transforms left \( U \)-modules that are invertible as \( S \)-modules into invertible \( U \)-bimodules. These results are based on the description of \( U \) as a quotient of the smash-product \( S \times L \) given in the following lemma. This description is established in [LOV17, Proposition 2.10] in the case of Lie-Rinehart algebras arising from Poisson algebras.

**Lemma 3.0.1.** The algebra \( S \rtimes L \) has the following properties.

1. The identity mappings \( \text{Id}_S : S \to S \) and \( \text{Id}_L : L \to L \) induce an isomorphism of \( R \)-algebras
\[ (S \rtimes L)/I \to U , \] (3.1)
where \( I \) is the two-sided ideal of the smash-product algebra \( S \times L \) generated by
\[ \{ s \otimes \alpha - 1 \otimes s \alpha \mid s \in S, \alpha \in L \} . \]
(2) If $L$ is projective as a left $S$-module, then $U$ is projective both as a left and as a right $S$-module.

Proof. (1) Recall (see [Rin63]) that

$$
[s + \alpha, t + \beta] = \alpha(t) - \beta(s) + [\alpha, \beta];
$$

then, $U$ is the factor $R$-algebra of the subalgebra of the universal enveloping algebra $U_R(S \otimes L)$ generated by the image of $S \otimes L$ by the two-sided ideal generated by the classes in $U_R(S \otimes L)$ of the following elements, for $s, t \in S$ and $\alpha \in L$,

$$
s \otimes t - st, \ s \otimes \alpha - s\alpha.
$$

Recall also that $S \times L$ is the $R$-algebra with underlying $R$-module

$$
S \otimes U_R(L),
$$

such that the images of $S \otimes 1$ and $1 \otimes U_R(L)$ are subalgebras, and the following hold, for all $s, t \in S$ and $\alpha, \beta \in S$,

$$
\begin{cases}
(s \otimes 1) \cdot (1 \otimes \alpha) = s \otimes \alpha \\
(1 \otimes \alpha) \cdot (s \otimes 1) = \alpha(s) \otimes 1 + s \otimes \alpha.
\end{cases}
$$

Therefore, the natural mappings $S \to U$ and $L \to U$ induce an $R$-algebra homomorphism from $S \times L$ to $U$. This homomorphism vanishes on $I$ whence the $R$-algebra homomorphism (3.1).

Besides, the universal property of $U$ stated in [Hae99] Section 2, p. 110 yields an $R$-algebra homomorphism

$$
U \to (S \times L)/I
$$

induced by the natural mappings $S \to (S \times L)/I$ and $L \to (S \times L)/I$. In view of the behaviour of (3.1) and (3.2) on the respective images of $S \cup L$, these algebra homomorphisms are inverse to each other.

(2) It is proved in [Rin63] Lemma 4.1 that $U$ is projective as a left $S$-module. Consider the increasing filtration of $U$ by the left $S$-submodules

$$
0 \subseteq F_0 U \subseteq F_1 U \subseteq \cdots
$$

where $F_p U$ is the image of $\otimes_{i=0}^p S \otimes L^{\otimes i}$ in $U$, for all $p \in \mathbb{N}$. In view of the equality

$$
as = sa + \alpha(s),
$$
in $U$ for all $s \in S$ and $\alpha \in L$, the left $S$-module $F_p U$ is also a right $S$-submodule of $U$, and $F_p U/F_{p-1} U$ is a symmetric $S$-bimodule for all $p \in \mathbb{N}$. Therefore, the considerations used in the proof of [Rin63] Lemma 4.1 may be adapted in order to prove that $U$ is projective as a right $S$-module. \hfill \Box

3.1. **Basic constructions of $U$-modules.** Left $S \times L$-modules are identified with $R$-modules $N$ endowed with a left $S$-module structure, and a left $L$-module structure such that, for all $n \in N$, $\alpha \in L$ and $s \in S$,

$$
\alpha \cdot (s \cdot n) = \alpha(s) \cdot n + s \cdot (\alpha \cdot n).
$$

Left $U$-modules are identified with left $S \times L$-modules $N$ such that, for all $n \in N$, $\alpha \in L$ and $s \in S$,

$$
s \cdot (\alpha \cdot n) = (s\alpha) \cdot n.
$$

Recall that the action of $L$ endows $S$ with a left $U$-module structure.

Right $S \times L$-modules are identified with the $R$-modules $M$ endowed with a right $S$-module structure and a right $L$-module structure such that, for all $m \in M$, $\alpha \in L$ and $s \in S$,

$$
(\alpha \cdot m) \cdot s = m \cdot \alpha(s) + (m \cdot s) \cdot \alpha.
$$
Right $U$-modules are identified with right $S \ltimes L$-modules $M$ such that, for all $m \in M$, $s \in S$ and $\alpha \in L$,

$$(m \cdot s) \cdot \alpha = m \cdot (s \alpha).$$

The following constructions are classical. The corresponding $U$-module structures are introduced in [Hue99] Section 2.

Let $M, M'$ be right $S \ltimes L$-modules. Let $N, N'$ be left $S \ltimes L$-module. Then,

- $N$ is a right $S \ltimes L$-module for the right $L$-module structure such that, for all $n \in N$, $s \in S$ and $\alpha \in L$,

$$(3.3) \quad n \cdot s = s \cdot n \text{ and } n \cdot \alpha = -\alpha \cdot n,$$

- $\text{Hom}_S(N, N')$ is a left $S \ltimes L$-module for the left $L$-module structure such that, for all $f \in \text{Hom}_S(N, N')$, $n \in N$ and $\alpha \in L$,

$$(3.4) \quad (\alpha \cdot f)(n) = \alpha \cdot f(n) - f(\alpha \cdot n),$$

moreover, this is a left $U$-module structure if $N$ and $N'$ are left $U$-modules,

- $\text{Hom}_S(M, M')$ is a left $S \ltimes L$-module for the left $L$-module structure such that, for all $f \in \text{Hom}_S(M, M')$, $m \in M$ and $\alpha \in L$,

$$(3.5) \quad (\alpha \cdot f)(m) = -f(m) \cdot \alpha + f(m \cdot \alpha),$$

- $\text{Hom}_S(N, S)$ is a right $S \ltimes L$-module for the right $L$-module structure such that, for all $f \in \text{Hom}_S(N, S)$, $n \in N$ and $\alpha \in L$,

$$(3.6) \quad (f \cdot \alpha)(n) = -\alpha f(n) + f(\alpha \cdot n),$$

- $N \otimes_S N'$ is a left $S \ltimes L$-module for the left $L$-module structure such that, for all $n \in N$, $n' \in N$ and $\alpha \in L$,

$$(3.7) \quad \alpha \cdot (n \otimes n') = \alpha \cdot n \otimes n' + n \otimes \alpha \cdot n',$$

moreover, this is a left $U$-module structure if $N$ and $N'$ are left $U$-modules,

- $M \otimes_S N$ is a left $S \ltimes L$-module for the left $L$-module structure such that, for all $m \in M$, $n \in N$ and $\alpha \in L$,

$$(3.8) \quad \alpha \cdot (m \otimes n) = -m \cdot \alpha \otimes n + m \otimes \alpha \cdot n.$$
Proof. (1) Given all \( s \in S \) and \( \alpha \in L \), denote
\[
s \otimes 1 - 1 \otimes s \in U^e \quad \text{and} \quad \alpha \otimes 1 - 1 \otimes \alpha \in U^e
\]
by \( ds \) and \( d\alpha \), respectively; in particular
\[
d\alpha \cdot ds = ds \cdot d\alpha + d(\alpha(s)),
\]
and, for all \( m \in M^S \),
\[
ds \cdot (d\alpha \cdot m) = d\alpha \cdot (ds \cdot m) - d(\alpha(s)) \cdot m = 0,
\]
which proves that \( d\alpha \cdot m \in M^S \). Therefore, (3.10) defines a left \( L \)-module structure on \( M^S \). Now, for all \( m \in M^S \), \( s \in S \) and \( \alpha \in L \),
\[
\alpha \cdot (s \otimes 1) \cdot m = d\alpha \cdot (s \otimes 1) \cdot m = (\alpha(s) \otimes 1 + sa \otimes 1 - s \otimes \alpha) \cdot m = (s \otimes 1)(\alpha \otimes 1 - 1 \otimes \alpha) \cdot m = (s \otimes 1)\alpha \cdot m + (s \otimes 1)\alpha \cdot m
\]
\[
(s \otimes 1) \cdot (\alpha \cdot m) = (s \otimes 1) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot m = (s \otimes 1) \cdot m - (s \otimes 1)(\alpha \otimes 1 - 1 \otimes \alpha) \cdot m = (s \otimes 1)\alpha \cdot m - (s \otimes 1)(1 \otimes s) \cdot m = (s \otimes 1 - 1 \otimes sa) \cdot m = (sa) \cdot m.
\]
Hence, this left \( L \)-module structure on \( M^S \) is a left \( U \)-module structure.

(2) By definition, \( \text{Hom}_{S^L}(S, M) \) is endowed with the left \( U \)-module structure such that (3.9) is an isomorphism in \( \text{Mod}(U) \). Let \( \varphi \in \text{Hom}_{S^L}(S, M) \), \( \alpha \in L \) and \( s \in S \). Then,
\[
(\alpha \cdot \varphi)(s) = (1 \otimes s) \cdot (\alpha \cdot \varphi(1)) = ((1 \otimes s)(\alpha \otimes 1 - 1 \otimes \alpha)) \cdot \varphi(1) = \varphi(1) \cdot (1 \otimes s) \cdot \alpha \cdot \varphi(1) = \alpha \cdot (1 \otimes s) \cdot \varphi(1) - (1 \otimes \alpha(s)) \cdot \varphi(1) = \alpha \cdot \varphi(1) - \varphi(\alpha(s)).
\]

Thus, the assignment \( M \mapsto M^S \) defines a functor
\[
G: \quad \text{Mod}(U^e) \to \text{Mod}(U)
\]
(3.11)
\[
M \mapsto M^S.
\]

3.3. The functor \( F = U \otimes_S - : \text{Mod}(U) \to \text{Mod}(U^e) \). Let \( N \in \text{Mod}(U) \). In view of [Hue99 (2.4)], \( UU \otimes_S N \) is a right \( U \)-module such that, for all \( u \in U \), \( n \in N \), \( s \in S \) and \( \alpha \in L \),
\[
(u \otimes n) \cdot s = u \otimes sn = us \otimes n \quad \text{and} \quad (u \otimes n) \cdot \alpha = u\alpha \otimes n - u \otimes \alpha \cdot n.
\]
Besides, \( U \otimes_S N \) is a left \( U \)-module such that, for all \( u, u' \in U \) and \( n \in N \),
\[
u' \cdot (u \otimes n) = u'u \otimes n.
\]
Therefore, \( U \otimes_S N \) is a \( U \)-bimodule, and hence a left \( U^e \)-module. These considerations define a functor
\[
F: \quad \text{Mod}(U) \to \text{Mod}(U^e)
\]
(3.12)
\[
N \mapsto U \otimes_S N.
3.4. The adjunction between $F$ and $G$.

**Proposition 3.4.1.** The functors $F = U \otimes_S -$ and $G = \text{Hom}_S(S, -)$ introduced in \ref{dual} and \ref{dual2} form an adjoint pair

\[
    \begin{array}{ccc}
        \text{Mod}U & \overset{\text{F}}{\longrightarrow} & \text{Mod}U^c \\
        & \downarrow{G} & \\
        & \text{Mod}U^c & \\
    \end{array}
\]

In particular, there is a functorial isomorphism, for all $M \in \text{Mod}(U^c)$ and $N \in \text{Mod}(U)$,

\[
    \text{Hom}_U(N, G(M)) \simto \text{Hom}_{U^c}(F(N), M) .
\]

**Proof.** Given $f \in \text{Hom}_U(N, G(M))$, denote by $\Phi(f)$ the well-defined mapping

\[
    U \otimes_S N \to M \\
    u \otimes n \mapsto (u \otimes 1) \cdot f(n);
\]

consider $F(N) (= U \otimes_S N)$ as a $U$-bimodule. Then, for all $u, u' \in U$, $n \in N$, $s \in S$ and $\alpha \in L$,

\[
    \Phi(f)(u' \cdot (u \otimes n)) = \Phi(f)(u'u \otimes n) = (u'u \otimes 1) \cdot f(n) = (u' \otimes 1) \cdot \Phi(f)(u \otimes n),
\]

\[
    \Phi(f)((u \otimes n) \cdot s) = \Phi(f)(u \otimes s \cdot n) = (u \otimes 1) \cdot f(s \cdot n) = (1 \otimes s) \cdot \Phi(f)(u \otimes n) = (\Phi(f)(u \otimes n)) \cdot s
\]

\[
    \Phi(f)((u \otimes n) \cdot \alpha) = \Phi(f)(u\alpha \otimes n - u \otimes \alpha \cdot n) = (u \alpha \otimes 1) \cdot f(n) - (u \otimes 1) \cdot f(\alpha \cdot n) = (u \otimes \alpha) \cdot f(\alpha) = (\Phi(f)(u \otimes n)) \cdot \alpha;
\]

in other words,

\[
    \Phi(f) \in \text{Hom}_{U^c}(F(N), M).
\]

Given $g \in \text{Hom}_{U^c}(F(N), M)$, then, for all $n \in N$ and $s \in S$,

\[
    (s \otimes 1 - 1 \otimes s) \cdot g(1 \otimes n) = g(s \otimes_S n - 1 \otimes_S s \cdot n) = 0;
\]

hence, denote by $\Psi(g)$ the well-defined mapping

\[
    N \to MS \\
    n \mapsto g(1 \otimes n).
\]

Therefore, for all $n \in N$, $s \in S$ and $\alpha \in L$,

\[
    \Psi(g)(s \cdot n) = g(1 \otimes s \cdot n) = g(s \otimes n) = g((s \otimes 1) \cdot (1 \otimes n)) = (s \otimes 1) \cdot g(1 \otimes n) = (s \otimes 1) \cdot \Psi(g)(n)
\]

\[
    \Psi(g)(\alpha \cdot n) = g(1 \otimes \alpha \cdot n) = g(\alpha \otimes n - (1 \otimes \alpha) \cdot (1 \otimes n)) = (\alpha \otimes 1) \cdot g(1 \otimes n) - (1 \otimes \alpha) \cdot g(1 \otimes n) = \alpha \cdot \Psi(g)(n);
\]

in other words,

\[
    \Psi(g) \in \text{Hom}_U(N, G(M)).
\]

By construction, $\Psi$ and $\Phi$ are inverse to each other. \qed
3.5. **Compatible left $S \times L$-modules.** As explained in Section 3.1, the main results of this article are expressed in terms of the action of $L$ on $\text{Ext}^\bullet_S(S, S^e)$ by Lie derivatives and will be presented in Section 3.2. Although this action does not define a $U$-module structure on $\text{Ext}^\bullet_S(S, S^e)$, it satisfies some compatibility with the $S$-module structure. The actions of $L$ satisfying such a compatibility have specific properties that are used in the rest of the article and which are summarised below.

Define a **compatible** left $S \times L$-module as a left $S \times L$-module $N$ such that, for all $n \in N$, $\alpha \in L$ and $s \in S$, the elements $sa \in L$ and $\alpha(s) \in S$ satisfy

\[(3.13) \quad (sa) \cdot n = s \cdot (\alpha \cdot n) - \alpha(s) \cdot n.\]

Note that a left $S \times L$-module is both compatible and a left $U$-module if and only if $L$ acts trivially, that is, by the zero action.

The two following lemmas present the properties of compatible left $S \times L$-modules used in the rest of the article.

**Lemma 3.5.1.** Let $M$ be a right $U$-module. Let $N$ be a compatible left $S \times L$-module. Then:

1. The right $S \times L$-module $N^\vee = \text{Hom}_S(N, S)$ is a right $U$-module,
2. The left $S \times L$-module $\text{Hom}_S(N^\vee, M)$ is a left $U$-module,
3. The left $S \times L$-module $M \otimes_S N$ is a left $U$-module,
4. The following canonical mapping is a morphism of left $U$-modules,

$$\theta: \quad M \otimes_S N \rightarrow \text{Hom}_S(N^\vee, M)$$

$$m \otimes n \mapsto (\theta_{m \otimes n}: \varphi \mapsto m \cdot \varphi(n)).$$

**Proof.** (1) Given $\varphi \in N^\vee$, $s \in S$ and $\alpha \in L$, then

$$\varphi \cdot (sa) = (\varphi \cdot s) \cdot \alpha.$$

Indeed, for all $n \in N$,

$$\begin{align*}
(\varphi \cdot (sa))(n) &= -(sa)(\varphi(n)) + \varphi((sa) \cdot n) \\
&= -s(\alpha(\varphi(n))) + \varphi(s \cdot (\alpha \cdot n) - \alpha(s) \cdot n) \\
&= -s\alpha(\varphi(n)) + \varphi(s \cdot (\alpha \cdot n) - \alpha(s) \cdot n) \\
&= ((\varphi \cdot s) \cdot \alpha)(n).
\end{align*}$$

(2) This is precisely [Hue99, (2.3)].

(3) The $S \times L$-module structure of $M \otimes_S N$ is described in (3.8). Given $m \in M$, $n \in N$, $s \in S$ and $\alpha \in L$, then

$$\begin{align*}
(sa) \cdot (m \otimes n) &= -m \cdot (sa) \otimes n + m \otimes (sa) \cdot n \\
&= -(m \cdot \alpha) \cdot s \otimes n + m \cdot \alpha(s) \otimes n + m \cdot s \cdot (\alpha \cdot n) - m \otimes \alpha(s) \cdot n \\
&= s \cdot (\alpha \cdot (m \otimes n)).
\end{align*}$$

(4) It suffices to prove that the given mapping is $L$-linear. Let $m \in M$, $n \in N$, $\alpha \in L$ and $\varphi \in \text{Hom}_S(N, S)$. Then,

$$\begin{align*}
(\alpha \cdot \theta_{m \otimes n})(\varphi) &= -\theta_{m \otimes (\varphi \cdot s \otimes n)} + \theta_{m \otimes n}(\varphi \cdot s) \\
&= -(m \cdot \alpha \cdot \varphi(n) - m \cdot \alpha(\varphi(n))) + m \cdot \varphi(n) - m \cdot \varphi(\alpha \cdot n) \\
&= -(m \cdot \alpha \cdot \varphi(n) - m \cdot \varphi(\alpha \cdot n) - m \cdot \alpha(n) + n \cdot \varphi(\alpha \cdot n)).
\end{align*}$$

thus, $\alpha \cdot \theta_{m \otimes n} = \theta_{\alpha(m \otimes n)}$. \(\square\)

Any left $S \times L$-module $N$ may be considered as a symmetric $S$-bimodule, or equivalently a right $S^e$-module, such that, for all $n \in N$ and $s, s' \in S$,

$$n \cdot (s \otimes s') = (ss') \cdot n;$$

accordingly, $N \otimes_S S^e$ is a right $U^e$-module in a natural way.
Lemma 3.5.2. Let \( N \) be a compatible left \( S \times L \)-module.

1. The right \( U^e \)-module \( N \otimes_S U^e \) is actually a \( U \)-\( U^e \)-bimodule such that for all \( n \in N \), \( u, v \in U \) and \( \alpha \in L \),

\[
\alpha \cdot (n \otimes (u \otimes v)) = \alpha \cdot n \otimes (u \otimes v) + n \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)).
\]

2. Let \( M \) be a right \( U \)-module. Then, there exists an isomorphism of left \( U^e \)-modules

\[
F(M \otimes_S N) \to M \otimes_U (N \otimes_S U^e)
\]

\[
v \otimes (m \otimes n) \mapsto m \otimes (n \otimes (1 \otimes v)).
\]

Proof. (1) Following part (3) of Lemma 3.5.1, there is a left \( U \)-module structure on \( U \otimes_S N \) such that, for all \( \alpha \in L \), \( n \in N \), \( u, v \in U \),

\[
\alpha \cdot (v \otimes n) = -\alpha \otimes n + v \otimes \alpha \cdot n.
\]

Therefore, there is a left \( U \)-module structure on \((U \otimes_S N) \otimes_S U \) (see (3.7)) such that, for all \( \alpha \in L \), \( n \in N \) and \( u, v \in U \),

\[
\alpha \cdot ((v \otimes n) \otimes u) = \alpha \cdot (v \otimes n) \otimes u + (v \otimes n) \otimes \alpha u = -(\alpha \otimes n) \otimes u + (v \otimes \alpha \cdot n) \otimes u + (v \otimes n) \otimes \alpha u.
\]

Under the canonical identification

\[
N \otimes_S U^e \to (U \otimes_S N) \otimes_S U
\]

\[
n \otimes (u \otimes v) \mapsto (v \otimes n) \otimes u,
\]

\( N \otimes_S U^e \) inherits of a left \( U \)-module structure which is the one claimed in the statement.

Now, \( N \otimes_S U^e \) inherits a right \( U^e \)-module structure from \( U^e \). This structure is compatible with the left \( U \)-module discussed previously so as to yield a left \( U \otimes (U^e)^\text{op} \)-module structure.

(2) Due to (1), there is a right \( U^e \)-module structure on \( M \otimes_U (N \otimes_S U^e) \). It is considered here as a left \( U^e \)-module structure such that, for all \( u, v, u', v' \in U \), \( m \in M \) and \( n \in N \),

\[
(u' \otimes v') \cdot (m \otimes (n \otimes (u \otimes v))) = m \otimes (n \otimes (uv' \otimes u'v));
\]

For the ease of reading, note that in \( F(M \otimes_S N) \),

\[
(1 \otimes \alpha) \cdot (v \otimes m \otimes n) = \nu v \otimes m \otimes n
\]

\[
(1 \otimes \alpha) \cdot (v \otimes m \otimes n) = \nu \otimes m \otimes n + v \otimes m \cdot \alpha \cdot n - v \otimes m \otimes \alpha \cdot n,
\]

and, in \( M \otimes_U (N \otimes_S U^e) \),

\[
m \cdot \alpha \otimes n \otimes u \otimes v = m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes \alpha u \otimes v - m \otimes n \otimes u \otimes \nu \alpha.
\]

The \( R \)-linear mapping from \( U \otimes M \otimes N \) to \( M \otimes_U (N \otimes_S U^e) \) given by

\[
v \otimes m \otimes n \mapsto m \otimes (n \otimes (1 \otimes v))
\]

induces a morphism of \( S \)-modules from \( U \otimes_S (M \otimes_S N) \) to \( M \otimes_U (N \otimes_S U^e) \) such as in the statement of the lemma. Denote it by \( \Psi' \):

\[
\Psi': U \otimes_S (M \otimes_S N) \to M \otimes_U (N \otimes_S U^e).
\]
This is a morphism of left $U^e$-modules. Indeed, for all $u, v \in U$, $m \in M$, $n \in N$ and $\alpha \in L$,
\[
\Psi'(u \otimes 1) \cdot (v \otimes m \otimes n)) = \Psi'(u v \otimes m \otimes n) = m \otimes n \otimes 1 \otimes uv
\]
\[
= (u \otimes 1) \cdot \Psi'(v \otimes m \otimes n)
\]
\[
\Psi'(1 \otimes \alpha) \cdot (v \otimes m \otimes n)) = \Psi'(v \alpha \otimes m \otimes n + v \otimes m \cdot \alpha \otimes n - v \otimes m \otimes \alpha \cdot n)
\]
\[
= m \otimes n \otimes 1 \otimes v + m \cdot \alpha \otimes n \otimes 1 \otimes v - m \otimes n \otimes \alpha \otimes v
\]
\[
= (1 \otimes \alpha) \cdot \Psi'(v \otimes m \otimes n)
\]
Consider the following morphism of $S$-modules
\[
\phi: M \otimes_S (N \otimes_S U^e) \rightarrow F(M \otimes_S N)
\]
\[
m \otimes (n \otimes (u \otimes v)) \mapsto (1 \otimes u) \cdot (v \otimes m \otimes n)
\]
Given $m \in M$, $n \in N$, $u, v \in U$ and $\alpha \in L$, then the image under $\phi$ of the term
\[
m \otimes \alpha \cdot n \otimes u \otimes v + m \otimes n \otimes au \otimes v - m \otimes n \otimes u \otimes va
\]
is equal to
\[
(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) + (1 \otimes au) \cdot (v \otimes m \otimes n) - (1 \otimes u) \cdot (va \otimes m \otimes n)
\]
which is equal to
\[
(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) + (1 \otimes u) \cdot (1 \otimes \alpha) \cdot (v \otimes m \otimes n) - (1 \otimes u) \cdot (va \otimes m \otimes n)
\]
In view of (3.15), this is equal to
\[
(1 \otimes u) \cdot (v \otimes m \otimes \alpha \cdot n) = \phi(m \cdot \alpha \otimes (n \otimes (u \otimes v)))
\]
Thus, $\phi$ induces a morphism of $S$-modules
\[
\Phi': M \otimes_U (N \otimes_S U^e) \rightarrow F(M \otimes_S N)
\]
\[
m \otimes (n \otimes (u \otimes v)) \mapsto (1 \otimes u) \cdot (v \otimes m \otimes n)
\]
It appears that $\Phi'$ is left and right inverse for $\Psi'$. Indeed,
- $\Phi' \circ \Psi' = \text{Id}_{F(M \otimes_S N)}$, and
- for all $u, v \in U$, $m \in M$ and $n \in N$,\[
\Psi' \circ \Phi'(m \otimes n \otimes u \otimes v) = \Psi'((1 \otimes u) \cdot (v \otimes m \otimes n))
\]
\[
= (1 \otimes u) \cdot \Psi'(v \otimes m \otimes n)
\]
\[
= (1 \otimes u) \cdot (m \otimes n \otimes 1 \otimes v)
\]
\[
= m \otimes n \otimes u \otimes v.
\]

3.6. Invertible $U$-bimodules. The following result is used in Section 5 in order to prove that $\text{Ext}_U(U, U^e)$ is invertible as a $U$-bimodule, under suitable conditions.

**Proposition 3.6.1.** Let $R$ be a commutative ring. Let $(S, L)$ be a Lie-Rinehart algebra over $R$. Denote by $U$ its enveloping algebra. Let $N$ be a left $U$-module. Assume that $N$ is invertible as an $S$-module. Then $F(N)$ is invertible as a $U$-bimodule.

The subsection is devoted to the proof of this proposition. Given a left $U$-module $N$, then $F(N) = U \otimes_S N$ as left $U$-modules. Hence, there is a functorial isomorphism
\[
(3.17) \quad \theta: \text{Hom}_S(N, U) \rightarrow \text{Hom}_U(F(N), U).
\]
Note:
Lemma 3.6.2. Let \( \psi \) be a left \( U \)-module (see (3.4)), and it inherits a right \( U \)-module structure from \( U_U \); by construction, these two structures form a \( U \)-bimodule structure.

\[ \psi \]

(1) By definition, \( \psi \) is a left \( U \)-module because, for all \( u \in U \), \( f \in \text{Hom}_S(N,U) \), \( u \cdot (\alpha \cdot f)(u \otimes n) = u(\alpha \cdot f)(n) \).

(2) By definition, \( \Phi \) is a morphism of right \( U \)-modules. It is also a morphism of left \( U \)-modules because, for all \( n \in N \), \( f \in \text{Hom}_S(N,U) \), \( u \in U \) and \( \alpha \in L \),

\[
\begin{align*}
\theta(\alpha \cdot f)(u \otimes n) &= u(\alpha \cdot f)(n) \\
&= u(\alpha f(n) - f(\alpha \cdot n)) = \theta(f)(ua \otimes n - u \otimes \alpha \cdot n) \\
&= \theta(f)(u \otimes n) \cdot \alpha = (\alpha \cdot \theta(f))(u \otimes n).
\end{align*}
\]

(3) By definition, \( \Phi \) is a morphism of right \( U \)-modules. It is also a morphism of left \( U \)-modules because, for all \( n \in N \), \( f \in \text{Hom}_S(N,U) \) and \( \alpha \in L \),

\[
\begin{align*}
\Phi(\alpha \cdot (n \otimes f)) &= \Phi(\alpha \cdot n \otimes f + n \otimes \alpha \cdot f) \\
&= (1 \otimes \alpha \cdot n) \otimes \theta(f) + (1 \otimes n) \otimes \theta(\alpha \cdot f) \\
&= (1 \otimes \alpha \cdot n) \otimes \theta(f) + (1 \otimes n) \cdot \alpha \otimes \theta(f) \\
&= (\alpha \otimes n) \otimes \theta(f) \\
&= \alpha \cdot \Phi(n \otimes f).
\end{align*}
\]

In order to prove that \( \Phi \) is bijective, consider the linear mapping

\[
\begin{align*}
\psi: \ F(N) \otimes \text{Hom}_S(F(N),U) &\rightarrow N \otimes \text{Hom}_S(N,U) \\
(u \otimes n) \otimes g &\mapsto u \cdot (n \otimes \theta^{-1}(g)).
\end{align*}
\]

Note that, for all \( u \in U \), \( \alpha \in L \), \( n \in N \) and \( g \in \text{Hom}_U(F(N),U) \),

\[
\begin{align*}
\psi((u \otimes n) \cdot \alpha \otimes g) &= \psi((u \alpha \otimes n) \otimes g - (u \otimes \alpha \cdot n) \otimes g) \\
&= u \alpha \cdot (n \otimes \theta^{-1}(g)) - u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) \\
&= u \cdot (\alpha \cdot n \otimes \theta^{-1}(g)) + n \otimes \alpha \cdot \theta^{-1}(g)) - u \cdot (\alpha \cdot n \otimes \theta^{-1}(g(g)) \\
&= u \cdot (n \otimes \theta^{-1}(\alpha \cdot g)) \quad \text{(see part (1))} \\
&= \psi((u \otimes n) \otimes \alpha \cdot g).
\end{align*}
\]

Hence, \( \psi \) induces a linear mapping

\[
\begin{align*}
\Psi: \ F(N) \otimes \text{Hom}_U(F(N),U) &\rightarrow N \otimes \text{Hom}_S(N,U) \\
(u \otimes n) \otimes g &\mapsto u \cdot (n \otimes \theta^{-1}(g)).
\end{align*}
\]
Now, by definition of $\Phi$ and $\Psi$,

$$\Psi \circ \Phi = \text{Id}_{N \otimes_S \text{Hom}_S(N,U)}.$$ 

Since

- $\Psi$ is a morphism of left $U$-modules by construction;
- as a left $U$-module, $F(N) \otimes_U \text{Hom}_U(F(N),U)$ is generated by the image of $(1 \otimes N) \otimes \text{Hom}_U(F(N),U)$; and
- for all $n \in N$ and $g \in \text{Hom}_U(F(N),U)$,

$$\Phi \circ \Psi((1 \otimes n) \otimes g) = (1 \otimes n) \otimes g;$$

the following holds

$$\Phi \circ \Psi = \text{Id}_{F(N) \otimes_U \text{Hom}_U(F(N),U)}.$$ 

Altogether, these considerations show that $\Phi$ is an isomorphism in $\text{Mod}(U^e)$.

(3) The diagram is commutative by definition of $\Phi$. \hfill $\square$

Like in the previous lemma, for all $N \in \text{Mod}(U)$, $\text{Hom}_S(N,U)$ is a $U$-bimodule, and hence $\text{Hom}_S(N,U) \otimes_S N$ is a $U$-bimodule by means of $[3.7]$ and the right $U$-module structure of $U$.

Lemma 3.6.3. Let $N$ be a left $U$-module. Then,

1. the mapping

$$\Phi': \text{Hom}_S(N,U) \otimes_S N \rightarrow \text{Hom}_U(F(N),U) \otimes_U F(N)$$

$$f \otimes n \mapsto \theta(f) \otimes (1 \otimes n)$$

is an isomorphism in $\text{Mod}(U^e)$; and

2. the diagram

$$\begin{array}{ccc}
\text{Hom}_S(N,U) \otimes_S N & \rightarrow & U \\
\Phi' \downarrow & & \downarrow \\
\text{Hom}_U(F(N),U) \otimes_U F(N) & \rightarrow & U 
\end{array}$$

with horizontal arrows given by evaluation, is commutative.

Proof. (1) First, since $F(N) = U \otimes S N$ in $\text{Mod}(U^e)$, then

$$\text{Hom}_U(F(N),U) \otimes_U F(N) \cong \text{Hom}_U(F(N),U) \otimes S N$$

as left $U$-modules. Under this identification, $\Phi'$ expresses as

$$\Phi': f \otimes n \mapsto \theta(f) \otimes n.$$ 

Therefore, $\Phi'$ is bijective because so is $\theta$.

Next, $\Phi'$ is a morphism of left $U$-modules because so is $\theta$. And it is a morphism of right $U$-modules because it is a morphism of right $S$-modules, and because, for all $f \in \text{Hom}_S(N,U)$, $n \in N$ and $\alpha \in L$,

$$\Phi'((f \otimes n) \cdot \alpha) = \Phi'(f \cdot \alpha \otimes n - f \otimes \alpha \cdot n)$$

$$= \theta(f \cdot \alpha) \otimes (1 \otimes n) - \theta(f) \otimes (1 \otimes \alpha \cdot n)$$

$$= \theta(f) \otimes \alpha \cdot (1 \otimes n) - \theta(f) \otimes (1 \otimes \alpha \cdot n)$$

$$= \theta(f) \otimes (1 \otimes n) \cdot \alpha$$

$$= \Phi'(f \otimes n) \cdot \alpha.$$ 

This proves (1).

(2) The diagram commutes by definition of $\Phi'$. \hfill $\square$
It is now possible to prove the result announced at the beginning of the subsection.

**Proof of Proposition 3.6.1**. Since \( N \) is invertible as an \( S \)-module, then the following evaluation mappings are bijective

\[
N \otimes_S \text{Hom}_S(N,U) \to U \quad \text{and} \quad \text{Hom}_S(N,U) \otimes_S N \to U.
\]

According to Lemmas 3.6.2 and 3.6.3, the following evaluation mappings are isomorphisms of \( U \)-bimodules

\[
F(N) \otimes_U \text{Hom}_U(F(N), U) \to U \quad \text{and} \quad \text{Hom}_U(F(N), U) \otimes_U F(N) \to U.
\]

Thus, \( F(N) \) is invertible as a \( U \)-bimodule. \( \square \)

4. **The Action of \( L \) on the Inverse Dualising Bimodule of \( S \)**

This section introduces an action of \( L \) on \( \text{Ext}_{S, S}^n(S, S^e) \) by means of Lie derivations, which is used to describe \( \text{Ext}_{U, U}^n(U, U^e) \) in the next section. When \( S \) is projective in \( \text{Mod}(R) \), then \( \text{Ext}_{S, S}^n(S, \cdot) \) is the Hochschild cohomology \( H^*(S; \cdot) \); in this setting, the Lie derivatives on \( H^*(S; S) \) and \( H_*(S; S) \) are defined in [Rin63, Section 9] and have a well-known expression in terms of the Hochschild resolution of \( S \). For the needs of the article, the definition is translated to arbitrary coefficients in terms of any projective resolution of \( S \) in \( \text{Mod}(S^e) \).

Hence, 4.1 introduces preliminary material, 4.2 deals with derivations on projective resolutions of \( S \) in \( \text{Mod}(S^e) \), 4.3 defines the Lie derivatives, 4.4 presents the action of \( L \) on \( \text{Ext}_{S, S}^n(S, S^e) \), and 4.5 discusses particular situations.

For the section, a projective resolution of \( S \) in \( \text{Mod}(S^e) \) is considered;

\[(P^\bullet, d) \to S.\]

Denote \( S \) by \( P^1 \) and the augmentation mapping \( P^0 \to S \) by \( d^0 \). For all \( M \in \text{Mod}(S^e) \) and \( s \in S \), denote by \( \lambda_s \) and \( \rho_s \) the multiplication mappings

\[
\lambda_s : M \to M, \quad m \mapsto (s \otimes 1) \cdot m
\]

and

\[
\rho_s : M \to M, \quad m \mapsto (1 \otimes s) \cdot m.
\]

4.1. **Data on the projective resolution.** For all \( s \in S \), the mappings \( \lambda_s, \rho_s \) on \( P^\bullet \) are morphisms of complexes of left \( S^e \)-modules and induce the same mapping

\[
S \to S
\]

\[
t \mapsto st
\]

in cohomology; Hence, there exists a morphism of graded left \( S^e \)-modules

\[
k_s : P^\bullet \to P^\bullet[-1]
\]

such that

\[
\lambda_s - \rho_s = k_s \circ d + d \circ k_s.
\]

**Lemma 4.1.1.** Let \( \partial : S \to S \) be an \( R \)-linear derivation. Let \( \psi : P^\bullet \to P^\bullet \) be a morphism of complexes of \( R \)-modules such that

- \( H^0(\psi) : S \to S \) is the zero mapping;
- there exists a morphism of graded left \( S^e \)-modules

\[
k : P^\bullet \to P^\bullet[-1]
\]

such that, for all \( p \in P^\bullet \) and \( s, t \in S \),

\[
\psi((s \otimes t) \cdot p) = (s \otimes t) \cdot \psi(p) - (1 \otimes \partial)(s \otimes t) \cdot (k \circ d + d \circ k)(p).
\]

Then, there exists a morphism of graded \( R \)-modules

\[
h : P^\bullet \to P^\bullet[-1]
\]
such that
- \( \psi = h \circ d + d \circ h; \) and
- for all \( s, t \in S \) and \( p \in P^* \),
\[
h((s \otimes t) \cdot p) = (s \otimes t) \cdot h(p) - (1 \otimes \partial)(s \otimes t) \cdot k(p) .
\]

**Proof.** The proof is an induction on \( n \leq 1 \), taking \( h^1 : S \to P^0 \) equal to 0. Let \( n \leq 0 \) and assume that there exist linear mappings, for all \( j \) such that \( n + 1 \leq j \leq 1 \),
\[
h^j : P^j \to P^{j-1}
\]
such that, for all \( j \) satisfying \( n + 1 \leq j \leq 0 \), \( p \in P^j \) and \( s, t \in S \),
\[
\psi^j = h^{j+1} \circ d^j + d^{j-1} \circ h^j
\]
\[
h^j((s \otimes t) \cdot p) = (s \otimes t) \cdot h^j(p) - (1 \otimes \partial)(s \otimes t) \cdot k^j(p).
\]

Let
\[
((p_i, \varphi^i))_{i \in I}
\]
be a coordinate system of the projective left \( S^e \)-module \( P^n \). That is, let \( p_i \in P^n \) and \( \varphi^i \in \text{Hom}_{S^e}(P^n, S^e) \) for all \( i \in I \), such that, for all \( p \in P^n \),
\[
p = \sum_{i \in I} \varphi^i(p) \cdot p_i,
\]
where \( \{ i \in I \mid \varphi^i(p) \neq 0 \} \) is finite. Since \( \psi : P^* \to P^* \) is a morphism of complexes, it follows from (4.4) that, for all \( i \in I \), there exists \( p_i' \in P^{n-1} \) such that
\[
\psi^n(p_i) = d^{n-1}(p_i') + h^{n+1} \circ d^n(p_i).
\]
Denote by \( h^n \) the linear mapping from \( P^n \) to \( P^{n-1} \) such that, for all \( p \in P^n \),
\[
h^n(p) = \sum_{i \in I} \varphi^i(p) \cdot p_i' - (1 \otimes \partial)(\varphi^i(p)) \cdot k^n(p_i).
\]
Then, for all \( p \in P^n \) and \( s, t \in S \),
\[
h^n((s \otimes t) \cdot p)
\]
\[
= \sum_{i \in I} (s \otimes t) \cdot \varphi^i(p) \cdot p_i' - (s \otimes t) \cdot (1 \otimes \partial)(\varphi^i(p)) \cdot k^n(p_i) - (1 \otimes \partial)(s \otimes t) \cdot \varphi^i(p) \cdot k^n(p_i)
\]
\[
= (s \otimes t) \cdot h^n(p) - (1 \otimes \partial)(s \otimes t) \cdot k^n(\sum_{i \in I} \varphi^i(p) \cdot p_i)
\]
\[
= (s \otimes t) \cdot h^n(p) - (1 \otimes \partial)(s \otimes t) \cdot k^n(p).
\]
Moreover,
\[
\psi^n = h^{n+1} \circ d^n + d^{n-1} \circ h^n.
\]
Indeed, for all \( p \in P^n \), \( p = \sum_{i \in I} \varphi^i(p) \cdot p_i \), and hence
\[
d^{n-1} \circ h^n(p) + h^{n+1} \circ d^n(p) = \sum_{i \in I} \varphi^i(p) \cdot d^{n-1}(p'_i) - (1 \otimes \partial)(\varphi^i(p)) \cdot d^{n-1} \circ k^n(p_i) + h^{n+1}(\sum_{i \in I} \varphi^i(p) \cdot d^n(p_i))
\]
\[
= \sum_{i \in I} \varphi^i(p) \cdot d^{n-1}(p'_i) - (1 \otimes \partial)(\varphi^i(p)) \cdot d^{n-1} \circ k^n(p_i) + \varphi^i(p) \cdot h^{n+1} \circ d^n(p_i) - (1 \otimes \partial)(\varphi^i(p)) \cdot k^{n+1} \circ d^n(p_i)
\]
\[
= \sum_{i \in I} \varphi^i(p) \cdot d^{n-1}(p'_i) + \varphi^i(p) \cdot h^{n+1} \circ d^n(p_i) + \psi^n(\varphi^i(p) \cdot p_i) - \varphi^i(p) \cdot \psi^n(p_i)
\]
\[
= \sum_{i \in I} \psi^n(\varphi^i(p) \cdot p_i) = \psi^n(p).
\]

\[\square\]

4.2. Derivations on the projective resolution. Let \( \partial : S \to S \) be an \( R \)-linear derivation. It defines an \( R \)-linear derivation on \( S^e \) denoted by \( \partial^e \),
\[
\partial^e : \quad S^e \to S^e
\]
\[
s \otimes t \quad \mapsto \quad \partial(s) \otimes t + s \otimes \partial(t).
\]

For every left \( S^e \)-module \( M \), a derivation of \( M \) relative to \( \partial \) is an \( R \)-linear mapping
\[
\partial_M : M \to M
\]
such that, for all \( m \in M \) and \( s, t \in S \),
\[
\partial_M((s \otimes t) \cdot m) = \partial^e(s \otimes t) \cdot m + (s \otimes t) \cdot \partial_M(m).
\]

A derivation of \( P^\bullet \) relative to \( \partial \) is a morphism of complexes of \( R \)-modules,
\[
\partial^\bullet : P^\bullet \to P^\bullet,
\]
such that \( \partial^n : P^n \to P^n \) is a derivation relative to \( \partial \) for all \( n \), and such that \( H^0(\partial^\bullet) = \partial \). Note that a morphism of complexes of \( R \)-modules \( \partial^\bullet : P^\bullet \to P^\bullet \) such that \( H^0(\partial^\bullet) = \partial \) is a derivation relative to \( \partial \) if and only if
\[
\left\{ \begin{align*}
\partial^\bullet \circ \lambda_s &= \lambda_{\partial(s)} + \lambda_s \circ \partial^\bullet, \\
\partial^\bullet \circ \rho_s &= \rho_{\partial(s)} + \rho_s \circ \partial^\bullet.
\end{align*} \right.
\]

(4.6)

Remark. For all derivations \( \partial_1^\bullet, \partial_2^\bullet : P^\bullet \to P^\bullet \) relative to \( \partial \), the difference
\[
\partial_1^\bullet - \partial_2^\bullet : P^\bullet \to P^\bullet
\]
is a null-homotopic morphism of complexes of left \( S^e \)-modules.

Lemma 4.2.1. There exists a mapping, which need not be linear,
\[\text{Der}_R(S) \to \text{Hom}_R(P^\bullet, P^\bullet)\]
\[
\partial \mapsto \partial^\bullet
\]
such that:

(1) For all \( \partial \in \text{Der}_R(S) \), the mapping \( \partial^\bullet \) is a derivation relative to \( \partial \).

(2) For all \( \partial_1, \partial_2 \in \text{Der}_R(S) \) and \( r \in R \), there exist morphisms of graded left \( S^e \)-modules
\[
\ell, \ell' : P^\bullet \to P^\bullet[-1]
\]
such that
\[
\left\{ \begin{align*}
[\partial_1, \partial_2]^\bullet &= [\partial_1^\bullet, \partial_2^\bullet] \\
(\partial_1 + r\partial_2)^\bullet &= (\partial_1^\bullet + r\partial_2^\bullet)
\end{align*} \right.
\]
\[
\ell \circ d + d \circ \ell = (\partial_1 + r\partial_2)^\bullet = (\partial_1^\bullet + r\partial_2^\bullet) = \ell' \circ d + d \circ \ell'.
\]
(3) For all \( s \in S \) and \( \partial \in \text{Der}_R(S) \), there exists a morphism of graded \( R \)-modules

\[
h: P^* \rightarrow P^*[-1]
\]

such that

\[
(s\partial)^* - \lambda_s \circ \partial^* = h \circ d + d \circ h
\]

and, for all \( p \in P^* \) and \( t_1, t_2 \in S \),

\[
h((t_1 \otimes t_2) \cdot p) = (t_1 \otimes t_2) \cdot h(p) - (t_1 \otimes \partial(t_2)) \cdot k_s(p).
\]

Recall that \( k_s: P^* \rightarrow P^*[-1] \) is a morphism of graded left \( S^e \)-modules such that \( \lambda_s - \rho_s = k_s \circ d + d \circ k_s \) (see (4.1) and (4.2)).

**Proof.** (1) Let \( \partial \in \text{Der}_R(S) \). For convenience, denote \( \partial \) by \( \partial^1 : S \rightarrow S \). The proof is an induction on \( n \leq 1 \). Let \( n \leq 0 \), and assume that a commutative diagram is given

\[
\begin{array}{ccccccccc}
P^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & \cdots & \xrightarrow{d} & P^0 & \xrightarrow{d^0} & P^1 & \xrightarrow{0} \\
\downarrow{\partial^{n+1}} & & \downarrow{\partial^n} & & \cdots & & \downarrow{\partial^1} & & \downarrow{0} \\
P^n & \xrightarrow{d^n} & P^{n+1} & \xrightarrow{d^{n+1}} & \cdots & \xrightarrow{d} & P^0 & \xrightarrow{d^0} & P^1 & \xrightarrow{0}
\end{array}
\]

where \( \partial^i: P^i \rightarrow P^i \) is a derivation relative to \( \partial \) for all \( i \in \{n+1, n+2, \ldots, 0\} \). Let

\[
\left((p_i, \varphi^i)\right)_{i \in I}
\]

be a coordinate system of the projective left \( S^e \)-module \( P^n \) (see the proof in 4.1).

Then, for all \( i \in I \), there exists \( p'_i \in P^n \) such that

\[
\partial^{n+1} \circ d^n(p_i) = d^n(p'_i).
\]

Denote by \( \partial^n \) the \( R \)-linear mapping from \( P^n \) to \( P^n \) such that, for all \( p \in P^n \),

\[
\partial^n(p) = \sum_{i \in I} \partial_i(\varphi^i(p)) \cdot p_i + \varphi^i(p) \cdot p'_i.
\]

Then, for all \( p \in P^n \),

\[
d^n \circ \partial^n(p) = \sum_{i \in I} \partial_i(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot d^n(p'_i)
\]

\[
= \sum_{i \in I} \partial_i(\varphi^i(p)) \cdot d^n(p_i) + \varphi^i(p) \cdot \partial^{n+1} \circ d^n(p_i)
\]

\[
= \partial^{n+1} \circ d^n(\sum_{i \in I} \varphi^i(p) \cdot p_i)
\]

\[
= \partial^{n+1} \circ d^n(p).
\]

Thus,

\[
d^n \circ \partial^n = \partial^{n+1} \circ d^n.
\]

Moreover, \( \partial^n \) is a derivation of \( P^n \) relative to \( \partial \) because \( \partial \) is a derivation of \( S^e \) and \( \varphi^1 \in \text{Hom}_{S^e}(P^n, S^e) \) for all \( i \in I \).

(2) Note that \([\partial_1, \partial_2]^* \) and \([\partial_1^*, \partial_2^*] \) (or, \((\partial_1 + r\partial_2)^* \) and \(\partial^* + r\partial^*_2 \)) are derivations of \( P^* \) relative to \([\partial_1, \partial_2] \) (or, to \(\partial_1 + r\partial_2 \), respectively). The conclusion therefore follows from the remark preceding Lemma 4.2.

(3) Denote by \( \psi \) the mapping \( (s\partial)^* \rightarrow \lambda_s \circ \partial^* \) given by

\[
P^* \rightarrow P^*
\]

\[
p \mapsto (s\partial)^*(p) - (s \otimes 1) \cdot \partial^*(p).\]
Then, for all \( p \in \mathcal{P}^* \) and \( t \in S \),
\[
\psi((t \otimes 1) \cdot p) = (s\partial)^*((t \otimes 1) \cdot p) - (s \otimes 1) \cdot \partial^*((t \otimes 1) \cdot p)
\]
\[
= (s\partial(t) \otimes 1) \cdot p + (t \otimes 1) \cdot (s\partial)^*(p) - (s \otimes 1) \cdot (\partial(t) \otimes 1) \cdot p - (s \otimes 1) \cdot (t \otimes 1) \cdot \partial^*(p)
\]
\[
= (t \otimes 1) \cdot \psi(p)
\]
and
\[
\psi((1 \otimes t) \cdot p) = (s\partial)^*((1 \otimes t) \cdot p) - (s \otimes 1) \cdot \partial^*((1 \otimes t) \cdot p)
\]
\[
= (1 \otimes s\partial(t)) \cdot p + (1 \otimes t) \cdot (s\partial)^*(p) - (s \otimes 1) \cdot (1 \otimes \partial(t)) \cdot p - (s \otimes 1) \cdot (1 \otimes t) \cdot \partial^*(p)
\]
\[
= (1 \otimes t) \cdot \psi(p) + (1 \otimes \partial(t)) \cdot (p_s - \lambda_s)(p)
\]
\[
= (1 \otimes t) \cdot \psi(p) - (1 \otimes \partial(t)) \cdot (k_s \circ d + d \circ k_s)(p).
\]
Hence, Lemma 4.1.1 may be applied, which yields (3). \( \square \)

**Remark.** Using the remark preceding Lemma 4.2.1, it may be checked that, although the mapping \( \text{Der}_R(S) \to \text{Hom}_R(\mathcal{P}^*, \mathcal{P}^*) \) of the lemma is not unique, two such mappings induce the same mapping from \( \text{Der}_R(S) \to H^0\text{Hom}_R(\mathcal{P}^*, \mathcal{P}^*) \), which is \( R \)-linear.

When \( S \) is projective in \( \text{Mod}(R) \), it is possible to be more explicit on a possible mapping \( \partial \mapsto \partial^* \). Indeed, the Hochschild complex \( B(S) = S^{\otimes n+2} \) is a projective resolution of \( S \). For all \( \partial \in \text{Der}_R(S) \), define the following mapping:
\[
L_{\partial}: \quad B(S) \to B(S)
\]
\[
(s_0 \cdots s_{n+1}) \mapsto \sum_{i=0}^{n+1} (s_0 \cdots \hat{s}_i \cdots s_{n+1}) \partial(s_i) \cdots |s_{i+1}| \cdots |s_n|.
\]
This is a derivation of \( B(S) \) relative to \( \partial \). It is direct to check that the mapping
\[
\text{Der}_R(S) \to \text{Hom}_R(B(S), B(S))
\]
\[
\partial \mapsto L_{\partial}
\]
is a morphism of Lie algebras over \( R \). Now, consider homotopy equivalences of complexes of \( S^c \)-modules,
\[
\mathcal{P}^* \xrightarrow{f/g} B(S),
\]
and, for all \( \partial \in \text{Der}_R(S) \), define \( \partial^* \) as
\[
\partial^* = g \circ L_{\partial} \circ f;
\]
this is a derivation relative to \( \partial \) because so is \( L_{\partial} \) and because \( f \) and \( g \) are morphisms of resolutions of \( S \) in \( \text{Mod}(S^c) \). The following mapping satisfies the conclusion of the preceding lemma, it is moreover \( R \)-linear.
\[
\text{Der}_R(S) \to \text{Hom}_R(\mathcal{P}^*, \mathcal{P}^*)
\]
\[
\partial \mapsto \partial^*.
\]
4.3. **Lie derivatives.** Consider a mapping $\partial \mapsto \partial^*$ such as in Lemma 4.2.1. Let $M$ be an $S$-bimodule and $\partial : S \to S$ be an $R$-linear derivation. Let $\partial_M : M \to M$ be a derivation relative to $\partial$. Given $n \in \mathbb{N}$ and $\psi \in \text{Hom}_S(P^{-n}, M)$, denote by $L_\partial(\psi)$ the mapping

$$L_\partial(\psi) = \partial_M \circ \psi - \psi \circ \partial^{-n}.$$  

This is a morphism in $\text{Mod}(S^e)$ because so is $\psi$ and because $\partial_M$ and $\partial^{-n}$ are derivations relative to $\partial$; moreover, it is a cocycle (or a coboundary) as soon as $\psi$ is because $\partial^* : P^* \to P^*$ is a morphism of complexes. Denote by $L_\partial$ the resulting mapping in cohomology

$$L_\partial : \text{Ext}^*_S(S, M) \to \text{Ext}^*_S(S, M)$$

such that for all $c \in \text{Ext}^*_S(S, M)$, say represented by a cocycle $\psi$, then $L_\partial(c)$ is represented by the cocycle $L_\partial(\psi)$. In the situations considered later in the article, there is no ambiguity on $\partial_M$, whence its omission in the notation.

Following similar considerations denote also by $L_\partial$ the mapping

$$L_\partial : \text{Tor}^*_S(S, M) \to \text{Tor}^*_S(S, M)$$

such that for all $\omega \in \text{Tor}^*_S(S, M)$, say represented by a cocycle $m \otimes p \in M \otimes_{S^e} P^*$ with sum sign omitted, $L_\partial(\omega)$ is represented by the cocycle

$$L_\partial(m \otimes p) := m \otimes \partial^*(p) + \partial_M(m) \otimes p.$$  

When $S$ is projective in $\text{Mod}(R)$, these operations may be written explicitly in terms of the Hochschild resolution. When $\psi$ is a Hochschild cocycle lying in $\text{Hom}_R(S^\otimes n, M)$, the mapping $L_\partial(\psi)$ is given by

$$(s_1 | \cdots | s_n) \mapsto \partial_M(f(s_1 | \cdots | s_n)) - \sum_{i=1}^{n} f(s_1 | \cdots | \partial(s_i) | \cdots | s_n).$$  

Likewise, the operation in Hochschild homology is induced by the following mapping at the level of Hochschild chains,

$$M \otimes S^\otimes n \to M \otimes S^\otimes n$$

$$(m|s_1|\cdots|s_n) \mapsto (\partial_M(m)|s_1|\cdots|s_n) + \sum_{i=1}^{n} (m|s_1|\cdots|\partial(s_i)|\cdots|s_n).$$

The operator $L_\partial$ is of course called the **Lie derivative** of $\partial$. When $M = S$ and $S$ is projective in $\text{Mod}(R)$, this is nothing else but the classical Lie derivative defined in [Rim3]. In view of the remark following Lemma 4.2.1 these constructions depend only on $\partial$ and $\partial_M$ and not on the choices of $P^*$ and the mapping $\partial \mapsto \partial^*$. In the sequel these constructions are considered mainly in the following cases:

- $M = S$ and $\partial_M = \partial$.
- $M = S^e$ and $\partial_M = \partial^e$.
- $M = \text{Ext}^*_S(S, S^e)$ ($n \in \mathbb{N}$) and $\partial_M = L_\partial$, which makes sense according to the result below.

In the sequel the following construction is also used. Consider $S$-bimodules $M, N$. Let $m, n \in \mathbb{N}$. Let $\partial \in \text{Der}_R(S)$ and let $\partial_M : M \to M$ and $\partial_N : N \to N$ be $R$-linear derivations relative to $\partial$. Then, for all $f \in \text{Hom}_R(\text{Ext}^*_S(S, M), \text{Tor}^*_S(S, N))$, define $L_\partial(f)$ as

$$L_\partial \circ f - f \circ L_\partial.$$  

Recall that for all $M \in \text{Mod}(S^e)$, the spaces $\text{Ext}^*_S(S, M)$ and $\text{Tor}^*_S(S, M)$ are left $S$-modules by means of $\lambda_s : M \to M$ for all $s \in S$; the corresponding multiplication by $s$ on these (co)homology spaces is denoted by $\lambda_s$. 


Lemma 4.3.1. Let $M \in \text{Mod}(S^*)$, $n \in \mathbb{N}$ and $s \in S$. Let $\partial, \partial' : S \rightarrow S$ be $R$-linear derivations. Let $\partial_M, \partial'_M : M \rightarrow M$ be $R$-linear derivations relative to $\partial$ and $\partial'$, respectively. Then, the following hold in $\text{Ext}_S^\bullet(S, M)$:

1. $\mathcal{L}_\partial \circ \lambda_s = \lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial$.
2. $\mathcal{L}_{[\partial, \partial']}$ is symmetric as an $S$-bimodule and let $\partial_N : N \rightarrow N$ be a derivation relative to $\partial$. Consider the contraction mapping

$$\text{Tor}_{m-n}^S(S, M) \rightarrow \text{Hom}_R(\text{Ext}_S^m(S, N), \text{Tor}_{m-n}^S(S, M \otimes_S N))$$

$$\omega \mapsto (c \mapsto \iota_c(\omega)).$$

If $m = n$, then it is $\mathcal{L}_\partial$-equivariant. When $S$ is projective in $\text{Mod}(R)$, it is $\mathcal{L}_\partial$ equivariant for all $m, n \in \mathbb{N}$;

3. If $M$ is symmetric as an $S$-bimodule, $\mathcal{L}_{\partial \partial'} = \lambda_s \circ \mathcal{L}_\partial$.

4. When $M = S^*$ (and $\partial_M = \partial^*$), the following equality holds in $\text{Ext}_S^\bullet(S, M)$:

$$\mathcal{L}_{\partial \partial'} = \lambda_s \circ \mathcal{L}_\partial - \lambda_{\partial(s)}.$$

Proof. (1) The equality is checked on cochains. Let $\psi \in \text{Hom}_{S^*}(P^{-n}, M)$. Then,

$$\mathcal{L}_\partial \circ \lambda_s(\psi) = \partial_M \circ \lambda_s \circ \psi - \lambda_s \circ \psi \circ \partial^*$$

$$= (\lambda_{\partial(s)} + \lambda_s \circ \partial_M) \circ \psi - \lambda_s \circ \psi \circ \partial^*$$

$$= (\lambda_{\partial(s)} + \lambda_s \circ \mathcal{L}_\partial(\psi)).$$

(2) Note that $\mathcal{L}_{[\partial, \partial']}$ is defined with respect to $[\partial_M, \partial'_M]$, which is a derivation of $M$ relative to $[\partial, \partial']$. Following Lemma 4.2.1 there exists a morphism of graded $S^*$-modules

$$\ell : P^* \rightarrow P^*[-1]$$

such that

$$[\partial, \partial']^* - [\partial^*, \partial^*] = \ell \circ d + d \circ \ell.$$ 

Let $\psi \in \text{Hom}_{S^*}(P^{-n}, M)$. If this is a cocycle, then

$$\mathcal{L}_{[\partial, \partial']}(\psi) = [\partial_M, \partial'_M] \circ \psi = \psi \circ (\partial^* \circ \partial^*) + \ell \circ d + d \circ \ell$$

$$= [\mathcal{L}_\partial, \mathcal{L}_\partial](\psi) \circ d + \psi \circ d \circ \ell,$$

which is cohomologous to $[\mathcal{L}_\partial \mathcal{L}_\partial](\psi)$. This proves (2).

(3) Note that the mapping

$$\partial_M \otimes S N : M \otimes_S N \rightarrow M \otimes_S N$$

$$x \otimes y \mapsto \partial_M(x) \otimes y + x \otimes \partial_N(y),$$

is a well-defined derivation relative to $\partial$, which defines $\mathcal{L}_\partial$ on

$$\text{Tor}_{m-n}^S(S, M \otimes_S N).$$

Assume first that $m = n$. Let $p^0$ be any element of the preimage of $1_S$ under the augmentation mapping $P^0 \rightarrow S$. Let $x \otimes p \in M \otimes_S P^{-m}$ and $\psi \in \text{Hom}_{S^*}(P^{-m}, N)$, and use the notation

$$\iota_\psi(x \otimes p) := (x \otimes \psi(p)) \otimes p^0.$$ 

Recall that the contraction mapping is induced by the mapping

$$M \otimes_S P^{-m} \rightarrow \text{Hom}_R(\text{Hom}_{S^*}(P^{-m}, N), (M \otimes_S N) \otimes_S P^0)$$

$$x \otimes p \mapsto \iota_\psi(x \otimes p)$$.
Denote $\mathcal{L}_\partial(\iota_\psi(x \otimes p)) - \iota(\mathcal{L}_\partial(\psi)(x \otimes p))$ by $\delta$. Then,

$$\delta = \mathcal{L}_\partial((x \otimes \psi(p)) \otimes p^0) - (x \otimes \mathcal{L}_\partial(\psi)(p)) \otimes p^0$$

$$= \partial_M(x \otimes \psi(p)) \otimes p^0 + x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(p) \otimes \partial^0(p^0)$$

$$- x \otimes \partial_N(\psi(p)) \otimes p^0 + x \otimes \psi(\partial^{-m}(p)) \otimes p^0$$

$$= \iota_\psi(\mathcal{L}_\partial(x \otimes p)) + x \otimes \psi(p) \otimes \partial^0(p^0).$$

Note that $\partial^0(p^0)$ lies in the image of $d: P^{-1} \to P^0$ because the image of $p^0$ under $P^0 \to S$ is 1 and $H^0(\partial^*) = \partial$. These considerations therefore prove (3) when $m = n$.

Now assume that $S$ is projective in $\text{Mod}(R)$. Then, the equivariance may be checked at the level of Hochschild (co)chains. Let $o = (x|s_1| \cdots |s_m) \in S^\otimes m$ and $\psi \in \text{Hom}_R(S^\otimes n, N)$. Then,

$$\mathcal{L}_\partial(\iota_\psi(o)) - \iota(\mathcal{L}_\partial(\psi)(o))$$

$$= \mathcal{L}_\partial(x \otimes \psi(s_1| \cdots |s_m))s_{n+1}| \cdots |s_m) - (x \otimes \mathcal{L}_\partial(\psi)(s_1| \cdots |s_m))s_{n+1}| \cdots |s_m)$$

$$= (\partial_M(x) \otimes \psi(s_1| \cdots |s_m))s_{n+1}| \cdots |s_m) + (x \otimes \partial_N(\psi(s_1| \cdots |s_m))s_{n+1}| \cdots |s_m)$$

$$+ \sum_{j=n+1}^m (x \otimes \psi(s_1| \cdots |s_{n})s_{n+1}| \cdots |s_{j}| \cdots |s_m)$$

$$- (x \otimes \partial_N(\psi(s_1| \cdots |s_{n}))s_{n+1}| \cdots |s_m)$$

$$+ \sum_{j=1}^n (x \otimes \psi(s_1| \cdots |\partial(s_{j})| \cdots |s_{n})s_{n+1}| \cdots |s_m)$$

$$= \iota_\psi(\mathcal{L}_\partial(o)),$$

which proves (3) for all $m, n \in \mathbb{N}$ when $S$ is projective in $\text{Mod}(R)$.

(4) Note that $\mathcal{L}_s\partial$ is defined with respect to the derivation $s\partial_M (= \lambda_s \circ \partial_M)$. Assume that $M$ is symmetric as an $S$-bimodule. Therefore, the mapping

$$\lambda_s \circ \partial^*: P^* \to P^*$$

is a derivation relative to $s\partial$. Let $\psi \in \text{Hom}_{S^e}(P^*, M)$ be a cocycle with cohomology class denoted by $c$. Since $\psi \circ \lambda_s = \lambda_s \circ \psi$,

$$\mathcal{L}_{s\partial}(\psi) = (\lambda_s \circ \partial_M) \circ \psi - \psi \circ (\lambda_s \circ \partial^*) = \lambda_s \circ \mathcal{L}_\partial(\psi).$$

Taking cohomology classes yields that $\mathcal{L}_{s\partial}(c) = \lambda_s \circ \mathcal{L}_\partial(c)$.

(5) Recall that, here, $\partial_M$ is taken equal to

$$(s\partial)^e: S^e \to S^e,$$

$$s_1 \otimes s_2 \mapsto s\partial(s_1) \otimes s_2 + s_1 \otimes s\partial(s_2).$$

Let $\psi \in \text{Hom}_{S^e}(P^{-n}, M)$ be a cocycle with cohomology class denoted by $c$. Let $h$ be as in part (3) of Lemma 4.2.1. Then,

$$\mathcal{L}_{s\partial}(\psi) = (s\partial)^e \circ \psi - \psi \circ (s\partial)^*$$

$$= (s\partial \otimes 1 + 1 \otimes s\partial) \circ \psi - \psi \circ (s\partial)^*$$

$$= \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \psi \circ (s\partial)^*.$$
Using (4.9), the equality becomes
\[ L_{s\partial}(\psi) = \lambda_s \circ (\partial \otimes 1) \circ \psi + \rho_s \circ (1 \otimes \partial) \circ \psi - \lambda_s \circ \psi \circ \partial^* - \psi \circ h \circ d - \psi \circ d \circ h. \]

Using \([\partial, \rho_s] = \rho_{\partial(s)}\), it then becomes
\[ L_{s\partial}(\psi) = \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^* - \psi \circ h \circ d = \lambda_s \circ (\partial \otimes 1) \circ \psi + (1 \otimes \partial) \circ \psi \circ (\rho_s - \lambda_s) + (1 \otimes \partial) \circ \psi \circ \lambda_s - \lambda_s \circ \psi \circ \partial^* - \psi \circ h \circ d. \]

Using (4.2), this becomes
\[ L_{s\partial}(\psi) = \lambda_s \circ (\partial \otimes 1) \circ \psi - \rho_{\partial(s)} \circ \psi - (1 \otimes \partial) \circ \psi \circ d \circ k_s \]
\[ = \lambda_s \circ (\partial \otimes 1 + 1 \otimes \partial) \circ \psi - \rho_{\partial(s)} \circ \psi - \lambda_s \circ \psi \circ \partial^* - (\psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s) \circ d. \]

Now, consider the following $R$-linear mapping denoted by $f$:
\[ \psi \circ h + (1 \otimes \partial) \circ \psi \circ k_s : P^{-n+1} \rightarrow S^e. \]
This is a morphism of $S$-bimodules. Indeed,
- it is a morphism of left $S$-modules because so are $\psi$, $1 \otimes \partial$, $k_s$ and $h$ (see (4.10));
- since $\psi$ and $k_s$ are morphisms of $S$-bimodules, then, for all $t \in S$,
\[ f \circ \rho_t = \psi \circ h \circ \rho_t + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \]
\[(4.10)\]
\[ = \psi \circ (\rho_t \circ h - \rho_{\partial(t)} \circ k_s) + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \]
\[ = \rho_t \circ \psi \circ h - \rho_{\partial(t)} \circ \psi \circ k_s + (1 \otimes \partial) \circ \rho_t \circ \psi \circ k_s \]
\[ = \rho_t \circ \psi \circ h + \rho_t \circ (1 \otimes \partial) \circ \psi \circ k_s \]
\[ = \rho_t \circ f. \]
Therefore, $L_{s\partial}(\psi)$ and $\lambda_s \circ L_{\partial}(\psi) - \rho_{\partial(s)} \circ \psi$ are cohomologous. Since so are $\lambda_{\partial(s)} \circ \psi$ and $\rho_{\partial(s)} \circ \psi$ it follows that
\[ L_{s\partial}(\psi) = \lambda_s \circ L_{\partial}(\psi) - \lambda_{\partial(s)}(e). \]

\[ \square \]

4.4. The action of $L$ on $\text{Ext}^*_S(S,S^e)$. According to Lemma 4.3.1, the mapping
\[ L \times \text{Ext}^*_S(S,S^e) \rightarrow \text{Ext}^*_S(S,S^e) \]
\[ (\alpha, e) \mapsto \alpha \cdot e := L_{\partial}(e) \]
endows $\text{Ext}^*_S(S,S^e)$ with a compatible left $S \times L$-module structure in the sense of (3.13), that is, a left $S \times L$-module structure such that, for all $e \in \text{Ext}^*_S(S,S^e)$, $\alpha \in L$ and $s \in S$,
\[ (s\alpha) \cdot e = s \cdot (\alpha \cdot e) - \alpha(s) \cdot e. \]
This left $S \rtimes L$-module structure on $\operatorname{Ext}^n_S(S,S^\vee)$ does not define a left $U$-module structure in general. However, Lemma 3.5.1 yields that $\operatorname{Ext}^n_S(S,S^\vee)$ is a right $U$-module by defining $\theta \cdot \alpha$, for all $\theta \in \operatorname{Ext}^n_S(S,S^\vee)$ and $\alpha \in L$, as

$$\theta \cdot \alpha : \operatorname{Ext}^n_S(S,S^\vee) \to S \quad e \mapsto -\alpha(\theta(e)) + \theta(\alpha \cdot e).$$

4.5. **Particular case of Van den Bergh and Calabi-Yau duality.** Recall that, whenever $\operatorname{Tor}^n_S(S,S) \simeq S$ as $S$-(bi)modules, a *volume form* is a free generator $\omega_S$ of $\operatorname{Tor}^n_S(S,S)$; and the associated divergence

$$\operatorname{div} : \operatorname{Der}_R(S) \to S$$

is defined such that, for all $\partial \in \operatorname{Der}_R(S)$,

$$(4.15) \quad \mathcal{L}_\partial(\omega_S) = \operatorname{div}(\partial)\omega_S.$$

When $S$ is Calabi-Yau in dimension $n$, any free generator $e_S$ of the left $S$-module $\operatorname{Ext}^n_S(S,S^\vee)$ defines an isomorphism of $S$-bimodules

$$\theta : S \to \operatorname{Ext}^n_S(S,S^\vee) \quad s \mapsto se_S.$$

In such a situation, the fundamental class $c_S \in \operatorname{Tor}^n_S(S,\operatorname{Ext}^n_S(S,S^\vee))$ (see Section 2.1) is a free generator of the left $S$-module $\operatorname{Tor}^n_S(S,\operatorname{Ext}^n_S(S,S^\vee))$; and hence the preimage $\omega_S$ of $c_S$ under the bijective mapping

$$\theta : \operatorname{Tor}^n_S(S,S) \to \operatorname{Tor}^n_S(S,\operatorname{Ext}^n_S(S,S^\vee)).$$

is a volume form for $S$, thus defining a divergence operator.

**Proposition 4.5.1.** The following properties hold.

1. Assume the following:
   - $R$ is Noetherian and $S$ is finitely generated as an $R$-algebra.
   - $S$ is projective in $\operatorname{Mod}(R)$.
   - $S$ has Van den Bergh duality with dimension $n$.

   Then there is an isomorphism of $S$-modules compatible with Lie derivatives

   $$\operatorname{Ext}^n_S(S,S^\vee) \simeq \Lambda^n_S \operatorname{Der}_R(S).$$

2. Assume that $S$ is Calabi-Yau in dimension $n$. Let $e_S$ be a free generator of the left $S$-module $\operatorname{Ext}^n_S(S,S^\vee)$. Let $\operatorname{div}$ be the resulting divergence operator. Then, for all $\partial \in \operatorname{Der}_R(S)$,

   $$(4.16) \quad \mathcal{L}_\partial(e_S) = -\operatorname{div}(\partial)e_S.$$

**Proof.** In both cases, $S$ lies in $\operatorname{per}(S^\vee)$. Denote the fundamental class of $S$ by $c_S$. In view of part (3) of Lemma 3.1, the definition of $c_S$ gives that

$$\mathcal{L}_\partial(c_S) = 0.$$  

(1) Denote $\operatorname{Ext}^n_S(S,S^\vee)$ by $D$. In view of Proposition 2.2.1 [HKR62, Theorem 3.1] applies and yields an isomorphism of $S$-modules

$$(4.17) \quad \operatorname{Tor}^n_S(S,S^\vee) \simeq \Lambda^n_S \Omega_{S/R}.$$

Following [Rim83, Section 9], this isomorphism is compatible with Lie derivatives. Identify $D^{-1}$ with $\operatorname{Hom}_S(D,S)$ and define $\partial_{D^{-1}}$ as follows, for all $\partial \in \operatorname{Der}_R(S)$,

$$\partial_{D^{-1}} : \operatorname{Hom}_S(D,S) \to \operatorname{Hom}_S(D,S) \quad f \mapsto \partial \circ f - f \circ \mathcal{L}_\partial.$$

The evaluation isomorphism

$$(4.19) \quad \operatorname{ev} : D \otimes_S D^{-1} \simeq S$$

...
is compatible with Lie derivatives in the following sense, where \( \partial \in \text{Der}_R(S) \),
\[
(4.20) \quad \partial \circ ev = ev \circ (\mathcal{L}_\partial \otimes \text{Id} + \text{Id} \otimes \partial_{D-1}).
\]

Besides, the duality isomorphism
\[
(4.21) \quad \iota_\gamma(c_S) : \text{Ext}^n_{S,D}(S,D^{-1}) \to \text{Tor}^n_S(S,D \otimes_S D^{-1})
\]

is compatible with the action of Lie derivatives because of (4.17) (see part (3) of Lemma 4.3.1). Combining (4.18), (4.19), (4.20) and (4.21) yields an isomorphism
\[
D^{-1} \simeq \Lambda^n_S \Omega_{S/R}.
\]

This proves (1).

(2) Keep the notation \( c_S, \omega_S, \theta, \theta_* \) for the objects defined from \( e_S \) before the statement of the proposition. Let \( \partial \in \text{Der}_R(S) \). There exists \( \lambda \in S \) such that
\[
\mathcal{L}_\partial(e_S) = \lambda e_S.
\]

Now, for all \( s \otimes p \in S \otimes_S P^n \),
\[
\mathcal{L}_\partial(\theta_*(s \otimes p)) = \mathcal{L}_\partial(se_S \otimes p) = \partial(s)e_S \otimes p + s\mathcal{L}_\partial(e_S) \otimes p + se_S \otimes \partial^*(p) = \theta_*(\mathcal{L}_\partial(s \otimes p)) + \lambda \theta_*(s \otimes p).
\]

Therefore,
\[
0 = \mathcal{L}_\partial(c_S) = \mathcal{L}_\partial(\theta_*(\omega_S)) = \theta_*(\mathcal{L}_\partial(\omega_S)) + \lambda \theta_*(\omega_S) = (\lambda + \text{div}(\partial))c_S.
\]

Since \( c_S \) is regular, \( \lambda = -\text{div}(\partial). \) \( \square \)

5. Proof of the main theorems

The main results of this article are proved in this section. For this purpose, a description of \( \text{Ext}^\bullet_{S,U}(U,U^c) \) is made in 5.1. The underlying \( S \)-module is expressed in terms of \( \text{Ext}^\bullet_{S,U}(S,S^c) \) and \( \text{Ext}^\bullet_{U,U}(S,U) \). The \( S \)-bimodule structure is described using the functor \( F \): \( \text{Mod}(U) \to \text{Mod}(U^c) \) and the action of \( L \) on \( \text{Ext}^\bullet_{S,U}(S,S^c) \) introduced in Section 4. This description is applied in 5.2 in order to prove Theorem 1. And Theorem 2 and Corollary 3 are proved in 5.3 and 5.4 by specialising to the situations where \( \text{Ext}^\bullet_{S,U}(S,S^c) \) and \( \text{Ext}^\bullet_{U,U}(S,U) \) are free, and where \( (S,L) \) arises from a Poisson bracket on \( S \), respectively.

Throughout the section, \( \text{Ext}^\bullet_{S,U}(S,S^c) \) is endowed with its compatible left \( S \times L \)-module structure introduced in 4.3.

5.1. The inverse dualising bimodule of \( U \). This subsection proves the following result.

**Proposition 5.1.1.** Let \( R \) be a commutative ring and \( d \in \mathbb{N} \). Let \( (S,L) \) be a Lie-Rinehart algebra over \( R \). Assume the following:

(a) \( S \) is flat as an \( R \)-module.
(b) For all \( n \in \mathbb{N} \), the \( S \)-module \( \text{Ext}^n_{S,U}(S,S^c) \) is projective.
(c) \( S \in \text{per}(S^c) \).
(d) \( L \) is finitely generated and projective with constant rank equal to \( d \) in \( \text{Mod}(S) \).

Then, \( \Lambda^n_S L^\vee \otimes_S \text{Ext}^\bullet_{S,U}(S,S^c) \) is a graded \( U \)-module such that, for all \( \alpha \in L \), \( c \in \text{Ext}^\bullet_{S,U}(S,S^c) \), and \( \varphi \in \Lambda^n_S L^\vee \),
\[
\alpha \cdot (\varphi \otimes c) = -\varphi \cdot \alpha \otimes c + \varphi \otimes \alpha \cdot c.
\]
Moreover, \( U \) is homologically smooth. Finally, there is an isomorphism of graded right \( U^e \)-modules
\[
\text{Ext}_{U^e}^*(U, U^e) \simeq F(\Lambda_0 S^d \otimes_S \text{Ext}_{S^d}^*(S, S^e)) .
\]

For this subsection, assume (a), (b), (c) and (d) are true, and consider
- a bounded resolution \( Q^\bullet \to S \) in \( \text{Mod}(U) \) by finitely generated and projective modules (see [Rin63, Lemma 4.1]),
- a bounded resolution \( \pi: P^\bullet \to S \) in \( \text{Mod}(S^e) \) by finitely generated and projective modules,
- an injective resolution \( j: U^e \to I^\bullet \) in \( \text{Mod}(U^e \otimes (U^e)^{op}) \).

Since \( S \) is flat over \( R \) and \( L \) is projective in \( \text{Mod}(S) \), part (2) of Lemma 3.0.1 gives that \( U^e \) is flat over \( R \). Therefore, the extension-of-scalars functor
\[
- \otimes U^e: \text{Mod}(U^e) \to \text{Mod}(U^e \otimes (U^e)^{op})
\]
is exact. Hence, the restriction-of-scalars-functor transforms injective \( U^e \)-bimodules into injective left \( U^e \)-modules. Thus, \( I^\bullet \) is an injective resolution of \( U^e \) in \( \text{Mod}(U^e) \). Therefore, there is an isomorphism of graded right \( U^e \)-modules
\[
\text{Ext}_{U^e}^*(U, U^e) \simeq H^*\text{Hom}_{U^e}(U, I^\bullet) .
\]
The right-hand side is a right \( U^e \)-module by means of \( I^\bullet \).

The proof of the above proposition is divided into separate lemmas.

**Lemma 5.1.2.** \( U \) is homologically smooth.

*Proof.* Since \( U \) is projective in \( \text{Mod}(S) \) (see part (2) of Lemma 3.0.1), the functor
\[
F: \text{Mod}(U) \to \text{Mod}(U^e)
\]
is exact. Moreover, \( F(S) \simeq U \) and \( S \in \text{per}(U) \). Therefore, in order to prove that \( U \) is homologically smooth, it suffices to prove that \( F(U) \in \text{per}(U^e) \), which is equivalent to \( F(U) \) being compact in the derived category \( D(U^e) \) of complexes of \( U \)-bimodules. Here is a proof of this fact. Let \( (M_k)_{k \in K} \) be a family in \( D(U^e) \), denote \( \oplus_{k \in K} M_k \) by \( M \), and consider fibrant resolutions of complexes of \( U \)-bimodules \( M_k \to i(M_k) \), for all \( k \in K \), and \( M \to i(M) \). Since \( S \) is homologically smooth, then \( S \) is compact in \( D(S^e) \), and hence the following natural mapping is a quasi-isomorphism,
\[
\bigoplus_{k \in K} \text{Hom}_{S^e}(P^\bullet, M_k) \to \text{Hom}_{S^e}(P^\bullet, M) .
\]
Since \( P^\bullet \) is a right bounded complex of projective \( S \)-bimodules, then the functor \( \text{Hom}_{S^e}(P^\bullet, -) \) preserves quasi-isomorphisms, and hence the following natural mapping is a quasi-isomorphism,
\[
\bigoplus_{k \in K} \text{Hom}_{S^e}(P^\bullet, i(M_k)) \to \text{Hom}_{S^e}(P^\bullet, i(M)) .
\]
Since \( U \) is projective over \( S \) on both sides, \( U^e \) is projective in \( \text{Mod}(S^e) \). Therefore, for all fibrant complexes \( I \) of \( U \)-bimodules, the functor \( \text{Hom}_{S^e}(-, I) \) preserves quasi-isomorphisms. Accordingly, the following natural mapping is a quasi-isomorphism:
\[
\bigoplus_{k \in K} \text{Hom}_{S^e}(S, i(M_k)) \to \text{Hom}_{S^e}(S, i(M)) .
\]
Since the pair \( (F, G) \) is adjoint and \( G \) is induced by the functor \( \text{Hom}_{S^e}(S, -) \), the following natural mapping is a quasi-isomorphism:
\[
\bigoplus_{k \in K} \text{Hom}_{U^e}(F(U), i(M_k)) \to \text{Hom}_{U^e}(F(U), i(M)) .
\]
Taking cohomology in degree 0 yields that the following natural mapping is bijective:

$$\bigoplus_{k \in K} D(U^e)(F(U), i(M_k)) \rightarrow D(U^e)(F(U), i(M)) .$$

This proves that $F(U)$ is compact in $D(U^e)$. Thus, $U$ is homologically smooth. □

The authors thank Bernhard Keller for having pointed out an incorrect argument in a previous version of this proof.

**Lemma 5.1.3.** There is an isomorphism of graded right $U^e$-modules

$$\text{Ext}^{\bullet}_{U^e}(U, U^e) \cong H^\bullet(\text{Hom}_U(Q^*, U) \otimes_U G(I^*)) .$$

**Proof.** Because of the isomorphism $F(S) \cong U$ in $\text{Mod}(U^e)$ and the adjunction $(F, G)$, there is a functorial isomorphism of complexes of right $U^e$-modules

$$\text{Hom}_{U^e}(U, I^*) \cong \text{Hom}_U(S, G(I^*)) .$$

Since $F$ is exact and the pair $(F, G)$ is adjoint, $G(I^*)$ is a left bounded complex of injective left $U$-modules. Hence, $\text{Hom}_{U^e}(\cdot, G(I^*))$ preserves quasi-isomorphisms. Thus, the quasi-isomorphism $Q^* \rightarrow S$ induces a quasi-isomorphism of complexes of right $U^e$-modules

$$\text{Hom}_{U^e}(S, G(I^*)) \rightarrow \text{Hom}_{U^e}(Q^*, G(I^*)) .$$

Since $Q^*$ is bounded and consists of finitely generated projective left $U$-modules, the following canonical mapping is a functorial isomorphism:

$$\text{Hom}_{U^e}(Q^*, U) \otimes_U G(I^*) \rightarrow \text{Hom}_{U^e}(Q^*, G(I^*)) .$$

Note that, whether in (5.3), (5.4), or (5.5), the involved right $U^e$-module structures are inherited from $I^*$. Thus, the announced isomorphism is proved. □

In order to examine the right-hand side of (5.2) by means of a spectral sequence, the following lemma describes $H^\bullet(G(I^*))$ as a graded $U-U^e$-bimodule.

**Lemma 5.1.4.** Consider $\text{Ext}^{\bullet}_{S^e}(S, S^e)$ as a left $S \rtimes L$-module as in Section 4.3. Then, there is a $U-U^e$-bimodule structure on $\text{Ext}^{\bullet}_{S^e}(S, S^e) \otimes_{S^e} U^e$ such that the $U^e$-module structure is inherited from $U^e$ and for all $\alpha \in L$, $c \in \text{Ext}^{\bullet}_{S^e}(S, S^e)$ and $u, v \in U$,

$$\alpha \cdot (c \otimes (u \otimes v)) = \alpha \cdot c \otimes (u \otimes v) + c \otimes ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)) .$$

For this structure, there is an isomorphism of graded $U-U^e$-bimodules

$$H^\bullet(G(I^*)) \cong \text{Ext}^{\bullet}_{S^e}(S, S^e) \otimes_{S^e} U^e .$$

**Proof.** The object $G(I^*)$ is $\text{Hom}_{S^e}(S, I^*)$ as a complex of $S$-modules, its right $U^e$-module structure is inherited from $I^*$, and the one of left $U$-module is given in Section 3.2.

First, since $U^e$ is projective in $\text{Mod}(S^e)$ and $I^*$ consists of injective left $U^e$-modules, then $I^*$ is a left bounded complex of injective left $S^e$-modules. Hence, $\text{Hom}_{S^e}(-, I^*)$ preserves quasi-isomorphisms. Thus, $\pi : P^* \rightarrow S$ induces a quasi-isomorphism of complexes of right $S^e$-modules

$$\pi' : \text{Hom}_{S^e}(S, I^*) \rightarrow \text{Hom}_{S^e}(P^*, I^*) .$$

For all $\alpha \in L$, let $\partial_\alpha : P^* \rightarrow P^*$ be a derivation relative to $\partial_\alpha : S \rightarrow S$ (see Section 4.2), and denote by $\delta_\alpha$ the mapping from $I^*$ to $I^*$ given by

$$i \mapsto (\alpha \otimes 1 - 1 \otimes \alpha) \cdot i .$$
Then, define $\alpha \cdot f$ and $\alpha \cdot g$, for all $f \in \Hom_{S^e}(S, \mathbb{I}^\bullet)$ and $g \in \Hom_{S^e}(\mathbb{P}^\bullet, \mathbb{I}^\bullet)$, by
\[
\begin{align*}
\alpha \cdot f &= \delta_\alpha \circ f - f \circ \partial_\alpha \\
\alpha \cdot g &= \delta_\alpha \circ g - g \circ \partial_\alpha ;
\end{align*}
\]
since $\pi \circ \partial_\alpha = \partial_\alpha \circ \pi$,
\[
\pi'(\alpha \cdot f) = \alpha \cdot \pi'(f).
\]
The hypotheses on $\mathbb{P}^\bullet$ yield an isomorphism of complexes of right $U^e$-modules,
\[
(5.7) \quad \ev : \Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} \mathbb{I}^\bullet \to \Hom_{S^e}(\mathbb{P}^\bullet, \mathbb{I}^\bullet).
\]
Endow the left-hand side term with the following action of $L$. For all $\alpha \in L$ and $\varphi \circ i \in \Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} \mathbb{I}^\bullet$, denote by $\alpha \cdot (\varphi \circ i)$ the (well-defined) element of $\Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} \mathbb{I}^\bullet$,
\[
\alpha \cdot \varphi \circ i = \varphi \circ (\delta_\alpha i).
\]
The assignment $\varphi \circ i \mapsto \alpha \cdot (\varphi \circ i)$ is a morphism of complexes of $R$-modules from $\Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} \mathbb{I}^\bullet$ to itself. In view of (4.8) and of the identity
\[
(\alpha \otimes 1 - 1 \otimes \alpha) \cdot ((s \otimes t) \cdot j) = \partial_\alpha (s \otimes t) \cdot j + (s \otimes t) \cdot (\alpha \otimes 1 - 1 \otimes \alpha) \cdot j
\]
in $\mathbb{I}^\bullet$, for all $s, t \in S$ and $j \in \mathbb{I}^\bullet$, the following holds
\[
(5.8) \quad \ev(\alpha \cdot (\varphi \circ i)) = \alpha \cdot \ev(\varphi \circ i).
\]
$\Hom_{S^e}(\mathbb{P}^\bullet, S^e)$ is also a bounded complex of projective right $S^e$-modules. Hence, the functor $\Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} -$ preserves quasi-isomorphisms. Thus, $j : U^e \to \mathbb{I}^\bullet$ induces a quasi-isomorphism of right $U^e$-modules,
\[
(5.9) \quad \Id \otimes j : \Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} U^e \to \Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} \mathbb{I}^\bullet.
\]
Endow the left-hand side term with the following action of $L$. For all $\alpha \in L$, $\varphi \in \Hom_{S^e}(\mathbb{P}^\bullet, S^e)$ and $u, v \in U$, denote by $\alpha \cdot (\varphi \circ (u \otimes v))$ the following (well-defined) element of $\Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} U^e$:
\[
\alpha \cdot \varphi \circ (u \otimes v) = \varphi \circ ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v)) ;
\]
The assignment $\varphi \circ (u \otimes v) \mapsto \alpha \cdot (\varphi \circ (u \otimes v))$ is a morphism of complexes of $R$-modules from $\Hom_{S^e}(\mathbb{P}^\bullet, S^e) \otimes_{S^e} U^e$ to itself, and
\[
(\Id \otimes j)(\alpha \cdot (\varphi \circ (u \otimes v))) = \alpha \cdot ((\Id \otimes j)(\varphi \circ (u \otimes v)))
\]
because $j : U^e \to \mathbb{I}^\bullet$ is a morphism of complexes of $U^e - U^e$-bimodules.

Since $U^e$ is projective in $\text{Mod}(S^e)$, there is an isomorphism of right $U^e$-modules,
\[
(5.10) \quad \Ext^*_{S^e}(S, S^e) \otimes_{S^e} U^e \cong \Ext^*_{S^e}(S, S^e) \otimes_{S^e} U^e .
\]
For all cocycles $\varphi \in \Hom_{S^e}(\mathbb{P}^\bullet, S^e)$, with cohomology class denoted by $c$, and for all $\alpha \in L$ and $u, v \in U$, the image under (5.10) of the cohomology class of
\[
\alpha \cdot (\varphi \circ (u \otimes v))
\]
is
\[
(5.11) \quad \alpha \cdot c \circ (u \otimes v) + c \circ ((\alpha \otimes 1 - 1 \otimes \alpha) \cdot (u \otimes v))
\]
where $\alpha \cdot c$ is defined in (4.4) (see (4.13)).

Combining (5.6), (5.7), (5.9), (5.10) yields an isomorphism of right $U^e$-modules,
\[
(5.12) \quad \Ext^*_S(S, S^e) \otimes_{S^e} U^e \cong \Hom(S^e, G(\mathbb{I}^\bullet)) ,
\]
such that, for all $\alpha \in L$, $c \in \Ext^*_S(S, S^e)$ and $u, v \in U$, if $\gamma$ denotes the image of $c \circ (u \otimes v)$ under (5.12), then $\alpha \cdot \gamma$ is the image of (5.11).

Thus, applying part (1) of Lemma 3.5.2 to $N = \Ext^*_S(S, S^e)$ yields the announced conclusion. \qed
Proof of Proposition 5.1.1. The statement relative to the left $U$-module structure on $\Lambda_S^d L^\vee \otimes \text{Ext}^*_S(S, S^e)$ follows from Lemma 3.5.1 and Lemma 5.1.2 shows that $U$ is homologically smooth. The (first quadrant, cohomological) spectral sequence converges to $H^\bullet(Hom_U(Q^\bullet, U) \otimes_U G(I^\bullet))$ and its $E_2^{p,q}$-term is, for all $p, q \in \mathbb{Z}$,

\begin{equation}
H^p_q(H^q(Hom_U(Q^\bullet, U) \otimes_U G(I^\bullet))).
\end{equation}

Since $\text{Hom}_U(Q^\bullet, U)$ consists of projective right $U$-modules, there is an isomorphism of right $U^e$-modules, for all $p, q \in \mathbb{Z}$,

\begin{equation}
\text{Hom}_U(Q^p, U) \otimes_U G(I^q) \cong \text{Hom}_U(Q^q, U) \otimes_U G(I^p)).
\end{equation}

The description of $H^\bullet(G(I^\bullet))$ made in Lemma 5.1.4 combines with (5.14) into the following isomorphism of right $U^e$-modules, for all $p, q \in \mathbb{Z}$,

\begin{equation}
H^q(H\text{Hom}_U(Q^p, U) \otimes_U G(I^q)) \cong \text{Hom}_U(Q^p, U) \otimes_U (\text{Ext}^q_{S^e}(S, S^e) \otimes_{S^e} U^e).
\end{equation}

Using Lemma 3.5.2 (part (2)), this isomorphism yields isomorphism of right $U^e$-modules, for all $p, q \in \mathbb{Z}$,

\begin{equation}
H^q(H\text{Hom}_U(Q^p, U) \otimes_U G(I^q)) \cong F(\text{Hom}_U(Q^p, U) \otimes_S \text{Ext}^q_{S^e}(S, S^e)).
\end{equation}

Given that $F$ is an exact functor, that $\text{Ext}^q_{S^e}(S, S^e)$ is projective in $\text{Mod}(S)$ for all $q$ and that $(S, L)$ has duality in dimension $d$, it follows from (5.16) that there is an isomorphism of right $U^e$-modules, for all $p, q \in \mathbb{Z}$,

\begin{equation}
H^p_q(H^q(H\text{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet))) \cong \begin{cases} F(\text{Ext}^q_{S^e}(S, S^e)) & \text{if } p = d, \\ 0 & \text{if } p \neq d. \end{cases}
\end{equation}

Therefore, the spectral sequence of the bicomplex (5.13) degenerates at $E_2$. Thus,

\begin{equation}
H^\bullet(H\text{Hom}_U(Q^\bullet, U) \otimes_U G(I^\bullet)) \cong F(\text{Ext}^q_{S^e}(S, S^e) \otimes_S \text{Ext}^\bullet_{S^e}(S, S^e)) \quad \text{in } \text{Mod}(S^e).
\end{equation}

The conclusion follows from [5.2] and from the isomorphism $\text{Ext}^d_{S^e}(S, U) \cong \Lambda_S^d L^\vee$ in $\text{Mod}(U)$ established in [Hue99] Theorem 2.10.

5.2. Proof of the main theorem.

Proof of Theorem 5.2. Following Proposition 5.1.1, $U$ is homologically smooth and there is an isomorphism of graded right $U^e$-modules

\begin{equation}
\text{Ext}^*_U(U, U^e) \cong F(\Lambda_S^d L^\vee \otimes_S \text{Ext}^\bullet_{S^e}(S, S^e)).
\end{equation}

According to Proposition 3.6.1, the functor $F$ transforms left $U$-modules that are invertible as $S$-modules into invertible $U$-bimodules. Note that

- $\Lambda_S^d L^\vee$ is invertible as an $S$-module because $L$ is projective with constant rank
- $\text{Ext}^n_{S^e}(S, S^e)$ is concentrated in degree $n$ and $\text{Ext}^n_{S^e}(S, S^e)$ is invertible as an $S$-module because $S$ has Van den Bergh duality.

Thus, $\text{Ext}^*_U(U, U^e)$ is concentrated in degree $n + d$ and $\text{Ext}^n_{S^e}(U, U^e)$ is invertible as a $U$-bimodule. Hence, $U$ has Van den Bergh duality in dimension $n + d$. 

\[\square\]
5.3. **Proof of Theorem 2.** The hypotheses of Theorem 2 are assumed throughout the subsection. Let \( \varphi_L \) be a free generator of the \( S \)-module \( \Lambda^d S L^{\nu} \). Let \( e_S \) be a free generator of the \( S \)-module \( \text{Ext}^*_S(S, S^n) \). Therefore, there exist mappings

\[
\lambda_L, \lambda_S: L \to S
\]
such that, for all \( \alpha \in L \),

\[
\begin{align*}
\alpha \cdot e_S &= \lambda_S(\alpha) \cdot e_S \\
\varphi_L \cdot \alpha &= \lambda_L(\alpha) \cdot \varphi_S.
\end{align*}
\]

Some basic properties of these are summarised below.

**Lemma 5.3.1.** Let \( \lambda \) be either one of \( \lambda_S \) or \( \lambda_L \). Then, for all \( \alpha, \beta \in L \) and \( s \in S \),

1. \( \lambda(s\alpha) = s\lambda(\alpha) - \alpha(s) \);
2. \( \lambda([\alpha, \beta]) = \alpha(\lambda(\beta)) - \beta(\lambda(\alpha)) \).

**Proof.** Assume that \( \lambda = \lambda_S \). Let \( s \in S \) and \( \alpha \in L \). Then, using [4.4]

\[
(s\alpha) \cdot e_S = s \cdot (\alpha \cdot e_S) - \alpha(s) \cdot e_S = (s\lambda(\alpha) - \alpha(s)) \cdot e_S,
\]

which proves (1), and

\[
\alpha \cdot (\beta \cdot e_S) = \alpha \cdot (\lambda(\beta) \cdot e_S)
\]

\[
= \alpha(\lambda(\beta)) \cdot e_S + \lambda(\beta) \cdot (\alpha \cdot e_S)
\]

\[
= (\alpha(\lambda(\beta)) + \lambda(\alpha)s\lambda(\beta)) \cdot e_S,
\]

from which (2) may be proved directly. The proof when \( \lambda = \lambda_L \) is analogous, using the right \( U \)-module structure of \( \Lambda^d S L^{\nu} \) instead of [4.4] \( \square \)

As proved later, the following automorphism is a Nakayama automorphism for \( U \).

**Lemma 5.3.2.** There exists a unique \( R \)-algebra homomorphism,

\[
\nu: U \to U,
\]

such that, for all \( s \in S \) and \( \alpha \in L \),

\[
\begin{align*}
\nu(s) &= s \\
\nu(\alpha) &= \alpha + \lambda_L(\alpha) - \lambda_S(\alpha)
\end{align*}
\]

This is an automorphism of \( R \)-algebra.

**Proof.** The uniqueness is immediate. For all \( \alpha \in L \), denote \( \alpha + \lambda_L(\alpha) - \lambda_S(\alpha) \) by \( \nu_\alpha \). Then, for all \( s \in S \) and \( \alpha, \beta \in L \),

\[
\begin{align*}
[\nu_\alpha, \nu_\beta] &= \lambda_L(\alpha) - \lambda_S(\alpha), \beta + \lambda_L(\beta) - \lambda_S(\beta)] \\
&= \lambda_L([\alpha, \beta]) \text{ by Lemma 5.3.1.}
\end{align*}
\]

\[
\nu_{s\alpha} = s\alpha + \lambda_L(s\alpha) - \lambda_S(s\alpha)
\]

\[
= s\alpha + s\lambda_L(\alpha) - s\lambda_S(\alpha) = s\nu_\alpha
\]

\[
[\nu_\alpha, s] = [\alpha + \lambda_L(\alpha) - \lambda_L(\alpha), s] = \alpha(s).
\]

This proves the existence of \( \nu \). Note that \( \nu \) preserves the filtration of \( U \) by the powers of \( L \) and that \( \text{gr}(\nu) \) is the identity mapping of \( U \). Accordingly, \( \nu \) is bijective. \( \square \)

Now it is possible to prove Theorem 2.
Proof of Theorem 2. From Theorem 1, $U$ has Van den Bergh duality in dimension $n + d$ and there is an isomorphism of $U$-bimodules

\[ \text{Ext}^{n+d}_{U^n}(U, U^c) \simeq F(\Lambda^d_S L^\vee \otimes_S \text{Ext}^n_{S^n}(S, S^c)), \]

where the tensor product inside $F(\bullet)$ is a left $U$-module by \[[3.8].\]

Recall that $\Lambda^d_S L^\vee$ and $\text{Ext}^n_{S^n}(S, S^c)$ are freely generated by $\varphi_L$ and $e_s$, respectively. Therefore, the following mapping is an isomorphism of left $U$-modules (see Section 3.3).

\[ \Phi: \quad U \to F(\Lambda^d_S L^\vee \otimes_S \text{Ext}^n_{S^n}(S, S^c)) \]

\[ u \mapsto u \otimes (\varphi_L \otimes e_s). \]

For all $s \in S$, $\alpha \in L$ and $u \in U$,

\[ \Phi(u)s = (u \otimes (\varphi_L \otimes e_s)) \cdot s = us \otimes (\varphi_L \otimes e_s) = \Phi(us), \]

\[ \Phi(u)\alpha = (u \otimes (\varphi_L \otimes e_s)) \cdot \alpha = u\alpha \otimes (\varphi_L \otimes e_s) - u \otimes (\varphi_L \otimes e_s) \]

\[ = u\alpha \otimes (\varphi_L \otimes e_s) - (u \otimes (\varphi_L \otimes e_s) = u \otimes (\varphi_L \otimes e_s) \]

\[ = (u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))) \otimes (\varphi_L \otimes e_s) = \Phi(u(\alpha + \lambda_L(\alpha) - \lambda_S(\alpha))). \]

Thus, denoting by $\nu$ the automorphism of $U$ considered in Lemma \[[5.3.2]; then, for all $u, v \in U$,

\[ \Phi(u) \cdot v = \Phi(u\nu(v)). \]

Combining \[[5.17], \[5.18\] and \[[5.19\] yields that there is an isomorphism of bimodules

\[ \text{Ext}^{n+d}_{U^n}(U, U^c) \simeq U^\nu. \]

Since $\lambda_S = -\text{div}$ (see Proposition \[4.5.1\]), this proves Theorem 2. \hfill \square

5.4. Case of Poisson algebras.

Proof of Corollary 7. From Proposition \[2.2.1\] $S$ has Van den Bergh duality in dimension $n$. Moreover, Proposition \[4.5.1\] yields an isomorphism of $S$-modules $\Lambda^d_S \text{Der}_R(S) \simeq \text{Ext}^n_{S^n}(S, S^c)$ which is compatible with the action of Lie derivations. Finally, according to \[[1.3\], the dualising module of $(S, \Omega_{S/R})$ is $\Lambda^d_S \text{Der}_R(S)$ with right $U$-module structure such that, for all $s \in S$ and $\varphi \in \Lambda^d_S \text{Der}_R(S)$,

\[ \varphi \cdot ds = -L_{(s, -)}(\varphi). \]

Using these considerations, the corollary follows from Theorems 1 and 2. \hfill \square

6. Examples

6.1. The case where $L$ is free as an $S$-module. In this subsection, it is assumed that $L$ is free as an $S$-module. Consider a basis $(\alpha_1, \ldots, \alpha_d)$ of $L$ over $S$. Denote the dual basis of $L^\vee$ by $(\alpha^*_1, \ldots, \alpha^*_d)$. In particular, $\Lambda^d_S L^\vee$ is free of rank one in $\text{Mod}(S)$, with a generator denoted by $\varphi_L$,

\[ \varphi_L = \alpha^*_1 \wedge \cdots \wedge \alpha^*_d. \]

For all $i \in \{1, \ldots, d\}$, consider the matrix of $\text{ad}_{\alpha_i}$, denoted by $(s^j_{i,k})_{j,k} \in M_d(S)$. Hence, for all $i, k \in \{1, \ldots, d\}$,

\[ [\alpha_i, \alpha_k] = \sum_{j=1}^d s^j_{i,k} \alpha_j. \]
In this situation, the action of $L$ on $\Lambda^*_S L$ by Lie derivatives specialises as follows. For all $i, j, k \in \{1, \ldots, d\}$,

\[
(\lambda_{\alpha_i}(\alpha_j))(\alpha_k) = \alpha_i(\alpha_j^*(\alpha_k)) - \alpha_j^*([\alpha_i, \alpha_k]) = -s^i_{j,k}.
\]

Hence, for all $i, j \in \{1, \ldots, d\}$,

\[
\lambda_{\alpha_i}(\alpha_j) = -\sum_{k=1}^d s^i_{j,k} \alpha_k.
\]

Thus, the right $U$-module structure of $\Lambda^d_S L^\vee$ is such that, for all $\alpha \in L$,

\[
\varphi_L \cdot \alpha = \text{Tr}(\text{ad}_\alpha)\varphi_L.
\]

Using this simplified description of $\Lambda^d_S L^\vee$ yields the following corollary of the main theorems of this article.

**Corollary 6.1.1.** Let $R$ be a commutative ring. Let $(S, L)$ be a Lie-Rinehart algebra of $R$. Denote by $U$ its enveloping algebra. Assume that

- $S$ is flat as an $R$-module,
- $S$ has Van den Bergh duality in dimension $n$,
- $L$ is free of rank $d$ as an $S$-module.

Let $(\alpha_1, \ldots, \alpha_d)$ be a basis of $L$ over $S$ as considered previously. Then, $U$ has Van den Bergh duality in dimension $n + d$ and there is an isomorphism of $U$-bimodules

\[
\text{Ext}^{n+d}_U(U, U^c) \simeq U \otimes_S \text{Ext}^n_S(S, S^c),
\]

where the left $U$-module structure on $U \otimes_S \text{Ext}^n_S(S, S^c)$ is the natural one and the right $U$-module structure is such that, for all $u \in U$, $e \in \text{Ext}^n_S(S, S^c)$ and $\alpha \in L$,

\[
(u \otimes e) \cdot \alpha = u\alpha \otimes e + u \otimes \text{Tr}(\text{ad}_\alpha)e - u \otimes L_{\alpha}(e).
\]

If, moreover, $S$ is Calabi-Yau, then $U$ is skew Calabi-Yau and each volume form on $S$ determines a Nakayama automorphism $\nu \in \text{Aut}_R(U)$ such that, for all $s \in S$ and $\alpha \in L$,

\[
\begin{cases}
\nu(s) = s \\
\nu(\alpha) = \alpha + \text{Tr}(\text{ad}_\alpha) + \text{div}(\partial_\alpha),
\end{cases}
\]

where $\text{div}$ denotes the divergence of the chosen volume form.

**Proof.** In view of (6.1), there is an isomorphism of right $U$-modules

\[
\Lambda^d_S L^\vee \simeq S,
\]

where the right $U$-module structure on the right-hand side term is such that, for all $\alpha \in L$,

\[
1 \cdot \alpha = \text{Tr}(\text{ad}_\alpha).
\]

The corollary therefore follows directly from Theorems 1 and 2. \qed

The previous corollary applies to any Lie-Rinehart algebra arising from a Poisson structure on $R[x_1, \ldots, x_n]$, $n \in \mathbb{N}\setminus\{0, 1\}$.

**Example 6.1.2.** Let $S = R[x, y]$. Let $P \subseteq S$. This defines a Poisson structure on $S$ such that

\[
\{x, y\} = P.
\]

Let $L := \Omega_{S/R}$ and consider $(S, L)$ as a Lie-Rinehart algebra over $R$ such that, for all $s, t \in S$,

- $[ds, dt] = d\{s, t\}$;
- $\partial_{ds} = \{s, -\}$. 
Then $(dx, dy)$ is a basis of $\Omega_{S/R}$ over $S$. Note that
\[
\begin{cases}
\text{Tr}(\text{ad}_{dx}) = \text{div}(\partial_{dx}) = \frac{\partial P}{\partial y} \\
\text{Tr}(\text{ad}_{dy}) = \text{div}(\partial_{dy}) = -\frac{\partial P}{\partial x}.
\end{cases}
\]
From Corollary 6.1.1, $U$ is skew Calabi-Yau in dimension $4$ and has a Nakayama automorphism $\nu \in \text{Aut}_R(S)$ such that
\[
\begin{cases}
\nu(x) = x, & \nu(dx) = dx + 2\frac{\partial P}{\partial y} \\
\nu(y) = y, & \nu(dy) = dy - 2\frac{\partial P}{\partial x}.
\end{cases}
\]
By considering the filtration of $U$ by the powers of the image of $L$ in $U$, with associated graded algebra the symmetric algebra of $L$ over $S$ (see [Rin63, Theorem 3.1]), it appears that $U \times = S \times = R \times$. Accordingly, $U$ has no nontrivial inner automorphism. Consequently, $U$ is Calabi-Yau if and only if $\nu = \text{Id}_U$, that is, if and only if $\text{char}(R) = 2$, or else $P \in R$.

Example 6.1.3. Let $S = R[x, y, z]$. Let $P_x, P_y, P_z \in S$ be such that
\[
\nabla \wedge \text{curl}(\overrightarrow{P}) = 0,
\]
where $\overrightarrow{P}$ denotes
\[
\begin{pmatrix}
P_x \\
P_y \\
P_z
\end{pmatrix}.
\]
Hence, the following defines a Poisson bracket on $S$,
\[
\{x, y\} = P_z, \quad \{y, z\} = P_x, \quad \{z, x\} = P_y.
\]
As in the previous example, let $(S, L := \Omega_{S/R})$ be the associated Lie-Rinehart algebra over $R$. As is well-known,
\[
\{x, -\} = P_x \frac{\partial}{\partial y} - P_y \frac{\partial}{\partial x}, \quad \{y, -\} = P_x \frac{\partial}{\partial z} - P_z \frac{\partial}{\partial x}, \quad \{z, -\} = P_y \frac{\partial}{\partial x} - P_x \frac{\partial}{\partial y}.
\]
Therefore, using the basis $(dx, dy, dz)$ of $\Omega_{S/R}$ over $S$,
\[
\begin{pmatrix}
\text{div}(\partial_{dx}) \\
\text{div}(\partial_{dy}) \\
\text{div}(\partial_{dz})
\end{pmatrix} = \begin{pmatrix}
\text{Tr}(\text{ad}_{dx}) \\
\text{Tr}(\text{ad}_{dy}) \\
\text{Tr}(\text{ad}_{dz})
\end{pmatrix} = \text{curl}(\overrightarrow{P}).
\]
Using Corollary 6.1.1 it follows that $U$ is skew Calabi-Yau in dimension $6$ and has a Nakayama automorphism $\nu \in \text{Aut}_R(S)$ such that
\[
\begin{pmatrix}
\nu(x) \\
\nu(y) \\
\nu(z)
\end{pmatrix} = \begin{pmatrix}
x \\
y \\
z
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
\nu(dx) \\
\nu(dy) \\
\nu(dz)
\end{pmatrix} = \begin{pmatrix}
dx \\
dy \\
dz
\end{pmatrix} + 2\text{curl}(\overrightarrow{P}).
\]
As in the previous example, there are no nontrivial inner automorphisms for $U$. Hence, $U$ is Calabi-Yau if and only if $\text{char}(R) = 2$, or else $\text{curl}(\overrightarrow{P}) = 0$. In particular, when $R$ contains $\mathbb{Q}$ as a subring, then $U$ is Calabi-Yau if and only if the Poisson bracket is Jacobian, that is, there exists $Q \in S$ such that $P = \text{grad}(Q)$.

By the Quillen-Suslin Theorem, when $R$ is a field and $n \in \mathbb{N}$, any $R[x_1, \ldots, x_n]$-module that is finitely generated and projective is free. Hence, Corollary 6.1.1 also applies to all Lie-Rinehart algebras of the shape $(R[x_1, \ldots, x_n], L)$, where $R$ is a field.
6.2. **On two dimensional Nambu-Poisson structures.** Following Corollary 1, $U$ is skew Calabi-Yau when $S$ is flat over $R$ and Calabi-Yau and $(S, L)$ is given by a Poisson bracket on $S$. Assuming these properties, this section computes a Nakayama automorphism of $U$ for a class of examples of two dimensional Nambu-Poisson structures (see [LGPV13, Section 8.3]).

Let $S = R[x, y, z]/(P)$ where $P = 1 + T$ for some $T ∈ R[x, y, z]$ which is $(p, q, r)$-homogeneous in the sense that $p, q, r ∈ R$ and $t := pα + qβ + rγ$ is a unit in $R$ which does not depend on the monomial $x^αy^βz^γ$ appearing in $T$. The hypotheses imply the following equality in $S$

\[
px \frac{∂P}{∂x} + qy \frac{∂P}{∂y} + rz \frac{∂P}{∂z} = -t \ (∈ R^x).
\]

Let $Q ∈ R[x, y, z]$ and endow $S$ with the Poisson structure such that

\[
\{x, y\} = Q \frac{∂P}{∂z}, \ \{y, z\} = Q \frac{∂P}{∂x}, \ \{z, x\} = Q \frac{∂P}{∂y}.
\]

Consider $(S, L := Ω_S/R)$ as a Lie-Rinehart algebra such that, for all $s, t, s' ∈ S$,

- $[ds, dt] = d\{s, t\}$,
- $(sdt)(s') = s\{t, s'\}$.

Consider the following 2-form on $S$

\[
ω_S = px dy ∧ dz + qy dz ∧ dx + rz dx ∧ dy.
\]

According to (6.2), $Ω_S/R$ is a projective $S$-module of rank 2. And the relation

\[
\frac{∂P}{∂x} dx + \frac{∂P}{∂y} dy + \frac{∂P}{∂z} dz = 0
\]

in $Ω_S/R$ yields the following relations in $Λ^2 S Ω_S/R$

\[
\frac{∂P}{∂x} dx ∧ dy = \frac{∂P}{∂z} dy ∧ dz,
\]

\[
\frac{∂P}{∂y} dy ∧ dz = \frac{∂P}{∂x} dz ∧ dx,
\]

\[
\frac{∂P}{∂z} dz ∧ dx = \frac{∂P}{∂y} dx ∧ dy.
\]

Combining with (6.2) yields

\[
dx ∧ dy = -t^{-1} \frac{∂P}{∂z} ω_S,
\]

\[
dy ∧ dz = -t^{-1} \frac{∂P}{∂x} ω_S,
\]

\[
dz ∧ dx = -t^{-1} \frac{∂P}{∂y} ω_S.
\]

Thus, $ω_S$ is a volume form of $S$.

In order to determine the divergence of $ω_S$, consider the derivations $δ_x, δ_y, δ_z ∈ Der_R(S)$ given by

\[
δ_x: \ x \mapsto 0, \ y \mapsto 0, \ z \mapsto -\frac{∂P}{∂z};
\]

\[
δ_y: \ x \mapsto -\frac{∂P}{∂x}, \ y \mapsto 0, \ z \mapsto \frac{∂P}{∂y};
\]

\[
δ_z: \ x \mapsto \frac{∂P}{∂x}, \ y \mapsto -\frac{∂P}{∂y}, \ z \mapsto 0.
\]

Note that

\[
\{x, -\} = Qδ_x, \ \{y, -\} = Qδ_y \text{ and } \{z, -\} = Qδ_z.
\]
Then,
\[
\iota_S(\omega_S) = \iota_S(pxdy \wedge dz + qyd\bar{z} \wedge dx + rzdx \wedge dy) = px\left(\frac{\partial P}{\partial z} dz + \frac{\partial P}{\partial y} dy\right) - qy\frac{\partial P}{\partial y} dx - rz\frac{\partial P}{\partial z} dx \]
\[
= tdx \quad \text{ (see (6.2)).}
\]
Therefore, using the symmetry between \(x, y\) and \(z\),
\[
\text{div}(\iota_x) = \text{div}(\iota_y) = \text{div}(\iota_z) = 0.
\]
Apply Lemma 5.3.1 taking into account that \(\lambda_S = -\text{div} \) (see (4.16); then,
\[
\text{div} \{x, -\} = \text{div}(Q\iota_x) = Q\text{div}(\iota_x) + \iota_x(Q).
\]
Therefore,
\[
(6.4) \quad \text{div} \{x, -\} = \frac{\partial Q}{\partial y} \frac{\partial P}{\partial z} - \frac{\partial Q}{\partial z} \frac{\partial P}{\partial y}.
\]
Applying Corollary 1 gives that the enveloping algebra \(U\) of \((S, \Omega_{S/R})\) is skew Calabi-Yau. It has a Nakayama automorphism \(\nu: U \to U\) such that, for all \(s \in S\),
\[
\begin{cases}
\nu(s) = s \\
\nu(dx) = dz \quad \text{ and } \\
\nu(dy) = dz \quad \text{ and } \\
\nu(dz) = dz
\end{cases}
\]
\[
\begin{pmatrix}
\nu(s) \\
\nu(dx) \\
\nu(dy) \\
\nu(dz)
\end{pmatrix} = \begin{pmatrix}
dz \\
dy \\
dz \\
2 \quad \rightarrow \text{grad}(Q) \wedge \rightarrow \text{grad}(P).
\end{pmatrix}
\]

References


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