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ON THE GEOMETRY OF K3 SURFACES WITH FINITE AUTOMORPHISM GROUP AND NO ELLIPTIC FIBRATIONS

XAVIER ROULLEAU

ABSTRACT. Nikulin [24], [25] and Vinberg [35] proved that there are only a finite number of lattices of rank ≥ 3 that are the Néron-Severi lattice of projective K3 surfaces with a finite automorphism group. The aim of this paper is to provide a more geometric description of such K3 surfaces X , when these surfaces have moreover no elliptic fibrations. In that case we show that such K3 surface is either a quartic with special hyperplane sections or a double cover of the plane branched over a smooth sextic curve which has special tangencies properties with some lines, conics or cuspidal cubic curves. We then study the converse i.e. if the geometric description we obtained characterizes these surfaces. In 4 cases the description is sufficient, in each of the 4 other cases there is exactly another one possibility which we study. We obtain that at least 5 moduli spaces of K3 surfaces (among the 8 we study) are unirational.

1. INTRODUCTION

The classification of projective complex K3 surfaces X that have a finite automorphism group has been established by Pyatetskii-Sapiro and Shafarevich [28] for Picard ranks $\rho = \text{rk}(\text{NS}(X))$ 1 and 2, by Vinberg [35] for rank 4 and Nikulin [24], [25] for the other ranks. This classification is done according to their Néron-Severi lattice $\text{NS}(X)$, which, according to the Torelli Theorem (see [28]) for K3 surfaces, strongly characterizes the surfaces. However in general for rank $\rho > 1$, we do not have a geometric construction of these surfaces. By geometric construction, we mean surfaces lying in a projective space or understood as cover of a known surface or, if there exist a fibration with a section, an explicit Weierstrass model.

By results of Nikulin, a K3 surface X with $\rho > 2$ has a finite number of automorphism if and only if the number of smooth rational curves (called (-2) -curves) is finite and non-zero. According to [3] and independently [20], these conditions are equivalent to requiring the surface X to be a Mori dream space i.e. that its Cox ring is finitely generated.

In this paper, we construct geometrically the K3 surfaces X which have a finite non-zero number of (-2) -curves and no elliptic fibrations (the case with

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fibrations is treated in [30]). From the results of Nikulin [25] and Vinberg [35], their Picard number ρ is 3 or 4 and there are 8 cases classified according to their Néron-Severi lattices, which are denoted S_1, \dots, S_6 for rank $\rho = 3$ and $L(24), L(27)$ for rank $\rho = 4$.

A conjecture which remained open for a quite long time is about the presence of infinitely many rational curves on a K3 surface. We observe that some of the last cases that remained open (and for which now the conjecture was proved, see e.g. [10] for a history and references) are the cases of K3 surfaces with Picard number 4, no elliptic fibrations and a finite automorphisms group. These K3 surfaces are exactly the surfaces we consider in Sections 4.1 and 4.2, and probably the difficulties in establishing the conjecture in these cases is related to their peculiar geometry. Also from the point of view of this paper, these are special K3 surfaces since both of them have a twin (see below for the definition) with different properties with respect to the presence of (-2) -curves.

In [17], Kondo studies the automorphism group of generic K3 surfaces with a finite number of (-2) -curves; for Picard rank ≤ 8 this group is either trivial or $\mathbb{Z}/2\mathbb{Z}$. We obtain:

Theorem 1. *Let X be a K3 surface of Picard number > 2 with finite automorphism group and no elliptic fibrations.*

- *The automorphism group of X is trivial if and only if X can be embedded as a smooth quartic in \mathbb{P}^4 . Then that embedding is unique and the quartic X is one of the following two types:*
 - i) Type S_2 ($\rho = 3$). There are three quadrics in \mathbb{P}^3 such that each intersects $X \hookrightarrow \mathbb{P}^3$ in the union of two smooth rational quartic curves. These 6 rational curves are the only (-2) -curves on X .*
 - ii) Type S_3 ($\rho = 3$). There exist two hyperplane sections such that each hyperplane section is a union of two smooth conics. These 4 conics are the only (-2) -curves on X .*
- *The group $\text{Aut}(X)$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if and only if X is a double cover of the plane branched over a smooth sextic curve C_6 . In that case the possibilities are:*
 - 1) Type S_1 ($\rho = 3$). The 6 (-2) -curves on X are pull-back of 3 conics that are 6-tangent to the sextic C_6 .*
 - 2) Type S_4 ($\rho = 3$). The 4 (-2) -curves on X are pull-back of a tritangent line and a 6-tangent conic to C_6 .*
 - 3) Type S_5 ($\rho = 3$). The 4 (-2) -curves on X are pull-back of two tritangent lines to C_6 .*
 - 4) Type S_6 ($\rho = 3$). The 6 (-2) -curves on X are pull-back of one 6-tangent conic and two cuspidal cubics that cut C_6 tangentially and at their cusps.*
 - 5) Type $L(24)$ ($\rho = 4$). The 6 (-2) -curves on X are pull-back of three tritangent lines to C_6 .*
 - 6) Type $L(27)$ ($\rho = 4$). The 8 (-2) -curves on X are pull-back of one tritangent line and three 6-tangent conics to C_6 .*

Here we say that a line (respectively a conic) is tritangent (respectively 6-tangent) to a sextic if the intersection multiplicity is even at every intersections points of the line (resp. the conic) and the sextic.

We implemented some algorithm in order to obtain classes of big and nef divisors on K3 surfaces. By exploiting the fact that the fundamental domain for the Weyl group is bounded, we can obtain all big and nef (or even ample) classes of square less than or equal to a given bound. In particular we can compute the first terms of the generating series

$$\Xi_X(T) = \sum_{D \in \text{NS}(X) \text{ big, nef}} T^{D^2}.$$

For example let X be a K3 surface of type S_2 in Theorem 1. Then

Theorem 2. *The generating series Ξ_X begins with*

$$\begin{aligned} & T^4 + 6T^{10} + 6T^{12} + T^{16} + 6T^{18} + 12T^{22} + 6T^{28} + 18T^{30} + 6T^{34} + 7T^{36} \\ & + 6T^{40} + 12T^{46} + 6T^{48} + 18T^{58} + T^{64} + 12T^{66} + 12T^{70} + 6T^{72} + 6T^{76} \\ & + 18T^{82} + 12T^{88} + 18T^{90} + 12T^{94} + 7T^{100} + 24T^{102} + 18T^{106} + 6T^{108} \dots \end{aligned}$$

Theorem 1 gives possible realizations of K3 surfaces X with a finite number of automorphisms and no elliptic fibrations as double cover of the plane branched over a particular smooth sextic curve, or as particular quartic surfaces in \mathbb{P}^3 . In the last section of this paper, we study the converse, i.e. if the geometric description of the K3 surfaces given in Theorem 1 are enough to characterize them as surfaces in the list $S_1, \dots, L(27)$. It turns out that in four cases among the eight, there are two possibilities. Let us give an example:

Proposition 3. *Let Y be the K3 surface which is the double cover of the plane $\pi : Y \rightarrow \mathbb{P}^2$ branched over a generic sextic curve that possess 3 tritangent lines. The pull-back of each 3 lines splits as the union of two (-2) -curves. Up to permutation, the intersection matrix of the six (-2) -curves over the lines is one of the following matrices*

$$\begin{pmatrix} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 0 & 1 \\ 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 0 & 3 & -2 \end{pmatrix}, \quad \begin{pmatrix} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 1 & 0 & 3 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 & 3 \\ 1 & 0 & 0 & 1 & 3 & -2 \end{pmatrix}.$$

In the first case, the surface Y has type $L(24)$. In the second case, Y contains also only six (-2) -curves, but there exists a fibration. Both cases exist.

Let X be a K3 surface of type in the list $S_1, \dots, L(27)$. Let us say that a K3 surface X' is a twin if it has the same geometric description as in Theorem 1, the same Picard number, but a different Néron-Severi lattice. We obtain:

Theorem 4. *The K3 surfaces of types $S_3, S_4, L(24), L(27)$ have twins and the Néron-Severi lattices of each twin is unique. Surfaces of types S_1, S_2, S_5, S_6 are characterized by their description in Theorem 1.*

The surfaces that are twin with surfaces of type $S_3, S_4, L(24), L(27)$ have some elliptic fibrations, three of these also have a finite automorphism group; these surfaces are discussed with more details in [30].

In the last section, we also give some explicit examples of K3 surfaces of various type, using the method of van Luijk [19] refined by Elsenhans and Jahnel [12] in order to check their Picard number. We moreover obtain that

Theorem 5. *The moduli spaces of K3 surfaces of type S_1, S_2, S_3, S_5 and S_6 are unirational.*

This is done by constructing the families of these examples. In the Appendix, we give various datas of the K3 surfaces we are considering, (such that their Néron-Severi lattices). The reader can find the Magma programs used for the computations into the ancillary file in the arXiv version of this paper.

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2. PRELIMINARIES

2.1. On the automorphisms of K3 surfaces with low Picard number.

Let X be a K3 surface with finite automorphism group and Picard number ≤ 8 . The following result is [17, Main Theorem, Lemma 2.3]:

Proposition 6. *The group of symplectic automorphisms is trivial. If moreover X is general then the group $\text{Aut}(X)$ is the trivial group or it is generated by a non symplectic involution.*

In case of Picard rank 3, one can remove the hypothesis that X is general. Indeed a non-symplectic automorphism should have order m such that $\varphi(m)$ divides $22 - 3 = 19$ (where φ is the totient Euler function), and that implies $m = 2$.

2.2. Linear series on K3 surfaces. To be self-contained, let us recall the following results of Saint Donat [33]:

Theorem 7. *Let X be a K3 surface that does not possess an elliptic pencil and let D be a big and nef divisor on X .*

- a) *One has $h^0(D) = 2 + \frac{1}{2}D^2$ and the linear system $|D|$ is base point free,*
- b) *the morphism $\varphi_{|D|}$ associated to $|D|$ is either 2 to 1 onto its image (hyperelliptic case) or it maps X birationally onto its image: in both the cases it contracts the (-2) -curves Γ such that $D\Gamma = 0$.*
- c) *if $|D|$ is hyperelliptic, then $\varphi_{|D|}$ is a double cover of \mathbb{P}^2 or of the Veronese surface in \mathbb{P}^5 .*
- d) *the linear system $|D|$ is hyperelliptic with $D^2 \geq 4$ if and only if there is a nef divisor B with $B^2 = 2$ and $D = 2B$.*
- e) *if $|D|$ is hyperelliptic and $D^2 = 2$, then $\varphi_{|D|}$ is a double cover of \mathbb{P}^2 .*

A K3 surface with no elliptic pencil cannot have nef classes with self-intersection 0.

2.3. Shimada algorithm to compute classes with fixed square and degree. Let X be a smooth surface. Let d, k be positive integers and let H be an ample class. The following is an algorithm for computing all classes D with $D^2 = d$ and degree $HD = k > 0$. First one projects onto the orthogonal H^\perp of H by the linear map

$$\pi : D \in \text{NS}(X) \rightarrow (H^2)D - (DH)H \in H^\perp.$$

The class D has square $D^2 = d$ and degree $k = HD$ if and only if

$$\pi(D)^2 = (H^2)^2 D^2 - (H^2)(DH)^2 = (H^2)^2 d - k^2 H^2 \text{ and } DH > 0.$$

Since H is ample, H^\perp is a negative definite lattice, thus when $d > 0$ and $k > 0$ are fixed, there are only a finite number of solutions $s \in H^\perp$ with $s^2 = (H^2)^2 d - k^2 H^2$, which can be found by a computer calculation. Then we take the reverse path to the Néron-Severi lattice by the application $s \rightarrow \frac{1}{H^2}(s + kH)$, discarding solutions D that are not in the integral lattice $\text{NS}(X)$ or such that $DH < 0$.

When X is a K3 surface with a finite number of (-2) -curves it is then easy to check if a divisor class D in $\text{NS}(X)$ is nef or even ample. If moreover the fundamental domain of the Weyl group is bounded, we explain in the next section how we can compute all classes of a given square, without restriction on their degree.

In fact one can understand the above algorithm as a easy proof that on a surface, the Hilbert Scheme of curves of given degree and square with respect to a polarization has only a finite number of irreducible components. The idea of it came from an algorithm of Lairez and Sertoz in [18] (following an idea of Deygtarev) for computing (-2) -curves, but soon after completion of this paper appeared [11] where it turns out that it was already described and used by Shimada in [31]. We used Magma software [4] for the computations.

We use the above algorithm to compute the (-2) -classes, and Vinberg's sieve (see [36]) to sort the (-2) -curves among the (-2) -classes.

2.4. Néron-Severi lattice and hyperbolic geometry. Let X be a surface of Picard number $\rho \geq 2$. Let $\mathbb{P}^{\rho-1}$ be the projectivisation of the real space $\text{NS}(X) \otimes \mathbb{R}$. The image \mathcal{L}_X in $\mathbb{P}^{\rho-1}$ of the positive cone $\{c \in \text{NS}(X)_\mathbb{R} \mid c^2 > 0\}$ is a ball of dimension $\rho - 1$. We will often not make a notational distinction between classes in $\text{NS}(X) \otimes \mathbb{R}$ and in $\mathbb{P}^{\rho-1}$, also we will often confuse a primitive effective divisor with its image in \mathcal{L}_X . For $D, D' \in \mathcal{L}_X$ let us define

$$\ell_X(D, D') = \frac{(DD')^2}{D^2 D'^2}.$$

By the Hodge Index Theorem $\ell_X(D, D') \geq 1$ with equality if and only if D and D' are proportional (observe that the function ℓ is homogeneous). In fact, the following function:

$$d_X(D, D') = \text{Arccosh} \left(\sqrt{\ell_X(D, D')} \right), \quad D, D' \in \mathcal{L}_X$$

is a distance on \mathcal{L}_X for which the geodesic between two points in \mathcal{L}_X is the intersection of \mathcal{L}_X with the line in $\mathbb{P}^{\rho-1}$ through these points (see [29, Chapter 3, Section 3.2]).

Let us suppose that the fundamental domain \mathcal{F}_X of the Weyl group is compact. Let us fix an ample element H in \mathcal{F}_X , let $d_{max}(\mathcal{F}_X, H)$ be the maximum of the distances between H and the points in \mathcal{F}_X and let us define $\ell(\mathcal{F}_X, H) = \cosh(d_{max}(\mathcal{F}_X, H))$. Then for any big and nef classes D in $\text{NS}(X)$, one has

$$(HD)^2 \leq \ell(\mathcal{F}_X, H)H^2D^2.$$

The consequence is that for the surfaces with compact \mathcal{F}_X , we can apply the algorithm in Section 2.3 to find all nef and big classes D in $\text{NS}(X)$ with fixed square $D^2 = d$: it is enough to search classes D with $D^2 = d$ up to degree $\sqrt{\ell(\mathcal{F}_X, H)dH^2}$ with respect to the ample class H . Then we can compute the generating series

$$\Theta_X(T) = \sum_{D \text{ big, nef, primitive}} T^{D^2}$$

of the primitive nef and big classes in $\text{NS}(X)$, up to some square.

The distances $\ell(\mathcal{F}_X, H)$ are computed by using the extremal rays r_1, \dots, r_k of the nef cone (the hyperplanes r_j^\perp are the facets of the effective cone). For example, in case of rank 3, the rays of the nef cone are elements that are orthogonal to two (-2) -curves A_i, A_j with $A_i A_j \in \{0, 1\}$.

Any even lattice of rank ≥ 5 represents 0 i.e. there is a non-trivial class c with $c^2 = 0$, thus the fundamental domain of a K3 surface with Picard number ≥ 5 cannot be bounded. Moreover if the fundamental domain of a K3 surface X is bounded, then X contains only a finite number of (-2) -curves. Indeed if there are an infinite number of (-2) -curves then the automorphism group of X is infinite by the Theorem of Sterk [34]. Therefore there exist big and nef classes which have an infinite orbit with the same square; but by the above discussion such a phenomenon is not possible on a bounded domain.

Finally, let us mention [32], where algorithms to study pull-back of divisors by double covers are developed and can be used in this paper.

3. RANK 3 AND COMPACT CASES

In this Section we study K3 surfaces with Picard rank 3 such that the fundamental polygon for the Weyl group is compact, in particular these surfaces have no elliptic fibrations. According to [25], there are 6 types of lattices S_1, \dots, S_6 for the Néron-Severi lattice of such surfaces; Nikulin described their associated fundamental domain \mathcal{F}_X .

3.1. Surfaces with Néron-Severi lattice of type S_1 . Let us study K3 surfaces X with a finite number of (-2) -curves and such that their Néron-Severi lattice is generated by L, A_1, A_2 , with L nef and A_1, A_2 two (-2) -curves, with intersection matrix

$$\begin{pmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

The surface X contains exactly 6 (-2) -curves A_1, \dots, A_6 . In the base $A_1, A_2, A_3 = L - 2A_1$, the classes of the (-2) -curves A_4, A_5, A_6 are

$$A_4 = A_1 - 2A_2 + 2A_3, \quad A_5 = 2A_1 - 3A_2 + 2A_3, \quad A_6 = 2A_1 - 2A_2 + A_3.$$

The intersection matrix of the 6 curves is

$$\begin{pmatrix} -2 & 0 & 4 & 6 & 4 & 0 \\ 0 & -2 & 0 & 4 & 6 & 4 \\ 4 & 0 & -2 & 0 & 4 & 6 \\ 6 & 4 & 0 & -2 & 0 & 4 \\ 4 & 6 & 4 & 0 & -2 & 0 \\ 0 & 4 & 6 & 4 & 0 & -2 \end{pmatrix}.$$

Using Section 2.4, we find that there is a unique big and nef class such that

$$D^2 = 2,$$

which is

$$D = A_1 - A_2 + A_3 = L - A_1 - A_2.$$

The degree of the curves A_i with respect to D is $DA_i = 2$ (thus D is ample). Since X has no elliptic fibration, the linear system $|D|$ is free and $\varphi_{|D|}$ is a double cover of the plane. We remark that

$$(3.1) \quad \begin{aligned} A_1 + A_4 &= 2D \\ A_2 + A_5 &= 2D \\ A_3 + A_6 &= 2D, \end{aligned}$$

so that $A_1 + A_4$, $A_2 + A_5$ and $A_3 + A_6$ are pull-back to X of conics C_1, C_2, C_3 in the plane. The branch curve B is a smooth sextic and, from equations (3.1), each conic C_j is tangent to B at 6 points.

Proposition 8. *The surface X is a double cover of \mathbb{P}^2 branched over a smooth sextic curve B . There exists 3 conics that are 6-tangent to B . The six (-2) -curves on X are the pull-back of these 3 conics.*

The automorphism group of X is $\mathbb{Z}/2\mathbb{Z}$.

The automorphism group of X is generated by the involution ι from the double-plane structure. From Equation 3.1, the action of ι^* on $\text{NS}(X)$ is by $\iota^*A_j = A_{j+3 \bmod 6}$. The Néron-Severi lattice contains exactly 6 big and nef classes D_i with $D_i^2 = 6$. Each of these classes contracts two disjoint (-2) -curves A_i, A_{i+1} (the index being taken mod 6) and they are the extremal points of the polyhedral fundamental domain \mathcal{F}_{S_1} (see [25, Figure 1]). These classes D_1, \dots, D_6 are:

$$\begin{aligned} D_1 &= 2A_1 + A_3, & D_2 &= A_1 + 2A_3, & D_3 &= 2A_1 - 3A_2 + 4A_3, \\ D_4 &= 4A_1 - 6A_2 + 5A_3, & D_5 &= 5A_1 - 6A_2 + 4A_3, & D_6 &= 4A_1 - 3A_2 + 2A_3, \end{aligned}$$

and their degree with respect to D is $DD_i = 6$ (moreover $\iota^*D_j = D_{j+3 \bmod 6}$). Therefore the distance between D and these divisors D_i equals to $\text{Arccosh}(\sqrt{3})$, which is the maximal distance between D and any points in \mathcal{F}_{S_1} . Using that, we find that the generating series

$$\Theta_X(T) = \sum_{D \text{ big, nef, primitive}} T^{D^2}$$

of the primitive nef and big classes begins with

$$\begin{aligned} T^2 + 6(T^4 + T^6 + T^{14} + T^{20} + 2T^{22} + T^{28} + T^{34} + T^{38} + 2T^{44} + 3T^{23} + T^{50} \\ + 2T^{52} + T^{60} + T^{62} + T^{68} + 2T^{70} + 2T^{76} + 2T^{78} + T^{82} + 3T^{86} + 2T^{92}) \dots \end{aligned}$$

The 6 classes $A_1 + A_3, A_1 + A_5, A_2 + A_4, A_2 + A_6, A_3 + A_5, A_4 + A_6$ are nef divisors of square 4. Each of these 6 classes is orthogonal to a unique (-2) -curve. Such a class C realizes X as a 1-nodal quartic in \mathbb{P}^3 . Projecting from a node, one get a double-cover from X to \mathbb{P}^2 . Since the automorphism group of X is $\mathbb{Z}/2\mathbb{Z}$, that double cover must be the one in Proposition 8, that phenomenon is a bit surprising, but is not a contradiction.

Also, equality $D = L - A_1 - A_2$ means that the model of $\varphi_{|D|}$ is geometrically obtained as a projection of the model in \mathbb{P}^4 given by $\varphi_{|L|}$ obtained by projecting from the line connecting the two nodes $\varphi_{|D|}(A_1)$ and $\varphi_{|D|}(A_2)$.

3.2. Surfaces with Néron-Severi lattice of type S_2 . Let us study K3 surfaces with a finite number of (-2) -curves and such that the Néron-Severi lattice is generated by L, A_1, A_2 with L nef and A_1, A_2 two (-2) -curves with intersection matrix

$$\begin{pmatrix} 36 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

Then X contains exactly 6 (-2) -curves A_1, \dots, A_6 . The (-2) -curve $A_3 = L - 5A_1 - 3A_2$ is such that A_1, A_2, A_3 generate $\text{NS}(X)$ and the classes of the 3 other (-2) -curves are

$$A_4 = A_1 - 2A_2 + 2A_3, \quad A_5 = 2A_1 - 3A_2 + 2A_3, \quad A_6 = 2A_1 - 2A_2 + A_3.$$

The intersection matrix of the 6 curves A_j is

$$\begin{pmatrix} -2 & 1 & 7 & 10 & 7 & 1 \\ 1 & -2 & 1 & 7 & 10 & 7 \\ 7 & 1 & -2 & 1 & 7 & 10 \\ 10 & 7 & 1 & -2 & 1 & 7 \\ 7 & 10 & 7 & 1 & -2 & 1 \\ 1 & 7 & 10 & 7 & 1 & -2 \end{pmatrix}.$$

Using Section 2.4, we obtain that the class

$$D = L - 4A_1 - 4A_2 = A_1 - A_2 + A_3.$$

is the unique big and nef class with square

$$D^2 = 4.$$

There are no big and nef divisors D' with $D'^2 < 4$. For each (-2) -curve A_j on X , we have $A_j D = 4$, thus D is ample. Since there is no elliptic fibration, by Theorem 7 the class D is a very ample. We remark that

$$\begin{aligned} A_1 + A_4 &= 2D \\ A_2 + A_5 &= 2D \\ A_3 + A_6 &= 2D. \end{aligned}$$

Therefore

Proposition 9. *There exists a unique embedding of X in \mathbb{P}^3 . For that embedding, there exist 3 quadrics Q_1, Q_2, Q_3 and each of them cuts $X \subset \mathbb{P}^3$ into the union of two (-2) -curves, each of degree 4.*

Suppose that there is a non-symplectic involution acting on X . Then its fixed locus has 1 dimensional components (see [2]). Since there is only one degree 4 polarization, the involution must preserve it. The fixed locus of a non symplectic involution on a smooth quartic surface contains either a line or a smooth quartic curve. The first case is impossible since the (-2) -curves on X have degree 4. By [2, Table 1], in the second case the Picard number of X would have rank at least 8, it is therefore also impossible. Thus according to Propositions 6:

Proposition 10. *The automorphism group of X is trivial.*

The linear maps $g, h : \text{NS}(X) \rightarrow \text{NS}(X)$ such that $g(A_1) = A_2$, $g(A_2) = A_3$, $g(A_3) = A_4$ and $h(A_1) = A_1$, $h(A_2) = A_6$, $h(A_3) = A_5$ are isometries (of respective order 6 and 2) of the lattice $\text{NS}(X)$. They verify $hgh = g^{-1}$, thus they generate the dihedral group \mathbb{D}_6 of order 12, the symmetry group of the hexagon. That group preserves the set of classes A_i , the ample cone and the fundamental domain for the Weyl group W_X , therefore there is no non-trivial element of \mathbb{D}_6 in the Weyl group W_X . Since the automorphism group of X is trivial, no isometry in \mathbb{D}_6 comes from an automorphism of X , therefore:

Corollary 11. *The index of the Weyl group W_X in $O(\text{NS}(X))$ is 12.*

Since $LA_1 = LA_2 = 0$ and $L = 5A_1 + 3A_2 + A_3$ is big and nef, the class L with $(L^2 = 36)$ is an extremal point of the fundamental polygon \mathcal{F}_X , (this is also in [25, Table 2]). The coordinates with respect to the basis (A_1, A_2, A_3) of the extremal classes $L = L_1, L_2, \dots, L_6$ of \mathcal{F}_X are:

$$(5, 3, 1), (13, -9, 5), (17, -21, 13), (13, -21, 17), (5, -9, 13), (1, 3, 5).$$

One has $L_i A_{2-i} = L_i A_{3-i} = 0$, where the indices are modulo 6. Since $DL_i = 36$ for all $i \in \{1, \dots, 6\}$, the maximum distance from D in \mathcal{F}_X equals

$$d_{max} = \text{Arccosh} \left(\sqrt{\frac{(LD)^2}{L^2 D^2}} \right) = \text{Arccosh}(3).$$

Using that result, one finds that the generating series Θ_X of the primitive nef and big classes begins with

$$T^4 + 6(T^{10} + T^{12} + T^{18} + 2T^{22} + T^{28} + 3T^{30} + T^{34} + T^{36} + 2T^{46} + 3T^{58} + 2T^{66} + 2T^{70} + T^{76} + 3T^{82} + 2T^{90} + 2T^{94} + T^{100}) \dots$$

3.3. Surfaces with Néron-Severi lattice of type S_3 . Let us study the K3 surfaces X such that the Néron-Severi lattice is generated by L, A_1, A_2 with L nef and A_1, A_2 two (-2) -curves with intersection matrix

$$\begin{pmatrix} 12 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

There are exactly 4 (-2) -curves on X :

$$A_1, A_2, A_3 = L - 3A_1 - 2A_2, A_4 = L - 2A_1 - 3A_2,$$

their intersection matrix is

$$\begin{pmatrix} -2 & 1 & 4 & 1 \\ 1 & -2 & 1 & 4 \\ 4 & 1 & -2 & 1 \\ 1 & 4 & 1 & -2 \end{pmatrix}.$$

The divisor

$$D = L - 2A_1 - 2A_2$$

satisfies

$$D^2 = 4$$

and it is ample since moreover

$$\forall i \in \{1, \dots, 4\}, A_i D = 2 > 0.$$

There are no big and nef divisors D' with $D'^2 < 4$.

Proposition 12. *The linear system $|D|$ defines an embedding $\phi_D : X \hookrightarrow \mathbb{P}^3$. The images of the four (-2) -curves by ϕ_D are 4 conics. Since*

$$A_1 + A_3 = D, \quad A_2 + A_4 = D,$$

the pairs of conics (A_1, A_3) and (A_2, A_4) are on the same hyperplanes P_1, P_2 respectively.

Proof. There are no elliptic fibrations, so that by Theorem 7, the divisor D defines an embedding. The remaining affirmations are immediately checked. \square

By [2, Table 1], a quartic surface that is invariant by an involution fixing an isolated point and a plane has Picard number at least 8. Suppose that the quadric $F \hookrightarrow \mathbb{P}^3$ is invariant by a involution $\iota : x \rightarrow (-x_1 : -x_2 : x_3 : x_4)$. Since the Picard number of X is 3, the involution must be non-symplectic. But then at least one of the fixed lines $x_1 = x_2 = 0$ or $x_3 = x_4 = 0$ must be contained in F . However the rational curves on F are conics, thus that case is also impossible. Since D is the unique big and nef divisor with $D^4 = 4$, we conclude that:

Corollary 13. *The automorphism group of the K3 surface X is trivial.*

There are 4 big and nef divisors of self-intersection 6, which are not ample. These divisors are

$$L - 2A_1 - A_2, \quad L - A_1 - 2A_2, \quad 2L - 5A_1 - 4A_2, \quad 2L - 6A_1 - 5A_2.$$

Each of these contracts a different (-2) -curve. There are 4 big and nef divisors of self-intersection 12:

$$D_1 = L, \quad D_2 = 3L - 8A_1 - 4A_2, \quad D_3 = 5L - 12A_1 - 4A_2, \quad D_4 = 3L - 4A_1 - 8A_2,$$

each of them contracts two (-2) -curves and these are the vertices of the fundamental polygon \mathcal{F}_{S_3} . Since $DD_i = 12$, the maximum distance from D and points in the polygon is $d_{max} = \text{Arccosh}(\sqrt{3})$. The generating series $\Theta_X(T)$ of the primitive nef and big classes begins with

$$T^4 + 4(T^6 + T^{10} + T^{12} + T^{22} + T^{30} + 2T^{34} + 2T^{42} + 2T^{46} + T^{52} + T^{58} + 2T^{66} + 2T^{70} + T^{76} + T^{78} + 2T^{82} + 2T^{84} + 4T^{94} + T^{100}) \dots$$

3.4. Surfaces with Néron-Severi lattice of type S_4 . Let us study the K3 surfaces X with Picard number 3 and Néron-Severi lattice generated by D, A_1, A_2 with D nef and A_1, A_2 two (-2) -curves with Gram matrix

$$\begin{pmatrix} 2 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}.$$

Then the K3 surface contains exactly four (-2) -curves, with coordinates

$$A_1, A_2, A_3 = D - A_1, A_4 = D - A_2,$$

their intersection matrix is

$$\begin{pmatrix} -2 & 1 & 3 & 1 \\ 1 & -2 & 1 & 6 \\ 3 & 1 & -2 & 1 \\ 1 & 6 & 1 & -2 \end{pmatrix}.$$

The curves A_1, A_2, A_3 generate the Néron-Severi lattice and one has $A_4 = 2A_1 - A_2 + 2A_3$. The divisor $D = A_1 + A_3$ is ample with $D^2 = 2$. One has

$$DA_1 = DA_3 = 1, DA_2 = DA_4 = 2.$$

Since there is no elliptic pencil, the linear system $|D|$ is base point free and X is a double cover of \mathbb{P}^2 branched over a smooth sextic curve C_6 . The divisor $A_1 + A_3$ is the pull-back of a tritangent line to C_6 . Since moreover

$$A_2 + A_4 = 2D,$$

the divisor $A_2 + A_4$ is the pull-back of a conic tangent to C_6 at 6 points. Thus:

Proposition 14. *The surface X is a double cover of \mathbb{P}^2 branched over a smooth sextic curve which admits a tritangent line and a 6-tangent conic. The (-2) -curves on X are the pull-back of the tritangent line and the conic. The automorphism group of X is $\mathbb{Z}/2\mathbb{Z}$.*

Since $A_1 + A_3$ and $A_2 + A_4$ are pull-back of curves in \mathbb{P}^2 , the involution of the double cover maps A_1, A_2, A_3, A_4 to A_3, A_4, A_1, A_2 . Each of the following big and nef divisors

$$\begin{aligned} D_1 &= 7A_1 + 5A_2 + 3A_3, & D_2 &= 3A_2 + 5A_3 + 7A_4 \\ D_3 &= 13A_1 - 5A_2 + 17A_3, & D_4 &= 17A_1 - 5A_2 + 13A_3. \end{aligned}$$

contracts two (-2) -curves A_s, A_t such that $A_s A_t = 1$, thus they are the vertices of the fundamental polygon \mathcal{F}_{S_4} . We have $D_i^2 = 60$ and $DD_i = 20$. The maximum distance from D to a point in \mathcal{F}_{S_4} is

$$d_{max} = \text{Arccosh} \left(\frac{(DD_1)^2}{D^2 D_1^2} \right)^{\frac{1}{2}} = \text{Arccosh} \left(\sqrt{\frac{10}{3}} \right).$$

The generating series $\Theta_X(T)$ of the primitive nef and big classes begins with

$$\begin{aligned} &T^2 + 2(T^4 + 2T^6 + T^{10} + T^{14} + 2T^{18} + 3T^{22} + 2T^{26} + T^{28} + 3T^{34} + 2T^{36} \\ &+ T^{38} + 2T^{42} + 5T^{46} + 2T^{52} + 4T^{54} + 5T^{58} + 2T^{60} + 2T^{62} + 2T^{66} + T^{68} \\ &+ 4T^{74} + 4T^{78} + 4T^{82} + 2T^{84} + 4T^{86} + 2T^{90} + T^{92} + 5T^{94} + 3T^{98}) + \dots \end{aligned}$$

The two square 4 divisors are $A_1 + A_2 + A_3$, $A_1 + A_3 + A_4$, they are orthogonal to respectively A_2, A_4 , and are exchanged by the covering involution.

3.5. Surfaces with Néron-Severi lattice of type S_5 . Let us study K3 surfaces such that their Néron-Severi lattice is generated by a divisor nef L and by two (-2) -curves A_1, A_2 with intersection matrix

$$(3.2) \quad \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}.$$

It contains exactly four (-2) -curves A_1, A_2, A_3, A_4 with intersection matrix

$$M(S_5) = \begin{pmatrix} -2 & 1 & 3 & 0 \\ 1 & -2 & 0 & 3 \\ 3 & 0 & -2 & 1 \\ 0 & 3 & 1 & -2 \end{pmatrix}$$

and generating $\text{NS}(X)$. Their classes are

$$A_1, A_2, A_3 = L - 2A_1 - A_2, A_4 = L - A_1 - 2A_2 = A_1 - A_2 + A_3.$$

The divisor $D = L - A_1 - A_2$ satisfies

$$D = A_1 + A_3 = A_2 + A_4$$

and

$$D^2 = 2$$

and for each (-2) -curve A_i we have $DA_i = 1$, thus it is ample. Since X has no elliptic fibration, the net $|D|$ is free and the morphism ϕ_D associated to $|D|$ is a double cover of the plane. Let C_6 be the sextic branch curve in \mathbb{P}^2 . It is a smooth curve such that there are two lines L_1, L_2 tangent to it at 3 points each and the pull-back of L_1 is $\pi^*L_1 = A_1 + A_3$, the pull-back of L_2 is $\pi^*L_2 = A_2 + A_4$ (so that $A_1A_3 = 3 = A_2A_4$). We thus have:

Proposition 15. *The surface X is a double cover of the plane branched over a smooth sextic curve which has two tritangent lines. The automorphism group of such a surface is $\mathbb{Z}/2\mathbb{Z}$.*

The involution exchanges A_1 with A_3 and A_2 with A_4 . We construct the family of such sextic curves in Section 5. We have $L^2 = 4$ and since the surface contains no elliptic curves, the linear system $|L|$ is base-point free and not hyperelliptic. Since $LA_1 = LA_2 = 0$ with $A_1A_2 = 1$, the surface X is the minimal resolution of a quartic surface with one cusp; the images of the curves A_3 and A_4 have degree 4. Also the divisor $L' = 3L - 4(A_1 + A_2)$ is nef, and $L'^2 = 4$. It plays a similar role for A_3, A_4 as L for A_1, A_2 . We have $LD = L'D = 4$

The divisors $M = 4L - A_1 + 3A_2$ and $M' = 2L + A_1 + 3A_2$ are nef with $M^2 = M'^2 = 12$ and $DM = DM' = 6$. They satisfy $MA_1 = MA_4 = 0$ and $M'A_2 = M'A_3 = 0$.

The divisors L, L', M, M' form the 4 vertices of the fundamental polygon \mathcal{F}_{S_5} . The maximum distance from D to a points in \mathcal{F}_{S_5} is

$$d_{max} = \text{Arccosh}(\max(\frac{(DL)^2}{D^2L^2}, \frac{(DM)^2}{D^2M^2})^{1/2}) = \text{Arccosh}(\sqrt{2}).$$

The generating series $\Theta_X(T)$ of the primitive nef and big classes on a K3 surface X with Néron-Severi lattice isometric to S_5 begins with

$$\begin{aligned} & T^2 + 2(T^4 + 2T^{10} + T^{12} + T^{14} + 2T^{22} + T^{26} + T^{28} + 2T^{30} + T^{34} \\ & + 2T^{38} + T^{44} + 3T^{46} + 2T^{50} + 4T^{58} + 3T^{62} + T^{66} + T^{68} + 2T^{70} \\ & + 3T^{74} + 2T^{76} + 3T^{82} + 2T^{86} + 2T^{92} + 3T^{94} + 3T^{98} + 2T^{100}) + \dots \end{aligned}$$

The two nef divisors of square 4 are orthogonal to A_1, A_2 and A_3, A_4 respectively.

3.6. Surfaces with Néron-Severi lattice of type S_6 . Let us study the K3 surfaces X with Picard number 3 and generated by a nef divisor D and (-2) -curves A_1, A_3 with Gram matrix

$$\begin{pmatrix} 2 & 2 & 3 \\ 2 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}.$$

The K3 surface contains exactly six (-2) -curves A_1, \dots, A_6 with intersection matrix

$$\begin{pmatrix} -2 & 6 & 1 & 5 & 5 & 1 \\ 6 & -2 & 5 & 1 & 1 & 5 \\ 1 & 5 & -2 & 11 & 0 & 9 \\ 5 & 1 & 11 & -2 & 9 & 0 \\ 5 & 1 & 0 & 9 & -2 & 11 \\ 1 & 5 & 9 & 0 & 11 & -2 \end{pmatrix}.$$

The (-2) -curves A_1, A_3, A_5 form a basis of $\text{NS}(X)$ and

$$A_2 = A_1 - 2A_3 + 2A_5, \quad A_4 = 3A_1 - 4A_3 + 3A_5, \quad A_6 = 3A_1 - 3A_3 + 2A_5.$$

The divisor

$$D = A_1 - A_3 + A_5$$

is the unique big and nef divisor such that $D^2 = 2$. We have $DA_1 = DA_2 = 2$ and moreover $DA_j = 3$ for $j \in \{3, \dots, 6\}$, thus D is ample. We observe moreover that

$$(3.3) \quad \begin{aligned} A_1 + A_2 &= 2D \\ A_3 + A_4 &= 3D \\ A_5 + A_6 &= 3D. \end{aligned}$$

Since X has no elliptic fibration, the net $|D|$ is free and the morphism ϕ_D associated to $|D|$ is a double cover of the plane. We have:

Proposition 16. *The branch locus C_6 of ϕ_D is a smooth sextic curve, there exists a 6-tangent conic C_o to C_6 and there are two cuspidal cubics E_1, E_2 such that the pull-back to X of C_o, E_1, E_2 are respectively $A_1 + A_2, A_3 + A_4, A_5 + A_6$.*

The tangent to the cusp p_i of E_i is distinct from the tangent to the sextic curve at p_i . The cubic E_i cuts C_6 tangentially at 8 other points than p_i .

The automorphism group of X is $\mathbb{Z}/2\mathbb{Z}$.

From Equation (3.3), the action of the involution on the Néron-Severi lattice is by exchanging A_1, A_3, A_5 with respectively A_2, A_4, A_6 .

Proof. It remains to prove the assertions on singularities of the cubics. The cubic curve cannot be nodal, otherwise there would be some ramification in that node and the pull-back would be irreducible. If the tangent to the cusp was the tangent to the sextic C_6 , then that would also create some ramification on the curve E_i .

By performing 4 successive blow-ups at p_i , taking the double cover and going to the minimal model X , we obtain that the local intersection number above p_i of the two (-2) -curves above E_i is equal to 3. Since there must be no ramification over E_i , the cubic E_i cuts C tangentially at 8 other points than p_i . \square

In base A_1, A_3, A_5 the following 4 divisors

$$D_1 = (34, -49, 41), D_2 = (10, 5, 3), D_3 = (34, -27, 19), D_4 = (10, -17, 25)$$

are nef and have square $D_i^2 = 132$. Moreover $D_1A_2 = D_1A_4 = 0$, $D_2A_1 = D_2A_3 = 0$, $D_3A_1 = D_3A_6 = 0$, $D_4A_2 = D_4A_5 = 0$. These divisors are 4 vertices of the fundamental domain, which is an hexagon. The divisors

$$D_5 = (2, 1, 5), D_6 = (20, -23, 17)$$

are nef and $D_5^2 = D_6^2 = 44$, $DD_5 = DD_6 = 22$. Moreover $D_5A_4 = D_5A_6 = 0 = D_6A_3 = D_6A_5$, so that D_5 and D_6 are the remaining vertices of the hexagon. The maximum distance between D and points in \mathcal{F}_{S_6} is

$$\text{Arccosh}(\max(\frac{44^2}{2 \cdot 132}, \frac{22^2}{2 \cdot 44})^{1/2}) = \text{Arccosh}\left(\sqrt{\frac{22}{3}}\right).$$

The generating series $\Theta_X(T)$ of the primitive nef and big classes begins with

$$\begin{aligned} & T^2 + 2(T^4 + 2T^6 + 3T^{10} + 2T^{12} + 3T^{14} + 2T^{18} + 4T^{26} + 2T^{28} + 2T^{30} + 7T^{34} \\ & \quad + 2T^{36} + 2T^{38} + 4T^{42} + T^{44} + 7T^{46} + 3T^{50} + 6T^{54} + 6T^{58} + 2T^{60} + 3T^{62} \\ & \quad + T^{68} + 8T^{70} + 3T^{74} + 5T^{76} + 6T^{78} + 7T^{82} + 4T^{86} + 6T^{90} + T^{92} + 11T^{94}) + \dots \end{aligned}$$

The two nef divisors of square 4 are orthogonal to A_1 and A_2 respectively.

4. RANK 4 AND COMPACT CASES

According to Vinberg's classification [35, Theorem 1], there are only two lattices of rank 4 with compact fundamental domain.

4.1. Lattice of rank 4 of type $L(24)$. Let us consider a $K3$ surface whose Néron-Severi lattice is the lattice denoted $L(24)$ in [35]. That lattice is generated by a divisor L and (-2) -curves A_1, A_2, A_3 with intersection matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}.$$

The surface X contains only three other (-2) -curves

$$A_4 = L - A_1, A_5 = L - A_2, A_6 = L - A_3,$$

and A_1, \dots, A_4 is a basis of $\text{NS}(X)$. The intersection matrix of the 6 curves is

$$\begin{pmatrix} -2 & 0 & 0 & 3 & 1 & 1 \\ 0 & -2 & 0 & 1 & 3 & 1 \\ 0 & 0 & -2 & 1 & 1 & 3 \\ 3 & 1 & 1 & -2 & 0 & 0 \\ 1 & 3 & 1 & 0 & -2 & 0 \\ 1 & 1 & 3 & 0 & 0 & -2 \end{pmatrix}.$$

The divisor L is ample. Since $L^2 = 2$ and there are no elliptic curves on X , the linear system $|L|$ defines a double cover $\phi_L : X \rightarrow \mathbb{P}^2$ of the plane \mathbb{P}^2 . One checks that $LA_i = 1$, and in fact

$$(4.1) \quad L = A_1 + A_4 = A_2 + A_5 = A_3 + A_6,$$

so that:

Proposition 17. *The surface X is a double cover of \mathbb{P}^2 . The sextic branch curve is smooth, has exactly 3 tritangent lines L_1, L_2, L_3 such that their pull-back to X are the six (-2) -curves on X . The automorphism group of X is $\mathbb{Z}/2\mathbb{Z}$.*

From Equation (4.1), the action of the involution on the Néron-Severi lattice is by exchanging A_1, A_2, A_3 with respectively A_4, A_5, A_6 .

There are 20 triple of (-2) -curves; the orthogonal complement of a triple A_i, A_j, A_k is generated by a big and nef divisor $D_{i,j,k}$ for only 8 such triples, which are

$$\begin{aligned} D_{123} &= 2L + A_1 + A_2 + A_3 & D_{156} &= 8L + 4A_1 - 3A_2 - 3A_3 \\ D_{456} &= 5L - A_1 - A_2 - A_3 & D_{234} &= 6L - 4A_1 + 3A_2 + 3A_3 \\ D_{126} &= 6L + 3A_2 + 3A_3 - 4A_4 & D_{246} &= 8L - 3A_1 + 4A_2 - 3A_3 \\ D_{135} &= 6L + 3A_2 - 4A_3 + 3A_4 & D_{345} &= 8L - 3A_1 - 3A_2 + 4A_3 \end{aligned}$$

These classes are the vertices of the fundamental polygon \mathcal{F}_X . The divisors D_{123} and D_{345} have self-intersection 14 and degree 7 with respect to D . The 6 other have self-intersection 28 and degree 14. The maximum distance between D and points in \mathcal{F}_X is

$$\ell = \text{Arccosh}\left(\max\left(\frac{7^2}{2 \cdot 14}, \frac{14^2}{2 \cdot 28}\right)^{1/2}\right) = \text{Arccosh}\left(\sqrt{\frac{7}{2}}\right).$$

The generating series $\Theta_X(T)$ of the primitive nef and big classes begins with

$$\begin{aligned} &T^2 + 2(3T^4 + 3T^6 + 3T^{10} + 3T^{12} + 4T^{14} + 3T^{18} + 3T^{20} + 9T^{22} + 6T^{26} + 6T^{28} \\ &+ 7T^{30} + 6T^{34} + 6T^{36} + 9T^{38} + 6T^{42} + 9T^{44} + 18T^{46} + 10T^{50} + 9T^{52} + 12T^{54} \\ &+ 18T^{58} + 9T^{60} + 15T^{62} + 9T^{66} + 9T^{68} + 12T^{70} + 22T^{74} + 15T^{76} + 15T^{78} + 15T^{82}) \dots \end{aligned}$$

The 6 nef divisors of square 4 are respectively orthogonal to the 6 pairs of curves A_i, A_j such that $A_i A_j = 1$.

4.2. Lattice of rank 4 of type $L(27)$. Consider the rank 4 lattice denoted $L(27)$ in [35], with intersection matrix

$$\begin{pmatrix} 12 & 2 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

This is a lattice of some K3 surface X that possess only 8 (-2) -curves A_1, \dots, A_8 . The intersection matrix of these curves is

$$\begin{pmatrix} -2 & 3 & 1 & 1 & 1 & 1 & 1 & 1 \\ 3 & -2 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -2 & 6 & 4 & 0 & 4 & 0 \\ 1 & 1 & 6 & -2 & 0 & 4 & 0 & 4 \\ 1 & 1 & 4 & 0 & -2 & 6 & 4 & 0 \\ 1 & 1 & 0 & 4 & 6 & -2 & 0 & 4 \\ 1 & 1 & 4 & 0 & 4 & 0 & -2 & 6 \\ 1 & 1 & 0 & 4 & 0 & 4 & 6 & -2 \end{pmatrix},$$

and the curves A_2, A_4, A_5 and A_7 are generators of Néron-Severi lattice $L(27)$. The surface X contains a unique big and nef divisor D with $D^2 = 2$, which is $D = A_1 + A_2$. We have

$$DA_1 = DA_2 = 1, DA_j = 2 \text{ for } j \geq 2,$$

thus D is ample. One can check that

$$(4.2) \quad A_1 + A_2 = D, \quad A_3 + A_4 = A_5 + A_6 = A_7 + A_8 = 2D.$$

The linear system $|D|$ is base point free and

Proposition 18. *The surface X is a double cover of \mathbb{P}^2 . The sextic branch curve is smooth, has exactly one tritangent line L_0 and three 6-tangent conics C_1, C_2, C_3 such that the pull-back to X of L_0, C_1, C_2, C_3 are the eight (-2) -curves on X . The automorphism group of X is $\mathbb{Z}/2\mathbb{Z}$.*

From Equation (4.2), the action of the involution on the Néron-Severi lattice is by exchanging A_1, A_3, A_5, A_7 with respectively A_2, A_4, A_6, A_8 . The fundamental domain has 12 vertices D_1, \dots, D_{12} , whose coordinates in base A_2, A_4, A_5, A_7 are:

$$\begin{array}{lll} (6, 3, 7, 2) & (6, -27, 22, 17) & (-6, -18, 23, 13) \\ (6, 3, 2, 7) & (6, -27, 17, 22) & (-6, -18, 13, 23) \\ (6, -12, 17, 7) & (-6, -3, 13, 8) & (-6, -33, 28, 23) \\ (6, -12, 7, 17) & (-6, -3, 8, 13) & (-6, -33, 23, 28). \end{array}$$

We have $D_i^2 = 60$ and $D_i D = 30$, thus the maximum distance between D and points in \mathcal{F}_X is $\text{Arccosh}(\sqrt{\frac{15}{2}})$. The generating series for the big and nef divisors begins with

$$\begin{aligned} T^2 &+ 2(3T^4 + 9T^6 + 7T^{10} + 12T^{12} + 9T^{14} + 6T^{18} + 3T^{20} + 13T^{22} + 12T^{26} + 9T^{28} \\ &+ 12T^{30} + 27T^{34} + 18T^{36} + 16T^{38} + 24T^{42} + 12T^{44} + 39T^{46} + 15T^{50} + 18T^{52} \\ &+ 36T^{54} + 31T^{58} + 27T^{60} + 22T^{62} + 30T^{66} + 15T^{68} + 36T^{70} + 30T^{74} + 36T^{76}) \dots \end{aligned}$$

The 6 nef divisors of square 4 are respectively orthogonal to the 6 curves A_3, \dots, A_8 .

5. CONSTRUCTIONS OF SURFACES

5.1. Construction of surfaces of type S_1 . Our previous results tell us that a surface of type S_1 is the double cover of \mathbb{P}^2 branched over a smooth sextic which has three 6-tangent conics. Conversely:

Proposition 19. *Let $\pi : Y \rightarrow \mathbb{P}^2$ be the double cover of \mathbb{P}^2 branched over a smooth sextic C_6 which has three 6-tangent smooth conics C_1, C_2, C_3 . Suppose that Y has Picard number 3. Then the Néron-Severi lattice of Y is isometric to S_1 and Y contains only 6 (-2) -curves.*

Proof. The pull-back of the conics splits as $\pi^*C_k = B_{2k-1} + B_{2k}$ (for $k \in \{1, 2, 3\}$) with $B_1B_2 = B_3B_4 = B_5B_6 = 6$. The involution coming from the double plane structure exchanges the pairs of curves (B_1, B_2) , (B_3, B_4) and (B_5, B_6) , thus

$$B_1(B_3 + B_4) = B_2(B_3 + B_4) = \frac{1}{2}\pi^*C_1\pi^*C_2 = 4,$$

and also $B_3(B_1 + B_2) = B_4(B_1 + B_2) = 4$. Let us define $B_1B_3 = a$ (with $a \in \{0, \dots, 4\}$). Then $B_1B_4 = 4 - a$, $B_2B_3 = 4 - a$, $B_2B_4 = a$. Continuing in the same way for the other intersections, one sees that there exist $a, b, c \in \{0, 1, \dots, 4\}$ such that

$$(B_iB_j)_{1 \leq i, j \leq 6} = \begin{pmatrix} -2 & 6 & a & \bar{a} & b & \bar{b} \\ 6 & -2 & \bar{a} & a & \bar{b} & b \\ a & \bar{a} & -2 & 6 & c & \bar{c} \\ \bar{a} & a & 6 & -2 & \bar{c} & c \\ b & \bar{b} & c & \bar{c} & -2 & 6 \\ \bar{b} & b & \bar{c} & c & 6 & -2 \end{pmatrix},$$

where $\bar{x} = 4 - x$. Up to exchanging B_3 with B_4 and B_5 with B_6 , we can suppose $a, b \in \{0, 1, 2\}$. By computing all minors of size 3, we obtain that the above intersection matrix has rank at least 3 (and at most 4), moreover it has rank 3 if and only if $a = b = 0$ and $c = 4$.

If this is the case, then the Néron-Severi lattice of the K3 surface X contains a sub-lattice isometric to S_1 , generated by $\frac{1}{2}(B_1 + B_2), B_1, B_3$. Since the discriminant group of S_1 does not contain isotropic elements, we conclude that $\text{NS}(X)$ is isometric to S_1 . \square

An example of a K3 surface with three 6-tangent conics has been constructed by Elsenhans and Jähnel as an auxiliary surface to obtain double plane K3 surfaces with Picard number 1 (see [12, 3.4.1]). Their idea is to consider a sextic curve C_6 with equation f_6 of the form

$$f_6 = q_1q_2q_3 - f_3^2,$$

where the q_j are quadrics and f_3 is a cubic. If the forms q_1, q_2, q_3 and f_3 are generic, then the curve $f_6 = 0$ is smooth and the intersection points of the cubic curve $F_3 = \{f_3 = 0\}$ and the quadric $Q_j = \{q_j = 0\}$ are transverse. For such point p of intersection, let u be a local parameter for Q_j and v

a local parameter for F_3 . From the equation f_6 , the parameter u is also a local parameter for the sextic, thus the local intersection multiplicity of C_6 and Q_j is ≥ 2 at p . Since there are 6 such points on Q_j , the quadric Q_j is 6-tangent to the sextic curve C_6 . In fact

Theorem 20. *Let C_6 be a generic smooth sextic curve which possesses three 6-tangent conics $C_j = \{q_j = 0\}$, $j \in \{1, 2, 3\}$. Then there exists a cubic form f_3 such that $C_6 = \{q_1q_2q_3 - f_3^2 = 0\}$.*

Remark. Another proof of that result is given in [1, Theorem 3.6].

Proof. Let $C_6 = \{q_1q_2q_3 - f_3^2 = 0\}$ be a generic sextic curve as above and let $Y \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 . Above each irreducible conics $q_i = 0$, ($1 \leq i \leq 3$), there are two (-2) -curves $B_{2i-1} + B_{2i}$ such that $B_{2i-1}B_{2i} = 6$. On the example of Proposition 22 below, the intersection matrix $(B_iB_j)_{1 \leq i, j \leq 6}$ is (up to permutation of B_{2i-1}, B_{2i}) the rank 3 intersection matrix of the 6 (-2) -curves on a K3 surface with Néron-Severi lattice isomorphic to S_1 . These intersection numbers remain the same for the (-2) -curves in that flat family of smooth surfaces Y . The Picard number of Y is at least 3 and if the equation of C_6 is moreover generic, the Picard number of Y is 3 and therefore by Proposition 19, one has $\text{NS}(Y) \simeq S_1$.

Let Φ be the map

$$\Phi : \begin{cases} H^0(\mathbb{P}^2, \mathcal{O}(2))^{\oplus 3} \oplus H^0(\mathbb{P}^2, \mathcal{O}(3)) & \rightarrow H^0(\mathbb{P}^2, \mathcal{O}(6)) \\ \tau = (q_1, q_2, q_3, f_3) & \rightarrow f_\tau = q_1q_2q_3 - f_3^2 \end{cases} .$$

That map is invariant under the action of

$$\Gamma : (q_1, q_2, q_3, f_3) \rightarrow (\lambda q_1, \mu q_2, \frac{1}{\lambda\mu} q_3, f_3)$$

for $\lambda, \mu \in \mathbb{C}^*$. Suppose that $f_\tau = f_{\tau'}$ for $\tau' = (q'_1, q'_2, q'_3, f'_3)$ and the form τ is generic so that the double cover Y branched over the sextic $f_\tau = 0$ has $\text{NS}(Y) \simeq S_1$. Since the conics $q_i = 0$ are the images of the 6 (-2) -curves on Y , up to rescaling by using a transformation Γ and up to taking a permutation of the q_j 's, one can suppose that $q'_1 = q_1$, $q'_2 = q_2$, and $q'_3 = \nu q_3$ for a scalar ν . Then one obtains the relations

$$q_1q_2q_3(1 - \nu) = (f_3 - f'_3)(f_3 + f'_3).$$

If $\nu \neq 1$, then the irreducible degree 2 form q_1 divides one of the two degree 3 factors $(f_3 - f'_3)$, $(f_3 + f'_3)$. But then q_2 or q_3 cannot be irreducible, which is a contradiction. Therefore $\nu = 1$ and $f_3 = \pm f'_3$. Thus the dimension of the (unirational) moduli of sextic curves with an equation of the form $q_1q_2q_3 - f_3^2 = 0$ is $6 \cdot 3 + 10 - 2 - 9 = 17$ dimensional. \square

In [30, Section 2], we proved that the moduli spaces of K3 surfaces with finite automorphism group and Picard number ≥ 3 are irreducible. From the above discussion, we get:

Corollary 21. *The moduli space of K3 surfaces of type S_1 is unirational.*

Proof. By Theorem 20, the rational map from $H^0(\mathbb{P}^2, \mathcal{O}(2))^{\oplus 3} \oplus H^0(\mathbb{P}^2, \mathcal{O}(3))$ to the moduli space of S_1 -polarized K3 surfaces defined by associating to

(q_1, q_2, q_3, f_3) the double cover surface branched over the sextic $\{q_1q_2q_3 - f_3^2 = 0\}$ is dominant. \square

In order to complete the proof of Theorem 20, let us choose the following equations:

$$\begin{aligned} q_1 &= 2(xy + y^2 + xz), & q_2 &= 4(x^2 + xy + z^2), \\ q_3 &= 5(x^2 + xy + yz + z^2), & f_3 &= x^2y + y^3 + y^2z + xz^2 + z^3. \end{aligned}$$

Then the associated sextic C_6 is smooth and the three conics $Q_j = \{q_j = 0\}$ are 6-tangent to it.

Proposition 22. *The surface Y which is the double cover of the plane branched over the curve $C_6 = \{q_1q_2q_3 - f_3^2 = 0\}$ has Picard number 3 and Néron-Severi lattice S_1 .*

For computing the Picard number, we use the method of van Luijk [19] refined by Elsenhans and Jahnel in [12, Section 3.3.1]. We first need to recall:

Theorem 23. *(Artin-Tate Conjecture; satisfied for K3 surfaces). Let Y be a K3 surface over \mathbb{F}_q . Denote by ρ the rank and by Δ_q the discriminant of the Picard group of Y/\mathbb{F}_q . Then*

$$\lim_{t \rightarrow q} \frac{P_W(Y, t)}{(t - q)^\rho} = q^{21-\rho} \#Br(Y) |\Delta_q|,$$

where $P_W(Y, t)$ is the second Weil polynomial and $Br(Y)$ denotes the Brauer group of Y/\mathbb{F}_q .

Proof. (Of Proposition 22). Let Y_{17} and Y_{29} be the reduction modulo 17 and 29 respectively of Y . Using Magma [4], one finds that the Weil polynomial of Y_{29} is

$$\begin{aligned} P_{29} &= (T - 29)^4(T^{18} + 34T^{17} + 1508T^{16} + 47096T^{15} + 1487729T^{14} + 36778612T^{13} \\ &\quad + 594823321T^{12} + 10706819778T^{11} - 68999505236T^{10} - 1000492825922T^9 \\ &\quad - 58028583903476T^8 + 7572730199403618T^7 + 353814783205469041T^6 \\ &\quad + 18398368726684390132T^5 + 625898351490474733529T^4 \\ &\quad + 16663261029844769954936T^3 + 448717814875105590929348T^2 \\ &\quad + 8508380105131809858775714T + 210457284365172120330305161). \end{aligned}$$

In that product, the degree 18 factor Q_{18} is irreducible and the roots of $Q_{18}(\frac{t}{29})$ are not roots of unity, thus by the Tate conjecture (which is satisfied for K3 surfaces), the Picard number of Y_{29} is 4 and this is also the geometric Picard number. We obtain that $\lim_{t \rightarrow 29} \frac{P_{29}(t)}{(t-29)^4} = 2^2 163 \cdot 29^{17}$. Since the order of the Brauer group is a square, we see that the class in \mathbb{Q}^* of the discriminant Δ_{29} modulo squares in \mathbb{Q}^* equals 163.

The Weil polynomial of Y_{17} is

$$\begin{aligned} P_{17} &= (T - 17)^4(T^{18} + 20T^{17} + 323T^{16} + 2890T^{15} + 34391T^{14} + 668168T^{13} \\ &\quad - 1419857T^{12} + 72412707T^{11} + 4513725403T^{10} + 139515148820T^9 \\ &\quad + 1304466641467T^8 + 6047981701347T^7 - 34271896307633T^6 \\ &\quad + 4660977897838088T^5 + 69332046230341559T^4 \\ &\quad + 1683778265594009290T^3 + 54386037978686500067T^2 \\ &\quad + 973223837513337369620t + 14063084452067724991009). \end{aligned}$$

Again the degree 18 factor R_{18} is such that $R_{18}(\frac{t}{17})$ has no cyclotomic factor, thus the surface Y_{17} has geometric Picard number 4. Since $\lim_{t \rightarrow 17} \frac{P_{17}(t)}{(t-17)^4} = 2^4 13 \cdot 17^{17}$, the class of the discriminant Δ_{17} modulo squares in \mathbb{Q}^* is 13. If the surface Y had Picard number 4, the discriminants Δ_{17} and Δ_{29} would belong in the same class modulo a square (see [12, Section 3.3.1]), thus the surface Y has (geometric) Picard number 3, and by Proposition 19, it has Néron-Severi lattice S_1 . \square

5.2. Construction of surfaces of type S_2 . We recall that a surface of type S_2 is a smooth quartic surface in \mathbb{P}^3 with Picard number 3 and three quadric sections, each of which is the union of two degree 4 smooth rational curves.

Conversely, let Y be a smooth quartic surface with three quadric sections Q_1, Q_2, Q_3 such that each decomposes into the union of two degree 4 smooth rational curves :

$$Q_1 = B_1 + B_2, Q_2 = B_3 + B_4, Q_3 = B_5 + B_6.$$

Because $Q_i^2 = 16$ and $B_k^2 = -2$, we get $B_1B_2 = B_3B_4 = B_5B_6 = 10$.

Proposition 24. *Suppose that the Picard number of Y equals to 3. Then the Néron-Severi lattice of Y is isometric to S_2 .*

Proof. Because $Q_sQ_t = 16$ for $1 \leq s, t \leq 3$, the intersection matrix of the 6 curves B_i , $1 \leq i \leq 6$ has the form

$$M = \begin{pmatrix} -2 & 10 & a_1 & b_1 & a_2 & b_2 \\ 10 & -2 & c_1 & d_1 & c_2 & d_2 \\ a_1 & c_1 & -2 & 10 & a_3 & b_3 \\ b_1 & d_1 & 10 & -2 & c_3 & d_3 \\ a_2 & c_2 & a_3 & c_3 & -2 & 10 \\ b_2 & d_2 & b_3 & d_3 & 10 & -2 \end{pmatrix}$$

where the a_i, b_i, c_i, d_i are integers in $\{0, 1, \dots, 16\}$ such that $a_i + b_i + c_i + d_i = 16$. Up to exchanging B_3 and B_4 and B_5 and B_6 , we can suppose that $a_1 \leq b_1$ and $a_2 \leq b_2$. Using a computer for checking the finite number of possibilities, one finds that the only solution for M to be of rank 3 (with $a_1 \leq b_1$ and $a_2 \leq b_2$) is

$$(a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2, a_3, b_3, c_3, d_3) = (1, 7, 7, 1, 1, 7, 7, 1, 7, 1, 7, 1).$$

Then the Néron-Severi lattice of Y contains S_2 . The discriminant group $A(S_2)$ of S_2 is generated by classes $\frac{1}{36}L$ and $\frac{1}{3}(A_1 - A_2)$. The unique non-trivial isotropic element in $A(S_2)$ is $a = \frac{1}{3}L$ (one has $a^2 = 4 = 0 \in \mathbb{Q}/2\mathbb{Z}$). The corresponding over-lattice is S_5 , but a K3 surface with Néron-Severi lattice S_5 does not have an ample class D with $D^2 = 4$, thus the Néron-Severi lattice of Y is S_2 . \square

Proposition 25. *The moduli space of K3 surfaces X with $\text{NS}(X) \simeq S_2$ is unirational.*

Proof. Consider two rational quartic curves C_1 and C_3 in \mathbb{P}^3 such that the degree of their intersection scheme $C_1 \cdot C_3$ is 1 (such rational quartic curves are

parametrized by a projective space). Let Y be a smooth quartic in the linear system of quartics containing C_1 and C_3 . By [15, Exercise IV.6.1], there exists a unique quadric Q (resp. Q') which is moreover smooth, containing the curve C_1 (resp. C_3). The intersection of Y and Q (resp. Q') is the union of C_1 (resp. C_3) and a degree 4 curve C_2 (resp. C_4). Since

$$8 = 2HC_1 = (C_1 + C_2)C_1$$

(for a hyperplane H), we get $C_1C_2 = 10$, and therefore from $(C_1 + C_2)^2 = 16$, we obtain $C_2^2 = -2$. Similarly, the intersection of Y with Q' is the union $C_3 + C_4$ with $C_3C_4 = 10$ and $C_4^2 = -2$. Since by construction $C_1C_3 = 1$, and moreover

$$8 = C_1(C_3 + C_4),$$

we get $C_1C_4 = 7$. From the construction, the curves C_1, \dots, C_4 and H generate a sub-lattice S_2 in the Néron-Severi lattice of Y . The Picard number of Y is therefore ≥ 3 and the generic surface has $\text{NS}(Y) \simeq S_2$. From that construction, it is clear that the moduli space is unirational. \square

To show the effectiveness of the construction, let us give an example of a K3 surface with Néron-Severi lattice S_2 .

Example 26. Let Y be the smooth quartic surface with equation

$$\begin{aligned} & -27x^4 - 246x^3y + 102xy^3 - 216y^4 - 108x^3z + 348x^2yz + 459xy^2z + 270y^3z \\ & -162x^2z^2 + 51xyz^2 - 27y^2z^2 - 108xz^3 - 27z^4 - 9x^3t - 9x^2yt + 522xy^2t \\ & -648y^3t + 46x^2zt - 369xyzt + 911y^2zt - 170xz^2t - 279yz^2t - 44z^3t \\ & -27x^2t^2 + 672xyt^2 - 84xzt^2 + 546yzt^2 + 24z^2t^2 + 72xt^3 = 0. \end{aligned}$$

Each quadrics $Q_1 = \{xy + y^2 + 2xz + yz + z^2 + 3xt + 3yt\}$ and $Q_2 = \{3xy - zt\}$ cuts Y into two smooth quartic rational curves B_1, B_2 and B_3, B_4 respectively. Moreover, one can check that $B_1B_3 = B_2B_4 = 1$ and $B_2B_3 = B_1B_4 = 7$, thus the curves B_1, \dots, B_4 and the class of the hyperplane section generate a lattice of type S_2 . If the Picard number of the surface Y equals 3, then S_2 is the Néron-Severi lattice of Y (by the same argument as in the proof of Proposition 24) and therefore there exists a third quadric Q_3 that cuts Y into the union of the two remaining (-2) -curves. Let Y_{23} and Y_{41} the reductions modulo 23 and 41 of the surface Y . Thanks to the computations [9] of Edgar Costa using algorithms developed in [6, 7, 8], both surfaces have Picard number and geometric Picard number 4. The discriminant of the Néron-Severi lattice of the surfaces Y_{23} and Y_{41} have classes modulo square -211 and -1 respectively, thus the Picard number of $Y_{\mathbb{C}}$ is 3 and Y is an example of surface of type S_2 .

5.3. Construction of surfaces of type S_3 . Let Y be a smooth quartic surface with two hyperplanes sections H_1, H_2 such that each of them is the union of two conics $H_1 = C_1 + C_2$ and $H_2 = C_3 + C_4$. Then

Proposition 27. *The intersection matrix of the curves C_1, \dots, C_4 is either*

$$M_1 = \begin{pmatrix} -2 & 4 & 1 & 1 \\ 4 & -2 & 1 & 1 \\ 1 & 1 & -2 & 4 \\ 1 & 1 & 4 & -2 \end{pmatrix} \text{ or } M_2 = \begin{pmatrix} -2 & 4 & 0 & 2 \\ 4 & -2 & 2 & 0 \\ 0 & 2 & -2 & 4 \\ 2 & 0 & 4 & -2 \end{pmatrix}.$$

Suppose that Y has Picard number 3. If the intersection matrix is M_1 then the Néron-Severi lattice of Y is isometric to S_3 , in particular Y has trivial automorphism group and the 4 conics are the only (-2) -curves on Y .

If the intersection matrix is M_2 then the 4 conics are also the only (-2) -curves on Y , this is case S_{114} in [25, Table 3].

Proof. Since the curves C_1, C_2 and C_3, C_4 are plane conics we have $C_1C_2 = 4 = C_3C_4$ and $C_jH_k = 2$. Let $a \in \{0, 1, 2\}$ be such that $a = C_1C_3$, then $C_1C_4 = 2 - a$, $C_3C_2 = 2 - a$ and $C_2C_4 = a$. Thus, up to exchanging C_3 and C_4 , the intersection matrix of the C_i is either M_1 or M_2 .

In the first case, the lattice generated by the 4 conics in Y is (isometric to) S_3 . Suppose that Y has Picard number 3. The discriminant group $A(S_3)$ of S_3 is generated by $\frac{1}{12}L, \frac{1}{3}(A_1 - A_2)$. There is no non-trivial isotropic element in $A(S_3)$, thus the Néron-Severi lattice of the quartic Y is S_3 .

In the second case, the rank 3 lattice generated by the 4 conics is denoted by S_{114} in [25, Table 3]. Still by [25, Table 3], there are two over-lattices S_{112} and S_{111} of the lattice S_{114} . But the surfaces with Néron-Severi lattice S_{112} or S_{111} contain only three (-2) -curves, thus the Néron-Severi lattice of Y is S_{114} , and by [25] such a surface contains only four (-2) -curves. \square

In the second case the fundamental domain \mathcal{F}_X is not compact, we also remark from the intersection matrix M_2 that it has an elliptic fibration; we give more details on that surface in [30].

Corollary 28. *The moduli space of K3 surfaces X with $\text{NS}(X) \simeq S_3$ is unirational.*

Proof. Let C_1 and C_3 be two conics on two fixed planes P_1, P_2 in \mathbb{P}^3 (we can choose $P_j = \{x_j = 0\}$, $j \in \{1, 2\}$), such that C_1, C_3 meet transversely at exactly one point. Let Y be a smooth quartic in the linear system of quartics containing C_1 and C_3 . The conics C_2, C_4 such that $P_1 \cdot Y = C_1 + C_2$, $P_2 \cdot Y = C_3 + C_4$ are smooth if Y is sufficiently generic. Then, because C_1, C_3 meet transversely at one point, the intersection matrix of C_1, \dots, C_4 is M_1 , thus the Picard number of Y is at least 3 and for a generic choice it is 3, then by Proposition 27, $\text{NS}(Y) \simeq S_3$. From that construction, it is clear that the moduli space is unirational. \square

To show the effectiveness of the construction, let us give an example of a K3 surface with Néron-Severi lattice S_3 .

Example 29. Consider the quartic Y with equation

$$\begin{aligned} & -10x^3z + 8x^3t - 180x^2t^2 - 9xy^3 - 2xy^2z - 10xy^2t + 9xyt^2 + 430xz^3 - 784xz^2t \\ & - 678xzt^2 + 680xt^3 + 3y^4 + 16y^3z - 43y^2z^2 - 1896y^2zt - 2825y^2t^2 - 4246yz^3 \\ & - 4700yz^2t + 23846yzt^2 + 32620yt^3 - 420z^4 + 376z^3t + 2048z^2t^2 - 376zt^3 - 1628t^4. \end{aligned}$$

One can check that the hyperplane sections $H_1 = \{x = 0\}$ and $H_2 = \{y = 0\}$ are the union of conics C_1, C_2, C_3, C_4 with intersection matrix M_1 . The reduction Y_{23} and Y_{73} modulo 23 and 73 of Y are smooth K3 surfaces. Thanks to the computations of E. Costa [9], the Picard number of both surfaces Y_{23} and Y_{73} is 4 and this is also their geometric Picard number. The class modulo squares of the discriminant of the Néron-Severi lattice of Y_{23} is $-2 \cdot 3 \cdot 11 \cdot 17$

whereas for Y_{73} , it is -6 , thus (see Proposition 22 and its proof) the surface Y has Picard number 3 and is an example of surface of type S_3 .

5.4. Surfaces of type S_4 . Let Y be a surface of which is the double cover of the plane $\pi : Y \rightarrow \mathbb{P}^2$ branched over a sextic curve C_6 that possess a tritangent line L and a 6-tangent conic C_o . Let $B_1 + B_2 = \pi^*L$ and $B_3 + B_4 = \pi^*C_o$ be the (-2) -curves over the line and the conic.

Proposition 30. *There are two possibilities for the intersection matrix of the (-2) -curves B_1, \dots, B_4 . Either it is*

$$M_1 = \begin{pmatrix} -2 & 3 & 1 & 1 \\ 3 & -2 & 1 & 1 \\ 1 & 1 & -2 & 6 \\ 1 & 1 & 6 & -2 \end{pmatrix} \text{ or it is } M_2 = \begin{pmatrix} -2 & 3 & 0 & 2 \\ 3 & -2 & 2 & 0 \\ 0 & 2 & -2 & 6 \\ 2 & 0 & 6 & -2 \end{pmatrix}.$$

Suppose that Y has Picard number 3 and the intersection matrix is M_1 . Then Y is a K3 surface of type S_4 .

If Y has Picard number 3 and the intersection matrix is M_2 , then the (-2) -curves B_1, \dots, B_4 are the only (-2) -curves on Y , this is case S_{113} in [25, Table 3].

In the second case the surface Y has not a bounded fundamental domain (see [25]). In fact, as it can be read of from the intersection matrix, the surface Y has two elliptic fibrations (see [30]). Both moduli of K3 surfaces have the same dimension i.e. 17.

Proof. It is easy to check that up to permuting B_3 and B_4 , the matrices M_1 and M_2 are the only possibilities for the intersection of the curves B_j . Let us suppose that Y has Picard number 3. If the intersection matrix is M_1 then the lattice generated by the (-2) -curves is S_4 ; the discriminant group of S_4 (of order 20) has no non-trivial isotropic element, thus there is no over-lattice containing S_4 and $\text{NS}(Y) \simeq S_4$.

Suppose that the intersection matrix is M_2 . The divisor $D = B_1 + B_2$ is ample with $D^2 = 2$ and the degrees of B_1, B_2, B_3, B_4 with respect to D are 1, 1, 2, 2 respectively. The lattice generated by the (-2) -curves is the lattice denoted S_{113} in [25, Table 3]. There is only one over-lattice of S_{113} , but by [25, Table 3], a surface with that lattice has only 3 (-2) -curves, therefore $\text{NS}(Y) \simeq S_{113}$. \square

It is not so easy to find an example of a K3 surface with Néron-Severi lattice S_4 . We found this one:

Example 31. Let f_3, q_2, q_4 be the forms

$$\begin{aligned} f_3 &= -xy^2 + 2y^3 + 5x^2z + 3xyz + yz^2 - 6z^3, & q_2 &= 2x^2 + xy + 2y^2 - 3z^2, \\ q_4 &= -3x^4 + 11x^3y + 13x^2y^2 + 17xy^3 + 2y^4 + 15x^2yz + 4xy^2z - y^3z \\ &+ 14x^2z^2 - 3xyz^2 - 2y^2z^2 + 2yz^3 - 12z^4. \end{aligned}$$

The conic $\{q_2 = 0\}$ is 6-tangent to the sextic $C_6 = \{q_2q_4 - f_3^2 = 0\}$. Moreover one can check that the line $L = \{y = 0\}$ is tritangent to C_6 at points $(-1 : 0 : 1)$, $(0 : 0 : 1)$, $(1 : 0 : 1)$. Let $Y \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 . Let Y_q be the reduction in finite field F_q of the surface

Y . Using Magma [4], one computes that surfaces Y_{312} and Y_{532} have Picard number 4, and this is also their geometric Picard number. Moreover, the class in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ of the discriminant of the Néron-Severi lattice of Y_{312} is -83 , whereas the class of the discriminant of Y_{532} is $-5 \cdot 53$, therefore, as in Proposition 22, the surface Y has Picard number 3. One can check moreover that the intersection of the (-2) -curves on Y is (up to ordering) the matrix M_1 , thus Y has type S_4 .

5.5. Construction of surfaces of type S_5 . Let Y be the double cover of the plane branched over a smooth sextic curve which has two tritangent lines L_1, L_2 .

Proposition 32. *Suppose that the Picard number of Y is 3. Then the Néron-Severi lattice of the K3 surface Y is isometric to S_5 .*

Proof. Let $\pi : Y \rightarrow \mathbb{P}^2$ be the double cover and let B_1, \dots, B_4 be the (-2) -curves such that

$$\pi^*L_1 = B_1 + B_2, \quad \pi^*L_2 = B_3 + B_4.$$

We must have $(B_1 + B_2)^2 = 2$ and $(B_3 + B_4)^2 = 2$, thus $B_1B_2 = B_3B_4 = 3$. Moreover using the involutions from the double plane structure, one find relations of type

$$B_1(B_3 + B_4) = \frac{1}{2}\pi^*L_1\pi^*L_2 = 1.$$

Finally, up to permuting B_3 and B_4 , the intersection matrix $(B_iB_j)_{i,j}$ is the one given for S_5 . Thus the rank 3 Néron-Severi lattice of Y contains S_5 . Since the discriminant group of S_5 has no non-trivial isotropic element, we have the reverse inclusion. \square

Let l_1, l_2, f_3, q_4 be two linear forms, a cubic form and a quartic and let $f_6 = l_1l_2q_4 - f_3^2$. Suppose that these forms are generic so that the sextic curve $C_6 = \{f_6 = 0\}$ is smooth. Then the sextic and the line $L_i = \{l_i = 0\}$ are tangent at the 3 intersection points of L_i and the cubic $E = \{f_3 = 0\}$. Conversely, we have:

Theorem 33. *Let $C = \{f_6 = 0\}$ be a generic smooth plane sextic curve which possesses two tritangent lines $L_1 = \{l_1 = 0\}$, $L_2 = \{l_2 = 0\}$. There exists a quartic form q_4 and a cubic form f_3 such that $f_6 = l_1l_2q_4 - f_3^2$.*

Proof. Let us study the dimension of the moduli space of smooth sextics C_6 which admit equations of the form $f_6 = l_1l_2q_4 - f_3^2$.

A K3 surface Y which is the double cover of the plane branched over a generic equation of the form $l_1l_2q_4 - f_3^2 = 0$ has Néron-Severi lattice of rank ≥ 3 . By the same argument as in the proof of Theorem 20, Example 35 below shows that the generic such surface Y has Picard number 3, and therefore $\text{NS}(Y) \simeq S_5$. In particular, the lines $l_i = 0$, $i \in \{1, 2\}$ are determined by Y : if (l, l', f, q) are forms of degree 1, 1, 3 and 4 respectively such that $l_1l_2q_4 - f_3^2 = ll'q - f^3$, then there exists scalars λ, μ such that $l = \lambda l_1$, $l' = \mu l_2$. The map Φ which to a quadruple (l_1, l_2, q_4, f_3) associate $l_1l_2q_4 - f_3^2$ is invariant under the action of the two dimensional family of automorphisms

$\Gamma : (l_1, l_2, q_4, f_3) \rightarrow (\lambda l_1, \mu l_2, \frac{1}{\lambda\mu} q_4, f_3)$, $\lambda, \mu \in \mathbb{C}^*$. Using such map Γ , one can suppose $l = l_1, l' = l_2$, and we have

$$(5.1) \quad l_1 l_2 q_4 - f_3^2 = l_1 l_2 q - f^2,$$

if and only if

$$l_1 l_2 (q_4 - q) = (f_3 + f)(f_3 - f).$$

Suppose there exists some quadric forms c_1, c_2 such that

$$l_1 c_1 = (f_3 + f), \quad l_2 c_2 = (f_3 - f),$$

then $f_3 = \frac{1}{2}(l_1 c_1 + l_2 c_2)$, but the curve C_6 would be singular at the intersection point of L_1 and L_2 , thus that case cannot happen. Up to exchanging f with $-f$ and after a short computation, it turns out that equation (5.1) is equivalent to the existence of a linear form l such that

$$\begin{cases} f = \pm(f_3 + l_1 l_2 l) \\ q = (q_4 + 2l f_3 + l_1 l_2 l^2). \end{cases}$$

The map Φ is a morphism from a $31 = 3+3+15+10$ dimensional vector space with 5 dimensional fibers (because of the choice of $l \in H^0(\mathbb{P}^2, \mathcal{O}(2))$ and the above parameters λ, μ), thus the image of Φ is a 26 dimensional variety in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(6))$. The moduli of smooth sextic curves which admits equations of the form $f_6 = l_1 l_2 q_4 - f_3^2$ is the quotient of that variety by $GL_3(\mathbb{C})$, thus it is 17 dimensional. By Proposition 32, the double cover of the plane branched over the generic curve C_6 is a K3 surface of type S_5 . The moduli space of K3 surfaces of type S_5 is irreducible and 17 dimensional. This moduli is also the moduli of sextic curves which possess two tritangent lines, thus the result. \square

From the proof of Theorem 33 one gets a dominant rational map from a 31 dimensional vector space to the moduli space of sextic with two tritangent lines, and therefore to the moduli of K3 surface of type S_5 . Thus we obtain

Corollary 34. *The moduli space of K3 surfaces of type S_5 is unirational.*

Let $C_6 = \{l_1 l_2 q_4 - f_3^2 = 0\}$ be a plane sextic curve with two tritangent lines and let $\pi : X \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 .

Remark. The sextic curve C_6 is tangent to the quartic curve $Q_4 = \{q_4 = 0\}$ at the 12 intersection points of Q_4 with $E = \{f_3 = 0\}$. The pull back $\pi^* Q_4$ splits: $\pi^* Q_4 = C_1 + C_2$ with $C_1^2 = C_2^2 = 4$ and $C_1 C_2 = 12$. If the Picard number of X is 3, one can check that the numerical equivalence classes of C_1 and C_2 are the two big and nef divisors of square 4 on X (see Section 3.5), and thus the linear systems $|C_1|$ and $|C_2|$ gives the two morphisms $X \rightarrow \mathbb{P}^3$. From the proof of Proposition 33, the pull-back of every quartic of the form $q_4 + 2l f_3 + l_1 l_2 l^2 = 0$ (where l is a linear form) also splits; that gives the linear systems $|C_1|$ and $|C_2|$.

In order to complete the proof of Theorem 33, let us give the following Example:

Example 35. By taking

$$\begin{aligned} l_1 &= 5x + z, \quad l_2 = 2z, \quad f_3 = 13x^2y + 7xy^2 + 4y^3 + 14x^2z + xyz + 20y^2z + 8xz^2, \\ q_4 &= 9x^4 + 4x^3y + 2x^2y^2 + 6xy^3 + 2y^4 + 12x^3z + 6x^2yz + 11xy^2z \\ &\quad + 7y^3z + 17x^2z^2 + 5xyz^2 + 16y^2z^2 + 16xz^3 + 3yz^3 + 8z^4, \end{aligned}$$

we obtain a smooth sextic curve $C_6 = \{l_1l_2q_4 - f_3^2 = 0\}$. Let Y be the double cover of the plane branched over C_6 . The (-2) -curves above the lines $l_i = 0$ generate a lattice isometric to S_5 . The reduction of Y modulo 29 and 53 have Picard number 4, and this is also the geometric Picard number. By using the same technics as in Proposition 22, we get in the first case that the class of the discriminant in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ is $11 \cdot 41$ and in the second case that the class of the discriminant is 59, thus the surface Y cannot have Picard number equal to 4 and it is a surface of type S_5 .

5.6. Construction of surfaces of type S_6 . Let $Y \rightarrow \mathbb{P}^2$ be the double cover of the plane branched over a generic sextic C_6 which possesses a 6-tangent conic Q_1 and two cuspidal cubics G_1, G_2 such that the irreducible components of the pull-back $\pi^*G_1 = B_3 + B_4$, $\pi^*G_2 = B_5 + B_6$ are smooth (-2) -curves.

Proposition 36. *The surface Y has type S_6 .*

Proof. Let B_1, B_2 be the (-2) -curves above the conic Q_1 . The intersection matrix of the curves B_1, \dots, B_6 is

$$\begin{pmatrix} -2 & 6 & u & \bar{u} & \bar{v} & v \\ 6 & -2 & \bar{u} & u & v & \bar{v} \\ u & \bar{u} & -2 & 11 & \tilde{a} & a \\ \bar{u} & u & 11 & -2 & a & \tilde{a} \\ \bar{v} & v & \tilde{a} & a & -2 & 11 \\ v & \bar{v} & a & \tilde{a} & 11 & -2 \end{pmatrix}$$

where $\bar{x} = 6 - x$ and $\tilde{a} = 9 - a$, and moreover $u, v \in \{0, \dots, 6\}$, $a \in \{0, \dots, 9\}$. Up to exchanging B_3 with B_4 and B_5 with B_6 , we can suppose that $u \leq 3$ and $v \leq 3$. One can then check that this matrix has rank 3 if and only if $u = v = 1$ and $a = 9$. This is then the intersection matrix of the Néron-Severi lattice of a K3 surface of type S_6 . \square

Let us take the notations of sub-section 3.6. The divisor $D_6 = A_1 + A_5$ is very ample, of square 6, with

$$D_6A_j = 3, 7, 1, 14, 3, 12, \quad \text{for } j = 1, \dots, 6.$$

The linear system $|D_6|$ defines a degree 6 model in \mathbb{P}^4 for which the curves A_1, A_5 are rational normal cubic curves contained in an hyperplane section, and A_3 is a line, which meets A_1 but not A_5 . Let us prove

Proposition 37. *The moduli space of K3 surfaces with $\text{NS}(X) \simeq S_6$ is unirational.*

Proof. Let $H \hookrightarrow \mathbb{P}^4$ be a hyperplane and let $C_1 \hookrightarrow H$ be a rational normal quartic curve. Let C_3 be a line not contained in H and that cuts (transversely) C_1 at one point. The linear system of quadrics (respectively cubics)

containing C_1 and C_3 has dimension 5 (respectively 21).

Let Y be the complete intersection K3 surface of a generic quadric and cubic in these linear systems. The hyperplane section of Y by H decomposes as $C_1 + C_5$, where C_5 is a degree 3 curve such that $C_5^2 = -2$. Since $(C_1 + C_5)C_2 = 1$, we have $C_2C_5 = 0$. The intersection matrix of (C_1, C_3, C_5) is the same as the curves (A_1, A_3, A_5) in sub-section 3.6, thus the lattice L_1 in $\text{NS}(Y)$ generated by C_1, C_2, C_3 is isometric to S_6 . The embedding $L_1 \hookrightarrow \text{NS}(Y)$ is primitive since L_1 has no over-lattice. The family of K3 surfaces Y we constructed is unirational. Moreover it contains the family of surfaces with $\text{NS}(Y) \simeq S_6$ as an open subset, since surfaces with $\rho > 3$ and containing strictly L_1 form an hypersurface in the moduli space of K3 with a polarization $L_1 \hookrightarrow \text{NS}(Y)$. We therefore conclude that the moduli of K3 surfaces with $\text{NS}(X) \simeq S_6$ is unirational. \square

5.7. Surfaces with Picard number 4 of type $L(24)$. Let $C_6 \hookrightarrow \mathbb{P}^2$ be a smooth plane sextic curve which admits 3 tritangent lines $L_j = \{l_j = 0\}$ $j \in \{1, 2, 3\}$, $l_j \in H^0(\mathbb{P}^2, \mathcal{O}(1))$. Let $\pi^* : Y \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 . The pull-back of the lines splits as $\pi^*L_j = B_{2j-1} + B_{2j}$ ($j \in \{1, 2, 3\}$) where B_1, \dots, B_6 are (-2) -curves.

Proposition 38. *Suppose that the Picard number of Y is 4. Up to exchanging B_3 with B_4 and B_5 with B_6 , the intersection matrix of the curves B_1, \dots, B_6 is one of the two following matrices:*

$$M_1 = \begin{pmatrix} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 0 & 1 \\ 1 & 0 & 3 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 3 \\ 1 & 0 & 1 & 0 & 3 & -2 \end{pmatrix}, M_2 = \begin{pmatrix} -2 & 3 & 0 & 1 & 0 & 1 \\ 3 & -2 & 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 1 & 0 & 3 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 & 3 \\ 1 & 0 & 0 & 1 & 3 & -2 \end{pmatrix}.$$

In the first case the K3 surface Y has type $L(24)$ studied in Section 4.1.

In the second case the (-2) -curves B_1, \dots, B_6 are also the only (-2) -curves on Y , which is a surface with finite automorphism group.

Proof. One has $B_1B_2 = B_3B_4 = B_5B_6 = 3$ and by the same computations as in Section 5.5, the other intersection number B_iB_j are in $\{-2, 0, 1\}$ and up to exchanging B_1, B_2 and B_3, B_4 , one can suppose $B_1B_3 = 0$, which forces $B_1B_4 = 0$, $B_2B_3 = 1$ and $B_2B_4 = 0$. Also up to exchanging B_5 with B_6 , we can suppose that their intersection with B_1, B_2 is as given by the two matrices. It remains two possibilities for the intersections of B_3, B_4 with B_5 and B_6 , which give matrices M_1 and M_2 . The case of matrix M_1 (the corresponding lattice has discriminant -28) is (up to permutation of the curves B_i) the intersection matrix given in Section 4.1.

If the intersection matrix is M_2 , the elements B_1, B_2, B_3, B_5 form a base of the lattice Λ generated by the B_i . The lattice Λ (of discriminant -27) is the lattice named $L(25)$ in [35], which verifies:

$$\Lambda \simeq \begin{bmatrix} 0 & 3 \\ 3 & -2 \end{bmatrix} \oplus \mathbf{A}_2(-1).$$

It has discriminant group $\mathbb{Z}/9\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ which has no non-trivial isotropic element, thus the Néron-Severi lattice of Y is Λ . Such a K3 surface also contains only a finite number of (-2) -curves (see [35]). \square

Remark 39. If Y is generic and its Néron-Severi lattice is M_2 , one can prove that the sextic curve possesses an equation of the form $l_1 l_2 l_3 g - f^2 = 0$, where the l_i are the equations of the tritangent lines and g, f are cubic forms (more details are given in [30]). The surface Y has two elliptic fibrations coming from the pull-back of the cubic curve $\{g = 0\}$ which is 9-tangent to the sextic curve.

It is not so easy to find an example of surface of type $L(24)$, here is an example:

Example 40. Let l_1, l_2, q_4, f be the forms

$$\begin{aligned} l_1 &= x + y + 2z, & l_2 &= -3x + 2y + z, \\ q_4 &:= 8x^4 + x^3y + x^2y^2 + 3xy^3 - 2y^4 - 20x^3z - 2x^2yz - xy^2z + 3y^3z \\ &\quad - 12x^2z^2 + xyz^2 + 4yz^3, \\ f &= 5x^3 - 3x^2y + xy^2 + 4y^3 + 2x^2z - 3xyz - 3y^2z + 5xz^2 + 4yz^2. \end{aligned}$$

The smooth sextic curve $C_6 = \{l_1 l_2 q_4 - f^2\}$ has three tritangent lines which are $l_1 = 0$, $l_2 = 0$ and the line $\{y = 0\}$. Let $Y \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 . Using the same technics as in Proposition 22, one computes that the Picard number and the geometric Picard number of its reduction Y_{13} modulo 13 is 4. Moreover, by the Artin-Tate formula, the discriminant of the Néron-Severi lattice of Y_{13} is -28 , thus Y is a surface of type $L(24)$ (alternatively, we computed the intersection matrix of the 6 (-2) -curves one the lines and checked that they generate the rank 4 lattice $L(24)$).

5.8. Surfaces with Picard number 4 of type $L(27)$. Let C_6 be a plane sextic curve which has one tritangent line $L_1 = \{l_1 = 0\}$ and three 6-tangent conics. Let $\pi : Y \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 .

Proposition 41. *The surface Y is either a K3 surface of type $L(27)$ or a K3 surface such that the Néron-Severi lattice has a base with intersection matrix*

$$M_2 = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 4 & 0 \\ 0 & 4 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}.$$

Both cases exists; in the second case Y has infinitely many (-2) -curves and automorphisms.

Proof. The pull-back of the conics splits: $\pi^*C_j = B_{2j-1} + B_{2j}$ ($j \in \{1, 2, 3\}$) with $B_1B_2 = B_3B_4 = B_5B_6 = 6$. Let B_7, B_8 be the (-2) -curves above the line L_1 . As in Sections 5.1 and 5.4, there exist $a, b, c \in \{0, 1, \dots, 4\}$ and

$u, v, w \in \{0, 1, 2\}$ such that

$$(B_i B_j)_{1 \leq i, j \leq 8} = \begin{pmatrix} -2 & 6 & a & \bar{a} & b & \bar{b} & u & \tilde{u} \\ 6 & -2 & \bar{a} & a & \bar{b} & b & \tilde{u} & u \\ a & \bar{a} & -2 & 6 & c & \bar{c} & v & \tilde{v} \\ \bar{a} & a & 6 & -2 & \bar{c} & c & \tilde{v} & v \\ b & \bar{b} & c & \bar{c} & -2 & 6 & w & \tilde{w} \\ \bar{b} & b & \bar{c} & c & 6 & -2 & \tilde{w} & w \\ u & \tilde{u} & v & \tilde{v} & w & \tilde{w} & -2 & 3 \\ \tilde{u} & u & \tilde{v} & v & \tilde{w} & w & 3 & -2 \end{pmatrix},$$

where $\bar{x} = 4 - x$ and $\tilde{w} = 2 - x$. Up to exchanging B_3 with B_4 , B_5 with B_6 and B_7 with B_8 , we can suppose $a, b \in \{0, 1, 2\}$ and $u \in \{0, 1\}$. The matrix $(B_i B_j)$ has rank 4 if and only if (a, b, c, u, v, w) is one of the following 6-tuples

$$\begin{aligned} T_1 &= (0, 0, 4, 1, 1, 1), & T_6 &= (2, 0, 4, 0, 2, 2), \\ T_2 &= (0, 0, 4, 0, 1, 0), & T_7 &= (0, 2, 4, 0, 2, 2), \\ T_3 &= (0, 0, 4, 1, 2, 0), & T_8 &= (0, 0, 2, 0, 2, 2), \\ T_4 &= (0, 0, 4, 1, 0, 2), & T_9 &= (0, 2, 0, 0, 2, 0), \\ T_5 &= (0, 0, 4, 0, 0, 1), & T_{10} &= (2, 0, 0, 0, 0, 2) \end{aligned}$$

The first case is the lattice $L(27)$. The lattices corresponding to cases T_2, \dots, T_5 are isometric and lead to a K3 surface with a Néron-Severi lattice of rank 4 and intersection matrix M_2 in some base. For that base, the class $D = (-1, 1, 1, 0)$ is ample and satisfies $D^2 = 2$; the classes $(0, 0, 0, 1)$, $(-1, 1, 1, -1)$ are the classes of irreducible (-2) -curves of degree 1 for D . The classes

$$\begin{aligned} &(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), \\ &(-3, 2, 2, 0), (-2, 1, 2, 0), (-2, 2, 1, 0) \end{aligned}$$

are the classes of irreducible (-2) -curves of degree 2 for D . Suppose that $|D|$ has some base points. Then there exists an elliptic curve E and (-2) -curve Γ such that $D = 2E + \Gamma$ and $\Gamma E = 1$. But that would imply $D\Gamma = 0$, which is impossible since D is ample, thus $|D|$ is base point free and it defines a double cover of \mathbb{P}^2 branched over a smooth sextic curve which has a tritangent line and three 6-tangent conics.

The cases T_5, \dots, T_{10} lead to the same rank 4 lattice with intersection matrix

$$\begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & -2 & 2 & 2 \\ 0 & 2 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix},$$

in some base. This is the Néron-Severi lattice of a K3 surface which contains only six (-2) -curves (case $2U \oplus [-2] \oplus [-2]$ in [35]), but this cannot happen in our situation, since the surface Y contains at least 8 (-2) -curves. \square

Let us give one example of surface of type $L(24)$:

Example 42. Let q_1, q_2, q_3 and f be the forms

$$\begin{aligned} q_1 &= 2x^2 + xy + 2y^2 - 3z^2, & q_2 &:= 3x^2 + 8xy + 10y^2 + 11yz \\ q_3 &= -25/3x^2 - xy - 3y^2 - 2xz - yz + 5z^2, \\ f &= 2x^3 - 3x^2y - 3x^2z - xyz + y^2z - 3xz^2 - 3yz^2. \end{aligned}$$

The conics $\{q_i = 0\}$, $i \in \{1, 2, 3\}$ are 6-tangent to the smooth sextic curve $C_6 = \{q_1 q_2 q_3 - f^2 = 0\}$, moreover the line $L = \{y = 0\}$ is tritangent to C_6 at points $(-1 : 0 : 1)$, $(0 : 0 : 1)$, $(1 : 0 : 1)$. Let $Y \rightarrow \mathbb{P}^2$ be the double cover branched over C_6 . One can check that the reduction Y_{19} modulo 19 of Y has Picard number 4, which is also the geometric Picard number. Using the Artin-Tate conjecture and Magma [4] as in Proposition 22, one obtains that the discriminant of the Néron-Severi lattice of Y_{19} is -60 (and the Brauer group is trivial), thus the surface Y is an example of surface of type $L(27)$ (note that the discriminant of the lattice $L(27)$ is -60 and the discriminant of the lattice associated to the matrix M_2 equals -52).

6. APPENDIX

For the convenience of the reader, we give a list of Gram matrices of the Néron-Severi lattices of the surfaces we consider, with their number of (-2) -curves, their automorphism group and their discriminant group:

	NS(X)	# of (-2) -curves	Aut(X)	Disc(X)
S_1	$\begin{pmatrix} 6 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{pmatrix}$	6	$\mathbb{Z}/2\mathbb{Z}$	$(\mathbb{Z}/2\mathbb{Z})^2 \times \mathbb{Z}/6\mathbb{Z}$
S_2	$\begin{pmatrix} 36 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$	6	1	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/36\mathbb{Z}$
S_3	$\begin{pmatrix} 12 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$	4	1	$\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/12\mathbb{Z}$
S_4	$\begin{pmatrix} 2 & 1 & 2 \\ 1 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix}$	4	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/20\mathbb{Z}$
S_5	$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$	4	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/12\mathbb{Z}$
S_6	$\begin{pmatrix} 2 & 2 & 3 \\ 2 & -2 & 1 \\ 3 & 1 & -2 \end{pmatrix}$	6	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/44\mathbb{Z}$
L_{24}	$\begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & -2 \end{pmatrix}$	6	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/14\mathbb{Z}$
L_{27}	$\begin{pmatrix} 12 & 2 & 0 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{pmatrix}$	8	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/30\mathbb{Z}$

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