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Well-posedness result for a system of random heat equation coupled with a multiplicative stochastic Barenblatt equation

Caroline Bauzet, Frédéric Lebon, Asghar Ali Maitlo and Aleksandra Zimmermann

Abstract. In this paper, a stochastic nonlinear evolution system under Neumann boundary conditions is investigated. Precisely, we are interested in finding an existence and uniqueness result for a system of random heat equation coupled with a Barenblatt's type equation with a multiplicative stochastic force in the sense of Itô. To do so, we investigate in a first step the case of an additive noise through a semi-implicit in time discretization of the system. This allows us to show the well-posedness of the system in the additive case. In a second step, the derivation of continuous dependence estimates of the solution with respect to the data allows us to show the desired existence and uniqueness result for the multiplicative case.

Keywords. Stochastic system, random heat equation, Barenblatt equation, additive noise, multiplicative noise, Itô integral, Neumann condition, time discretization, fixed point, maximal monotone operators.

1. Introduction

We consider the following system, coupling a heat equation with Barenblatt's one, perturbed firstly by an additive noise:

$$\left\{ \begin{array}{l} \partial_t \vartheta + \partial_t \left(\chi - \int_0^\cdot h dw \right) - \Delta \vartheta = 0 \text{ in } (0, T) \times D \times \Omega, \\ \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot h dw \right) \right) - \Delta \chi = \vartheta \text{ in } (0, T) \times D \times \Omega, \\ \nabla \chi \cdot \mathbf{n} = \nabla \vartheta \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial D \times \Omega, \\ \chi(0, \cdot) = \chi_0 \text{ and } \vartheta(0, \cdot) = \vartheta_0, \end{array} \right. \quad (1.1)$$

and secondly by a multiplicative one:

$$\left\{ \begin{array}{l} \partial_t \vartheta + \partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) dw \right) - \Delta \vartheta = 0 \text{ in } (0, T) \times D \times \Omega, \\ \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) dw \right) \right) - \Delta \chi = \vartheta \text{ in } (0, T) \times D \times \Omega, \\ \nabla \chi \cdot \mathbf{n} = \nabla \vartheta \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial D \times \Omega, \\ \chi(0, \cdot) = \chi_0 \text{ and } \vartheta(0, \cdot) = \vartheta_0, \end{array} \right. \quad (1.2)$$

where $T > 0$, D denotes a smooth and bounded domain of \mathbb{R}^d with $d \geq 1$, \mathbf{n} is the outward normal vector to the boundary ∂D , χ_0 and ϑ_0 are given initial conditions. We consider a standard adapted one-dimensional continuous Brownian motion

$$w = \{w_t, \mathcal{F}_t, 0 \leq t \leq T\}$$

defined on a complete probability space (Ω, \mathcal{F}, P) with a countably generated σ -field denoted \mathcal{F} and a filtration $(\mathcal{F}_t)_{t \geq 0}$ satisfying usual conditions (see [14], [20] for further informations on stochastic calculus). Let us precise that the additive and multiplicative stochastic integrals $\int_0^\cdot h dw$ and $\int_0^\cdot \mathcal{H}(\chi) dw$ are considered in the sense of Itô.

We assume the following assumptions:

$$H_1: h \in \mathcal{N}_w^2(0, T, H^1(D))^\dagger.$$

$$H_2: \tilde{\alpha} = I_d + \alpha \text{ where } I_d : \mathbb{R} \rightarrow \mathbb{R} \text{ is the identity function and } \alpha : \mathbb{R} \rightarrow \mathbb{R} \text{ is a Lipschitz-continuous, coercive and non-decreasing function such that } \alpha(0) = 0.$$

$$H_3: \chi_0, \vartheta_0 \in H^1(D).$$

$$H_4: \mathcal{H} : H^1(D) \rightarrow H^1(D) \text{ is a Lipschitz-continuous mapping with Lipschitz constant } C_{\mathcal{H}} > 0.$$

1.1. State of the art

In the deterministic case (i.e. when $h = \mathcal{H} = 0$), one application of such nonlinear evolution system is the description of phase transition phenomena, including irreversible phase changes (for instance, solidification of glue, cooking an egg...) see [12] for further details.

Let us mention that Barenblatt's type equations, (namely $f(\partial_t \chi) - \Delta \chi = 0$ with f a non-decreasing function), were initially studied by G.I. BARENBLATT for the theory of fluids in elasto-plastic porous medium [6], under the assumption that the porous medium is irreversibly deformable. After that, number of researches was carried out around such equations for detail one is referred to [15, 16, 17]. Moreover these type of equations has been studied in various areas : for irreversible phase change modeling [21], for reaction-diffusion with absorption problems in Biochemistry [21], for irreversible damage and fracture evolution analysis [10, 11, 19] and recently for constrained stratigraphic problems in Geology [1, 2, 3, 4, 24].

[†]For a given separable Hilbert space X , we denote by $\mathcal{N}_w^2(0, T, X)$ the space of predictable X -valued processes endowed with the norm $\|\phi\|_{\mathcal{N}_w^2(0, T, X)}^2 = E \left[\int_0^T \|\phi\|_X^2 dt \right]$ (see Da Prato-Zabczyk [14] p.94).

Concerning the study of Barenblatt equations with a stochastic force term, a few papers has been written. Up to our knowledge, none of them proposes the study of the coupling with a random heat equation. Let us mention the study [5], where the authors were concerned by Barenblatt equation with stochastic coefficients. In [7], the authors proposed an existence and uniqueness result for a stochastic Barenblatt equation (in the sense of Itô as in the present paper) under Dirichlet boundary conditions in the additive and multiplicative case. After that, well-posedness theory for stochastic abstract problems of Barenblatt's type has been investigated in [9]. More recently, an extension of [7] has been proposed in [8], by considering Neumann boundary conditions and additionally the presence of a nonlinear source term.

1.2. Goal of the study

In the study of composite or bonded structures, temperature effects in the evolution of damage at the interface can not be ignored, it is even a fundamental coupling [11, 25]. Additionally, the introduction of stochastic and random effects is also important from a modeling point of view in order to take into account several phenomena such as microscopic fluctuations, random forcing, effects of interscale interactions... In this direction, the aim of the present work is to study the coupling between a stochastic Barenblatt equation and a random heat one under Neumann boundary conditions. The idea is to extend the work done in [8] on a stochastic Barenblatt equation by proposing an existence and uniqueness result for such a coupling system.

1.3. General notations

For the sake of clarity, let us make precise some useful notations :

- . $Q = (0, T) \times D$.
- . $x.y$ the usual scalar product of x and y in \mathbb{R}^d .
- . $\mathcal{D}(D) = \mathcal{C}_c^\infty(D)$ and $\mathcal{D}'(D)$ the space of distributions on D .
- . $\|\cdot\|$ and (\cdot, \cdot) respectively the usual norm and the scalar product in $L^2(D)$.
- . $E[\cdot]$ the expectation, *i.e.* the integral over Ω with respect to the probability measure P .
- . $C_\alpha > 0$ the Lipschitz constant of α .
- . $\bar{C}_\alpha > 0$ the coerciveness constant of α which satisfies for any x, y in \mathbb{R} ,

$$(\alpha(x) - \alpha(y))(x - y) \geq \bar{C}_\alpha(x - y)^2.$$

- . $\bar{C}_{\tilde{\alpha}} > 0$ the coerciveness constant of $\tilde{\alpha}$ which satisfies for any x, y in \mathbb{R} ,

$$(\tilde{\alpha}(x) - \tilde{\alpha}(y))(x - y) \geq \bar{C}_{\tilde{\alpha}}(x - y)^2.$$

1.4. Concept of solution and main results of the paper

Let us introduce the concept of solutions we are interested in for System (1.1) and System (1.2).

Definition 1.1. Any pair of predictable processes $(\vartheta, \chi) \in (\mathcal{N}_w^2(0, T, H^1(D)))^2$ such that $\vartheta \in L^2(\Omega, H^1(Q))$ and $\partial_t(\chi - \int_0^\cdot h dw) \in L^2(\Omega \times Q)$, is a solution of System (1.1) if t -almost everywhere in $(0, T)$, P -almost surely in Ω , the following variational formulations hold: for any $v \in H^1(D)$,

$$\int_D \partial_t \vartheta v dx + \int_D \partial_t (\chi - \int_0^\cdot h dw) v dx + \int_D \nabla \vartheta \cdot \nabla v dx = 0 \quad (1.3)$$

$$\int_D \tilde{\alpha} \left(\partial_t (\chi - \int_0^\cdot h dw) \right) v dx + \int_D \nabla \chi \cdot \nabla v dx = \int_D \vartheta v dx, \quad (1.4)$$

with $\chi(0, \cdot) = \chi_0 \in H^1(D)$ and $\vartheta(0, \cdot) = \vartheta_0 \in H^1(D)$.

Definition 1.2. Any pair of predictable processes $(\vartheta, \chi) \in (\mathcal{N}_w^2(0, T, H^1(D)))^2$ such that $\vartheta \in L^2(\Omega, H^1(Q))$ and $\partial_t(\chi - \int_0^\cdot \mathcal{H}(\chi) dw) \in L^2(\Omega \times Q)$, is a solution of System (1.2) if t -almost everywhere in $(0, T)$, P -almost surely in Ω , the following variational formulations hold: for any $v \in H^1(D)$,

$$\int_D \partial_t \vartheta v dx + \int_D \partial_t (\chi - \int_0^\cdot \mathcal{H}(\chi) dw) v dx + \int_D \nabla \vartheta \cdot \nabla v dx = 0$$

$$\int_D \tilde{\alpha} \left(\partial_t (\chi - \int_0^\cdot \mathcal{H}(\chi) dw) \right) v dx + \int_D \nabla \chi \cdot \nabla v dx = \int_D \vartheta v dx,$$

with $\chi(0, \cdot) = \chi_0 \in H^1(D)$ and $\vartheta(0, \cdot) = \vartheta_0 \in H^1(D)$.

Remark 1.3. We will see later on that the respective solutions of (1.1) and (1.2) belong to the space $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$. Thus, they satisfy the initial condition in the following sense:

$$\begin{aligned} & \text{P-a.s, in } \Omega & \chi(t=0, \cdot) &= \lim_{t \rightarrow 0} \chi(t, \cdot) \text{ in } L^2(D) \\ \text{and } & \text{P-a.s, in } \Omega & \vartheta(t=0, \cdot) &= \lim_{t \rightarrow 0} \vartheta(t, \cdot) \text{ in } L^2(D). \end{aligned}$$

The results we want to prove in the sequel are the following :

Theorem 1.4. *Under assumptions H_1 to H_3 , there exists a unique pair (ϑ, χ) solution of System (1.1) in the sense of Definition 1.1.*

Moreover, the unique solution of (1.1) satisfies the following stability result which asserts the continuous dependence of the solution with respect to the integrand h in the stochastic noise :

Proposition 1.5. *Consider h, \hat{h} in $\mathcal{N}_w^2(0, T, H^1(D))$ and denote by (ϑ, χ) and $(\hat{\vartheta}, \hat{\chi})$ the associated solutions to the System (1.1) in the sense of Definition 1.1 with the respective set of data (ϑ_0, χ_0, h) and $(\vartheta_0, \chi_0, \hat{h})$. Then, there exists a constant $C_\alpha^T > 0$ which only depends on T, C_α and \hat{C}_α such that for any t in $[0, T]$*

$$\begin{aligned} & E \left[\|(\vartheta - \hat{\vartheta})(t)\|^2 \right] + E \left[\|\nabla(\vartheta - \hat{\vartheta})(t)\|^2 \right] \\ & + \frac{1}{4} E \left[\|(\chi - \hat{\chi})(t)\|^2 \right] + \frac{1}{4} E \left[\|\nabla(\chi - \hat{\chi})(t)\|^2 \right] \\ & \leq C_\alpha^T \left(\|h - \hat{h}\|_{L^2(\Omega \times Q_t)}^2 + \|\nabla(h - \hat{h})\|_{L^2(\Omega \times Q_t)}^2 \right), \end{aligned}$$

where $Q_t = (0, t) \times D$.

Theorem 1.6. *Under Assumptions H_2 to H_4 , there exists a unique pair (ϑ, χ) solution of Problem (1.2) in the sense of Definition 1.2.*

1.5. Outline of the paper

The paper is organized as follows. Firstly, we are interested in showing the existence of a couple (ϑ, χ) solution of System (1.1). To do so, the approach is the following one: we approximate our “additive” stochastic system by using an implicit time discretization scheme with a parameter $\Delta t > 0$.

After deriving stability estimates satisfied by the time-approximations of the couple (ϑ, χ) , our aim is to pass to the limit on the obtained discrete system with respect to the time-step Δt . Note that due to the random variable, classical results of compactness do not hold, and the main difficulty is in the identification of the nonlinear term’s limit associated with the discretization of $\alpha(\partial_t(\chi - \int_0^\cdot h dw))$. Using arguments on maximal monotone operator, one is able to handle this difficulty.

Secondly, the uniqueness result for (1.1) is proven by using classical energy estimates well known for the heat equation and adapted to the random and stochastic case. This allows us to show additionally at the limit on the discretization parameter Δt that the couple (ϑ, χ) depends continuously on the data. Finally, exploiting this stability result of the solution with respect to the data, we are able to extend (thanks to a fixed point argument) our result of existence and uniqueness to the multiplicative case, that is the well-posedness of Problem (1.2).

2. Time approximation of the additive case

The result of existence of a solution for Problem (1.1) is based on an implicit time discretization scheme for the deterministic part and an explicit one for the Itô part. To do so, let us introduce notations used for the discretization procedure.

2.1. Notations and preliminary results

We consider X a separable Banach space, $N \in \mathbb{N}^*$, set $\Delta t = \frac{T}{N}$ and $t_n = n\Delta t$ with $n \in \{0, \dots, N\}$. For any sequence $(x_n)_{0 \leq n \leq N} \subset X$, let us denote by

$$\begin{aligned} x^{\Delta t} &= \sum_{k=0}^{N-1} x_{k+1} \mathbb{1}_{[t_k, t_{k+1})}, \\ x_{\Delta t} &= \sum_{k=0}^{N-1} x_k \mathbb{1}_{[t_k, t_{k+1})} = x^{\Delta t}(\cdot - \Delta t), \\ \tilde{x}^{\Delta t} &= \sum_{k=0}^{N-1} \left[\frac{x_{k+1} - x_k}{\Delta t} (\cdot - t_k) + x_k \right] \mathbb{1}_{[t_k, t_{k+1})}, \\ \frac{\partial \tilde{x}^{\Delta t}}{\partial t} &= \sum_{k=0}^{N-1} \frac{x_{k+1} - x_k}{\Delta t} \mathbb{1}_{[t_k, t_{k+1})}, \end{aligned}$$

with the convention that $t_{-1} = -\Delta t$, for $t < 0$, $\tilde{x}^{\Delta t}(t_0) = x_0$ and $x^{\Delta t}(t_N) = \tilde{x}^{\Delta t}(t_N) = x_N$. Elementary calculations yield for an arbitrary constant $C > 0$ independent of Δt

$$\begin{aligned} \|x^{\Delta t}\|_{L^2(0,T;X)}^2 &= \Delta t \sum_{k=1}^N \|x_k\|_X^2 \quad ; \quad \|\tilde{x}^{\Delta t}\|_{L^2(0,T;X)}^2 \leq C \Delta t \sum_{k=0}^N \|x_k\|_X^2; \\ \|x^{\Delta t} - \tilde{x}^{\Delta t}\|_{L^2(0,T;X)}^2 &= \Delta t \sum_{k=0}^{N-1} \|x_{k+1} - x_k\|_X^2; \\ \|x^{\Delta t}(\cdot - \Delta t) - x^{\Delta t}\|_{L^2(0,T;X)}^2 &= \Delta t \sum_{k=0}^{N-1} \|x_{k+1} - x_k\|_X^2; \\ \left\| \frac{\partial \tilde{x}^{\Delta t}}{\partial t} \right\|_{L^2(0,T;X)}^2 &= \frac{1}{\Delta t} \sum_{k=0}^{N-1} \|x_{k+1} - x_k\|_X^2; \\ \|x^{\Delta t}\|_{L^\infty(0,T;X)} &= \max_{k=1,\dots,N} \|x_k\|_X \quad \text{and} \quad \|\tilde{x}^{\Delta t}\|_{L^\infty(0,T;X)} = \max_{k=0,\dots,N} \|x_k\|_X. \end{aligned}$$

We will use the following notations for the discretization of the data for any n in $\{0, \dots, N\}$:

$$w_n = w(t_n), \quad h_n = \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} h(s, \cdot) ds, \quad B_n = \sum_{k=0}^{n-1} (w_{k+1} - w_k) h_k,$$

with the convention that $t_{-1} = -\Delta t$ and $h(s, \cdot) = 0$ if $s < 0$.

Remark 2.1. As h is predictable with values in $H^1(D)$ then h_n belongs to $L^2((\Omega, \mathcal{F}_{t_n}); H^1(D))$ for any n in $\{0, \dots, N\}$.

Remark 2.2. For any n in $\{0, \dots, N\}$, $B_n = \int_0^{t_n} h_{\Delta t}(s) dw(s)$.

Indeed, as h_k is \mathcal{F}_{t_k} -measurable, one has

$$B_n = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} h_k dw(s) = \int_0^{t_n} \sum_{k=0}^{n-1} h_k \mathbf{1}_{[t_k, t_{k+1}[}(s) dw(s) = \int_0^{t_n} h_{\Delta t}(s) dw(s).$$

Lemma 2.3. *There exists a constant $C \geq 0$ independent of Δt such that for any n in $\{0, \dots, N\}$*

$$E \left[\sum_{k=0}^n \|h_k\|_{H^1(D)}^2 \right] \leq \frac{C}{\Delta t}.$$

Proof. The proof of this result can be found in [8] (Lemma 2.3). \square

Lemma 2.4. *The sequence $(h_{\Delta t})$ converges to h in $\mathcal{N}_w^2(0, T, H^1(D))$ as the time discretization parameter Δt tends to 0.*

Proof. See SIMON [22], Lemma 12 p.52. \square

Proposition 2.5. *The sequences $(B^{\Delta t})$ and $(\tilde{B}^{\Delta t})$ converge to $\int_0^\cdot h dw$ in $L^2(0, T; L^2(\Omega, H^1(D)))$ as the time discretization parameter Δt tends to 0.*

Proof. The proof of this result can be found in [8] (Proposition 2.5). \square

Remark 2.6. If one assumes that h belongs to $\mathcal{N}_w^2(0, T, H^2(D))$, one shows in the same manner that $B^{\Delta t}$ converges strongly to $\int_0^\cdot h dw$ in $L^2((0, T) \times \Omega, H^2(D))$ as Δt tends to 0.

2.2. Discretization schemes

Let N be a positive integer and $n \in \{0, \dots, N\}$. Using the notations of the previous section, the discretization scheme for (1.3) is the following one: for a given small positive parameter Δt , for ϑ_n, χ_n in $L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$ and $\chi_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$, our aim is to find ϑ_{n+1} in $L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$, such that P -a.s in Ω and for any v in $H^1(D)$

$$\int_D \left(\frac{\vartheta_{n+1} - \vartheta_n}{\Delta t} \right) v dx + \int_D \left(\frac{\chi_{n+1} - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t} \right) v dx + \int_D \nabla \vartheta_{n+1} \cdot \nabla v dx = 0. \quad (2.1)$$

Similarly the discretization scheme for (1.4) is the following one: for a given small positive parameter Δt , for $\chi_n \in L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$ and $\vartheta_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$, our aim is to find $\chi_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$, such that P -a.s in Ω and for any v in $H^1(D)$

$$\int_D \tilde{\alpha} \left(\frac{\chi_{n+1} - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t} \right) v dx + \int_D \nabla \chi_{n+1} \cdot \nabla v dx = \int_D \vartheta_{n+1} v dx. \quad (2.2)$$

With the notations introduced in the previous section, we propose the following discretization of the variational problems (1.3) and (1.4) : t -almost everywhere in $(0, T)$, P -almost surely in Ω and for any v in $H^1(D)$

$$\int_D \partial_t(\tilde{\vartheta}^{\Delta t}) v dx + \int_D \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) v dx + \int_D \nabla \tilde{\vartheta}^{\Delta t} \cdot \nabla v dx = 0 \quad (2.3)$$

$$\int_D \tilde{\alpha} \left(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) \right) v dx + \int_D \nabla \tilde{\chi}^{\Delta t} \cdot \nabla v dx = \int_D \tilde{\vartheta}^{\Delta t} v dx. \quad (2.4)$$

Firstly, we show that the discrete system composed by the approximation schemes (2.1) and (2.2) is well-defined. Secondly, our aim is to derive boundedness results for the approximate sequences $\vartheta^{\Delta t}$, $\tilde{\vartheta}^{\Delta t}$, $\chi^{\Delta t}$ and $\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}$.

Proposition 2.7. *Set $N \in \mathbb{N}^*$, $n \in \{0, \dots, N\}$ and $\vartheta_n, \chi_n \in L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$, $\chi_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$. If we assume that $\Delta t \leq 1$, then there exists a unique $\vartheta_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$ satisfying (2.1), P -a.s in Ω and for any v in $H^1(D)$.*

Proof. A direct application of Lax-Milgram Theorem gives us the result. \square

Proposition 2.8. *Set $N \in \mathbb{N}^*$, $n \in \{0, \dots, N\}$ and $\chi_n \in L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$, $\vartheta_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$. If we assume that $\Delta t < 1$, then there exists a unique $\chi_{n+1} \in L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$ satisfying (2.2), P -a.s in Ω and for any v in $H^1(D)$.*

Proof. The proof is mostly the same as in [8], Proposition 2.7, so we refer the reader to this paper. \square

Proposition 2.9. *Set $N \in \mathbb{N}^*$, $n \in \{0, \dots, N\}$ and $\vartheta_n, \chi_n \in L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$. Assume that $\Delta t < \bar{C}_{\tilde{\alpha}}$, then there exists a unique pair $(\vartheta_{n+1}, \chi_{n+1})$ belonging to $L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)) \times L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$ and satisfying P -a.s in Ω and for any v in $H^1(D)$, (2.1-2.2).*

Proof. Set $N \in \mathbb{N}^*$, $n \in \{0, \dots, N\}$ and $\vartheta_n, \chi_n \in L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$. We introduce the following functionals

$$f : L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D)) \rightarrow L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$$

$$\tilde{\chi} \mapsto \vartheta^f,$$

where ϑ^f satisfies, P -a.s in Ω and for any v in $H^1(D)$

$$\int_D \left(\frac{\vartheta^f - \vartheta_n}{\Delta t} \right) v dx + \int_D \left(\frac{\tilde{\chi} - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t} \right) v dx$$

$$+ \int_D \nabla \vartheta^f \cdot \nabla v dx = 0. \quad (2.5)$$

Thanks to Proposition 2.7, f is well defined. Similarly, we introduce

$$g : L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D)) \rightarrow L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$$

$$\tilde{\vartheta} \mapsto \chi^g,$$

where χ^g satisfies, P -a.s in Ω and for any v in $H^1(D)$

$$\int_D \tilde{\alpha} \left(\frac{\chi^g - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t} \right) v dx + \int_D \nabla \chi^g \cdot \nabla v dx = \int_D \tilde{\vartheta} v dx. \quad (2.6)$$

Thanks to Proposition 2.8, g is well defined. Let us prove that the composition $g \circ f$ is a strict contraction in $L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$. On the one hand, note that (2.5) can be written P -a.s in Ω and for any v in $H^1(D)$ as

$$\int_D \vartheta^f v dx + \Delta t \int_D \nabla \vartheta^f \cdot \nabla v dx$$

$$= \int_D (\vartheta_n - \tilde{\chi} + \chi_n + h_n(w_{n+1} - w_n)) v dx. \quad (2.7)$$

Set $\tilde{\chi}_1, \tilde{\chi}_2$ in $L^2((\Omega, \mathcal{F}_{t_n}); L^2(D))$ and define $\vartheta_1^f = f(\tilde{\chi}_1)$, $\vartheta_2^f = f(\tilde{\chi}_2)$. Then using (2.7), one gets P -a.s in Ω and for any v in $H^1(D)$

$$\int_D (\vartheta_1^f - \vartheta_2^f) v dx + \Delta t \int_D \nabla (\vartheta_1^f - \vartheta_2^f) \cdot \nabla v dx = \int_D (\tilde{\chi}_2 - \tilde{\chi}_1) v dx. \quad (2.8)$$

By choosing $v = (\vartheta_1^f - \vartheta_2^f)$ in (2.8) we obtain

$$\int_D |\vartheta_1^f - \vartheta_2^f|^2 dx + \Delta t \int_D |\nabla(\vartheta_1^f - \vartheta_2^f)|^2 dx = \int_D (\tilde{\chi}_2 - \tilde{\chi}_1)(\vartheta_1^f - \vartheta_2^f) dx.$$

Then

$$\int_D |\vartheta_1^f - \vartheta_2^f|^2 dx + 2\Delta t \int_D |\nabla(\vartheta_1^f - \vartheta_2^f)|^2 dx \leq \int_D |\tilde{\chi}_1 - \tilde{\chi}_2|^2 dx.$$

By taking the expectation, one gets

$$E \left[\int_D |\vartheta_1^f - \vartheta_2^f|^2 dx \right] + 2\Delta t E \left[\int_D |\nabla(\vartheta_1^f - \vartheta_2^f)|^2 dx \right] \leq E \left[\int_D |\tilde{\chi}_1 - \tilde{\chi}_2|^2 dx \right],$$

which implies that

$$E \left[\|\vartheta_1^f - \vartheta_2^f\|^2 \right] \leq E \left[\|\tilde{\chi}_1 - \tilde{\chi}_2\|^2 \right]. \quad (2.9)$$

On the other hand, by defining $\tilde{\chi}_1^g = g(\vartheta_1^f)$ and $\tilde{\chi}_2^g = g(\vartheta_2^f)$, (2.6) gives P -a.s in Ω and for any v in $H^1(D)$

$$\begin{aligned} & \int_D \left[\tilde{\alpha} \left(\frac{\tilde{\chi}_1^g - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t} \right) - \tilde{\alpha} \left(\frac{\tilde{\chi}_2^g - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t} \right) \right] v dx \\ & + \int_D \nabla(\tilde{\chi}_1^g - \tilde{\chi}_2^g) \cdot \nabla v dx = \int_D (\vartheta_1^f - \vartheta_2^f) v dx. \end{aligned} \quad (2.10)$$

By choosing $v = \tilde{\chi}_1^g - \tilde{\chi}_2^g$ in (2.10) and using the coercivity of $\tilde{\alpha}$, we obtain by taking the expectation

$$\begin{aligned} & \left(\frac{\bar{C}_{\tilde{\alpha}}}{\Delta t} - \frac{1}{2} \right) E \left[\int_D |\tilde{\chi}_1^g - \tilde{\chi}_2^g|^2 dx \right] \\ & + E \left[\int_D |\nabla(\tilde{\chi}_1^g - \tilde{\chi}_2^g)|^2 dx \right] \leq \frac{1}{2} E \left[\int_D |\vartheta_1^f - \vartheta_2^f|^2 dx \right], \end{aligned}$$

and so

$$2 \left(\frac{\bar{C}_{\tilde{\alpha}}}{\Delta t} - \frac{1}{2} \right) E \left[\|\tilde{\chi}_1^g - \tilde{\chi}_2^g\|^2 \right] \leq E \left[\|\vartheta_1^f - \vartheta_2^f\|^2 \right]. \quad (2.11)$$

By comparing (2.9) and (2.11), we get

$$E \left[\|\tilde{\chi}_1^g - \tilde{\chi}_2^g\|^2 \right] \leq \frac{1}{2 \left(\frac{\bar{C}_{\tilde{\alpha}}}{\Delta t} - \frac{1}{2} \right)} E \left[\|\tilde{\chi}_1 - \tilde{\chi}_2\|^2 \right]. \quad (2.12)$$

Under the assumption $\Delta t < \bar{C}_{\tilde{\alpha}}$, the function $g \circ f$ which maps $L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$ in itself is a strict contraction and admits a unique fixed point in $L^2((\Omega, \mathcal{F}_{t_{n+1}}); L^2(D))$. Using this, there exists a unique pair $(\vartheta_{n+1}, \chi_{n+1})$ in $L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D)) \times L^2((\Omega, \mathcal{F}_{t_{n+1}}); H^1(D))$ satisfying P -a.s in Ω and for any v in $H^1(D)$, (2.1) and (2.2). \square

2.3. First estimates on the approximate sequences

Our aim is to find boundedness results for the sequences $\tilde{\vartheta}^{\Delta t}, \vartheta^{\Delta t}, \tilde{\chi}^{\Delta t}, \chi^{\Delta t}$ and $\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}$.

Proposition 2.10. *There exists a constant $C > 0$ independent of Δt such that*

$$\|\tilde{\vartheta}^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))}, \|\vartheta^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))} \leq C, \quad (2.13)$$

$$\|\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t}\|_{L^2(\Omega \times Q)} \leq C\sqrt{\Delta t}, \quad (2.14)$$

$$\|\nabla \tilde{\vartheta}^{\Delta t}\|_{L^2(\Omega \times Q)}, \|\nabla \vartheta^{\Delta t}\|_{L^2(\Omega \times Q)} \leq C, \quad (2.15)$$

$$\|\nabla \tilde{\chi}^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))}, \|\nabla \chi^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))} \leq C, \quad (2.16)$$

$$\|\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega \times Q)} \leq C. \quad (2.17)$$

Proof. Set $N \in \mathbb{N}^*$, $n \in \{0, \dots, N-1\}$ and $k \in \{0, \dots, n\}$. Consider the variational formulations (2.1) and (2.2) with the couple of indexes $(k+1, k)$. By adding (2.1) with the test function $v = \vartheta_{k+1}$ and (2.2) with the test function $v = \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t}$, one gets

$$\begin{aligned} & \int_D \left(\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right) \vartheta^{k+1} dx + \|\nabla \vartheta_{k+1}\|^2 + \left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \\ & + \int_D \alpha \left(\frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right) \times \left(\frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right) dx \\ & + \int_D \nabla \chi_{k+1} \cdot \nabla \left(\frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right) dx = 0. \end{aligned}$$

Using the coerciveness property of α , one gets

$$\begin{aligned} & \int_D \vartheta^{k+1} \left(\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right) dx + \|\nabla \vartheta_{k+1}\|^2 + (\bar{C}_\alpha + 1) \left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \\ & + \int_D \nabla \chi_{k+1} \cdot \nabla \left(\frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right) dx \leq 0. \end{aligned}$$

Then

$$\begin{aligned} & \int_D \vartheta^{k+1} \left(\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right) dx + \|\nabla \vartheta_{k+1}\|^2 + (\bar{C}_\alpha + 1) \left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \\ & + \int_D \nabla \chi_{k+1} \cdot \nabla \frac{\chi_{k+1} - \chi_k}{\Delta t} dx \\ & \leq \int_D \nabla(\chi_{k+1} - \chi_k) \cdot \nabla h_k \frac{w_{k+1} - w_k}{\Delta t} dx + \int_D \nabla \chi_k \cdot \nabla h_k \frac{w_{k+1} - w_k}{\Delta t} dx. \end{aligned}$$

Using the formula

$$a(a-b) = \frac{1}{2}\{a^2 - b^2 + (a-b)^2\}$$

with $a = \vartheta_{k+1}$ (respectively $a = \nabla\chi_{k+1}$) and $b = \vartheta_k$ (respectively $b = \nabla\chi_k$), one gets for any $\epsilon > 0$

$$\begin{aligned}
& \frac{1}{2\Delta t} \left[\|\vartheta_{k+1}\|^2 - \|\vartheta_k\|^2 + \|\vartheta_{k+1} - \vartheta_k\|^2 \right] + \|\nabla\vartheta_{k+1}\|^2 \\
& + (\bar{C}_\alpha + 1) \left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \\
& + \frac{1}{2\Delta t} \left[\|\nabla\chi_{k+1}\|^2 - \|\nabla\chi_k\|^2 + \|\nabla(\chi_{k+1} - \chi_k)\|^2 \right] \\
\leq & \frac{\epsilon}{2\Delta t} \|\nabla(\chi_{k+1} - \chi_k)\|^2 + \frac{|w_{k+1} - w_k|^2}{2\epsilon\Delta t} \|\nabla h_k\|^2 \\
& - \int_D \nabla\chi_k \cdot \nabla h_k \frac{w_{k+1} - w_k}{\Delta t} dx.
\end{aligned}$$

Then, since $\nabla\chi_k$ and ∇h_k are \mathcal{F}_{t_k} -measurable, by taking the expectation one gets

$$\begin{aligned}
& \frac{1}{2\Delta t} E \left[\|\vartheta_{k+1}\|^2 - \|\vartheta_k\|^2 + \|\vartheta_{k+1} - \vartheta_k\|^2 \right] + E \left[\|\nabla\vartheta_{k+1}\|^2 \right] \\
& + (\bar{C}_\alpha + 1) E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{2\Delta t} \right\|^2 \right] \\
& + \frac{1}{2\Delta t} E \left[\|\nabla\chi_{k+1}\|^2 - \|\nabla\chi_k\|^2 + \|\nabla(\chi_{k+1} - \chi_k)\|^2 \right] \\
\leq & \frac{\epsilon}{2\Delta t} E \left[\|\nabla(\chi_{k+1} - \chi_k)\|^2 \right] + \frac{1}{2\epsilon} E \left[\|\nabla h_k\|^2 \right].
\end{aligned}$$

In this way

$$\begin{aligned}
& E \left[\|\vartheta_{k+1}\|^2 \right] - E \left[\|\vartheta_k\|^2 \right] + E \left[\|\vartheta_{k+1} - \vartheta_k\|^2 \right] + 2\Delta t E \left[\|\nabla\vartheta_{k+1}\|^2 \right] \\
& + 2(\bar{C}_\alpha + 1)\Delta t E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \right] \\
& + E \left[\|\nabla\chi_{k+1}\|^2 \right] - E \left[\|\nabla\chi_k\|^2 \right] + (1 - \epsilon) E \left[\|\nabla(\chi_{k+1} - \chi_k)\|^2 \right] \\
\leq & \frac{\Delta t}{\epsilon} E \left[\|\nabla h_k\|^2 \right].
\end{aligned}$$

By summing from $k = 0$ to n , one gets

$$\begin{aligned}
& \sum_{k=0}^n E \left[\|\vartheta_{k+1}\|^2 \right] - \sum_{k=0}^n E \left[\|\vartheta_k\|^2 \right] + \sum_{k=0}^n E \left[\|\vartheta_{k+1} - \vartheta_k\|^2 \right] + 2 \sum_{k=0}^n \Delta t E \left[\|\nabla \vartheta_{k+1}\|^2 \right] \\
& + 2(\bar{C}_\alpha + 1) \sum_{k=0}^n \Delta t E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - \frac{w_{k+1} - w_k}{\Delta t} h_k \right\|^2 \right] \\
& + \sum_{k=0}^n E \left[\|\nabla \chi_{k+1}\|^2 \right] - \sum_{k=0}^n E \left[\|\nabla \chi_k\|^2 \right] + (1 - \epsilon) \sum_{k=0}^n E \left[\|\nabla(\chi_{k+1} - \chi_k)\|^2 \right] \\
& \leq \frac{1}{\epsilon} \sum_{k=0}^n \Delta t E \left[\|\nabla h_k\|^2 \right].
\end{aligned}$$

By taking $\epsilon = \frac{1}{2}$ and using Lemma 2.3, there exists a constant $C > 0$ independent of Δt such that

$$\begin{aligned}
& E \left[\|\vartheta_{n+1}\|^2 \right] + \sum_{k=0}^n E \left[\|\vartheta_{k+1} - \vartheta_k\|^2 \right] + \sum_{k=0}^n \Delta t E \left[\|\nabla \vartheta_{k+1}\|^2 \right] \\
& + \sum_{k=0}^n \Delta t E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \right] + E \left[\|\nabla \chi_{n+1}\|^2 \right] \\
& + \frac{1}{2} \sum_{k=0}^n E \left[\|\nabla(\chi_{k+1} - \chi_k)\|^2 \right] \leq C,
\end{aligned} \tag{2.18}$$

and we get directly the announced estimates:

$$\begin{aligned}
\|\tilde{\vartheta}^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))}, \|\vartheta^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))} & \leq C, \\
\|\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t}\|_{L^2(\Omega \times Q)} & \leq C\sqrt{\Delta t}, \\
\|\nabla \tilde{\vartheta}^{\Delta t}\|_{L^2(\Omega \times Q)}, \|\nabla \vartheta^{\Delta t}\|_{L^2(\Omega \times Q)} & \leq C, \\
\|\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega \times Q)} & \leq C, \\
\|\nabla \tilde{\chi}^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))}, \|\nabla \chi^{\Delta t}\|_{L^\infty(0,T,L^2(\Omega \times D))} & \leq C.
\end{aligned}$$

Additionally, let us note that one can also deduce from (2.18) the following bound:

$$\|\nabla(\tilde{\chi}^{\Delta t} - \chi^{\Delta t})\|_{L^2(\Omega \times Q)} \leq C\sqrt{\Delta t}. \tag{2.19}$$

□

From these first estimates, one can deduce directly the following ones.

Proposition 2.11. *There exists a constant $C > 0$ independent of Δt such that*

$$\|\tilde{\chi}^{\Delta t} - \chi^{\Delta t}\|_{L^2((0,T)\times\Omega, H^1(D))} \leq C\sqrt{\Delta t}, \quad (2.20)$$

$$\|\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t}(\cdot - \Delta t)\|_{L^2(\Omega\times Q)} \leq C\sqrt{\Delta t}, \quad (2.21)$$

$$\|\tilde{\chi}^{\Delta t} - \tilde{\chi}^{\Delta t}(\cdot - \Delta t)\|_{L^2((0,T)\times\Omega, H^1(D))} \leq C\sqrt{\Delta t}, \quad (2.22)$$

$$\|\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))} \leq C, \quad (2.23)$$

$$\|\nabla(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega\times Q)} \leq C, \quad (2.24)$$

$$\|\tilde{\chi}^{\Delta t}\|_{L^2((0,T)\times\Omega, H^1(D))}, \|\chi^{\Delta t}\|_{L^2((0,T)\times\Omega, H^1(D))} \leq C, \quad (2.25)$$

$$\|\tilde{\chi}^{\Delta t}(\cdot - \Delta t)\|_{\mathcal{N}_w^2(0,T, H^1(D))}, \|\tilde{\vartheta}^{\Delta t}(\cdot - \Delta t)\|_{\mathcal{N}_w^2(0,T, H^1(D))} \leq C. \quad (2.26)$$

Proof. Using (2.18), we have

$$\begin{aligned} \|\tilde{\chi}^{\Delta t} - \chi^{\Delta t}\|_{L^2(\Omega\times Q)}^2 &= \Delta t \sum_{k=0}^{N-1} E \left[\|\chi_{k+1} - \chi_k\|^2 \right] \\ &\leq \Delta t \sum_{k=0}^{N-1} E \left[2\Delta t^2 \left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 + 2\|h_k(w_{k+1} - w_k)\|^2 \right] \\ &= 2\Delta t^2 \sum_{k=0}^{N-1} \Delta t E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \right] + 2 \sum_{k=0}^{N-1} \Delta t^2 E \left[\|h_k\|^2 \right] \\ &\leq C\Delta t, \end{aligned}$$

combining it with the previous estimate (2.19), one deduces that (2.20) holds.

Note that

$$\tilde{\vartheta}^{\Delta t} - \tilde{\vartheta}^{\Delta t}(\cdot - \Delta t) = (\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t}) + (\vartheta^{\Delta t} - \vartheta^{\Delta t}(\cdot - \Delta t)) + (\vartheta^{\Delta t}(\cdot - \Delta t) - \tilde{\vartheta}^{\Delta t}(\cdot - \Delta t))$$

and that there exists a constant $C > 0$ independent of Δt such that

$$\|\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t}\|_{L^2(\Omega\times Q)}^2, \|\vartheta^{\Delta t} - \vartheta^{\Delta t}(\cdot - \Delta t)\|_{L^2(\Omega\times Q)}^2 \text{ and } \|\vartheta^{\Delta t}(\cdot - \Delta t) - \tilde{\vartheta}^{\Delta t}(\cdot - \Delta t)\|_{L^2(\Omega\times Q)}^2$$

are controlled by

$$C\Delta t \sum_{k=0}^{N-1} E \left[\|\vartheta_{k+1} - \vartheta_k\|^2 \right].$$

Then, owing to (2.18), one gets directly (2.21). Now, using the same kind of decomposition for $\tilde{\chi}^{\Delta t} - \tilde{\chi}^{\Delta t}(\cdot - \Delta t)$, one shows that (2.22) holds. Additionally for any n in $\{0, \dots, N-1\}$ since $B_0 = 0$ one has

$$\begin{aligned} &E \left[\|\chi_{k+1} - B_{n+1}\|^2 \right] \\ &\leq 2\|\chi_0\|^2 + 2T \sum_{k=0}^n \Delta t E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \right] \end{aligned}$$

combining this with (2.18), we show that $\|\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega\times D))} \leq C$. Let us now prove that $\|\nabla(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega\times Q)}$ is bounded independently of Δt . Using (2.16), it remains to show that $\nabla\tilde{B}^{\Delta t}$ is bounded in $L^2(\Omega \times Q)$.

Due to Lemma 2.3 and the fact that $E[(w_{j+1} - w_j)^2] = \Delta t$ for any $j \in \{0, \dots, N-1\}$, one has

$$\begin{aligned}
\|\nabla \tilde{B}^{\Delta t}\|_{L^2(\Omega \times Q)}^2 &\leq \Delta t \sum_{k=0}^N E \left[\int_D \left(\sum_{j=0}^k (w_{j+1} - w_j) \nabla h_j \right)^2 dx \right] \\
&= \Delta t \sum_{k=0}^N \sum_{j=0}^k \int_D E[(w_{j+1} - w_j)^2] E[(\nabla h_j)^2] dx \\
&= \Delta t \sum_{k=0}^N \Delta t E \sum_{j=0}^k \|h_j\|_{H^1(D)}^2 \\
&\leq C,
\end{aligned}$$

and the result holds. Using the fact that $\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}$ and $\tilde{B}^{\Delta t}$ are bounded in $L^2(\Omega \times Q)$, one gets that $\tilde{\chi}^{\Delta t}$ is also bounded in $L^2(\Omega \times Q)$. Finally, combining this with (2.16), one obtains the boundedness of $\tilde{\chi}^{\Delta t}$ in $L^2((0, T) \times \Omega, H^1(D))$. Thanks to (2.20)-(2.22), one gets the same result for $\tilde{\chi}^{\Delta t}(\cdot - \Delta t)$ and $\chi^{\Delta t}$ which gives (2.25).

Note that $\tilde{\chi}^{\Delta t}(\cdot - \Delta t)$ and $\tilde{\vartheta}^{\Delta t}(\cdot - \Delta t)$ are bounded in $L^2((0, T) \times \Omega, H^1(D))$ respectively due to (2.22)-(2.25) and (2.13)-(2.15)-(2.21). Thus, they belong to $\mathcal{N}_w^2(0, T, H^1(D))$ as continuous and adapted processes. Finally, (2.26) holds. \square

2.4. Second estimates on the approximate sequences

Proposition 2.12. *There exists a constant $C > 0$ independent of Δt such that*

$$\|\nabla \tilde{\vartheta}^{\Delta t}\|_{L^\infty(0, T, L^2(\Omega \times D))}, \|\nabla \vartheta^{\Delta t}\|_{L^\infty(0, T, L^2(\Omega \times D))} \leq C, \quad (2.27)$$

$$\|\nabla(\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t})\|_{L^2(\Omega \times Q)} \leq C\sqrt{\Delta t}, \quad (2.28)$$

$$\|\nabla(\tilde{\vartheta}^{\Delta t} - \tilde{\vartheta}^{\Delta t}(\cdot - \Delta t))\|_{L^2(\Omega \times Q)} \leq C\sqrt{\Delta t}, \quad (2.29)$$

$$\|\partial_t \tilde{\vartheta}^{\Delta t}\|_{L^2(\Omega \times Q)} \leq C. \quad (2.30)$$

Proof. Set $N \in \mathbb{N}^*$, $n \in \{0, \dots, N-1\}$ and $k \in \{0, \dots, n\}$. We consider the variational formulation (2.1) with the couple of indexes $(k+1, k)$ and choose the particular test function $v = \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t}$ to get P -almost surely in Ω

$$\begin{aligned}
&\left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|^2 + \int_D \left(\frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right) \times \left(\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right) dx \\
&+ \int_D \nabla \vartheta_{k+1} \cdot \nabla \left(\frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right) dx = 0.
\end{aligned}$$

Then, for any $\delta > 0$, we have

$$\begin{aligned} & \left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|^2 + \frac{1}{2\Delta t} \left[\|\nabla \vartheta_{k+1}\|^2 - \|\nabla \vartheta_k\|^2 + \|\nabla(\vartheta_{k+1} - \vartheta_k)\|^2 \right] \\ & \leq \frac{1}{2\delta} \left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 + \frac{\delta}{2} \left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|^2. \end{aligned}$$

By choosing $\delta = 1$, taking the expectation and summing from $k = 0$ to n , one gets

$$\begin{aligned} & \sum_{k=0}^n \Delta t E \left[\left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|^2 \right] + \sum_{k=0}^n E \left[\|\nabla \vartheta_{k+1}\|^2 \right] - \sum_{k=0}^n E \left[\|\nabla \vartheta_k\|^2 \right] \\ & + \sum_{k=0}^n E \left[\|\nabla(\vartheta_{k+1} - \vartheta_k)\|^2 \right] \leq \sum_{k=0}^n \Delta t E \left[\left\| \frac{\chi_{k+1} - \chi_k}{\Delta t} - h_k \frac{w_{k+1} - w_k}{\Delta t} \right\|^2 \right]. \end{aligned}$$

Thanks to (2.18), one concludes that

$$\sum_{k=0}^n \Delta t E \left[\left\| \frac{\vartheta_{k+1} - \vartheta_k}{\Delta t} \right\|^2 \right] + E \left[\|\nabla \vartheta_{n+1}\|^2 \right] + \sum_{k=0}^n E \left[\|\nabla(\vartheta_{k+1} - \vartheta_k)\|^2 \right] \leq C.$$

Finally, we have directly the announced estimates

$$\begin{aligned} \|\nabla \tilde{\vartheta}^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega \times D))}, \|\nabla \vartheta^{\Delta t}\|_{L^\infty(0,T;L^2(\Omega \times D))} & \leq C, \\ \|\nabla(\tilde{\vartheta}^{\Delta t} - \vartheta^{\Delta t})\|_{L^2(\Omega \times Q)} & \leq C\sqrt{\Delta t}, \\ \|\partial_t \tilde{\vartheta}^{\Delta t}\|_{L^2(\Omega \times Q)} & \leq C. \end{aligned}$$

Arguing as for the obtention of (2.21), one shows finally that

$$\|\nabla(\tilde{\vartheta}^{\Delta t} - \tilde{\vartheta}^{\Delta t}(\cdot - \Delta t))\|_{L^2(\Omega \times Q)} \leq C\sqrt{\Delta t}.$$

□

2.5. Weak convergence results on the approximate sequences

Due to Propositions 2.10, 2.11 and 2.12, we obtain the following convergence results.

Proposition 2.13. *Up to subsequences denoted in the same way, there exists ϑ belonging to $\mathcal{N}_w^2(0, T, H^1(D)) \cap L^2(\Omega, H^1(Q))$ such that*

- (i) $\vartheta^{\Delta t}, \vartheta^{\Delta t} \rightharpoonup \vartheta$ in $L^2((0, T) \times \Omega, H^1(D))$,
- (ii) $\nabla \tilde{\vartheta}^{\Delta t}, \nabla \vartheta^{\Delta t} \xrightarrow{*} \nabla \vartheta$ in $L^\infty(0, T; L^2(\Omega \times D))$,
- (iii) $\partial_t \tilde{\vartheta}^{\Delta t} \rightharpoonup \partial_t \vartheta$ in $L^2(\Omega \times Q)$,
- (iv) $\tilde{\vartheta}^{\Delta t}(0) \rightharpoonup \vartheta(0)$ in $L^2(\Omega \times D)$.

Proof.

(i) Thanks to (2.13), (2.14), (2.15), (2.21), (2.26), (2.28) and (2.29), there

exists ϑ in $L^2((0, T) \times \Omega, H^1(D))$ such that, up to subsequences denoted in the same way, we have

$$\tilde{\vartheta}^{\Delta t}, \vartheta^{\Delta t}, \vartheta^{\Delta t}(\cdot - \Delta t) \rightharpoonup \vartheta \text{ in } L^2((0, T) \times \Omega, H^1(D)).$$

Since $\tilde{\vartheta}^{\Delta t}(\cdot - \Delta t)$ belongs to the Hilbert space $\mathcal{N}_w^2(0, T, H^1(D))$ endowed with the norm of $L^2((0, T) \times \Omega, H^1(D))$, one gets that ϑ is also in $\mathcal{N}_w^2(0, T, H^1(D))$.

(ii) Using (2.27)-(2.28), one gets directly that up to subsequences denoted in the same way,

$$\nabla \tilde{\vartheta}^{\Delta t}, \nabla \vartheta^{\Delta t} \xrightarrow{*} \nabla \vartheta \text{ in } L^\infty(0, T; L^2(\Omega \times D)).$$

(iii) (2.30) gives us directly the announced result.

(iv) Since $L^2(\Omega, H^1(Q))$ is continuously embedded in $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$, one gets that ϑ belongs to $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$. Thus, ϑ is an element of $\mathcal{C}([0, T], L^2(\Omega \times D))$ and we have and that

$$\tilde{\vartheta}^{\Delta t}(0) \rightharpoonup \vartheta(0) \text{ in } L^2(\Omega \times D).$$

□

Proposition 2.14. *Up to subsequences denoted in the same way, there exist χ belonging to $\mathcal{N}_w^2(0, T, H^1(D)) \cap L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$ and $\tilde{\chi}$ in $L^2(\Omega \times Q)$ such that*

- (i) $\tilde{\chi}^{\Delta t}, \chi^{\Delta t} \rightharpoonup \chi$ in $L^2((0, T) \times \Omega, H^1(D))$,
- (ii) $\nabla \tilde{\chi}^{\Delta t}, \nabla \chi^{\Delta t} \xrightarrow{*} \nabla \chi$ in $L^\infty(0, T; L^2(\Omega \times D))$,
- (iii) $\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t} \rightharpoonup \chi - \int_0^\cdot h dw$ in $L^2(\Omega, H^1(Q))$,
- (iv) $\alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})) \rightharpoonup \tilde{\chi}$ in $L^2(\Omega \times Q)$,
- (v) $(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})(0) \rightharpoonup \chi(0)$ in $L^2(\Omega \times D)$.

Proof. (i) Thanks to (2.20)-(2.22)-(2.25) and (2.26), there exists χ in $L^2((0, T) \times \Omega, H^1(D))$ such that, up to subsequences denoted in the same way, we have

$$\tilde{\chi}^{\Delta t}, \chi^{\Delta t}, \tilde{\chi}^{\Delta t}(\cdot - \Delta t) \rightharpoonup \chi \text{ in } L^2((0, T) \times \Omega, H^1(D)).$$

Since $\tilde{\chi}^{\Delta t}(\cdot - \Delta t)$ belongs to the Hilbert space $\mathcal{N}_w^2(0, T, H^1(D))$ endowed with the norm of $L^2((0, T) \times \Omega, H^1(D))$, one gets that χ is also in $\mathcal{N}_w^2(0, T, H^1(D))$.

(ii) Using (2.16)-(2.20), one gets directly that up to subsequences denoted in the same way,

$$\nabla \tilde{\chi}^{\Delta t}, \nabla \chi^{\Delta t} \xrightarrow{*} \nabla \chi \text{ in } L^\infty(0, T; L^2(\Omega \times D)).$$

(iii) Thanks to (2.17)-(2.23)-(2.24), there exists ζ in $L^\infty(0, T; L^2(\Omega \times D))$ and $L^2(\Omega, H^1(Q))$ such that, up to a subsequence,

$$\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t} \rightharpoonup \zeta \text{ in } L^2(\Omega, H^1(Q)) \text{ and } \tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t} \xrightarrow{*} \zeta \text{ in } L^\infty(0, T; L^2(\Omega \times D)).$$

Using Proposition 2.5, one gets by uniqueness of the limit that

$$\zeta = \chi - \int_0^\cdot h dw.$$

(iv) Due to the Lipschitz property of α and (2.17), $\alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}))$ is bounded in $L^2(\Omega \times Q)$ and there exists $\bar{\chi}$ in the same space such that, up to a subsequence

$$\alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})) \rightharpoonup \bar{\chi} \text{ in } L^2(\Omega \times Q).$$

(v) Since $L^2(\Omega, H^1(Q))$ is continuously embedded in $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$, one gets that $\chi - \int_0^\cdot h dw$ belongs to $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$. Moreover, as the Itô integral of an $\mathcal{N}_w^2(0, T, L^2(D))$ process is a continuous square integrable $L^2(D)$ -valued martingale (see [14]), $\int_0^\cdot h dw$ is in $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$. Thus χ belongs to $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$ and finally χ is an element of $\mathcal{C}([0, T], L^2(\Omega \times D))$. Particularly, we have

$$\tilde{\chi}^{\Delta t}(0) - \tilde{B}^{\Delta t}(0) \rightharpoonup (\chi - \int_0^\cdot h dw)(0) = \chi(0) \text{ in } L^2(\Omega \times D).$$

□

Using these convergence results, let us derive some properties satisfied by the weak limits ϑ and χ .

2.6. Properties of the weak limits ϑ and χ

Proposition 2.15. $\vartheta(0) = \vartheta_0$ and $\chi(0) = \chi_0$ in $L^2(D)$.

Proof. Thanks to Proposition 2.13 and Proposition 2.14, we have

$$\tilde{\vartheta}^{\Delta t}(0) \rightharpoonup \vartheta(0) \text{ and } (\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})(0) \rightharpoonup \chi(0) \text{ in } L^2(\Omega \times D).$$

Note that $\tilde{\vartheta}^{\Delta t}(0) = \vartheta_0$ and $(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})(0) = \chi_0$. Using the fact that χ_0 and ϑ_0 are deterministic, one concludes that the announced result holds in $L^2(D)$. □

Proposition 2.16. *The following results hold*

(i) *The application $t \in [0, T] \mapsto E[\|\nabla \vartheta(t)\|^2] \in \mathbb{R}$ is continuous.*

(ii) *ϑ belongs to the space $\mathcal{C}([0, T], L^2(\Omega, H^1(D)))$.*

Proof. (i) Note that P -almost surely in Ω , ϑ satisfies the heat equation

$$\begin{cases} \partial_t \vartheta - \Delta \vartheta &= -\partial_t U, \\ \vartheta(0, \cdot) &= \vartheta_0, \end{cases}$$

where $U = \chi - \int_0^\cdot h dw$. Since $\vartheta_0 \in H^1(D)$, the study of the heat equation gives us the following energy equality (see [13] Theorem X.11 p.220), for any $t \in [0, T]$ by denoting $Q_t = (0, t) \times D$:

$$\int_{Q_t} |\partial_t \vartheta|^2 ds dx + \int_{Q_t} \partial_t U \partial_t \vartheta ds dx + \frac{1}{2} \|\nabla \vartheta(t)\|^2 = \frac{1}{2} \|\nabla \vartheta_0\|^2.$$

Then by taking the expectation :

$$\begin{aligned} E \left[\int_{Q_t} |\partial_t \vartheta|^2 ds dx \right] + E \left[\int_{Q_t} \partial_t U \partial_t \vartheta ds dx \right] \\ + \frac{1}{2} E \left[\|\nabla \vartheta(t)\|^2 \right] = \frac{1}{2} E \left[\|\nabla \vartheta_0\|^2 \right]. \end{aligned} \quad (2.31)$$

Using (2.31) and the properties of the Lebesgue integral, one gets the continuity of

$$t \in [0, T] \mapsto E \left[\|\nabla \vartheta(t)\|^2 \right] \in \mathbb{R}.$$

(ii) Firstly, since ϑ belongs to $L^2(\Omega, H^1(Q))$, it is also an element of $\mathcal{C}([0, T], L^2(\Omega, L^2(D)))$. Combining this with the continuity result proved in (i), one gets that the application

$$t \in [0, T] \mapsto E \left[\|\vartheta(t)\|_{H^1(D)}^2 \right] \in \mathbb{R}$$

is also continuous. Secondly, that thanks to the following embedding (see [18] Lemme 8.1 p.297) :

$$L^\infty(0, T; L^2(\Omega, H^1(D))) \cap \mathcal{C}([0, T], L^2(\Omega, L^2(D))) \subset \mathcal{C}_w([0, T], L^2(\Omega, H^1(D)))^\dagger$$

one shows that ϑ also belongs to $\mathcal{C}([0, T], L^2(\Omega, H^1(D)))$. Thus, combining this with the above continuity result, one concludes that ϑ is an element of $\mathcal{C}([0, T], L^2(\Omega, H^1(D)))$. \square

3. Proof of Theorem 1.4

Thanks to the weak convergence results stated in the previous section, passing to the limit in (2.3)-(2.4) with respect to Δt is now possible and gives, using the separability of $H^1(D)$, t -almost everywhere in $(0, T)$, P -almost surely in Ω and for any v in $H^1(D)$

$$\begin{aligned} \int_D \partial_t \vartheta v dx + \int_D \partial_t \left(\chi - \int_0^\cdot h dw \right) v dx + \int_D \nabla \vartheta \cdot \nabla v dx = 0 \\ \int_D \partial_t \left(\chi - \int_0^\cdot h dw \right) v dx + \int_D \bar{\chi} v dx + \int_D \nabla \chi \cdot \nabla v dx = \int_D \vartheta v dx. \end{aligned}$$

Then it remains to identify the nonlinear weak limit $\bar{\chi}$ in $L^2(\Omega \times Q)$ of $\alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}))$. To do so, we suppose in a first step (only for technical reasons) that h belongs to $\mathcal{N}_w^2(0, T, H^2(D))$ by following the idea of [8]. In a second step (Subsection 3.2), we will see how to get back the well posedness result when h is in $\mathcal{N}_w^2(0, T, H^1(D))$.

$^\dagger \mathcal{C}_w([0, T], L^2(\Omega, H^1(D)))$ denotes the set of functions defined on $[0, T]$ with values in $L^2(\Omega, H^1(D))$ which are weakly continuous.

3.1. Existence result for (1.1) when $h \in \mathcal{N}_w^2(0, T, H^2(D))$

Proposition 3.1. *Assume that h belongs to $\mathcal{N}_w^2(0, T, H^2(D))$. Then, up to a subsequence*

$$\alpha \left(\partial_t (\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) \right) \rightharpoonup \alpha \left(\partial_t (\chi - \int_0^\cdot h dw) \right) \text{ in } L^2(\Omega \times Q).$$

Proof. Set n in $\{0, \dots, N-1\}$. We introduce for the sequel the notations

$$U = \chi - \int_0^\cdot h dw \text{ and } U_{n+1} = \chi_{n+1} - \sum_{k=0}^n (w_{k+1} - w_k) h_k.$$

Firstly, we consider (2.2) with the test function

$$v = \frac{U_{n+1} - U_n}{\Delta t} = \frac{\chi_{n+1} - \chi_n}{\Delta t} - h_n \frac{w_{n+1} - w_n}{\Delta t}.$$

Thus, one gets P -a.s in Ω :

$$\begin{aligned} & \int_D \left(\frac{U_{n+1} - U_n}{\Delta t} \right)^2 dx + \int_D \alpha \left(\frac{U_{n+1} - U_n}{\Delta t} \right) \left(\frac{U_{n+1} - U_n}{\Delta t} \right) dx \\ & + \int_D \nabla U_{n+1} \cdot \nabla \left(\frac{U_{n+1} - U_n}{\Delta t} \right) dx \\ & = \sum_{k=0}^n (w_{k+1} - w_k) \int_D \Delta h_k \left(\frac{U_{n+1} - U_n}{\Delta t} \right) dx + \int_D \left(\frac{U_{n+1} - U_n}{\Delta t} \right) \vartheta_{n+1} dx. \end{aligned} \quad (3.1)$$

Secondly, (2.1) with the test function $v = \vartheta_{n+1}$ gives P -a.s in Ω :

$$\int_D \left(\frac{\vartheta_{n+1} - \vartheta_n}{\Delta t} \right) \vartheta_{n+1} dx + \int_D \left(\frac{U_{n+1} - U_n}{\Delta t} \right) \vartheta_{n+1} dx + \int_D |\nabla \vartheta_{n+1}|^2 dx = 0$$

and then

$$\begin{aligned} & \int_D \left(\frac{U_{n+1} - U_n}{\Delta t} \right) \vartheta_{n+1} dx \\ & = -\frac{1}{2\Delta t} \left[\|\vartheta_{n+1}\|^2 - \|\vartheta_n\|^2 + \|\vartheta_{n+1} - \vartheta_n\|^2 \right] - \|\nabla \vartheta_{n+1}\|^2. \end{aligned}$$

Injecting this in (3.1), we obtain

$$\begin{aligned} & \Delta t \int_D \left(\frac{U_{n+1} - U_n}{\Delta t} \right)^2 dx + \Delta t \int_D \alpha \left(\frac{U_{n+1} - U_n}{\Delta t} \right) \left(\frac{U_{n+1} - U_n}{\Delta t} \right) dx \\ & + \frac{1}{2} (\|\nabla U_{n+1}\|^2 - \|\nabla U_n\|^2) + \frac{1}{2} (\|\vartheta_{n+1}\|^2 - \|\vartheta_n\|^2) \\ & + \frac{1}{2} \|\vartheta_{n+1} - \vartheta_n\|^2 + \Delta t \|\nabla \vartheta_{n+1}\|^2 \\ & \leq \Delta t \int_D \Delta B_{n+1} \frac{U_{n+1} - U_n}{\Delta t} dx. \end{aligned}$$

By adding from $n = 0$ to $N - 1$, we get

$$\begin{aligned} & \Delta t \sum_{n=0}^{N-1} \int_D \left(\frac{U_{n+1} - U_n}{\Delta t} \right)^2 dx + \Delta t \sum_{n=0}^{N-1} \int_D \alpha \left(\frac{U_{n+1} - U_n}{\Delta t} \right) \left(\frac{U_{n+1} - U_n}{\Delta t} \right) dx \\ & + \frac{1}{2} \sum_{n=0}^{N-1} (\|\nabla U_{n+1}\|^2 - \|\nabla U_n\|^2) + \frac{1}{2} \sum_{n=0}^{N-1} (\|\vartheta_{n+1}\|^2 - \|\vartheta_n\|^2) + \Delta t \sum_{n=0}^{N-1} \|\nabla \vartheta_{n+1}\|^2 \\ & \leq \Delta t \sum_{n=0}^{N-1} \int_D \Delta B_{n+1} \frac{U_{n+1} - U_n}{\Delta t} dx, \end{aligned}$$

and then

$$\begin{aligned} & \int_Q |\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})|^2 dt dx + \int_Q \alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})) \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx \\ & + \frac{1}{2} (\|\nabla U_N\|^2 - \|\nabla U_0\|^2) + \frac{1}{2} (\|\vartheta_N\|^2 - \|\vartheta_0\|^2) + \int_Q |\nabla \vartheta^{\Delta t}|^2 dt dx \\ & \leq \int_Q \Delta B^{\Delta t} \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx. \end{aligned}$$

Noticing that $\nabla \tilde{U}^{\Delta t}(T) = \nabla U_N$ and that $\tilde{\vartheta}^{\Delta t}(T) = \vartheta_N$, we finally get after taking the expectation

$$\begin{aligned} & E \left[\int_Q |\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})|^2 dt dx \right] + \frac{1}{2} E \left[\|\nabla \tilde{U}^{\Delta t}(T)\|^2 \right] \\ & + \frac{1}{2} E \left[\|\tilde{\vartheta}^{\Delta t}(T)\|^2 \right] + E \left[\int_Q |\nabla \vartheta^{\Delta t}|^2 dt dx \right] \\ & + E \left[\int_Q \alpha \left(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) \right) \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx \right] \\ & \leq E \left[\int_Q \Delta B^{\Delta t} \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx \right] + \frac{1}{2} E \left[\|\nabla U_0\|^2 \right] + \frac{1}{2} E \left[\|\vartheta_0\|^2 \right]. \quad (3.2) \end{aligned}$$

Now, passing to the superior limit in (3.2), we have using Proposition 2.13 and Proposition 2.14

$$\begin{aligned} & \liminf_{\Delta t \rightarrow 0} \|\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})\|_{L^2(\Omega \times Q)}^2 + \frac{1}{2} \liminf_{\Delta t \rightarrow 0} E \left[\|\nabla \tilde{U}^{\Delta t}(T)\|^2 \right] \\ & + \frac{1}{2} \liminf_{\Delta t \rightarrow 0} E \left[\|\tilde{\vartheta}^{\Delta t}(T)\|^2 \right] + \liminf_{\Delta t \rightarrow 0} \|\nabla \vartheta^{\Delta t}\|_{L^2(\Omega \times Q)}^2 \\ & + \limsup_{\Delta t \rightarrow 0} E \left[\int_Q \alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})) \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx \right] \\ & \leq E \left[\int_Q \int_0^t \Delta h dw \partial_t U dt dx \right] + \frac{1}{2} E \left[\|\nabla \chi_0\|^2 \right] + \frac{1}{2} E \left[\|\vartheta_0\|^2 \right]. \end{aligned}$$

Indeed, due to Remark 2.6, $B^{\Delta t}$ converges strongly in $L^2((0, T) \times \Omega, H^2(D))$ to $\int_0^{\cdot} h dw$ and so, by continuity of the Laplace operator, $\Delta B^{\Delta t}$ converges

strongly in $L^2(\Omega \times Q)$ to $\Delta \int_0^\cdot h dw$. Moreover, following [20] (Lemma 2.4.1 p.35),

$$\Delta \int_0^\cdot h dw = \int_0^\cdot \Delta h dw,$$

and the convergence holds. Note that thanks to the following embedding (see [18] Lemme 8.1 p.297) :

$$L^\infty(0, T; L^2(\Omega, H^1(D))) \cap \mathcal{C}([0, T], L^2(\Omega, L^2(D))) \subset \mathcal{C}_w([0, T], L^2(\Omega, H^1(D)))^\dagger$$

one gets that for all times t in $[0, T]$, $\tilde{U}^{\Delta t}(t)$ belongs to $L^2(\Omega, H^1(D))$ and $\tilde{U}^{\Delta t}(t) \rightharpoonup U(t)$ in $L^2(\Omega, H^1(D))$. Then owing to the lower semi-continuity of the $L^2(\Omega, H^1(D))$ -norm

$$\liminf_{\Delta t \rightarrow 0} E \left[\|\nabla \tilde{U}^{\Delta t}(T)\|^2 \right] \geq E \left[\|\nabla U(T)\|^2 \right].$$

Using the same arguments on $\tilde{\vartheta}^{\Delta t}$, one gets also that

$$\liminf_{\Delta t \rightarrow 0} E \left[\|\tilde{\vartheta}^{\Delta t}(T)\|^2 \right] \geq E \left[\|\vartheta(T)\|^2 \right].$$

Finally, the lower semi-continuity of the $L^2(\Omega \times Q)$ -norm gives us

$$\begin{aligned} & \|\partial_t(\chi - \int_0^\cdot h dw)\|_{L^2(\Omega \times Q)}^2 + \frac{1}{2} E \left[\|\nabla U(T)\|^2 \right] \\ & + \frac{1}{2} E \left[\|\vartheta(T)\|^2 \right] + \|\nabla \vartheta\|_{L^2(\Omega \times Q)}^2 \\ & + \limsup_{\Delta t \rightarrow 0} E \left[\int_Q \alpha(\partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t})) \partial_t(\tilde{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx \right] \\ & \leq E \left[\int_Q \int_0^t \Delta h dw \partial_t U dt dx \right] + \frac{1}{2} \|\nabla \chi_0\|^2 + \frac{1}{2} \|\vartheta_0\|^2. \end{aligned} \quad (3.3)$$

Note that P -almost surely in Ω , U satisfies the heat equation

$$\begin{cases} \partial_t U - \Delta U & = g, \\ U(0, \cdot) & = \chi_0, \end{cases}$$

where $g = \vartheta - \bar{\chi} + \int_0^\cdot \Delta h dw$. Since $\chi_0 \in H^1(D)$, the following energy equality holds for any $t \in [0, T]$ (see [13] Theorem X.11 p.220):

$$\begin{aligned} & \int_{Q_t} |\partial_t U|^2 ds dx + \int_{Q_t} \bar{\chi} \partial_t U ds dx + \frac{1}{2} \|\nabla U(t)\|^2 \\ & = \int_{Q_t} \int_0^s \Delta h dw \partial_t U ds dx + \int_{Q_t} \vartheta \partial_t U ds dx + \frac{1}{2} \|\nabla \chi_0\|^2, \end{aligned}$$

$^\dagger \mathcal{C}_w([0, T], L^2(\Omega, H^1(D)))$ denotes the set of functions defined on $[0, T]$ with values in $L^2(\Omega, H^1(D))$ which are weakly continuous.

where $Q_t = (0, t) \times D$. Then by taking the expectation :

$$\begin{aligned} & E \left[\int_{Q_t} |\partial_t U|^2 ds dx \right] + E \left[\int_{Q_t} \bar{\chi} \partial_t U ds dx \right] + \frac{1}{2} E \left[\|\nabla U(t)\|^2 \right] \\ &= E \left[\int_{Q_t} \int_0^s \Delta h dw \partial_t U ds dx \right] + E \left[\int_{Q_t} \vartheta \partial_t U ds dx \right] + \frac{1}{2} \|\nabla \chi_0\|^2. \end{aligned} \quad (3.4)$$

In the same manner, note that ϑ satisfies P -almost surely in Ω the heat equation

$$\begin{cases} \partial_t \vartheta - \Delta \vartheta &= -\partial_t U, \\ \vartheta(0, \cdot) &= \vartheta_0 \in H^1(D), \end{cases}$$

and so the following energy equality holds for any $t \in [0, T]$ (see [23] Lemma 1.2 p260) :

$$\frac{1}{2} E \left[\|\vartheta(t)\|^2 \right] + \|\nabla \vartheta\|_{L^2(\Omega \times Q_t)}^2 + E \left[\int_{Q_t} \vartheta \partial_t U ds dx \right] = \frac{1}{2} \|\vartheta(0)\|^2. \quad (3.5)$$

In this way, by injecting (3.4)-(3.5) written with $t = T$ in (3.3) we finally have

$$\begin{aligned} & \limsup_{\Delta t \rightarrow 0} E \left[\int_Q \alpha \left(\partial_t (\bar{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) \right) \partial_t (\bar{\chi}^{\Delta t} - \tilde{B}^{\Delta t}) dt dx \right] \\ & \leq E \left[\int_Q \bar{\chi} \partial_t \left(\chi - \int_0^t h dw \right) dt dx \right]. \end{aligned}$$

As $\alpha : \mathbb{R} \rightarrow \mathbb{R}$ is a Lipschitz-continuous nondecreasing function, the operator

$$\begin{aligned} A_\alpha : L^2(\Omega \times Q) &\rightarrow L^2(\Omega \times Q) \\ u &\mapsto \alpha(u), \end{aligned}$$

is maximal monotone and one gets that $\bar{\chi} = \alpha \left(\partial_t \left(\chi - \int_0^\cdot h dw \right) \right)$ (see LIONS [18] p.172). \square

Proposition 3.2. *Assume that h belongs to $\mathcal{N}_w^2(0, T, H^2(D))$. Then, the following results hold*

- (i) *The application $t \in [0, T] \mapsto E \left[\|\nabla \chi(t)\|^2 \right] \in \mathbb{R}$ is continuous.*
- (ii) *χ belongs to the space $\mathcal{C}([0, T], L^2(\Omega, H^1(D)))$.*

Proof. (i) Using (3.4) and Lebesgue's theorem, one shows the continuity of

$$t \in [0, T] \mapsto E \left[\|\nabla U(t)\|^2 \right] \in \mathbb{R},$$

where $U = \chi - \int_0^\cdot h dw$. Moreover, since $\int_0^\cdot h dw$ is in $\mathcal{C}([0, T], L^2(\Omega, H^1(D)))$, one gets that the application

$$t \in [0, T] \mapsto E \left[\left\| \int_0^t \nabla h(s) dw(s) \right\|^2 \right] \in \mathbb{R}$$

is continuous and the announced result holds.

(ii) Due to (3.4), Lebesgue's theorem and the fact that $U = \chi - \int_0^\cdot h dw$ is an element of $\mathcal{C}([0, T], L^2(\Omega, L^2(D)))$, one gets the continuity of the application

$$t \in [0, T] \mapsto E \left[\|U(t)\|_{H^1(D)}^2 \right] \in \mathbb{R}.$$

Note that as mentioned in the proof of Proposition 3.1, U belongs to the space $\mathcal{E}_w([0, T], L^2(\Omega, H^1(D)))$. Combining this with the above continuity result, one concludes that U is in $\mathcal{C}([0, T], L^2(\Omega, H^1(D)))$ and due to the regularity of Itô integral, it is the same for χ . \square

Proposition 3.3. *Assume that h belongs to $\mathcal{N}_w^2(0, T, H^2(D))$. Then the couple (ϑ, χ) given by Proposition 2.13 and Proposition 2.14 is a solution of System (1.1) in the sense of Definition 1.1.*

Proof. Firstly, note that using Proposition 2.13, Proposition 2.14, Proposition 2.16 and Proposition 3.2, ϑ and χ own regularities required by Definition 1.1. Secondly, they satisfy respectively the initial conditions $\vartheta(0, \cdot) = \vartheta_0$ and $\chi(0, \cdot) = \chi_0$ in $L^2(D)$ owing to Proposition 2.15. Thirdly, thanks to Proposition 3.1, by passing to the limit in (2.3)-(2.4) with respect to Δt and using the separability of $H^1(D)$, one gets t -almost everywhere in $(0, T)$, P -almost surely in Ω and for any v in $H^1(D)$

$$\begin{cases} \int_D \partial_t \vartheta v dx + \int_D \partial_t \left(\chi - \int_0^\cdot h dw \right) v dx + \int_D \nabla \vartheta \cdot \nabla v dx = 0 \\ \int_D \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot h dw \right) \right) v dx + \int_D \nabla \chi \cdot \nabla v dx = \int_D \vartheta v dx. \end{cases}$$

Hence, (ϑ, χ) is a solution of (1.1) in the sense of Definition 1.1 \square

We now have all the necessary tools to show the result of existence and uniqueness for Problem (1.1) stated in Theorem 1.4.

3.2. Existence result for (1.1) when $h \in \mathcal{N}_w^2(0, T, H^1(D))$

Assume that h belongs to $\mathcal{N}_w^2(0, T, H^1(D))$. Owing to the density of $\mathcal{C}_c^\infty(\overline{D})$ in $H^1(D)$, we propose to approach h by a sequence $(h_n)_{n \in \mathbb{N}} \subset \mathcal{N}_w^2(0, T, \mathcal{C}_c^\infty(\overline{D}))$. Using Proposition 3.3, one is able to define the following sequences :

Definition 3.4. Set $n, m \in \mathbb{N}$ and consider ϑ_0, χ_0 in $H^1(D)$ and h_n, h_m belonging to $\mathcal{N}_w^2(0, T, \mathcal{C}_c^\infty(\overline{D}))$. We define the couples (ϑ_n, χ_n) and (ϑ_m, χ_m) as solutions of Problem (1.1) in the sense of Definition 1.1 with the respective sets of data $(\vartheta_0, \chi_0, h_n)$ and $(\vartheta_0, \chi_0, h_m)$.

For any $n, m \in \mathbb{N}$, we introduce the notations

$$U_n = \chi_n - \int_0^\cdot h_n dw \text{ and } U_m = \chi_m - \int_0^\cdot h_m dw.$$

In what follows, our aim is to show that $(\vartheta_n)_{n \in \mathbb{N}}, (\chi_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ are Cauchy sequences in suitable spaces.

Lemma 3.5. *The sequences $(\vartheta_n)_{n \in \mathbb{N}}$, $(\chi_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ introduced in Definition 3.4 satisfy the following properties*

- (i) $(\vartheta_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H^1(D)) \cap L^2(\Omega, H^1(Q))$.
- (ii) $(\chi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H^1(D))$.
- (iii) $(U_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, H^1(Q))$.
- (iv) For any t in $[0, T]$, $(\vartheta_n(t))_{n \in \mathbb{N}}$ and $(\chi_n(t))_{n \in \mathbb{N}}$ are Cauchy sequences in $L^2(\Omega, H^1(D))$.

Proof. Since the couples (ϑ_n, χ_n) and (ϑ_m, χ_m) satisfy the variational formulation (1.3) respectively with h_n and h_m , one gets by subtracting them, t -almost everywhere in $(0, T)$, P -almost surely in Ω and for any v in $H^1(D)$ that

$$\begin{aligned} & \int_D \partial_t(\vartheta_n - \vartheta_m)v dx + \int_D \partial_t(U_n - U_m)v dx \\ & + \int_D \nabla(\vartheta_n - \vartheta_m) \cdot \nabla v dx = 0. \end{aligned} \quad (3.6)$$

For a fixed t in $[0, T]$, we consider in (3.6) the test function

$$v = \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t}.$$

By noticing that

$$\begin{aligned} & \int_D \nabla(\vartheta_n(t) - \vartheta_m(t)) \cdot \nabla \left(\vartheta_n(t) - \vartheta_m(t) - (\vartheta_n(t - \Delta t) - \vartheta_m(t - \Delta t)) \right) dx \\ & = \frac{1}{2} \left[\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2 - \|\nabla(\vartheta_n - \vartheta_m)(t - \Delta t)\|^2 \right] \\ & + \frac{1}{2} \|\nabla(\vartheta_n - \vartheta_m)(t) - \nabla(\vartheta_n - \vartheta_m)(t - \Delta t)\|^2, \end{aligned}$$

we thus obtain

$$\begin{aligned} & \int_D \partial_t(\vartheta_n - \vartheta_m) \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t} dx \\ & + \int_D \partial_t(U_n - U_m) \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t} dx \\ & \frac{1}{2\Delta t} \left[\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2 - \|\nabla(\vartheta_n - \vartheta_m)(t - \Delta t)\|^2 \right] \\ & + \frac{1}{2\Delta t} \|\nabla(\vartheta_n - \vartheta_m)(t) - \nabla(\vartheta_n - \vartheta_m)(t - \Delta t)\|^2 = 0. \end{aligned}$$

By taking the expectation and the integral from Δt to T , one gets

$$\begin{aligned} & \int_{\Delta t}^T E \left[\int_D \partial_t(\vartheta_n - \vartheta_m) \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t} dx \right] dt \\ & + \int_{\Delta t}^T E \left[\int_D \partial_t(U_n - U_m) \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t} dx \right] dt \\ & + \frac{1}{2\Delta t} \int_{\Delta t}^T E \left[\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2 - \|\nabla(\vartheta_n - \vartheta_m)(t - \Delta t)\|^2 \right] dt \\ & + \frac{1}{2\Delta t} \int_{\Delta t}^T E \left[\|\nabla(\vartheta_n - \vartheta_m)(t) - \nabla(\vartheta_n - \vartheta_m)(t - \Delta t)\|^2 \right] dt = 0. \end{aligned}$$

Then, a change of variables gives us

$$\begin{aligned} & \int_{\Delta t}^T E \left[\int_D \partial_t(\vartheta_n - \vartheta_m) \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t} dx \right] dt \\ & + \int_{\Delta t}^T E \left[\int_D \partial_t(U_n - U_m) \frac{(\vartheta_n - \vartheta_m)(t) - (\vartheta_n - \vartheta_m)(t - \Delta t)}{\Delta t} dx \right] dt \\ & + \frac{1}{2\Delta t} \int_{T-\Delta t}^T E \left[\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2 \right] dt \leq \frac{1}{2\Delta t} \int_0^{\Delta t} E \left[\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2 \right] dt. \end{aligned}$$

By using Proposition 2.16, we obtain by passing to the limit with Δt and using the initial values:

$$\begin{aligned} & E \left[\int_Q |\partial_t(\vartheta_n - \vartheta_m)|^2 dx dt \right] + E \int_Q \partial_t(U_n - U_m) \partial_t(\vartheta_n - \vartheta_m) dx \\ & + \frac{1}{2} E \left[\|\nabla(\vartheta_n - \vartheta_m)(T)\|^2 \right] \leq \frac{1}{2} E \left[\|\nabla(\vartheta_n - \vartheta_m)(0)\|^2 \right] = 0. \end{aligned}$$

By denoting $Q_t = (0, t) \times D$, using the identity $ab = \frac{1}{2}[(a+b)^2 - a^2 - b^2]$ in the second term of the above equation and discarding nonnegative terms one has (since T is arbitrary) for any $t \in [0, T]$

$$\begin{aligned} & \|\partial_t(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q_t)}^2 + E \left[\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2 \right] \\ & \leq \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2. \end{aligned} \quad (3.7)$$

In the same manner, using the test function $\vartheta_n - \vartheta_m$ in (3.6), one shows the following inequality for any $t \in [0, T]$

$$\begin{aligned} & E \left[\|\vartheta_n - \vartheta_m(t)\|^2 \right] + 2\|\nabla(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & \leq \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 + \|\vartheta_n - \vartheta_m\|_{L^2(\Omega \times Q_t)}^2. \end{aligned} \quad (3.8)$$

Similarly, exploiting the fact that (ϑ_n, χ_n) and (ϑ_m, χ_m) satisfy additionally the variational formulation (1.4) respectively with h_n and h_m , one gets by subtracting them, t -almost everywhere in $(0, T)$, P -almost surely in Ω and

for any v in $H^1(D)$ that

$$\begin{aligned} & \int_D \partial_t(U_n - U_m)v dx + \int_D (\alpha(\partial_t(U_n)) - \alpha(\partial_t(U_m)))v dx \\ & + \int_D \nabla(\chi_n - \chi_m) \cdot \nabla v dx = \int_D (\vartheta_n - \vartheta_m)v dx. \end{aligned} \quad (3.9)$$

For a fixed t in $[0, T]$, by taking in (3.9) the test function

$$v = \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t},$$

we get

$$\begin{aligned} & \int_D \partial_t(U_n - U_m) \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t} dx \\ & + \frac{1}{\Delta t} \int_D \nabla(\chi_n(t) - \chi_m(t)) \cdot \nabla \left(\chi_n(t) - \chi_m(t) - (\chi_n(t - \Delta t) - \chi_m(t - \Delta t)) \right) dx \\ & - \frac{1}{\Delta t} \int_D \nabla(\chi_n(t) - \chi_m(t)) \cdot \nabla \left(\int_{t-\Delta t}^t (h_n - h_m) dw \right) dx \\ & + \int_D (\alpha(\partial_t U_n) - \alpha(\partial_t U_m)) \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t} dx \\ & = \int_D (\vartheta_n - \vartheta_m) \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t} dx. \end{aligned}$$

Then, by taking the expectation, the integral from Δt to T and using changes of variables one arrives at (see [8], Proof of Theorem 1.4 for details)

$$\begin{aligned} & \int_{\Delta t}^T E \left[\int_D \partial_t(U_n - U_m) \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t} dx \right] dt \\ & + \frac{1}{2\Delta t} \int_{\Delta t}^T E \left[\|\nabla(\chi_n - \chi_m)(t)\|^2 \right] dt - \frac{1}{2\Delta t} \int_0^{\Delta t} E \left[\|\nabla(\chi_n - \chi_m)(t)\|^2 \right] dt \\ & + \int_{\Delta t}^T E \left[\int_D (\alpha(\partial_t U_n) - \alpha(\partial_t U_m)) \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t} dx \right] dt \\ & \leq \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q)}^2 \\ & + \int_D (\vartheta_n - \vartheta_m) \frac{(U_n - U_m)(t) - (U_n - U_m)(t - \Delta t)}{\Delta t} dx. \end{aligned}$$

By passing to the limit with Δt in this inequality, using Proposition 3.2 and the initial value we obtain:

$$\begin{aligned} & E \left[\int_Q |\partial_t(U_n - U_m)|^2 dx dt \right] + \frac{1}{2} E \left[\|\nabla(\chi_n - \chi_m)(T)\|^2 \right] \\ & + E \left[\int_Q (\alpha(\partial_t U_n) - \alpha(\partial_t U_m)) \partial_t(U_n - U_m) dx dt \right] \\ & \leq \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q)}^2 + \int_D (\vartheta_n - \vartheta_m) \partial_t(U_n - U_m) dx. \end{aligned}$$

Then, due to the coercivity of α , one also has for any $t \in [0, T]$, by still denoting $Q_t = (0, t) \times D$,

$$\begin{aligned} & (\bar{C}_\alpha + \frac{1}{2}) \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2} E [\|\nabla(\chi_n - \chi_m)(t)\|^2] \\ & \leq \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2} \|\vartheta_n - \vartheta_m\|_{L^2(\Omega \times Q_t)}^2. \end{aligned} \quad (3.10)$$

In the same manner, using the test function $U_n - U_m$ in (3.9), one shows the following inequality for any $t \in [0, T]$

$$\begin{aligned} & \frac{1}{2} E [\|(U_n - U_m)(t)\|^2] + \frac{1}{2} \|\nabla(\chi_n - \chi_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & \leq C_\alpha^2 \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2} \|U_n - U_m\|_{L^2(\Omega \times Q_t)}^2 \\ & \quad + \frac{T}{2} \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 + \|\vartheta_n - \vartheta_m\|_{L^2(\Omega \times Q_t)}^2. \end{aligned} \quad (3.11)$$

Now, by adding (3.8) and (3.11), one gets

$$\begin{aligned} & E [\|(\vartheta_n - \vartheta_m)(t)\|^2] + \frac{1}{2} E [\|(U_n - U_m)(t)\|^2] \\ & \leq (C_\alpha^2 + 1) \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{T}{2} \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & \quad + \frac{1}{2} \|U_n - U_m\|_{L^2(\Omega \times Q_t)}^2 + 2 \|\vartheta_n - \vartheta_m\|_{L^2(\Omega \times Q_t)}^2. \end{aligned} \quad (3.12)$$

From (3.10)

$$\|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 \leq \frac{1}{\bar{C}_\alpha + \frac{1}{2}} \left\{ \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2} \|\vartheta_n - \vartheta_m\|_{L^2(\Omega \times Q_t)}^2 \right\},$$

substituting it in (3.12), we obtain

$$\begin{aligned} & E [\|(\vartheta_n - \vartheta_m)(t)\|^2] + \frac{1}{2} E [\|(U_n - U_m)(t)\|^2] \\ & \leq \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q)}^2 \\ & \quad + \frac{1}{2} \|U_n - U_m\|_{L^2(\Omega \times Q_t)}^2 + \left(\frac{C_\alpha^2 + 1}{2\bar{C}_\alpha + 1} + 2 \right) \|\vartheta_n - \vartheta_m\|_{L^2(\Omega \times Q_t)}^2. \end{aligned}$$

By denoting for any t in $[0, T]$

$$\begin{aligned} y(t) &= E [\|(\vartheta_n - \vartheta_m)(t)\|^2] + \frac{1}{2} E [\|(U_n - U_m)(t)\|^2], \\ K_\alpha^{n,m}(t) &= \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \\ \text{and } \hat{C}_\alpha &= \frac{C_\alpha^2 + 1}{2\bar{C}_\alpha + 1} + 2, \end{aligned}$$

we have for any t in $[0, T]$

$$y(t) \leq K_\alpha^{n,m}(t) + \hat{C}_\alpha \int_0^t y(s) ds \leq K_\alpha^{n,m}(T) + \hat{C}_\alpha \int_0^t y(s) ds. \quad (3.13)$$

Firstly, Grönwall's Lemma then asserts that for any t in $[0, T]$

$$E [|\!(\vartheta_n - \vartheta_m)(t)\!|^2] + \frac{1}{2} E [|\!(U_n - U_m)(t)\!|^2] \leq K_\alpha^{n,m}(T) e^{\hat{C}_\alpha t}. \quad (3.14)$$

Thus, by taking the supremum over $[0, T]$ in (3.14), one obtains the estimate

$$\begin{aligned} & \sup_{t \in [0, T]} E [|\!(\vartheta_n - \vartheta_m)(t)\!|^2] + \sup_{t \in [0, T]} \frac{1}{2} E [|\!(U_n - U_m)(t)\!|^2] \\ & \leq \left(2\hat{C}_\alpha + \frac{T}{2} \right) e^{\hat{C}_\alpha T} \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q)}^2. \end{aligned} \quad (3.15)$$

Secondly, using (3.15) in (3.10) allows us to affirm that

$$\begin{aligned} & \left(\bar{C}_\alpha + \frac{1}{2} \right) \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q)}^2 + \frac{1}{2} \sup_{t \in [0, T]} E [|\!\|\nabla(\chi_n - \chi_m)(t)\!\|^2] \\ & \leq \left(\left(\hat{C}_\alpha + \frac{T}{4} \right) T e^{\hat{C}_\alpha T} + 1 \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q)}^2. \end{aligned} \quad (3.16)$$

Thirdly, thanks to (3.16) in (3.7), we obtain

$$\begin{aligned} & \|\partial_t(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q)}^2 + \sup_{t \in [0, T]} E [|\!\|\nabla(\vartheta_n - \vartheta_m)(t)\!\|^2] \\ & \leq \frac{1}{\bar{C}_\alpha + \frac{1}{2}} \left(\left(\hat{C}_\alpha + \frac{T}{4} \right) T e^{\hat{C}_\alpha T} + 1 \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q)}^2. \end{aligned} \quad (3.17)$$

Finally, since $(h_n)_n$ is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H^1(D))$ and owing to (3.15), (3.16) and (3.17), one concludes that

- (i) $(\vartheta_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H^1(D)) \cap L^2(\Omega, H^1(Q))$.
- (ii) $(\chi_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathcal{N}_w^2(0, T, H^1(D))$.
- (iii) $(U_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, H^1(Q))$.
- (iv) For any t in $[0, T]$, $(\vartheta_n(t))_{n \in \mathbb{N}}$ and $(\chi_n(t))_{n \in \mathbb{N}}$ are Cauchy sequences in $L^2(\Omega, H^1(D))$.
- (v.) $(\partial_t \vartheta_n)_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega \times Q)$.

□

As mentioned by DA PRATO-ZABCZYK [14], $\mathcal{N}_w^2(0, T, H^1(D))$ is complete, thus due to Lemma 3.5, the following convergence results hold directly.

Corollary 3.6. *There exist ϑ in $\mathcal{N}_w^2(0, T, H^1(D)) \cap L^2(\Omega, H^1(Q))$ and χ in $\mathcal{N}_w^2(0, T, H^1(D))$ such that the sequences $(\vartheta_n)_{n \in \mathbb{N}}$, $(\chi_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ (introduced in Definition 3.4) satisfy the following convergence results*

$$\begin{aligned} \vartheta_n &\rightarrow \vartheta && \text{in } \mathcal{N}_w^2(0, T, H^1(D)) \text{ and } L^2(\Omega; H^1(Q)), \\ \chi_n &\rightarrow \chi && \text{in } \mathcal{N}_w^2(0, T, H^1(D)), \\ \partial_t \vartheta_n &\rightarrow \partial_t \vartheta && \text{in } L^2(\Omega \times Q) \\ \partial_t U_n &\rightarrow \partial_t \left(\chi - \int_0^\cdot h dw \right) && \text{in } L^2(\Omega \times Q) \\ \forall t \in [0, T], \vartheta_n(t, \cdot) &\rightarrow \vartheta(t, \cdot) && \text{in } L^2(\Omega, H^1(D)), \\ \forall t \in [0, T], \chi_n(t, \cdot) &\rightarrow \chi(t, \cdot) && \text{in } L^2(\Omega, H^1(D)). \end{aligned}$$

Note that since $(\vartheta_n)_{n \in \mathbb{N}}$ converges in $L^2(\Omega; H^1(Q))$, it also converges in $L^2(\Omega, \mathcal{C}([0, T], L^2(D)))$ and, using the regularity of the Itô integral, the same is true for $(\chi_n)_{n \in \mathbb{N}}$. Thus, using Corollary 3.6, we get that t -almost everywhere in $(0, T)$, P -almost surely in Ω and for any v in $H^1(D)$

$$\begin{aligned} \int_D \partial_t \vartheta v dx + \int_D \partial_t \left(\chi - \int_0^\cdot h dw \right) v dx + \int_D \nabla \vartheta \cdot \nabla v dx &= 0 \\ \int_D \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot h dw \right) \right) v dx + \int_D \nabla \chi \cdot \nabla v dx &= \int_D \vartheta v dx, \end{aligned}$$

and we have the existence result when $h \in \mathcal{N}_w^2(0, T, H^1(D))$ as announced in Theorem 1.4.

3.3. Uniqueness result for (1.1)

Theorem 3.7. *For h in $\mathcal{N}_w^2(0, T, H^1(D))$ and initial data $(\chi_0, \vartheta_0) \in H^1(D)^2$ the solution in the sense of Definition 1.1 of System (1.1) is unique.*

Proof. We consider h in $\mathcal{N}_w^2(0, T, H^1(D))$, and (χ, ϑ) , $(\hat{\chi}, \hat{\vartheta})$ two solutions in the sense of Definition 1.1 of System (1.1). Using the notations $U = \chi - \int_0^\cdot h dw$ and $\hat{U} = \hat{\chi} - \int_0^\cdot h dw$, one has

$$\left\{ \begin{array}{ll} \partial_t(\vartheta - \hat{\vartheta}) + \partial_t(U - \hat{U}) - \Delta(\vartheta - \hat{\vartheta}) = 0 & \text{in } (0, T) \times D \times \Omega, \\ \partial_t(U - \hat{U}) - \Delta(U - \hat{U}) + \alpha(\partial_t U) - \alpha(\partial_t \hat{U}) = \vartheta - \hat{\vartheta} & \text{in } (0, T) \times D \times \Omega, \\ \nabla(U - \hat{U}) \cdot \mathbf{n} = \nabla(\vartheta - \hat{\vartheta}) \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial D \times \Omega, \\ (U - \hat{U})(0, \cdot) = 0 \text{ and } (\vartheta - \hat{\vartheta})(0, \cdot) = 0. & \end{array} \right.$$

Denoting $\xi = \vartheta - \hat{\vartheta}$ and $u = U - \hat{U}$, it follows that (ξ, u) is the solution of the following system of heat equations

$$\left\{ \begin{array}{ll} \partial_t \xi - \Delta \xi = -\partial_t u & \text{in } (0, T) \times D \times \Omega, \\ \partial_t u - \Delta u = \alpha(\partial_t \hat{U}) - \alpha(\partial_t U) + \xi & \text{in } (0, T) \times D \times \Omega, \\ \nabla u \cdot \mathbf{n} = \nabla \xi \cdot \mathbf{n} = 0 & \text{on } (0, T) \times \partial D \times \Omega, \\ u(0, \cdot) = 0 \text{ and } \xi(0, \cdot) = 0. & \end{array} \right.$$

Following the same arguments as in (3.5), note that ξ satisfies the energy equality

$$\frac{1}{2}E [\|\xi(t)\|^2] + E \left[\int_{Q_t} |\nabla \xi|^2 ds dx \right] = \frac{1}{2} \|\xi(0)\|^2 - E \left[\int_{Q_t} \xi \partial_t u ds dx \right],$$

for any t in $[0, T]$ where $Q_t = (0, t) \times D$.

Then

$$E [\|\xi(t)\|^2] + 2\|\nabla \xi\|_{L^2(\Omega \times Q_t)}^2 \leq \|\partial_t u\|_{L^2(\Omega \times Q_t)}^2 + \|\xi\|_{L^2(\Omega \times Q_t)}^2. \quad (3.18)$$

As in (3.4), one has for any t in $[0, T]$

$$\begin{aligned} E \left[\int_{Q_t} |\partial_t u|^2 ds dx \right] + \frac{1}{2}E [\|\nabla u(t)\|_{L^2(D)}^2] &= \frac{1}{2}\|\nabla u(0)\|_{L^2(D)}^2 \\ + E \left[\int_{Q_t} \xi \partial_t u ds dx \right] - E \left[\int_{Q_t} (\alpha(\partial_t U) - \alpha(\partial_t \hat{U})) \partial_t u ds dx \right]. \end{aligned}$$

Note that this allows us to affirm thanks to Lebesgue's theorem that the application $t \in [0, T] \mapsto E [\|\nabla u(t)\|^2] \in \mathbb{R}$ is continuous.

Due to the coercivity property of α , one gets for any t in $[0, T]$

$$(\bar{C}_\alpha + \frac{1}{2})\|\partial_t u\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2}E [\|\nabla u(t)\|_{L^2(D)}^2] \leq \frac{1}{2}\|\xi\|_{L^2(\Omega \times Q_t)}^2,$$

and then

$$\|\partial_t u\|_{L^2(\Omega \times Q_t)}^2 \leq \frac{1}{1 + 2\bar{C}_\alpha} \|\xi\|_{L^2(\Omega \times Q_t)}^2. \quad (3.19)$$

On the one hand, going back to (3.18), in virtue of (3.19), we have for any t in $[0, T]$

$$E [\|\xi(t)\|^2] \leq \left(1 + \frac{1}{1 + 2\bar{C}_\alpha}\right) \int_0^t E [\|\xi(s)\|^2] ds,$$

and Grönwall's Lemma allows us to assert that $\xi = 0$, thus $\vartheta = \tilde{\vartheta}$.

On the other hand, the study of the heat equation also provides the following estimate on u for any t in $[0, T]$:

$$\begin{aligned} \frac{1}{2}E [\|u(t)\|^2] - \frac{1}{2}\|u(0)\|^2 + \|\nabla u\|_{L^2(\Omega \times Q_t)}^2 \\ \leq \|u\|_{L^2(\Omega \times Q_t)}^2 + \frac{C_\alpha^2}{2}\|\partial_t u\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2}\|\xi\|_{L^2(\Omega \times Q_t)}^2, \end{aligned}$$

which gives using (3.19) and the fact that $\xi = 0$

$$\frac{1}{2}E [\|u(t)\|^2] \leq \|u\|_{L^2(\Omega \times Q_t)}^2.$$

According to Grönwall's Lemma, $u = 0$ which implies that $\chi = \tilde{\chi}$ and the uniqueness result holds for (1.1). \square

4. Proof of Proposition 1.5

The proof of Lemma 3.5 allows us to show directly the following continuous dependence result on the sequences $(\vartheta_n)_{n \in \mathbb{N}}$, $(\chi_n)_{n \in \mathbb{N}}$ given by Definition 3.4 with respect to the sequence of integrands $(h_n)_{n \in \mathbb{N}}$ in the stochastic noise.

4.1. Preliminary result

Lemma 4.1. *There exists a constant $C_\alpha^T > 0$ which only depends on T , C_α and \bar{C}_α such that the sequences $(\vartheta_n)_{n \in \mathbb{N}}$, $(\chi_n)_{n \in \mathbb{N}}$ and $(U_n)_{n \in \mathbb{N}}$ introduced in Definition 3.4 satisfy the following inequality for any t in $[0, T]$*

$$\begin{aligned} & \|\partial_t(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q_t)}^2 + \left(\bar{C}_\alpha + \frac{1}{2}\right) \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & + E [\|(\vartheta_n - \vartheta_m)(t)\|^2] + E [\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2] \\ & + \frac{1}{4} E [\|(\chi_n - \chi_m)(t)\|^2] + \frac{1}{4} E [\|\nabla(\chi_n - \chi_m)(t)\|^2] \\ & \leq C_\alpha^T \left(\|h_n - h_m\|_{L^2(\Omega \times Q_t)}^2 + \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \right), \end{aligned} \quad (4.1)$$

where $Q_t = (0, t) \times D$.

Proof. Owing to (3.13) and by using again Grönwall's Lemma one gets for any t in $[0, T]$

$$\begin{aligned} & E [\|(\vartheta_n - \vartheta_m)(t)\|^2] + \frac{1}{2} E [\|(U_n - U_m)(t)\|^2] \\ & \leq K_\alpha^{n,m}(t) + \int_0^t \hat{C}_\alpha K_\alpha^{n,m}(s) e^{\hat{C}_\alpha(t-s)} ds, \end{aligned}$$

where

$$K_\alpha^{n,m}(t) = \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \text{ and } \hat{C}_\alpha = \frac{C_\alpha^2 + 1}{2\bar{C}_\alpha + 1} + 2.$$

Thus we obtain

$$\begin{aligned} & E [\|(\vartheta_n - \vartheta_m)(t)\|^2] + \frac{1}{2} E [\|(U_n - U_m)(t)\|^2] \\ & \leq (1 + \hat{C}_\alpha T e^{\hat{C}_\alpha T}) \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2, \end{aligned}$$

And using this in (3.10) allows us to affirm for any t in $[0, T]$ that

$$\begin{aligned} & \left(\bar{C}_\alpha + \frac{1}{2}\right) \|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2} E [\|\nabla(\chi_n - \chi_m)(t)\|^2] \\ & \leq \left\{ \frac{T}{2} (1 + \hat{C}_\alpha T e^{\hat{C}_\alpha T}) \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) + 1 \right\} \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2. \end{aligned}$$

Now by injecting this last inequality in (3.7), we obtain for any t in $[0, T]$

$$\begin{aligned} & \|\partial_t(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q_t)}^2 + E [\|\nabla(\vartheta_n - \vartheta_m)(t)\|^2] \\ & \leq \frac{1}{\bar{C}_\alpha + \frac{1}{2}} \left\{ \frac{T}{2} (1 + \hat{C}_\alpha T e^{\hat{C}_\alpha T}) \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) + 1 \right\} \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2. \end{aligned}$$

Finally, by noticing that for any t in $[0, T]$

$$E [|\!(\chi_n - \chi_m)(t)\!|^2] \leq 2E [|\!(U_n - U_m)(t)\!|^2] + 2\|h_n - h_m\|_{L^2(\Omega \times Q_t)}^2$$

one gets

$$\begin{aligned} & E [|\!(\vartheta_n - \vartheta_m)(t)\!|^2] + \frac{1}{4}E [|\!(\chi_n - \chi_m)(t)\!|^2] \\ & + \|\partial_t(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q_t)}^2 + E [|\!\nabla(\vartheta_n - \vartheta_m)(t)\!|^2] \\ & + (\bar{C}_\alpha + \frac{1}{2})\|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 + \frac{1}{2}E [|\!\nabla(\chi_n - \chi_m)(t)\!|^2] \\ \leq & \left(\frac{1}{\bar{C}_\alpha + \frac{1}{2}} + 1\right) \left\{ \frac{T}{2} (1 + \hat{C}_\alpha T e^{\hat{C}_\alpha T}) \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) + 1 \right\} \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & + (1 + \hat{C}_\alpha T e^{\hat{C}_\alpha T}) \left(\frac{C_\alpha^2 + 1}{\bar{C}_\alpha + \frac{1}{2}} + \frac{T}{2} \right) \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & + \frac{1}{2}\|h_n - h_m\|_{L^2(\Omega \times Q_t)}^2 \end{aligned}$$

and thus the existence of a constant C_α^T which only depends on $T, C_\alpha, \bar{C}_\alpha$ and \hat{C}_α such that

$$\begin{aligned} & \|\partial_t(\vartheta_n - \vartheta_m)\|_{L^2(\Omega \times Q_t)}^2 + (\bar{C}_\alpha + \frac{1}{2})\|\partial_t(U_n - U_m)\|_{L^2(\Omega \times Q_t)}^2 \\ & + E [|\!(\vartheta_n - \vartheta_m)(t)\!|^2] + E [|\!\nabla(\vartheta_n - \vartheta_m)(t)\!|^2] \\ & + \frac{1}{4}E [|\!(\chi_n - \chi_m)(t)\!|^2] + \frac{1}{4}E [|\!\nabla(\chi_n - \chi_m)(t)\!|^2] \\ \leq & C_\alpha^T \left(\|h_n - h_m\|_{L^2(\Omega \times Q_t)}^2 + \|\nabla(h_n - h_m)\|_{L^2(\Omega \times Q_t)}^2 \right). \end{aligned}$$

□

4.2. Proof of Proposition 1.5

Using a Cauchy sequence argument as in Subsection 3.2 and recalling the uniqueness result of Theorem 3.7, one gets by passing to the limit in the inequality (4.1) above the stability result announced in Proposition 1.5. More precisely, for $h, \hat{h} \in \mathcal{N}_w^2(0, T, H^1(D))$ there exist $(h_n)_{n \in \mathbb{N}}$ and $(\hat{h}_n)_{n \in \mathbb{N}}$ in $\mathcal{N}_w^2(0, T, C_c^\infty(\bar{D}))$ such that $h_n \rightarrow h, \hat{h}_n \rightarrow \hat{h}$ in $\mathcal{N}_w^2(0, T, H^1(D))$ for $n \rightarrow \infty$. We fix initial data $(\vartheta_0, \chi_0) \in H^1(D)^2$ and consider solutions $(\vartheta_n, \chi_n), (\hat{\vartheta}_n, \hat{\chi}_n)$ with data $(h_n, \vartheta_0, \chi_0), (\hat{h}_n, \vartheta_0, \chi_0)$ respectively. Plugging $(\vartheta_n, \chi_n) = (\vartheta_n, \chi_n), (\vartheta_m, \chi_m) = (\hat{\vartheta}_n, \hat{\chi}_n)$ into (4.1) and using the convergence results

from Corollary 3.6, we can pass to the limit and obtain

$$\begin{aligned}
& \|\partial_t(\vartheta - \hat{\vartheta})\|_{L^2(\Omega \times Q_t)}^2 + \left(\bar{C}_\alpha + \frac{1}{2}\right) \|\partial_t(U - \hat{U})\|_{L^2(\Omega \times Q_t)}^2 \\
& + E \left[\|\vartheta - \hat{\vartheta}(t)\|^2 \right] + E \left[\|\nabla(\vartheta - \hat{\vartheta})(t)\|^2 \right] \\
& + \frac{1}{4} E \left[\|\chi - \hat{\chi}(t)\|^2 \right] + \frac{1}{4} E \left[\|\nabla(\chi - \hat{\chi})(t)\|^2 \right] \\
& \leq C_\alpha^T \left(\|h - \hat{h}\|_{L^2(\Omega \times Q_t)}^2 + \|\nabla(h - \hat{h})\|_{L^2(\Omega \times Q_t)}^2 \right),
\end{aligned}$$

where (ϑ, χ) , $(\hat{\vartheta}, \hat{\chi})$ are the unique solutions with data (h, ϑ_0, χ_0) , $(\hat{h}, \vartheta_0, \chi_0)$ respectively, $U := \chi - \int_0^\cdot h \, dw$, $\hat{U} := \hat{\chi} - \int_0^\cdot \hat{h} \, dw$.

5. Proof of Theorem 1.6

Under Assumptions H_2 to H_4 , we are interested in the following system with multiplicative noise:

$$\left\{ \begin{array}{l} \partial_t \vartheta + \partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) \, dw \right) - \Delta \vartheta = 0 \text{ in } (0, T) \times D \times \Omega, \\ \tilde{\alpha} \left(\partial_t \left(\chi - \int_0^\cdot \mathcal{H}(\chi) \, dw \right) \right) - \Delta \chi = \vartheta \text{ in } (0, T) \times D \times \Omega, \\ \nabla \chi \cdot \mathbf{n} = \nabla \vartheta \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial D \times \Omega, \\ \chi(0, \cdot) = \chi_0 \text{ and } \vartheta(0, \cdot) = \vartheta_0. \end{array} \right.$$

Using Theorem 1.4, we define the application

$$\begin{aligned}
f : \mathcal{N}_w^2(0, T, H^1(D)) & \rightarrow \mathcal{N}_w^2(0, T, H^1(D)) \times \mathcal{N}_w^2(0, T, H^1(D)), \\
S & \mapsto (\vartheta_S, \chi_S),
\end{aligned}$$

where (ϑ_S, χ_S) is the solution of the following system with additive noise

$$\left\{ \begin{array}{l} \partial_t \vartheta_S + \partial_t \left(\chi_S - \int_0^\cdot \mathcal{H}(S) \, dw \right) - \Delta \vartheta_S = 0 \text{ in } (0, T) \times D \times \Omega, \\ \tilde{\alpha} \left(\partial_t \left(\chi_S - \int_0^\cdot \mathcal{H}(S) \, dw \right) \right) - \Delta \chi_S = \vartheta_S \text{ in } (0, T) \times D \times \Omega, \\ \nabla \chi_S \cdot \mathbf{n} = \nabla \vartheta_S \cdot \mathbf{n} = 0 \text{ on } (0, T) \times \partial D \times \Omega, \\ \chi_S(0, \cdot) = \chi_0 \text{ and } \vartheta_S(0, \cdot) = \vartheta_0, \end{array} \right.$$

in the sense of Definition 1.1 with $h = \mathcal{H}(S)$. Additionally, we consider the following projection application

$$\begin{aligned}
g : \mathcal{N}_w^2(0, T, H^1(D)) \times \mathcal{N}_w^2(0, T, H^1(D)) & \rightarrow \mathcal{N}_w^2(0, T, H^1(D)), \\
(u, v) & \mapsto v.
\end{aligned}$$

Our aim is to show that the composition $g \circ f$ admits a unique fixed-point in $\mathcal{N}_w^2(0, T, H^1(D))$. The idea is to exploit the fact that, for any $a > 0$, the

following exponential weight in time norm

$$\begin{aligned} \mathcal{N}_w^2(0, T, H^1(D)) &\rightarrow \mathbb{R}_+ \\ v &\mapsto \int_0^T e^{-at} E[\|v(t)\|_{H^1(D)}^2] dt \end{aligned}$$

provides an equivalent norm to the usual one on $\mathcal{N}_w^2(0, T, H^1(D))$.

Set $a > 0$, consider S and \hat{S} in $\mathcal{N}_w^2(0, T, H^1(D))$ and define $f(S) = (\vartheta_S, \chi_S)$ and $f(\hat{S}) = (\vartheta_{\hat{S}}, \chi_{\hat{S}})$. According to [20] Lemma 2.41 p.35, note that $\mathcal{H}(S)$ and $\mathcal{H}(\hat{S})$ belong to $\mathcal{N}_w^2(0, T, H^1(D))$. Using Proposition 1.5, one gets for any t in $[0, T]$

$$\begin{aligned} &E[\|(\vartheta_S - \vartheta_{\hat{S}})(t)\|^2] + E[\|\nabla(\vartheta_S - \vartheta_{\hat{S}})(t)\|^2] \\ &+ \frac{1}{4} E[\|(\chi_S - \chi_{\hat{S}})(t)\|^2] + \frac{1}{4} E[\|\nabla(\chi_S - \chi_{\hat{S}})(t)\|^2] \\ &\leq C_\alpha^T \left(\|\mathcal{H}(S) - \mathcal{H}(\hat{S})\|_{L^2(\Omega \times Q_t)}^2 + \|\nabla(\mathcal{H}(S) - \mathcal{H}(\hat{S}))\|_{L^2(\Omega \times Q_t)}^2 \right) \\ &\leq C_\alpha^T \int_0^t E[\|\mathcal{H}(S) - \mathcal{H}(\hat{S})\|_{H^1(D)}^2] ds. \end{aligned} \quad (5.1)$$

We fix $t > 0$. Multiplying (5.1) with e^{-at} and integrating over $(0, T)$, we obtain

$$\begin{aligned} &\int_0^T e^{-at} E[\|(\chi_S - \chi_{\hat{S}})(t)\|^2] dt + \int_0^T e^{-at} E[\|\nabla(\chi_S - \chi_{\hat{S}})(t)\|^2] dt \\ &\leq 4C_\alpha^T \int_0^T e^{-at} \int_0^t E[\|(\mathcal{H}(S) - \mathcal{H}(\hat{S}))(s)\|_{H^1(D)}^2] ds dt. \end{aligned}$$

By using an integration by parts one gets

$$\begin{aligned} &\int_0^T e^{-at} E[\|(\chi_S - \chi_{\hat{S}})(t)\|^2] dt + \int_0^T e^{-at} E[\|\nabla(\chi_S - \chi_{\hat{S}})(t)\|^2] dt \\ &\leq 4C_\alpha^T C_{\mathcal{H}}^2 \int_0^T e^{-at} \int_0^t E[\|(S - \hat{S})(s)\|_{H^1(D)}^2] ds dt \\ &= 4C_\alpha^T C_{\mathcal{H}}^2 \times \frac{1}{a} \int_0^T e^{-at} E[\|(S - \hat{S})(t)\|_{H^1(D)}^2] dt \\ &\quad - 4C_\alpha^T C_{\mathcal{H}}^2 \times \left(\frac{1}{a} e^{-aT} \int_0^T E[\|(S - \hat{S})(t)\|_{H^1(D)}^2] dt \right) \\ &\leq 4C_\alpha^T C_{\mathcal{H}}^2 \times \frac{1}{a} \int_0^T e^{-at} E[\|(S - \hat{S})(t)\|_{H^1(D)}^2] dt. \end{aligned}$$

Finally

$$\begin{aligned} &\int_0^T e^{-at} E[\|(g \circ f)(S) - (g \circ f)(\hat{S})\|_{H^1(D)}^2] dt \\ &\leq 4C_\alpha^T C_{\mathcal{H}}^2 \times \frac{1}{a} \int_0^T e^{-at} E[\|(S - \hat{S})(t)\|_{H^1(D)}^2] dt. \end{aligned}$$

Under the condition $a > 4C_\alpha^T C_{\mathcal{H}}^2$, $g \circ f$ is a contractive mapping, it has a unique fixed-point and the result holds.

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References

- [1] S. N. Antontsev, G. Gagneux, R. Luce, and G. Vallet. New unilateral problems in stratigraphy. *M2AN Math. Model. Numer. Anal.*, 40(4):765–784, 2006.
- [2] S. N. Antontsev, G. Gagneux, R. Luce, and G. Vallet. A non-standard free boundary problem arising from stratigraphy. *Anal. Appl. (Singap.)*, 4(3):209–236, 2006.
- [3] S. N. Antontsev, G. Gagneux, A. Mokrani, and G. Vallet. Stratigraphic modelling by the way of a pseudoparabolic problem with constraint. *Adv. Math. Sci. Appl.*, 19(1):195–209, 2009.
- [4] S. N. Antontsev, G. Gagneux, and G. Vallet. On some problems of stratigraphic control. *Prikl. Mekh. Tekhn. Fiz.*, 44(6):85–94, 2003.
- [5] Adimurthi, N. Seam, and G. Vallet. On the equation of Barenblatt-Sobolev. *Commun. Contemp. Math.*, 13(5):843–862, 2011.
- [6] G. I. Barenblatt. *Scaling, self-similarity, and intermediate asymptotics*, volume 14 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press, Cambridge, 1996. With a foreword by Ya. B. Zeldovich.
- [7] C. Bauzet, J. Giacomoni, and G. Vallet. On a class of quasilinear Barenblatt equations. *Revista Real Academia de Ciencias de Zaragoza*, 38:35–51, 2012.
- [8] C. Bauzet, F. Lebon, and A. A.Maitlo. The Neumann Problem for a Barenblatt equation with a multiplicative stochastic force and a nonlinear source term. *NonLinear Differential Equations and Applications, Vol. 26(3)*, 2019.
- [9] C. Bauzet and G. Vallet. On abstract Barenblatt equations. *Differ. Equ. Appl.*, 3(4):487–502, 2011.
- [10] E. Bonetti and G. Bonfanti, and F. Lebon, and R. Rizzoni. A model of imperfect interface with damage. *Meccanica*, 52:1911–1922, 2017.
- [11] E. Bonetti and G. Bonfanti, and F. Lebon. Derivation of imperfect interface models coupling damage and temperature. *Computers and Mathematics with Applications*, 77:2906–2016, 2019.
- [12] G. Bonfanti, M. Frémond, and F. Luterotti. Global solution to a nonlinear system for irreversible phase changes. *Advances in Mathematical Sciences and Applications*, Gakkōtoshō, Tokyo, Vol.10(1),1–23, 2000.
- [13] H. Brezis. *Analyse fonctionnelle*. Collection Mathématiques Appliquées pour la Maîtrise. Masson, Paris, 1983. Théorie et applications.

- [14] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.
- [15] J. Hulshof and J. L. Vázquez. Self-similar solutions of the second kind for the modified porous medium equation. *European J. Appl. Math.*, 5(3):391–403, 1994.
- [16] N. Igbida. Solutions auto-similaires pour une équation de Barenblatt. *Rev. Mat. Apl.*, 17(1):21–36, 1996.
- [17] S. Kamin, L. A. Peletier, and J. L. Vázquez. On the Barenblatt equation of elastoplastic filtration. *Indiana Univ. Math. J.*, 40(4):1333–1362, 1991.
- [18] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod, 1969.
- [19] A. Orefice and G. Mancusi, and S. Dumond, and F. Lebon. *An Experimental/Numerical Study on the Interfacial Damage of Bonded Joints for Fibre-Reinforced Polymer Profiles at Service Conditions*. Technologies, 4(20) 18p., 2016.
- [20] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.
- [21] M. Ptashnyk. Degenerate quasilinear pseudoparabolic equations with memory terms and variational inequalities. *Nonlinear Anal.*, 66(12):2653–2675, 2007.
- [22] J. Simon. Una generalización del teorema de Lions-Tartar. *Bol. Soc. Esp. Mat. Apl.*, 40:43–69, 2007.
- [23] R. Temam. Navier-Stokes Equations Theory and numerical analysis. *North-Holland Publishing company, Amsterdam, New York, Oxford*, Vol. 2, 1979.
- [24] G. Vallet. Sur une loi de conservation issue de la géologie. *C. R. Math. Acad. Sci. Paris*, 337(8):559–564, 2003.
- [25] S. Zhu and C. Cai. Interface damage and its effect on vibrations of slab track under temperature and vehicle dynamic loads. *International Journal of Non-Linear Mechanics*, 58:222–232, 2014.

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