HAL
open science

Degrees of Irreducible Morphisms over Perfect Fields

Claudia Chaio, Patrick Le Meur, Sonia Trepode

To cite this version:

Claudia Chaio, Patrick Le Meur, Sonia Trepode. Degrees of Irreducible Morphisms over Perfect Fields. Algebras and Representation Theory, 2019, 22 (2), pp.495-515. 10.1007/s10468-018-9781-9 . hal-02361291

HAL Id: hal-02361291

https://hal.science/hal-02361291

Submitted on 13 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

DEGREES OF IRREDUCIBLE MORPHISMS OVER PERFECT FIELDS

CLAUDIA CHAIO, PATRICK LE MEUR, AND SONIA TREPODE

Abstract. The module category of any artin algebra is filtered by the powers of its radical, thus defining an associated graded category. As an extension of the degree of irreducible morphisms, this text introduces the degree of morphisms in the module category. When the ground ring is a perfect field, and the given morphism behaves nicely with respect to covering theory (as do irreducible morphisms with indecomposable domain or indecomposable codomain), it is shown that the degree of the morphism is finite if and only if its induced functor has a representable kernel. This gives a generalisation of Igusa and Todorov result, about irreducible morphisms with finite left degree and over an algebraically closed field. As a corollary, generalisations of known results on the degrees of irreducible morphisms over perfect fields are given. Finally, this study is applied to the composition of paths of irreducible morphisms in relationship to the powers of the radical.

1. Introduction

Let A be an artin algebra over an artin commutative ring $\Bbbk$ and $\operatorname{mod} A$ its category of finitely generated modules. The radical (rad) of $\operatorname{mod} A$ is related to many useful tools to understand this category. In particular, it is deeply connected to Auslander-Reiten theory based on irreducible morphisms. Recall that rad denotes the ideal in $\operatorname{mod} A$ generated by non-isomorphisms between indecomposable modules. Its powers rad^ℓ ($\ell \geqslant 0$) are defined inductively by $\operatorname{rad}^0 = \operatorname{mod} A$ and $\operatorname{rad}^{\ell+1} = \operatorname{rad}^\ell \cdot \operatorname{rad} = \operatorname{rad} \cdot \operatorname{rad}^\ell$. The infinite radical $\operatorname{rad}^\infty$ is defined as $\operatorname{rad}^\infty = \cap_{\ell \geqslant 0} \operatorname{rad}^\ell$. In particular, a morphism in $\operatorname{mod} A$ between indecomposable modules is irreducible if and only if it lies in $\operatorname{rad} \backslash \operatorname{rad}^2$ or, equivalently, it lies in rad and its image in $\operatorname{rad}/\operatorname{rad}^2$ is non-zero.

The purpose of this article is to generalise the results on degrees and composition of irreducible morphisms in $\operatorname{mod} A$ proven in [4] to the context of finite-dimensional $\Bbbk$-algebras over a perfect field.

These new results are presented in connection with the filtration $(\operatorname{rad}^\ell)_{\ell \geqslant 0}$ of $\operatorname{mod} A$, with a point of view that may be interesting for future investigations. Results on compositions have already provided useful information on the Auslander-Reiten structure of A (see for instance [11]). Such results involve the degree of irreducible morphisms, introduced by S. Liu in [10]. These are defined in terms of indices n such that the composition of the given irreducible morphism with some morphism in rad^n and not in rad^{n+1} lies in rad^{n+2}.

K. Igusa and G. Todorov proved in [8, Theorem 6.2] that, when A is an artin algebra of finite representation type, or a finite-dimensional algebra over an algebraically closed field, and when $f\colon X \to Y$ is an irreducible morphism with X or else Y indecomposable such that the left degree of f is finite then the kernel of the functor $\operatorname{Hom}_A(-, f)$ is representable.

One of the aims of this paper is to extend such results when A is a finite-dimensional algebra over a perfect field. More precisely, the previously mentioned result of K. Igusa and G. Todorov asserts that,

Date: May 20, 2018.

2010 *Mathematics Subject Classification.* 16G10; 16G60; 16G70.

Key words and phrases. Representation theory; Finite dimensional algebras; Auslander-Reiten theory; Irreducible morphisms; Degrees of morphisms; Covering theory.

under the same conditions, if the inclusion morphism $i\colon \operatorname{Ker}(f) \to X$ lies in $\operatorname{rad}^n \backslash \operatorname{rad}^{n+1}$ for some integer n, then the following sequence is exact for all integers $\ell \geqslant 1$, and indecomposables $Z \in \operatorname{mod} A$,

$$0 \to \frac{\operatorname{rad}^{\ell-n}}{\operatorname{rad}^{\ell-n+1}}(Z, \operatorname{Ker}(f)) \xrightarrow{-\cdot i} \frac{\operatorname{rad}^{\ell}}{\operatorname{rad}^{\ell+1}}(Z, X) \xrightarrow{-\cdot f} \frac{\operatorname{rad}^{\ell+1}}{\operatorname{rad}^{\ell+2}}(Z, Y)\,.$$

In order to generalise the above result, we introduce, following S. Liu, the notion of left and right degree of any morphism f from X to Y in $\operatorname{mod} A$ such that $f \in \operatorname{rad}^n \backslash \operatorname{rad}^{n+1}$, for some positive integer n.

More precisely, we define $d_\ell(f)$ to be the least integer m such that there exists (Z, g) with $Z \in \operatorname{mod} A$ indecomposable and $g\colon Z \to X$ lying in $\operatorname{rad}^m \backslash \operatorname{rad}^{m+1}$ and such that $fg \in \operatorname{rad}^{m+n+1}$ (and ∞ if such (Z, g) does not exist). In other words, $d_\ell(f)$ is the least integer d such that the morphism of functors $\frac{\operatorname{rad}^d(-,X)}{\operatorname{rad}^{d+1}(-,X)} \to \frac{\operatorname{rad}^{n+d}(-,Y)}{\operatorname{rad}^{n+d+1}(-,Y)}$ induced by f is not a monomorphism. Note that, when f is replaced by any f' such that $f - f' \in \operatorname{rad}^{n+1}$, then the following morphism

$$\oplus_{d \geqslant 0} \frac{\operatorname{rad}^d(-, X)}{\operatorname{rad}^{d+1}(-, X)} \longrightarrow \oplus_{d \geqslant 0} \frac{\operatorname{rad}^{n+d}(-, Y)}{\operatorname{rad}^{n+d+1}(-, Y)} \tag{1}$$

between contravariant functors from $\operatorname{mod} A$ to the category of graded $\Bbbk$-vector spaces remains unchanged, and hence $d_\ell(f) = d_\ell(f')$. The *right degree* $d_r(f)$ is defined dually.

The results of this article are based on functors with the covering property, which exist for any algebra A over a perfect field $\Bbbk$, see ([5]). Hence,

from now on, $\Bbbk$ is assumed to be a perfect field.

Given an Auslander-Reiten component Γ of $\operatorname{mod} A$, given a $\Bbbk$-linear category with length $\mathcal{C}$ and given a well-behaved functor $F\colon \mathcal{C} \to \Bbbk(\Gamma)$, a morphism $X \to Y$ in $\operatorname{mod} A$ is called *homogeneous* if there exist tuples $(x_s)_s$ and $(y_t)_t$ of objects in $\mathcal{C}$ and morphisms $(\varphi_{s,t})_{s,t} \in \oplus_{s,t}\mathcal{C}(x_s, y_t)$ such that $X = \oplus_x Fx_s$, $Y = \oplus_t Fy_t$ and for every s, t, the component $Fx_s \to Fy_t$ of the given morphism $X \to Y$ is equal to $F(\varphi_{s,t})$. In such a case, the morphism is called homogeneous with respect to $(y_t)_t$ (or to $(x_s)_s$) if it is necessary to work with a predetermined decomposition $Y = \oplus_t Fy_t$ (or $X = \oplus_s Fx_s$, respectively). Note that, if the given morphism $X \to Y$ lies in rad^m for some m, then it is necessary that $\varphi_{s,t} \in \mathcal{R}^m\mathcal{C}$ for every s, t.

Given $d \in \mathbb{N}$, a morphism $f\colon X \to Y$ is called homogeneous *up to* rad^{d+1} if it is the sum of two morphisms from X to Y, the former being homogeneous and the latter lying in rad^{d+1}.

Now, we state the first result of the article.

Theorem A (Proposition 3.4.1)**.** *Let $f\colon X \to Y$ be a morphism in $\operatorname{mod} A$. Assume that*

- *f lies in $\operatorname{rad}^d \backslash \operatorname{rad}^{d+1}$ and is homogeneous up to rad^d,*
- *f has finite left degree denoted by n.*

Then,

(1) there exists $f'\colon X \to Y$ such that $f - f' \in \operatorname{rad}^{d+1}$ and such that the inclusion morphism $i\colon \operatorname{Ker}(f') \to X$ lies in $\operatorname{rad}^n \backslash \operatorname{rad}^{n+1}$,

(2) there exists a direct sum decomposition $\operatorname{Ker}(f') = (\oplus_{m \geqslant n} K^{(m)}) \oplus K^{(\infty)}$ such that the restriction of the inclusion morphism i to any indecomposable direct summand of $K^{(m)}$ (or, to $K^{(\infty)}$) lies in $\operatorname{rad}^m \backslash \operatorname{rad}^{m+1}$ (or, to $\operatorname{rad}^\infty$, respectively),

(3) for every integer $\ell \geqslant n$ and every indecomposable $Z \in \operatorname{mod} A$, the following sequence

$$0 \to \bigoplus_{n \leqslant m \leqslant \ell} \frac{\operatorname{rad}^{\ell-m}}{\operatorname{rad}^{\ell-m+1}}(Z, K^{(m)}) \xrightarrow{-\cdot i} \frac{\operatorname{rad}^{\ell}}{\operatorname{rad}^{\ell+1}}(Z, X) \xrightarrow{-\cdot f} \frac{\operatorname{rad}^{\ell+d}}{\operatorname{rad}^{\ell+d+1}}(Z, Y)$$

is exact.

A morphism in $\mathrm{rad}^d\backslash\mathrm{rad}^{d+1}$ is not necessarily homogeneous up to rad^{d+1}, but is a sum of such morphisms. Any irreducible morphism with indecomposable domain (or codomain) is homogeneous up to rad^2.

As a consequence of Theorem A, it is possible to extend the above mentioned result of K. Igusa and G. Todorov to the context of finite-dimensional algebras over perfect fields.

Theorem B (Theorem 4.1.1)**.** *Let $f\colon X \to Y$ be an irreducible morphism. Assume that X, or else Y, is indecomposable and that f has finite left degree denoted by n. Then, there exists $f'\colon X \to Y$ irreducible such that*

(1) $f - f' \in \mathrm{rad}^2$,
(2) the inclusion morphism $i\colon \mathrm{Ker}(f') \to X$ lies in $\mathrm{rad}^n\backslash\mathrm{rad}^{n+1}$,
(3) for every integer $\ell \geqslant n$ and every indecomposable $Z \in \mathrm{mod}\, A$, the following sequence is exact

$$0 \to \frac{\mathrm{rad}^{\ell-n}}{\mathrm{rad}^{\ell-n+1}}(Z, \mathrm{Ker}(f')) \xrightarrow{-\cdot i} \frac{\mathrm{rad}^{\ell}}{\mathrm{rad}^{\ell+1}}(Z, X) \xrightarrow{-\cdot f} \frac{\mathrm{rad}^{\ell+1}}{\mathrm{rad}^{\ell+2}}(Z, Y)$$

is exact. In addition, if f is freely irreducible, then f' may be chosen equal to f.

Following [5, Section 2.5], a morphism $f\colon X \to \oplus_{i=1}^r X_i^{n_i}$ with X, X_i indecomposable is called *freely irreducible* if, for every $i \in \{1, \dots, r\}$, the n_i-tuple $(\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}})$ of residue classes in $\mathrm{irr}(X, X_i)$ is free over $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$. Note that freely irreducible morphisms are particular cases of strongly irreducible morphisms introduced in [5]. In particular, when X is indecomposable, f is freely irreducible if the division algebra $\mathrm{End}_A(X)/\mathrm{rad}(X, X)$ is trivial. When $\Bbbk$ is an algebraically closed field, the irreducible morphism in Theorem B is automatically freely irreducible. In this setting, Statements (1) and (2) of Theorem B where proved in [4] as well as the particular case $\ell = n$ of Statement (3). Actually, it is possible to derive from the above theorem extensions of the results in [4] (there, $\Bbbk$ is an algebraically closed field) to finite dimensional algebras over perfect fields. For instance

- if $f\colon X \to Y$ is a freely irreducible morphism where X, or else Y is indecomposable, then $d_\ell(f)$ is finite (equal to some given n) if and only if f is not a monomorphism and the inclusion morphism $\mathrm{Ker}(f) \to X$ lies in $\mathrm{rad}^n\backslash\mathrm{rad}^{n+1}$ (see Corollary 4.1.2),
- given two irreducible morphisms $f_1, f_2\colon X \to Y$ where X is indecomposable and the division algebra $\mathrm{End}_A(X)/\mathrm{rad}(X, X)$ is trivial, if $d_\ell(f_1) < \infty$, then $d_\ell(f_1) = d_\ell(f_2)$ and $\mathrm{Ker}(f_1) \simeq \mathrm{Ker}(f_2)$,
- the algebra A is of finite representation type if and only if, for every indecomposable injective $I \in \mathrm{mod}\, A$, the irreducible morphism $I \to I/\mathrm{soc}(I)$ has finite left degree (see Theorem 4.1.1 for a richer statement).

Finally, one of the original motivations for studying the kernel of (1) was to determine when the composition of n irreducible morphisms between indecomposable A-modules lies in rad^{n+1}. A first approach of this problem is given in [4] when $\Bbbk$ is algebraically closed and extended to the case where $\Bbbk$ is a perfect field in [5]. The above results provide another approach to this problem.

Theorem C. *Let A be a finite dimensional $\Bbbk$-algebra over a perfect field. Let $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ be a chain of irreducible morphisms between indecomposable A-modules. For each t, let $f'_t\colon X_{t-1} \to X_t$ be such as f' in Theorem B when $f = f_t$. Consider the following assertions.*

(i) $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$.
(ii) There exists $t \in \{1, \dots, n\}$ such that $d_\ell(f_t) \leqslant t - 1$, and there exists $h \in \mathrm{rad}^{t-1-d_\ell(f_t)}(X_0, \mathrm{Ker}(f'_t))$ not lying in $\mathrm{rad}^{t-d_\ell(f_t)}$, and such that $f_1 \cdots f_{t-1} - hi \in \mathrm{rad}^t$ (where $i\colon \mathrm{Ker}(f'_t) \to X_{t-1}$ is the inclusion morphism).

(iii) There exists $t \in \{1, \dots, n\}$ such that $d_l(f_t) \leq t-1$, there exists a path of length $t-1-d_\ell(f_t)$ from X_0 to $\mathrm{Ker}(f'_t)$ and with nonzero composition, and there exists a path $X_0 \to X_1 \to \cdots \to X_{t-1} \to X_t$ with zero composition.

Then (i) and (ii) are equivalent and imply (iii). If, moreover, $\dim_{\Bbbk} \mathrm{irr}(X_{s-1}, X_s) = 1$ for every $s \in \{1, \dots, t\}$, then (iii) implies (i) and (ii).

The text is organised as follows. Section 2 sets some usual conventions on modules and gives some technical lemmas. It also introduces categories with length and functors with a covering property defined on these categories. These notions are essential in the proofs of the above mentioned results. Section 3 is devoted to proving Theorem A. This is applied in section 4 to irreducible morphisms. There, Theorem B and the above mentioned results related to degrees of irreducible morphisms are proved. Finally, Theorem C is proved in Section 4.

Unless otherwise stated, $\Bbbk$ denotes a perfect field and A is a finite dimensional algebra over $\Bbbk$.

2. Preliminaries

This section introduces basic material used in the proofs of the main results of the text. The conventions on modules are set in 2.1. Next, 2.2 collects a couple of useful results on kernels of irreducible morphisms. The proofs of the main results are based on functors with the covering property defined on categories with length. The latter are defined in 2.3 and the former are introduced in 2.4.

2.1. **Conventions on modules.** Let $\operatorname{ind} A$ be a full subcategory of $\operatorname{mod} A$ which contains exactly one representative of each isomorphism class of indecomposable modules. Given modules $X, Y \in \operatorname{ind} A$, the quotient vector space $\mathrm{rad}(X,Y)/\mathrm{rad}^2(X,Y)$ is denoted by $\mathrm{irr}(X,Y)$ and called the *space of irreducible morphisms* from X to Y. It is naturally an $\mathrm{End}_A(X)/\mathrm{rad}(X,X) - \mathrm{End}_A(Y)/\mathrm{rad}(Y,Y)$-bimodule. The division $\Bbbk$-algebra $\mathrm{End}_A(X)/\mathrm{rad}(X,X)$ is denoted by κ_X. Recall that the *Auslander-Reiten quiver of* A is the quiver $\Gamma(\operatorname{mod} A)$ with vertices the modules in $\operatorname{ind} A$, such that there is an arrow (and exactly one) $X \to Y$ if and only if $\mathrm{irr}(X,Y) \neq 0$ for every pair of vertices $X, Y \in \Gamma$. The Auslander-Reiten translation is denoted by $\tau_A = D\mathrm{Tr}$. If Γ is a connected component of $\Gamma(\operatorname{mod} A)$ (or an *Auslander-Reiten component*, for short), the full subcategory of $\operatorname{ind} A$ with objects the modules in Γ is denoted by $\operatorname{ind} \Gamma$. Let $f\colon X \to \oplus_{i=1}^r X_i^{n_i}$ be an irreducible morphism where $X \in \operatorname{ind} A$, $X_1, \dots, X_r \in \operatorname{ind} A$ are pairwise non isomorphic and $n_1, \dots, n_r \geqslant 1$. Recall from the introduction that, f is freely irreducible if, for every $i \in \{1, \dots, r\}$, the n_i-tuple $(\overline{f_{i,1}}, \dots, \overline{f_{i,n_i}})$ of residue classes in $\mathrm{irr}(X, X_i)$ is free over $\kappa_X \otimes_{\Bbbk} \kappa_{X_i}^{\mathrm{op}}$. It is always free over $\kappa_{X_i}^{\mathrm{op}}$. Note that f is freely irreducible under any of the following conditions: if $\kappa_X \simeq \Bbbk$; or if $\Bbbk$ is an algebraically closed field. With dual considerations is defined freely irreducible morphisms with indecomposable codomain.

Let $X \to Y$ be an arrow in $\Gamma(\operatorname{mod} A)$ with valuation denoted by (a, b). In particular, a equals $\dim_{\kappa_X} \mathrm{irr}(X,Y)$. Since $\kappa_X \otimes_{\Bbbk} \kappa_Y^{\mathrm{op}}$ is semisimple $\Bbbk$-algebra (because $\Bbbk$ is perfect), a is at least $\ell(\mathrm{irr}(X,Y))$ and at most $\dim_{\Bbbk} \kappa_Y \cdot \ell(\mathrm{irr}(X,Y))$, where ℓ denotes the length over $\kappa_X \otimes_{\Bbbk} \kappa_Y^{\mathrm{op}}$. Dual identities apply to b. Note that, if $a = 1$ or $b = 1$, then $\ell(\mathrm{irr}(X,Y)) = 1$. This occurs when, for instance, the arrow $X \to Y$ has finite left degree (see [11, Section 1.6]).

The following lemma will be useful later on.

Lemma 2.1.1. *Let $\{M_i\}_i$ be a family of indecomposable modules and let $M = \oplus_i M_i$. For every i, let $\{\lambda_{i,j}\}_j$ be a family of morphisms $M_i \to M$. Assume that $[\sum_j \lambda_{i,j}\ ;\ i]^T\colon \oplus_i M_i \to M$ is an isomorphism. Then, there exists an index j_i, for every i, such that the resulting morphism $[\lambda_{i,j_i}\ ;\ i]^T\colon \oplus_i M_i \to M$ is an isomorphism.*

Proof. Clearly it suffices to treat the case where $M = M_1 \oplus M_2$, where $\{\lambda_{1,j}\}_j$ consists of two terms, say u, v, and where $\{\lambda_{2,j}\}_j$ consists of only one term, say w. The hypothesis then says that $[u+v, w]^T \colon M_1 \oplus M_2 \to M$ is an isomorphism and the conclusion says that at least one of the two morphisms $[u, w]^T$ or $[v, w]^T$ is an isomorphism.

Denote by $\overline{M}$ the factor module $M/\mathrm{Im}(w)$ and by $\pi \colon M \to \overline{M}$ the canonical surjection. Let $[s, t] \colon M \to M_1 \oplus M_2$ be the inverse morphism of $[u + v, w]^T$. This amounts to the following equalities

(1) $s \cdot (u + v) + tw = \mathrm{Id}_M$,

(2) $\begin{pmatrix} (u+v)s & (u+v)t \\ ws & wt \end{pmatrix} = \begin{pmatrix} \mathrm{Id}_{M_1} & 0 \\ 0 & \mathrm{Id}_{M_2} \end{pmatrix}$.

In particular, $s \colon M \to M_1$ induces a morphism $\overline{s} \colon \overline{M} \to M_1$ such that $\pi\overline{s} = s$. It satisfies

- $\pi\overline{s} \cdot (u + v)\pi = s \cdot (u + v)\pi = (s \cdot (u + v) + tw)\pi = \pi$,
- $(u + v)\pi\overline{s} = (u + v)s = \mathrm{Id}_{M_1}$.

Hence, $(u + v)\pi \colon M_1 \to \overline{M}$ is an isomorphism with inverse $\overline{s}$. Since M_1 is indecomposable, at least one of the two morphisms $u\pi \colon M_1 \to \overline{M}$ or $v\pi \colon M_1 \to \overline{M}$ is an isomorphism. Assume the former, then $\overline{M} = \mathrm{Im}(u\pi)$. Accordingly, $M = \mathrm{Im}(u) + \mathrm{Im}(w)$. Therefore, $[u, w]^T \colon M \to M$ is a surjection, and hence an isomorphism. □

2.2. **Properties on kernels of irreducible morphisms.** The following lemma compares the kernel of an irreducible morphism with the kernel of the corestriction to a proper direct summand of its codomain. The reader may formulate the dual version of this lemma with a completely analogous proof.

Lemma 2.2.1. *Let $f = [f_1, f_2] \colon X \to Y_1 \oplus Y_2$ be an irreducible morphism where $X \in \mathrm{ind}\, A$ and $Y_1, Y_2 \in \mathrm{mod}\, A$ are nonzero. Let $i \in \{1, 2\}$. If f is an epimorphism then*

(a) $\mathrm{Ker}(f) \not\simeq \mathrm{Ker}(f_i)$,
(b) $\mathrm{Ker}(f)$ *and* $\mathrm{Ker}(f_i)$ *are non injective,*
(c) $\mathrm{Ker}(f_i)$ *is non simple and the middle term of an almost split sequence starting at* $\mathrm{Ker}(f_i)$ *is indecomposable.*

Proof. There is no loss of generality assuming that $i = 1$. Note that f_1 is an epimorphism because so is f. There are short exact sequences $0 \to \mathrm{Ker}(f) \to X \xrightarrow{f} Y_1 \oplus Y_2 \to 0$ and $0 \to \mathrm{Ker}(f_1) \to X \xrightarrow{f_1} Y_1 \to 0$ in $\mathrm{mod}\, A$. Since $\dim_{\Bbbk} Y_1 < \dim_{\Bbbk}(Y_1 \oplus Y_2)$, a length argument shows that $\mathrm{Ker}(f) \not\simeq \mathrm{Ker}(f_1)$. This proves (a). Using the same exact sequences, the fact that $X \in \mathrm{ind}\, A$ entails that $\mathrm{Ker}(f)$ and $\mathrm{Ker}(f_1)$ are non injective. This proves (b). In order to prove (c) it suffices to prove that $\mathrm{Ker}(f_1)$ is non simple, thanks to [9] (see also [2]). Note that $\mathrm{Ker}(f) = \mathrm{Ker}(f_1) \cap \mathrm{Ker}(f_2)$ is a proper (and indecomposable) submodule of $\mathrm{Ker}(f_1)$. Hence $\mathrm{Ker}(f_1)$ is non simple. This proves (c). □

The following lemma is proved in [6, Theorem 3.2] for algebras over algebraically closed fields. The result given there still works for algebras over artin rings. Its main argument is recalled below for the convenience of the reader. As usual, given a morphism $X \to Y$ in $\mathrm{mod}\, A$, a *kernel (morphism)* for f is a morphism $K \to X$ such that the induced sequence of functors $0 \to \mathrm{Hom}_A(-, K) \to \mathrm{Hom}_A(-, X) \to \mathrm{Hom}_A(-, Y)$ is exact.

Lemma 2.2.2. *Assume that A is an artin algebra over an artin ring. Let $0 \to K \xrightarrow{i} X \xrightarrow{f} Y \to 0$ be an exact sequence in $\mathrm{mod}\, A$ such that X is indecomposable and f is irreducible. Assume that there exists $n \in \mathbb{N}$ such that $i \in \mathrm{rad}^n \backslash \mathrm{rad}^{n+1}$.*

(1) For any morphism $K \to X$ lying in $\mathrm{rad}^n \backslash \mathrm{rad}^{n+1}$, there exists an automorphism $X \to X$ such that the composition morphism $K \to X \to X$ is a kernel morphism of f.

(2) If X is indecomposable, then there exists a path $X_0 = K \to X_1 \to \cdots \to X_{n-1} \to X_n = X$ of irreducible morphisms between indecomposables and with composition equal to a kernel morphism of f.

Proof. (1) This is obtained upon applying [1, Proposition 5.7, p. 173] to the given morphism $u\colon K \to X$. The cited result asserts that either u factors through i, or i factors through u.

(2) The class of i modulo rad^{n+1} is a non trivial sum of compositions of paths of length n. Since $i \notin \operatorname{rad}^{n+1}$, at least one of these paths has composition not lying in rad^{n+1}. Using (1), such a path fits the conclusion (up to composition of the last morphism of that path with an automorphism of X). □

2.3. **Categories with length.** Some proofs in this text make use of specific functors taking values in $\operatorname{ind} A$. These functors are defined over categories with length. This subsection defines these categories.

Let $\mathcal{C}$ be a $\Bbbk$-linear category. Assume that $\dim_{\Bbbk}\mathcal{C}(x,y) < \infty$ for every $x, y \in \mathcal{C}$. Assume also that distinct objects in $\mathcal{C}$ are not isomorphic. Define $\mathcal{RC}$ to be the ideal of $\mathcal{C}$ consisting of those morphisms that are not invertible. The powers of $\mathcal{RC}$ are defined recursively by $\mathcal{R}^0\mathcal{C} = \mathcal{C}$, $\mathcal{R}^1\mathcal{C} = \mathcal{RC}$ and $\mathcal{R}^{\ell+1}\mathcal{C} = \mathcal{R}^{\ell}\mathcal{C} \cdot \mathcal{RC} = \mathcal{RC} \cdot \mathcal{R}^{\ell}\mathcal{C}$. A morphism in $\mathcal{C}$ is called *irreducible* if it lies in $\mathcal{RC}$ and not in $\mathcal{R}^2\mathcal{C}$. A *path* (of irreducible morphisms) in $\mathcal{C}$ is a sequence $x_0 \xrightarrow{\varphi_1} x_1 \to \cdots \to x_{\ell-1} \xrightarrow{\varphi_n} x_n$ where $x_0, \ldots, x_n \in \mathcal{C}$ and φ_i is an irreducible morphism from x_{i-1} to x_i for every i; By definition the length of such a path is n.

A category *with length* is a $\Bbbk$-linear category $\mathcal{C}$ as above such that

(a) for every $x, y \in \mathcal{C}$, the paths of irreducible morphisms from x to y all have the same length,
(b) $\bigcap_{n\geqslant 0} \mathcal{R}^n\mathcal{C} = 0$ (in particular, any morphism in $\mathcal{C}$ is a sum of compositions of paths of irreducible morphisms in $\mathcal{C}$).

Note that, for such a category, $\mathcal{C}(x,x)$ is a division $\Bbbk$-algebra for every x.

As an example, the mesh category of any modulated translation quiver ([7, Section 1.2]) with length is a category with length. Recall that a quiver with length is a quiver such that any two parallel paths have the same length. The categories with length used in this text are all of this shape. However, the presentation here sticks to the general setting in order to avoid unnecessary technicalities.

The following proposition is used throughout the text. It follows from the definitions.

Proposition 2.3.1. *Let $\mathcal{C}$ be a $\Bbbk$-linear category with length and $x, y \in \mathcal{C}$. If there exists a path of irreducible morphisms from x to y in $\mathcal{C}$ and with length denoted by ℓ, then*

(1) $\mathcal{C}(x,y) = \mathcal{RC}(x,y) = \cdots = \mathcal{R}^{\ell}\mathcal{C}(x,y)$,
(2) $\mathcal{R}^i\mathcal{C}(x,y) = 0$ for every $i > \ell$,
(3) if there exists an irreducible morphism $x \to y$, then $\mathcal{C}(x,y)\backslash\{0\}$ consists of irreducible morphisms,
(4) for every $z \in \mathcal{C}$, if there exists a path of irreducible morphisms from y to z with length denoted by ℓ', then $\mathcal{R}^i\mathcal{C}(x,z) = 0$, for every $i > \ell + \ell'$.

2.4. **Functors with the covering property.** Let Γ be an Auslander-Reiten component of A. Let $\mathcal{C}$ be a $\Bbbk$-linear category with length and $F\colon \mathcal{C} \to \operatorname{ind}\Gamma$ be a $\Bbbk$-linear functor. This subsection introduces the class of functors F that are used in the proof of the main results of this text.

2.4.1. *Definition and basic features.* By definition, F is said to have the *covering property* if it satisfies the following conditions:

(a) for every $X \in \Gamma$, there exists $x \in \mathcal{C}$ such that $Fx = X$,
(b) $F(\mathcal{RC}) \subseteq \operatorname{rad}$,

(c) for every $x, y \in \mathcal{C}$ and $n \geqslant 0$, the two following maps induced by F are bijective

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n \Bbbk(\widetilde{\Gamma})(x,z)/\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(x,z) \quad \to \quad \mathrm{rad}^n(Fx,Fy)/\mathrm{rad}^{n+1}(Fx,Fy)$$

$$\bigoplus_{Fz=Fy} \mathfrak{R}^n \Bbbk(\widetilde{\Gamma})(z,x)/\mathfrak{R}^{n+1}\Bbbk(\widetilde{\Gamma})(z,x) \quad \to \quad \mathrm{rad}^n(Fy,Fx)/\mathrm{rad}^{n+1}(Fy,Fx),$$

(d) for every $x, y \in \mathcal{C}$, the two following maps induced by F are injective

$$\bigoplus_{Fz=Fy} \Bbbk(\widetilde{\Gamma})(x,z) \to \mathrm{Hom}_A(Fx,Fy) \quad \textit{and} \quad \bigoplus_{Fz=Fy} \Bbbk(\widetilde{\Gamma})(z,x) \to \mathrm{Hom}_A(Fy,Fx),$$

(e) for every $x \in \mathcal{C}$ and $Y \in \Gamma$, there exists at most one $y \in \mathcal{C}$ such that $Fy = Y$ and $\mathcal{C}(x,y)$ contains an irreducible morphism,

(f) for every $X \in \Gamma$ and $y \in \mathcal{C}$, there exists at most one $x \in \mathcal{C}$ such that $Fx = X$ and $\mathcal{C}(x,y)$ contains an irreducible morphism.

When F has the covering property, it induces a $\Bbbk$-algebra isomorphism

$$\mathcal{C}(x,x)/\mathcal{R}\mathcal{C}(x,x) \xrightarrow{\sim} \mathrm{End}_A(Fx)/\mathrm{rad}(Fx,Fx)$$

for every $x \in \mathcal{C}$. Also it induces a linear isomorphism $\mathcal{R}\mathcal{C}(x,y)/\mathcal{R}^2\mathcal{C}(x,y) \xrightarrow{\sim} \mathrm{irr}(Fx,Fy)$ for every $x, y \in \mathcal{C}$ such that there exists an irreducible morphism $x \to y$ in $\mathcal{C}$.

Well-behaved functors in the sense of [4, 5] are examples of functors with the covering property where $\mathcal{C}$ equals the mesh category of the universal cover of the modulated translation quiver Γ. The following result plays a central role in the present text.

Proposition 2.4.1 (Proposition 2.5 of [5])**.** *Let A be a finite dimensional algebra over a perfect field $\Bbbk$. Let Γ be an Auslander-Reiten component of A.*

(1) There exists a $\Bbbk$-linear functor $F\colon \mathcal{C} \to \mathrm{ind}\,\Gamma$ with the covering property.

(2) Let $X \in \Gamma$ and let $f\colon X \to \oplus_{i=1}^r X_i^{n_i}$ be a freely irreducible morphism where $X_1, \ldots, X_r$ are pairwise non isomorphic. There exists a $\Bbbk$-linear functor $F\colon \mathcal{C} \to \mathrm{ind}\,\Gamma$ with the covering property, and there exist $x \in F^{-1}X$, $x_i \in F^{-1}X_i$ $(1 \leqslant i \leqslant r)$ such that each component $X \to X_i$ of f lies in the image of the mapping $\mathcal{C}(x,x_i) \to \mathrm{Hom}_A(X,X_i)$ induced by F.

2.4.2. *Homogeneous morphisms.* The notion of homogeneous morphism that is considered in this text is relative to a given functor $F\colon \mathcal{C} \to \mathrm{ind}\,\Gamma$ with the covering property. It is used to express the assumptions in the main results of this text.

Recall from the introduction that, a morphism $X \to Y$ in $\mathrm{mod}\,A$ is called homogeneous if there exist tuples $(x_s)_s$ and $(y_t)_t$ of objects in $\mathcal{C}$ and a tuple of morphisms $(\varphi_{s,t})_{s,t} \in \oplus_{s,t}\mathcal{C}(x_s,y_t)$ such that $X = \oplus_x Fx_s$, such that $Y = \oplus_t Fy_t$ and such that, for every s,t, the component $Fx_s \to Fy_t$ of the given morphism $X \to Y$ is equal to $F(\varphi_{s,t})$. In such a case, the morphism is called homogeneous with respect to $(y_t)_t$ (or to $(x_s)_s$) if it is necessary to work with a predetermined decomposition $Y = \oplus_t Fy_t$ (or $X = \oplus_s Fx_s$, respectively). Note that, if the given morphism $X \to Y$ lies in rad^m for some m, then it is necessary that $\varphi_{s,t} \in \mathcal{R}^m\mathcal{C}$ for every s,t.

Given $d \in \mathbb{N}$, a morphism $f\colon X \to Y$ is called homogeneous *up to* rad^{d+1} if it is the sum of two morphisms from X to Y, the former being homogeneous and the latter lying in rad^{d+1}. If the former morphism is equal to $[F(\varphi_{s,t})\ ;\ s,t]\colon X \to Y$ (with the previous notation), then each $\varphi_{s,t}$ may be chosen to be zero when it lies in $\mathcal{R}^{d+1}\mathcal{C}$. Note that this condition makes the decomposition of f unique. In such a case, the morphism $[F(\varphi_{s,t})\ ;\ s,t]\colon X \to Y$ is called the *homogeneous part* of f.

For instance, for every $X \in \Gamma$, any irreducible morphism $X \to \oplus_t Y_t$ is homogeneous up to rad^2 (with respect to any given $x \in \mathcal{C}$ such that $Fx = X$, and for any given $F\colon \mathcal{C} \to \mathrm{ind}\,\Gamma$). Also, when

this irreducible morphism is freely irreducible, then there exists a functor with the covering property for which that irreducible morphism is homogeneous (see Proposition 2.4.1).

The following lemma is useful in the proof of the main results of this text. It deals with homogeneous monomorphisms. Consider the following setting. Let $i\colon K \to X$ be a monomorphism in $\operatorname{mod} A$ where every indecomposable direct summand of X lies in Γ. Let $X = \oplus_s X_s$ be a direct sum decomposition into indecomposable modules (with $X_s \in \Gamma$, for every s) and, for every s, let $x_s \in \mathcal{C}$ be such that $Fx_s = X_s$. Fix a direct sum decomposition $K = K^{(\infty)} \oplus \left(\oplus_{m\geqslant 0} \oplus_r K_r^{(m)}\right)$ such that each $K_r^{(m)}$ is indecomposable, such that the restriction $K^{(\infty)} \to X$ of i lies in $\operatorname{rad}^\infty$ and such that each restriction $K_r^{(m)} \to X$ of i lies in $\operatorname{rad}^m \backslash \operatorname{rad}^{m+1}$. For simplicity, $\oplus_r K_r^{(m)}$ is denoted by $K^{(m)}$ for every m.

Lemma 2.4.2. *Under the previous setting, assume that the restriction $\oplus_{m,r} K_r^{(m)} \to X$ of i is homogeneous with respect to $(x_s)_s$. Then, for every $\ell \geqslant n$, $Z \in \Gamma$ and $h \in \operatorname{Hom}_A(Z, K)$, the following conditions are equivalent*

(i) $hi \in \operatorname{rad}^\ell$,
(ii) for every m, the component $Z \to K^{(m)}$ of h lies in $\operatorname{rad}^{\ell-m}$.

Proof. Obviously, $(ii) \implies (i)$. Assume (i). There is no loss of generality in assuming that $K^{(\infty)} = 0$. Since i is homogeneous, there exists $k_r^{(m)} \in \mathcal{C}$ such that $Fk_r^{(m)} = K_r^{(m)}$, for every (m, r), and there exists, for every m, r, s, a morphism $\iota_{r,s}^{(m)} \in \mathcal{R}^m\mathcal{C}(k_r^{(m)}, x_s)$ that is zero when it lies in rad^{m+1} and such that $i = [F(\iota_{r,s}^{(m)})\ ;\ m, r, s]$. For every m, r, denote by $h_r^{(m)}\colon Z \to K_r^{(m)}$ the corresponding component of h; There exists $(\eta_{z,r}^{(m)})_{z\in F^{-1}Z} \in \oplus_z \mathcal{C}(z, k_r^{(m)})$ such that

- $h_r^{(m)} - \sum_z F(\eta_{z,r}^{(m)}) \in \operatorname{rad}^{\ell-m}$,
- for every z, the morphism $\eta_{z,r}^{(m)}$ is 0 if it lies in $\mathcal{R}^{\ell-m}\mathcal{C}$.

Then $(h_r^{(m)} - \sum_z F(\eta_{z,r}^{(m)}))F(\iota_{r,s}^{(m)})$ lies in rad^ℓ and equals $h_r^{(m)}F(\iota_{r,s}^{(m)}) - \sum_z F(\eta_{z,r}^{(m)}\iota_{r,s}^{(m)})$. Since hi lies rad^ℓ, then so does $\sum_{r,m} h_r^{(m)}F(\iota_{r,s}^{(m)})$ for every s. Accordingly, $\sum_z F\left(\sum_{r,m} \eta_{z,r}^{(m)}\iota_{r,s}^{(m)}\right)$ lies in rad^ℓ for every s, and hence $\sum_{r,m} \eta_{z,r}^{(m)}\iota_{r,s}^{(m)}$ lies in $\mathcal{R}^\ell\mathcal{C}(z, x_s)$ for every z, s.

Now, for every r, m, z, s, if both $\eta_{z,r}^{(m)}$ and $\iota_{r,s}^{(m)}$ are nonzero, then there exists a path of irreducible morphisms of length at most $\ell - 1$ from z to x_s in $\mathcal{C}$, and hence $\mathcal{R}^\ell\mathcal{C}(z, x_s) = 0$. Thus $\sum_{r,m} \eta_{z,r}^{(m)}\iota_{r,s}^{(m)} = 0$ for every z, s. Applying F to this equality yields that the following composite morphism is zero

$$Z \xrightarrow{[\sum_z F(\eta_{z,r}^{(m)})\ ;\ r,m]} \oplus_{r,m} K_r^{(m)} \xrightarrow{i} X\,.$$

Given that i is a monomorphism, it follows that $\sum_z F(\eta_{z,r}^{(m)}) = 0$ for every m, r, and hence $\eta_{z,r}^{(m)} = 0$ for every m, r, z. Accordingly, $h_r^{(m)} \in \operatorname{rad}^{\ell-m}$ for every m, r. □

3. Investigation of the left degree

This section investigates the left degree of a morphism $f\colon X \to Y$ under certain conditions expressed in terms of functors with the covering property. The purpose of the section is to prove Proposition 3.4.1: Assuming that f lies in $\operatorname{rad}^d \backslash \operatorname{rad}^{d+1}$ and is homogeneous up to rad^d for some $d \in \mathbb{N}$ and that f has finite left degree denoted by n, there exists a direct sum decomposition $\operatorname{Ker}(f') = K^{(\infty)} \oplus \left(\oplus_{m\geqslant n} \oplus_r K_r^{(m)}\right)$ of the kernel of the homogeneous part f' of f such as in the setting of Lemma 2.4.2 and such that for

every $Z \in \operatorname{ind} A$ and every $\ell \geqslant m$, the following sequence is exact

$$0 \to \bigoplus_{n \leqslant m \leqslant \ell} \frac{\operatorname{rad}^{\ell-m}}{\operatorname{rad}^{\ell-m+1}}(Z, K^{(m)}) \to \frac{\operatorname{rad}^{\ell}}{\operatorname{rad}^{\ell+1}}(Z, X) \to \frac{\operatorname{rad}^{\ell+d}}{\operatorname{rad}^{\ell+d+1}}(Z, Y)\,.$$

The general setting in which this result is valid is presented in 3.1 and is assumed throughout the section. Next, a key lemma is proved in 3.2. Then 3.3 is devoted to the construction of the above mentioned direct sum decomposition of $\operatorname{Ker}(f')$. Finally, 3.4 is devoted to Proposition 3.4.1.

3.1. **Setting of the study.** Given $f\colon X \to Y$ lying in $\operatorname{rad}^d \backslash \operatorname{rad}^{d+1}$ for some integer $d \geqslant 1$, recall that $d_\ell(f) = n$ if n is the least integer such that there exists $Z \in \operatorname{ind} A$ and $g \in \operatorname{Hom}_A(Z, X)$ verifying $g \in \operatorname{rad}^n \backslash \operatorname{rad}^{n+1}$ and $gf \in \operatorname{rad}^{n+d+1}$.

Let Γ be an Auslander-Reiten component of A. Let $F\colon \mathcal{C} \to \operatorname{ind}\Gamma$ be a functor with the covering property for Γ. Let $f\colon X \to Y$ be a morphism in $\operatorname{mod} A$ satisfying the following conditions

- every indecomposable direct summand of $X \oplus Y$ lies in Γ,
- f lies in rad^d and is homogeneous up to rad^{d+1} for some $d \in \mathbb{N}$,
- the left degree of f is finite and denoted by n.

Let $X = \oplus_s X_s$ and $Y = \oplus_t Y_t$ be direct sum decompositions such that there exist tuples of objects $(x_s)_s$ and $(y_t)_t$ in $\mathcal{C}$ and a tuple of morphisms $(\varphi_{s,t})_{s,t} \in \oplus_{s,t} \mathcal{R}^d\mathcal{C}(x_s, y_t)$ verifying

- $X_s = Fx_s$ and $Y_t = Fy_t$ for every s, t,
- for every s, t, the morphism $\varphi_{s,t}$ is zero if it lies in $\mathcal{R}^{d+1}\mathcal{C}$,
- $f - [F(\varphi_{s,t})\ ;\ s, t]$ lies in rad^{d+1}.

Recall that the morphism $[F(\varphi_{s,t})\ ;\ s, t]$ is called the homogeneous part of f. In what follows it is denoted by f'. Recall also that the homogeneity condition made on f is valid in any of the following cases

- f is irreducible (see 2.4.1),
- for every s, t, the component $X_s \to Y_t$ of f is the sum of a morphism in rad^{d+1} and of the composition of a path of irreducible morphisms between indecomposables and with length d,
- for every s, t, the vector space $\operatorname{Hom}_A(X_s, Y_t)/\operatorname{rad}^\infty(X_s, Y_t)$ has dimension at most 1.

The objective of the section is to establish some general properties on the left degree of f. These are expressed in terms of f'.

3.2. **First interpretation of the finiteness of left degree.** The following lemma investigates the shape of a morphism lying in $\operatorname{rad}^\ell \backslash \operatorname{rad}^{\ell+1}$ and having composition with f lying in $\operatorname{rad}^{\ell+d+1}$.

Lemma 3.2.1. *Let $Z \in \Gamma$, $g \in \operatorname{Hom}_A(Z, X)$ and $\ell \geqslant n$ be such that $g \in \operatorname{rad}^\ell \backslash \operatorname{rad}^{\ell+1}$ and $gf \in \operatorname{rad}^{\ell+d+1}$. For every s, let $(\gamma_{z,s})_{z \in F^{-1}Z} \in \oplus_z \mathcal{R}^\ell\mathcal{C}(z, x_s)$ be such that g is the sum of $g' := [\sum_z F(\gamma_{z,s})\ ;\ s]\colon Z \to X$ and a morphism in $\operatorname{rad}^{\ell+1}$ and such that, for every z, s, the morphism $\gamma_{z,s}$ is zero when it lies in $\mathcal{R}^{\ell+1}\mathcal{C}$. Then*

(a) $\sum_s \gamma_{z,s}\varphi_{s,t} = 0$ in $\operatorname{Hom}_A(Z, Y_t)$ for every z, t, and hence
(b) $g'f' = 0$.

Proof. It follows from the hypotheses that $g'f' - gf \in \operatorname{rad}^{\ell+d+1}$. Since, moreover, $[\sum_z F(\gamma_{z,s})\ ;\ s]\, f' = [\sum_z F(\sum_s \gamma_{z,s}\varphi_{s,t})\ ;\ t]$, then $\sum_s \gamma_{z,s}\varphi_{s,t} \in \mathcal{R}^{\ell+d+1}\mathcal{C}$ for every z, t. Given s, t, z, the morphism $\gamma_{z,s}\varphi_{s,t}$ is a linear combination of compositions of paths of irreducible morphisms of length $\ell + d$ in $\mathcal{C}$; Thus $\sum_s \gamma_{z,s}\varphi_{s,t} = 0$ for every z, t (see Proposition 2.3.1). This proves (a). And (b) is a direct consequence of (a). $\square$

3.3. **Decomposition of the kernel.** This subsection collects technical properties on the inclusion morphism $i\colon \mathrm{Ker}(f') \to X$. Its aim is to show the existence of a direct sum decomposition of $\mathrm{Ker}(f')$ fitting the setting of Lemma 2.4.2. First, in 3.3.1, it is proved that $i \in \mathrm{rad}^n \backslash \mathrm{rad}^{n+1}$. From this property is derived a first direct sum decomposition of $\mathrm{Ker}(f')$ that is close to the desired one. After introducing useful notation on i in 3.3.2, the needed decomposition of $\mathrm{Ker}(f')$ is then obtained in 3.3.3 by taking the direct image of the one in 3.3.1 under a suitable automorphism of $\mathrm{Ker}(f')$. Finally, it is proved in 3.3.4 that i is homogeneous with respect to the resulting decomposition.

3.3.1. *A first decomposition of the kernel* $\mathrm{Ker}(f')$. The following lemma gives a first property on the inclusion morphism $i\colon \mathrm{Ker}(f') \to X$.

Lemma 3.3.1. *The inclusion morphism* $\mathrm{Ker}(f') \to X$ *lies in* $\mathrm{rad}^n \backslash \mathrm{rad}^{n+1}$.

Proof. Since f has finite left degree, there exists Z, g fitting the hypotheses of Lemma 3.2.1 with $\ell = n$. Using the notation introduced there, the morphism $[\sum_z F(\gamma_{z,s})\ ;\ s]\colon Z \to X$ lies in $\mathrm{rad}^n \backslash \mathrm{rad}^{n+1}$ and factors through i. Therefore $i \notin \mathrm{rad}^{n+1}$. Moreover, $i \in \mathrm{rad}^n$ because $if' = 0$ and $d_\ell(f') = n$. □

Hence, there exists a direct sum decompositions

$$\left\{ \begin{array}{l} \mathrm{Ker}(f') = K^{(\infty)} \oplus \left(\oplus_{n \leqslant m} K^{(m)}\right) \\ K^{(m)} = \oplus_r K_r^{(m)} \text{ for every } m \end{array} \right. \tag{2}$$

such that

(a) each $K_r^{(m)}$ is indecomposable,
(b) the inclusion morphism $K^{(\infty)} \to X$ lies in rad^∞,
(c) for every m, r, the inclusion morphism $K_r^{(m)} \to X$ lies in $\mathrm{rad}^m \backslash \mathrm{rad}^{m+1}$.

For every such set of decompositions (2) satisfying (a), (b) and (c), denote by α_K the sequence indexed by the integers not smaller than n and with value at any $m \geqslant n$ equal to the number of terms in the direct sum decomposition $K^{(m)} = \oplus_r K_r^{(m)}$. In what follows, it is assumed that

(d) the decomposition (2) satisfying (a), (b) and (c) is such that the sequence α_K is minimal for the lexicographic order.

In the sequel, the inclusion morphisms $K_r^{(m)} \to X$ and $K^{(m)} \to X$ are denoted by $i_r^{(m)}$ and $i^{(m)}$, respectively.

3.3.2. *Notation for the inclusion morphism* $\mathrm{Ker}(f') \to X$. Recall that i denotes the inclusion morphism $\mathrm{Ker}(f') \to X$. In order to prove the main result of this section, it is useful to show the existence of an automorphism σ of $\mathrm{Ker}(f')$ such that each restriction $K^{(m)} \to X$ of σi is homogeneous with respect to $(x_s)_s$. For that purpose, some notation is introduced below. For every m, r, s, there exists $(\iota_{z,r,s}^{(m)})_{z \in F^{-1} K_r^{(m)}} \in \oplus_z \mathcal{R}^m \mathcal{C}(z, x_s)$ such that

- $i_r^{(m)} - [\sum_z F(\iota_{z,r,s}^{(m)})\ ;\ s]$ lies in rad^{m+1} (recall that $X = \oplus_s X_s$),
- $\iota_{z,r,s}^{(m)}$ is zero whenever it lies in $\mathcal{R}^{m+1}\mathcal{C}$.

The following lemma collects useful factorisation properties of these morphisms.

Lemma 3.3.2. *The families of morphisms* $\{i_r^{(m)}\}_{r,m}$ *and* $\{\iota_{z,r,s}^{(m)}\}_{m,z,r,s}$ *have the following properties.*

(1) For every z, m, r *(with* $Fz = K_r^{(m)}$*), there exists* $\sigma_{z,r}^{(m)}\colon K_r^{(m)} \to \mathrm{Ker}(f')$ *such that* $[F(\iota_{z,r,s}^{(m)})\ ;\ s] = \sigma_{z,r}^{(m)} i$.
(2) For every (m, r)*, there exists* $\tau_r^{(m)}\colon K_r^{(m)} \to \mathrm{Ker}(f')$ *such that* $i_r^{(m)} - [\sum_z F(\iota_{z,r,s}^{(m)})\ ;\ s] = \tau_r^{(m)} i$.
(3) Moreover, $\tau_r^{(m)} + \sum_z \sigma_{z,r}^{(m)}$ *is the inclusion morphism* $K_r^{(m)} \to \mathrm{Ker}(f')$.

Proof. (1) Fix m and r. Since $i_r^{(m)} f' = 0$ and $f' = [F(\varphi_{s,t})\ ;\ s,t]$, the morphism $[\sum_z F(\sum_s \iota_{z,r,s}^{(m)} \varphi_{s,t})\ ;\ t]$ from $K_r^{(m)}$ to Y lies in rad^{m+d+1}. In view of Propositions 2.3.1 and 2.4.1, it follows that $\sum_s \iota_{z,r,s}^{(m)} \varphi_{s,t}$ lies in $\mathcal{R}^{m+d+1}\mathcal{C}$ for every z, s, t, and hence is zero (note that $\iota_{z,r,s}^{(m)} \varphi_{s,t}$ is a linear combination of compositions of paths of irreducible morphisms with length $m + d$). Thus $[F(\iota_{z,r,s}^{(m)})\ ;\ s]f'$ is zero, and therefore $[F(\iota_{z,r,s}^{(m)})\ ;\ s]$ factors through i for every z.

(2) Note that $\left(i_r^{(m)} - [\sum_z F(\iota_{z,r,s}^{(m)})\ ;\ s]\right) f'$ is zero because so are $i_r^{(m)} f'$ and $[F(\iota_{z,r,s}^{(m)})\ ;\ s]f'$ (for every z). Hence, there exists $\tau_r^{(m)} \colon K_r^{(m)} \to \mathrm{Ker}(f')$ such that $i_r^{(m)} - [\sum_z F(\iota_{z,r,s}^{(m)})\ ;\ s] = \tau_r^{(m)} i$.

(3) According to (1) and (2), the composition $(\tau_r^{(m)} + \sum_z \sigma_{z,r}^{(m)})i$ equals $i_r^{(m)}$. Whence the conclusion. □

Note that $\tau_r^{(m)} i = \left(\sum_z \sigma_{z,r}^{(m)}\right) i - i_r^{(m)} \in \mathrm{rad}^{m+1}$ for every m.

3.3.3. *The direct sum decomposition of* $\mathrm{Ker}(f')$. The following lemma shows the existence of an automorphism σ of $\mathrm{Ker}(f')$ such that each restriction $K_r^{(m)} \to X$ of σi is homogeneous with respect to $(x_s)_s$.

Lemma 3.3.3. *Let $i \colon \mathrm{Ker}(f') \to X$ be the inclusion morphism. There exists an automorphism σ of $\mathrm{Ker}(f')$ and, for every (m, r), there exist*

- $k_r^{(m)} \in F^{-1} K_r^{(m)}$,
- $\iota_{r,s}^{(m)} \in \mathcal{R}^m \mathcal{C}(k_r^{(m)}, x_s)$ *(for every s) not lying in $\mathcal{R}^{m+1}\mathcal{C} \backslash \{0\}$,*

such that the restriction of $\sigma i \colon \mathrm{Ker}(f') \to X$ to $K_r^{(m)}$ equals $[F(\iota_{r,s}^{(m)})\ ;\ s]$. In particular, each restriction $K_r^{(m)} \to X$ of σi is homogeneous with respect to $(x_s)_s$.

Proof. Apply Lemma 2.1.1 to the following data

- take M to be $\mathrm{Ker}(f')$,
- take $M = \oplus_i M_i$ to be a direct sum decomposition into indecomposables subordinated to the decomposition $\mathrm{Ker}(f') = K^{(\infty)} \oplus \left(\oplus_{m \geqslant n} \oplus_r K_r^{(m)}\right)$,
- for every i such that M_i is a summand of $K^{(\infty)}$, take $\{\lambda_{i,j}\}_j$ to be the family consisting of a single element, namely the inclusion morphism $M_i \to \mathrm{Ker}(f')$,
- for every i such that $M_i = K_r^{(m)}$ for some (m, r), take $\{\lambda_{i,j}\}_j$ to be the family consisting of the morphisms $\sigma_{z,r}^{(m)}$ (for $z \in F^{-1} K_r^{(m)}$) together with the morphism $\tau_r^{(m)}$ (see Lemma 3.3.2).

According to Lemma 2.1.1, there exists an automorphism $\sigma \colon \mathrm{Ker}(f') \to \mathrm{Ker}(f')$ such that

- its restriction to $K^{(\infty)}$ is the inclusion morphism,
- for every (m, r), either there exists $k_r^{(m)} \in F^{-1} K_r^{(m)}$ such that the restriction of σ to $K_r^{(m)}$ is $\sigma_{k_r^{(m)},r}^{(m)}$, or else the restriction of σ to $K_r^{(m)}$ is $\tau_r^{(m)}$; In the former case, the composite morphism $K_r^{(m)} \xrightarrow{\sigma_{|K_r^{(m)}}} \mathrm{Ker}(f') \xrightarrow{i} X$ equals $[F(\iota_{k_r^{(m)},r,s}^{(m)})\ ;\ s]$ and lies in rad^m. In the latter case, it equals $i_r^{(m)} - [\sum_{Fz = K_r^{(m)}} F(\iota_{z,r,s}^{(m)})\ ;\ s]$ and lies in rad^{m+1}. In view of the minimality condition (d) assumed on the set of decompositions (2) (see 3.3.1), it is necessary that the former case always occurs and never does the latter case.

Now, denote by $\sigma_r^{(m)}$ and $\iota_{r,s}^{(m)}$ the morphisms $\sigma_{k_r^{(m)},r}^{(m)}$ and $\iota_{k_r^{(m)},r,s}^{(m)}$, respectively. Thus $\sigma_r^{(m)} i = [F(\iota_{r,s}^{(m)})\ ;\ s]$. Note that, since $\iota_{r,s}^{(m)}$ is zero whenever it lies in $\mathcal{R}^{m+1}\mathcal{C}$ and since σi is a monomorphism, it is necessary

that $\iota_{r,s}^{(m)} \notin \mathcal{R}^{m+1}\mathcal{C}$. The conclusion of the lemma therefore follows from the previous considerations and from those in 3.3.2. □

3.3.4. *Homogeneity of the inclusion morphism* $\mathrm{Ker}(f') \to X$. Replacing the set of decompositions (2) by their respective direct images under any automorphism σ of $\mathrm{Ker}(f')$ such as in Lemma 3.3.3 yields the following result.

Proposition 3.3.4. *There exist direct sum decompositions*

$$\begin{cases} \mathrm{Ker}(f') = K^{(\infty)} \oplus \left(\oplus_{n \leqslant m} K^{(m)}\right) \\ K^{(m)} = \oplus_r K_r^{(m)} \text{ for every } m \end{cases}$$

such that

(1) *each* $K_r^{(m)}$ *is indecomposable,*
(2) *the inclusion morphism* $K^{(\infty)} \to X$ *lies in* rad^∞*,*
(3) *for every* m, r*, the inclusion morphism* $K_r^{(m)} \to X$ *lies in* $\mathrm{rad}^m \backslash \mathrm{rad}^{m+1}$ *and is homogeneous with respect to* $(x_s)_s$*.*

Proof. This follows readily from Lemma 3.3.3 after taking into account the comment that precedes the statement of the proposition. □

The following consequence of the previous result is useful in the proof of the main results of this text.

Corollary 3.3.5. *Let* $i\colon \mathrm{Ker}(f') \to X$ *be the inclusion morphism. Let* $(K_r^{(m)})_{m,r}$ *be such as in Proposition 3.3.4. For every* $\ell \geqslant n$*,* $Z \in \Gamma$ *and* $h \in \mathrm{Hom}_A(Z, K)$ *the following conditions are equivalent*

(i) $hi \in \mathrm{rad}^\ell$*,*
(ii) *for every* m, r*, the component* $Z \to K_r^{(m)}$ *of* h *lies in* $\mathrm{rad}^{\ell-m}$*.*

Proof. This follows from Lemma 2.4.2 and Proposition 3.3.4. □

3.4. **The kernel characterisation for homogeneous morphisms.** It is now possible to prove the main result of this section. It translates the finiteness of the left degree of f in terms of exact sequences. As a reminder, here is the setting under which this result is valid: Γ is an Auslander-Reiten component of A and $F\colon \mathcal{C} \to \mathrm{ind}\,\Gamma$ is a functor with the covering property relatively to Γ; a morphism $f\colon X \to Y$ is given such that every indecomposable direct summand of $X \oplus Y$ lies in Γ; it is assumed that f lies in $\mathrm{rad}^d \backslash \mathrm{rad}^{d+1}$ and is homogeneous up to rad^{d+1} for some $d \in \mathbb{N}$, and that f has finite left degree denoted by n; finally, the homogeneous part of f is denoted by f', the inclusion morphism $\mathrm{Ker}(f') \to X$ is denoted by i and a direct sum decomposition of $\mathrm{Ker}(f')$ is given such as in Proposition 3.3.4. For every m, the inclusion morphism $K^{(m)} \to X$ is denoted by $i^{(m)}$.

Proposition 3.4.1. *For every* $Z \in \mathrm{ind}\,A$ *and every integer* $\ell \geqslant n$*, the following sequence is exact*

$$(\mathcal{E}_{Z,\ell}) \qquad 0 \to \bigoplus_{n \leqslant m \leqslant \ell} \frac{\mathrm{rad}^{\ell-m}}{\mathrm{rad}^{\ell-m+1}}(Z, K^{(m)}) \to \frac{\mathrm{rad}^\ell}{\mathrm{rad}^{\ell+1}}(Z, X) \to \frac{\mathrm{rad}^{\ell+d}}{\mathrm{rad}^{\ell+d+1}}(Z, Y)\,.$$

Proof. Note that the arrows of the sequence are induced by the restrictions $i^{(m)}\colon K^{(m)} \to X$ of i and by $f\colon X \to Y$. The exactness at $\oplus_{n \leqslant m \leqslant \ell} \frac{\mathrm{rad}^{\ell-m}}{\mathrm{rad}^{\ell-m+1}}(Z, K^{(m)})$ follows from Corollary 3.3.5. To prove the exactness at $\frac{\mathrm{rad}^\ell}{\mathrm{rad}^{\ell+1}}(Z, X)$, consider $g \in \mathrm{rad}^\ell(Z, X)$ such that $gf \in \mathrm{rad}^{\ell+d+1}(Z, Y)$.

Following Lemma 3.2.1, there exists $g' \in \mathrm{rad}^\ell(Z, X)$ such that $g - g' \in \mathrm{rad}^{\ell+1}$ and $g'f' = 0$. Note that, following Proposition 3.3.4, the inclusion morphism $\mathrm{Ker}(f') \to X$ may be written as $[i^{(m)}\ ;\ n \leqslant m \leqslant \ell]^T + a$ where $a \in \mathrm{rad}^{\ell+1}$. Accordingly, there exists $(h^{(m)})_{n \leqslant m \leqslant \ell} \in \oplus_m \mathrm{Hom}_A(Z, K^{(m)})$ such that $g' - \sum_m h^{(m)} i^{(m)} \in \mathrm{rad}^{\ell+1}$.

Now, Corollary 3.3.5 entails that $h^{(m)} \in \operatorname{rad}^{\ell-m}$ because $g' \in \operatorname{rad}^{\ell}$. Denote by $\overline{g}$ and $\overline{h^{(m)}}$ the classes of g and $h^{(m)}$ in $\frac{\operatorname{rad}^{\ell}}{\operatorname{rad}^{\ell+1}}(Z,X)$ and $\frac{\operatorname{rad}^{\ell-m}}{\operatorname{rad}^{\ell-m+1}}(Z,K^{(m)})$, respectively. Taking into account that $i^{(m)} \in \operatorname{rad}^{m}$ for every m, it follows from the previous considerations that $\overline{g}$ is the image of $(\overline{h^{(m)}})_{n\leqslant m\leqslant \ell}$, which finishes proving that $(\mathcal{E}_{Z,\ell})$ is exact. □

4. Applications to irreducible morphisms

This section applies the considerations of the previous section to some structure results on irreducible morphisms. In 4.1, the main result of the previous section is applied to irreducible morphisms $X \to Y$ where X or Y is indecomposable. And some consequences are derived. Next, in 4.2, the kernels of irreducible morphisms $X \to Y$ are compared when their left degree is finite. Finally, these results are applied in 4.3 to characterise algebras of finite representation type in terms of left degrees of irreducible morphisms. The results of this section extend the results proved in [4] to algebras over perfect fields and also strengthens them.

4.1. **Irreducible morphisms with finite (left) degree.** Since the kernel of an irreducible morphism is indecomposable, Proposition 3.4.1 specialises to irreducible morphisms as follows.

Theorem 4.1.1. *Let $f\colon X \to Y$ be an irreducible morphism such that X or Y is indecomposable. Assume that $d_\ell(f)$ is finite and denoted by n. Then, there exists an irreducible morphism $f'\colon X \to Y$ such that*

(1) $f - f' \in \operatorname{rad}^2$,
(2) the inclusion morphism $i\colon \operatorname{Ker}(f') \to X$ lies in $\operatorname{rad}^n\backslash\operatorname{rad}^{n+1}$ and is the composition of n irreducible morphisms between indecomposables,
(3) for every $\ell \geqslant n$, $Z \in \operatorname{ind} A$, the following sequence is exact

$$0 \to \frac{\operatorname{rad}^{\ell-n}}{\operatorname{rad}^{\ell-n+1}}(Z,\operatorname{Ker}(f')) \xrightarrow{-\cdot i} \frac{\operatorname{rad}^{\ell}}{\operatorname{rad}^{\ell+1}}(Z,X) \xrightarrow{-\cdot f} \frac{\operatorname{rad}^{\ell+1}}{\operatorname{rad}^{\ell+2}}(Z,Y)\,.$$

If f is freely irreducible, then f' may be chosen equal to f.

Proof. Assume that X is indecomposable (the remaining case is dealt with using dual considerations). Let Γ be the Auslander-Reiten component containing X. Let $F\colon \mathcal{C} \to \operatorname{ind}\Gamma$ be a functor with the covering property for which f is homogeneous up to rad^2 (see Proposition 2.4.1). Note that, when f is freely irreducible, it may be assumed to be homogeneous. Let f' be the homogeneous part of f. In particular, $f = f'$ if f is freely irreducible. The conclusion therefore follows from Lemma 2.2.2 and Proposition 3.4.1. □

The previous result provides the following characterisation of when the left degree is finite in terms of the kernel of the considered freely irreducible morphism. This characterisation is proved in [4] for irreducible morphisms with indecomposable domain or with indecomposable codomain.

Corollary 4.1.2. *Let $X, Y \in \operatorname{mod} A$ be such that X or Y is indecomposable. Let $f\colon X \to Y$ be a freely irreducible morphism. Let $i\colon \operatorname{Ker}(f) \to X$ be the inclusion morphism. Then, for every integer $n \geqslant 1$, the following conditions are equivalent.*

(i) $d_\ell(f)$ is finite and equal to n,
(ii) $i \in \operatorname{rad}^n\backslash\operatorname{rad}^{n+1}$.

Proof. Both (i) and (ii) imply that $d_\ell(f)$ is finite. Hence $d_\ell(f)$ may be assumed to be finite. Therefore, part (2) of Theorem 4.1.1 applies with $f = f'$. This proves the equivalence. □

The following Corollary extends [4, Corollary 3.2, Corollary 3.8] to the setting of algebras over perfect fields with an analogous proof. The details are given for the convenience of the reader. The Corollary relates the finiteness of left and right degrees for a given irreducible morphism.

Corollary 4.1.3. *Let $f\colon X \to Y$ be an irreducible morphism with $X \in \operatorname{ind} A$ or $Y \in \operatorname{ind} A$.*

(1) If $d_\ell(f) < \infty$ then f is not a monomorphism and $d_r(f) = \infty$. In particular every left minimal almost split morphism with non-injective domain has infinite left degree.
(2) If $d_r(f) < \infty$ then f is not an epimorphism and $d_\ell(f) = \infty$. In particular every right minimal almost split morphism with non-projective codomain has infinite right degree.
(3) If A is of finite representation type then the following conditions are equivalent
  (a) $d_\ell(f) < \infty$,
  (b) $d_r(f) = \infty$,
  (c) f is an epimorphism.

Proof. Assume that $d_\ell(f) < \infty$. By Theorem 4.1.1, there exists an irreducible morphism $f'\colon X \to Y$ such that the inclusion morphism $\operatorname{Ker}(f') \to X$ does not lie in $\operatorname{rad}^{1+d_\ell(f)}$. In particular, f' is not a monomorphism, and hence X has larger length than Y does. Therefore f is not a monomorphism, and hence it is an epimorphism. Using dual considerations shows that, if $d_r(f) < \infty$, then f is a monomorphism and not an epimorphism. Whence (1) and (2). And (3) follows from (1) and (2) because, when A is of finite representation type, then any nonzero morphism lies in $\operatorname{rad}^n \backslash \operatorname{rad}^{n+1}$ for some $n \in \mathbb{N}$. □

4.2. **Application to the kernel of irreducible morphisms.** The objective of this subsection is to prove Proposition 4.2.1: two irreducible morphisms $X \to Y$ where $X \in \operatorname{ind} A$ is such that $\kappa_X \simeq \Bbbk$ have isomorphic kernels (or, cokernels) whenever at least one of them has finite left degree (or, right degree, respectively). For this purpose, the following result is needed. It is proved in [4, Proposition 3.3] when $\Bbbk$ is algebraically closed and part of the statement is proved in in [11, 1.10] when $\Bbbk$ is an artin ring.

Proposition 4.2.1. *Let $f\colon X \to Y$ be an irreducible morphism with $Y \in \operatorname{ind} A$ non-projective. Assume that the almost split sequence in* $\operatorname{mod} A$

$$\begin{tikzcd} & & X' \arrow[dr, "f'"] & & \\ 0 \arrow[r] & \tau_A Y \arrow[ur, "g"] \arrow[dr, "g'"'] & & Y \arrow[r] & 0 \\ & & X \arrow[ur, "f"'] & & \end{tikzcd}$$

is such that $X' \neq 0$. Then $d_\ell(f) < \infty$ if and only if $d_\ell(g) < \infty$. In such a case $d_\ell(g) = d_\ell(f) - 1$.

Proof. It was proved in [11, 1.10] that if $d_\ell(f) < \infty$ then $d_\ell(g) \leqslant d_\ell(f) - 1$. Conversely assume that $d_\ell(g) = n < \infty$. Then there exists $Z \in \operatorname{ind} A$ and $h \in \operatorname{Hom}_A(Z, \tau_A Y)$ such that $h \in \operatorname{rad}^n \backslash \operatorname{rad}^{n+1}$ and $hg \in \operatorname{rad}^{n+2}$. In particular $hg' \in \operatorname{rad}^{n+1}$. On the one hand $d_\ell([g, g']) = \infty$ because $[g, g']$ is left minimal almost split (see Corollary 4.1.3). Therefore $h[g, g'] = [hg, hg'] \in \operatorname{rad}^{n+1} \backslash \operatorname{rad}^{n+2}$, and hence $hg' \notin \operatorname{rad}^{n+2}$. On the other hand $hg'f = -hgf' \in \operatorname{rad}^{n+3}$. Thus $d_\ell(f) \leqslant n + 1 = d_\ell(g) + 1$. □

Now it is possible to prove the main result of this subsection. It is proved in [4, Corollary 3.6] when $\Bbbk$ is algebraically closed (under which condition the residue field of any indecomposable module is trivial).

Proposition 4.2.2. *Let $f_1, f_2\colon X \to Y$ be irreducible morphisms. Assume that $X \in \operatorname{ind} A$ and $\kappa_X \simeq \Bbbk$, or else $Y \in \operatorname{ind} A$ and $\kappa_Y \simeq \Bbbk$. If $d_\ell(f_1) < \infty$ then $d_\ell(f_1) = d_\ell(f_2)$ and $\operatorname{Ker}(f_1) \simeq \operatorname{Ker}(f_2)$.*

Proof. First suppose that $X \in \operatorname{ind} A$ and $\kappa_X \simeq \Bbbk$. In particular, both f_1 and f_2 are freely irreducible. Denote by $i\colon \operatorname{Ker}(f_1) \to X$ and $j\colon \operatorname{Ker}(f_2) \to X$ the inclusion morphisms. Let $Y = \oplus_t Y_t$ be a decomposition such that $Y_t \in \operatorname{ind} A$ for every t, an write $f_1 = [f_{1,t}\ ;\ t]$ and $f_2 = [f_{2,t}\ ;\ t]$ accordingly. For every t, the arrow $X \to Y_t$ in $\Gamma(\operatorname{mod} A)$ has finite degree at most $d_\ell(f_1)$. Since $\kappa_X \simeq \Bbbk$, it follows that $X \to Y_t$ has valuation $(b, 1)$ for some $b \geqslant 1$ (see 2.1). Therefore, there exists $a_t \in \operatorname{Aut}_A(Y_t)$ such that $f_{2,t} - f_{1,t}a_t \in \operatorname{rad}^2$. Since $if_1 = 0$, it follows that $i[f_{1,t}a_t\ ;\ t] = 0$, and hence $if_2 \in \operatorname{rad}^{2+d_\ell(f_1)}$. This shows that f_2 has finite left degree bound by $d_\ell(f_1)$. Now, f_1 and f_2 play symmetric roles. Similar considerations as above therefore show that $jf_1 \in \operatorname{rad}^{d_\ell(f_2)+2}$ and $d_\ell(f_1) \leqslant d_\ell(f_2)$. Hence, $d_\ell(f_1) = d_\ell(f_2)$ and $jf_1 \in \operatorname{rad}^{2+d_\ell(f_1)}$. Applying part (3) of Theorem 4.1.1 to f_1 then shows that there exists a section $\sigma\colon \operatorname{Ker}(f_2) \to \operatorname{Ker}(f_1)$ such that $j - \sigma i \in \operatorname{rad}^{d_\ell(f_1)+1}$. In particular, σ is an isomorphism because both $\operatorname{Ker}(f_1)$ and $\operatorname{Ker}(f_2)$ are indecomposable. This finishes proving the proposition when $X \in \operatorname{ind} A$ and $\kappa_X \simeq \Bbbk$.

Now, assume that $Y \in \operatorname{ind} A$ and $\kappa_Y \simeq \Bbbk$. If $d_\ell(f_1) = 1$, then f_1 is right minimal almost split, and hence so is f_2. The conclusion is then immediate. From now on, assume that $d_\ell(f_1) > 1$. In particular, neither f_1 nor f_2 is minimal right almost split. From part (3) of Theorem 4.1.1, f_1 is not a monomorphism. Hence, Y is not projective, and there exist almost split sequences

$$\begin{tikzcd} & & X' \arrow[dr] & & \\ 0 \arrow[r] & \tau_A Y \arrow[ur, "g_1"] \arrow[dr] & & Y \arrow[r] & 0 \\ & & X \arrow[ur, "f_1"'] & & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} & & X' \arrow[dr] & & \\ 0 \arrow[r] & \tau_A Y \arrow[ur, "g_2"] \arrow[dr] & & Y \arrow[r] & 0 \\ & & X \arrow[ur, "f_2"'] & & \end{tikzcd}$$

where $X' \neq 0$. First note that $\operatorname{Ker}(f_1) \simeq \operatorname{Ker}(g_1)$. Indeed, by diagram chasing, the natural monomorphism $\operatorname{Ker}(f_1) \to X$ induces an monomorphism $\operatorname{Ker}(f_1) \to \operatorname{Ker}(g_1)$. Moreover $\operatorname{Ker}(f_1)$ and $\operatorname{Ker}(g_1)$ have the same length. Therefore $\operatorname{Ker}(f_1) \simeq \operatorname{Ker}(g_1)$. Similar considerations show that $\operatorname{Ker}(f_2) \simeq \operatorname{Ker}(g_2)$. Next if follows from Proposition 4.2.1 that $d_\ell(g_1) = d_\ell(f_1) - 1$. Since $g_1, g_2\colon \tau_A Y \to X'$ are irreducible, and since $\kappa_{\tau_A Y} \simeq \kappa_Y \simeq \Bbbk$, the first part of the proof yields that $d_\ell(g_1) = d_\ell(g_2)$ and $\operatorname{Ker}(g_1) \simeq \operatorname{Ker}(g_2)$. As a consequence, $\operatorname{Ker}(f_1) \simeq \operatorname{Ker}(f_2)$. Finally, applying again Proposition 4.2.1 gives $d_\ell(f_2) = d_\ell(g_2) + 1$, and hence $d_\ell(f_2) = d_\ell(f_1)$. □

4.3. **Application to the finite representation type.** To end this section, an application of Theorem 4.1.1 to a characterisation of algebras of finite representation type is given below in terms of left and right degrees. In the following theorem, the equivalence of conditions (a) to (e) was proved in [4, 4], and the assertions (f) and (g) where proved in [3], in the case where $\Bbbk$ is algebraically closed.

Theorem 4.3.1. *Let A be a finite dimensional algebra over a perfect field $\Bbbk$. The following conditions are equivalent*

(a) A is of finite representation type,
(b) for every indecomposable projective A-module P, the inclusion $\operatorname{rad}(P) \hookrightarrow P$ has finite right degree,
(c) for every indecomposable injective A-module I, the quotient $I \twoheadrightarrow I/\operatorname{soc}(I)$ has finite left degree,
(d) for every irreducible epimorphism $X \to Y$ with X or Y indecomposable, the left degree is finite,
(e) for every irreducible monomorphism $X \to Y$ with X or Y indecomposable, the right degree is finite.

If A is of finite representation type, then

(f) there exists an indecomposable injective A-module I with associated irreducible epimorphism $\pi\colon I \twoheadrightarrow I/\operatorname{soc}(I)$ such that $d_\ell(f) \leqslant d_\ell(\pi)$ for every irreducible epimorphism $f\colon X \to Y$ with X or Y indecomposable.

(g) there exists an indecomposable projective A-module P with associated irreducible monomorphism $\iota\colon \mathrm{rad}(P) \hookrightarrow P$ such that $d_r(f) \leqslant d_r(\iota)$ for every irreducible monomorphism $f\colon X \to Y$ with X or Y indecomposable.

Proof. The proof of the equivalence of conditions (a) to (e) follows from the considerations in [4, Lemma 4.1, Lemma 4.2, Theorem A] provided that Theorem 4.1.1 is used here instead of [4, Proposition 3.4] there, and that Corollary 4.1.3 is used here instead of [4, Corollary 3.8] there.

There only remains to prove (f) and (g) assuming that A is of finite representation type. Since (g) is dual to (f) and since there are only finitely many isomorphism classes of indecomposable injective modules, it suffices to prove that, for every irreducible epimorphism $f\colon X \to Y$, there exists an indecomposable injective module I such that $d_\ell(f) \leqslant d_\ell(\pi)$. Apply Theorem 4.1.1 to such an f: Since $d_\ell(f) < \infty$, there exists an irreducible morphism $f'\colon X \to Y$ such that $f - f' \in \mathrm{rad}^2$ and such that the inclusion morphism $i\colon \mathrm{Ker}(f') \to X$ lies in $\mathrm{rad}^{d_\ell(f)} \backslash \mathrm{rad}^{d_\ell(f)+1}$. Let S be a simple direct summand of $\mathrm{soc}(\mathrm{Ker}(f'))$. Let I be its injective hull. Therefore there exists a morphism $X \to I$ making the following diagram commute

$$\begin{tikzcd} S \arrow[rr, hook] \arrow[dr, hook] & & I \\ & \mathrm{Ker}(f') \arrow[r, hook, "i"'] & X \arrow[u] \end{tikzcd} .$$

Note that all irreducible morphisms $I \to I/S$ have their kernel isomorphic to S. Therefore, considering (c), and applying Theorem 4.1.1 to $\pi\colon I \to I/S$ yields that the inclusion morphism $S \to I$ lies in $\mathrm{rad}^{d_\ell(\pi)} \backslash \mathrm{rad}^{d_\ell(\pi)+1}$. Thus, $d_\ell(f) \leqslant d_\ell(\pi)$. □

5. Application to compositions of irreducible morphisms

In this section, by a *path* is meant a path of irreducible morphisms between indecomposable modules. The objective of this section is to investigate the paths $f_1, \dots, f_n$ such that $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$. The main result is Theorem C. The following proposition shows a first part of it.

Proposition 5.0.2. *Let $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ be a path. For each t, let $f'_t\colon X_{t-1} \to X_t$ be such as f' in Theorem B when $f = f_t$. The following are equivalent.*

(i) $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$.

(ii) There exists $t \in \{1, \dots, n\}$ such that $d_\ell(f_t) \leqslant t-1$, and there exists $h \in \mathrm{rad}^{t-1-d_\ell(f_t)}(X_0, \mathrm{Ker}(f'_t))$ not lying in $\mathrm{rad}^{t-d_\ell(f_t)}$, and such that $f_1 \cdots f_{t-1} - hi \in \mathrm{rad}^t$ (where $i\colon \mathrm{Ker}(f'_t) \to X_{t-1}$ is the inclusion morphism).

In particular, if $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$ and if t is an integer such as in (ii), then the following holds.

(iii) There exists a path of length $t - 1 - d_\ell(f_t)$ from X_0 to $\mathrm{Ker}(f'_t)$ and with nonzero composition.

Proof. Assume (ii). Since $if_t = 0$, then

$$f_1 \cdots f_n = (f_1 \cdots f_{t-1} - hi) \cdot f_t \cdots f_n \,.$$

By assumption, $f_1 \cdots f_{t-1} - hi$ lies in rad^t. Consequently, $f_1 \cdots f_n \in \mathrm{rad}^n + 1$. This proves that (ii) implies (i).

Assume that $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$. There is no loss of generality in assuming that $f_1 \cdots f_{n-1} \notin \mathrm{rad}^n$. Therefore f_n has finite left degree. Now, consider the exact sequence obtained upon applying Theorem 4.1.1 to $f := f_n$. This shows assertion (ii). Thus, $(i) \implies (ii)$. The last assertion of the corollary is obtained by considering any decomposition of h into a sum of compositions of paths of length at least d. □

The following example shows that, in the previous result, (iii) need not imply (i).

Example 5.0.3. *Consider the artin $\mathbb{R}$-algebra of finite representation type*

$$A = \begin{pmatrix} \mathbb{C} & 0 \\ \mathbb{C} & \mathbb{R} \end{pmatrix}$$

where $\mathbb{R}$ is the field of real numbers and $\mathbb{C}$ the field of complex numbers.

The Auslander-Reiten quiver without considering valuations in the arrows is the following

$$\begin{array}{ccccccc} S_2 & & \dots & & P_1/S_2 & & \\ & \overset{f}{\searrow} & & \overset{g}{\nearrow} & & \overset{h}{\searrow} & \\ & & P_1 & & & & \tau^{-1}P_1 \\ & \overset{f'}{\nearrow} & & \overset{g'}{\searrow} & & \overset{h'}{\nearrow} & \\ S_2 & & \dots & & P_1/S_2 & & \end{array}$$

where the two copies of S_2 and P_1/S_2 are identified.

The path $S_2 \xrightarrow{f'} P_1 \xrightarrow{g} P_1/S_2$ is a pre-sectional path, since $S_2 \oplus S_2$ is a summand of the domain of the right almost split morphism for P_1. Moreover, $f'g \in \mathrm{rad}^2 \backslash \mathrm{rad}^3$. In fact, if $f'g \in \mathrm{rad}^3$, since $fg = 0$ then $d_l((f, f')) = 1$ a contradiction to the fact that (f, f') is not a surjective right almost split morphism.

Observe that the above path satisfies (iii) in Proposition 5.0.2, but it is not in rad^3.

Remark 5.0.4. *Let $X_0 \xrightarrow{f_1} X_1 \to \cdots \to X_{n-1} \xrightarrow{f_n} X_n$ be a path.*

(1) Assume that there exists an integer t such that the composition of any path $X_0 \to X_1 \to \cdots \to X_{t-1} \to X_t$ is nonzero and such that $d_\ell(f_s) \geqslant s$ for every $s \geqslant t+1$. Then, the former condition entails that $f_1 \cdots f_t \in \mathrm{rad}^t \backslash \mathrm{rad}^{t+1}$ (see [5, Proposition 3]*). And the latter condition then implies that $f_1 \cdots f_n \notin \mathrm{rad}^{n+1}$.*

(2) Following [5, Section 3]*, if there exists a path $X_0 \to X_1 \to \cdots \to X_{n-1} \to X_n$ with composition equal to 0, and if each one of the $\Bbbk$-vector spaces $\mathrm{irr}(X_{i-1}, X_i)$ is one dimensional, then $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$.*

It is now straightforward to prove the second main result of this text.

Proof of Theorem C. The equivalence $(i) \Leftrightarrow (ii)$ is provided by Proposition 5.0.2.

Assume that $f_1 \cdots f_n \in \mathrm{rad}^{n+1}$. Applying Proposition 5.0.2 yields an integer t such that $f_1 \cdots f_t \in \mathrm{rad}^{t+1}$ and $d_\ell(f_t) \leqslant t-1$, and a path of length $t - 1 - d_\ell(f_t)$ from X_0 to $\mathrm{Ker}(f_t)$ and with nonzero composition. Applying [5, Section 3] to the path $(f_1, \dots, f_t)$ then yields a path $X_0 \to \cdots \to X_t$ with zero composition. This shows that (i) and (ii) imply (iii).

Assume (iii) and assume that $\dim_{\Bbbk} \mathrm{irr}(M_{s-1}, M_s) = 1$ for every $s \in \{1, \dots, t\}$. Then $f_1 \cdots f_t \in \mathrm{rad}^{t+1}$ according to part (2) of Lemma 2.2.2. □

References

[1] M. Auslander, I. Reiten, and S. O. Smalø. *Representation theory of Artin algebras*, volume 36 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995.

[2] S. Brenner. On the kernel of an irreducible map. *Linear Algebra Appl.*, 365:91–97, 2003. Special issue on linear algebra methods in representation theory.

[3] C. Chaio. Problems solved by using degrees of irreducible morphisms. In *Expository lectures on representation theory*, volume 607 of *Contemp. Math.*, pages 179–203. Amer. Math. Soc., Providence, RI, 2014.

[4] C. Chaio, P. Le Meur, and S. Trepode. Degrees of irreducible morphisms and finite representation type. *J. Lond. Math. Soc. (2)*, 84(1):35–57, 2011.
[5] C. Chaio, P. Le Meur, and S. Trepode. Covering techniques for Auslander-Reiten theory. *J. Pure Appl. Algebra*, 2018. https://doi.org/10.1016/j.jpaa.2018.04.013.
[6] F. U. Coelho and D. D. da Silva. Relative degrees of irreducible morphisms. *J. Algebra*, 428:471–489, 2015.
[7] K. Igusa and G. Todorov. A characterization of finite Auslander-Reiten quivers. *J. Algebra*, 89(1):148–177, 1984.
[8] K. Igusa and G. Todorov. Radical layers of representable functors. *J. Algebra*, 89(1):105–147, 1984.
[9] H. Krause. The kernel of an irreducible map. *Proc. Amer. Math. Soc.*, 121(1):57–66, 1994.
[10] S. Liu. Degrees of irreducible maps and the shapes of Auslander-Reiten quivers. *J. London Math. Soc. (2)*, 45(1):32–54, 1992.
[11] S. Liu. Shapes of connected components of the Auslander-Reiten quivers of Artin algebras. In *Representation theory of algebras and related topics (Mexico City, 1994)*, volume 19 of *CMS Conf. Proc.*, pages 109–137. Amer. Math. Soc., Providence, RI, 1996.

(Claudia Chaio) Centro Marplatense de Investigaciones Matemáticas, FCEyN, Universidad Nacional de Mar del Plata, CONICET. Funes 3350, 7600 Mar del Plata, Argentina
E-mail address: algonzal@mdp.edu.ar

(Patrick Le Meur) Laboratoire de Mathématiques, Université Blaise Pascal & CNRS, Complexe Scientifique Les Cézeaux, BP 80026, 63171 Aubière cedex, France
Current address: Université Paris Diderot, Sorbonne Université, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, F-75013, Paris, France
E-mail address: patrick.le-meur@imj-prg.fr

(Sonia Trepode) Centro Marplatense de Investigaciones Matemáticas, FCEyN, Universidad Nacional de Mar del Plata,CONICET. Funes 3350, 7600 Mar del Plata, Argentina
E-mail address: strepode@mdp.edu.ar