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# Combinatorial description of the principal congruence subgroup $\Gamma(2)$ in $SL(2, \mathbb{Z})$

Flavien Mabilat

**Abstract.** We characterize sequences of positive integers  $(c_1, c_2, \dots, c_n)$  for which the  $2 \times 2$  matrix  $\begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \dots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix}$  belongs to the principal congruence subgroup of level 2 in  $SL(2, \mathbb{Z})$ . The answer is given in terms of dissections of a convex  $n$ -gon into a mixture of triangles and quadrilaterals.

## 1 Introduction

The classical *modular group*

$$SL(2, \mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and its quotient by the center,  $PSL(2, \mathbb{Z}) := SL(2, \mathbb{Z}) / \{\pm \text{Id}\}$ , play a central role in several classical areas, such as the theory of continued fractions, hyperbolic geometry, and the theory of modular forms. The group  $SL(2, \mathbb{Z})$  naturally acts on the upper half-plane, and perhaps the most remarkable fact about it is that the quotient by this action is the moduli space of elliptic curves (this fact explains the name “modular group” due to Klein). The structure of the modular group and its subgroups has been thoroughly studied, see [18]. An important class of subgroups,  $\Gamma(N)$ , are called *principal congruence subgroups of level  $N$* . These are defined as follows:

$$\Gamma(N) := \{A \in SL(2, \mathbb{Z}) \mid A = \text{Id} \pmod{N}\},$$

where  $N$  is a positive integer.

This note is about a relation of the modular group to combinatorics. The idea is based on the fact that every element of  $SL(2, \mathbb{Z})$  has a (canonical) presentation

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(i.e. a description by means of generators and relations) by a sequence of *positive integers*. This has been known for a long time (cf. [19]), but started to be exploited only very recently; see [17],[15]. One uses a general principle that positive integers must count some (geometric, combinatorial, etc.) objects.

Our approach is based on the work of Coxeter [8] and Conway-Coxeter [6], [7]. Coxeter and Conway used the notion of *frieze pattern* to characterize sequences of positive integers  $(c_1, \dots, c_n)$  such that

$$\begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix} = -\text{Id}, \quad (1)$$

and that satisfy an extra condition of *total positivity*, formulated as the positivity of the entries of Coxeter's *frieze pattern*, or just *frieze* for short. All positive solutions of equation (1) were classified in [17]. For a detailed explanation of the total positivity property, see [17],[15] (and also [5]).

Our goal is twofold. We give a short overview of the combinatorial approach to the modular group, that we believe should be better known. We prove a new theorem that gives a combinatorial description of  $\Gamma(2)$ , the principal congruence subgroup of level 2.

## 2 Sequences of positive integers

The group  $\text{SL}(2, \mathbb{Z})$  has two generators whose standard choice is

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These generators satisfy the relations:  $S^2 = (TS)^3 = -\text{Id}$ , and this is a complete set of relations in  $\text{SL}(2, \mathbb{Z})$ . This classical fact can be found in many textbooks; for a particularly elementary proof, see [1]. It readily implies that every element  $A$  of  $\text{SL}(2, \mathbb{Z})$  can be written, for some positive integer  $n$ , in the form

$$A = \pm T^{c_1} S T^{c_2} S \cdots T^{c_n} S = \pm \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix}, \quad (2)$$

where  $c_1, \dots, c_n$  are positive integers; see [18],[19],[17], and the explanation below. We will use the notation

$$M(c_1, \dots, c_n) := \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix},$$

For the generators one easily checks

$$T = -M(2, 1, 1), \quad T^{-1} = -M(1, 1, 2, 1), \quad \text{and} \quad S = -M(1, 1, 2, 1, 1).$$

A decomposition  $A = \pm M(c_1, \dots, c_n)$  with each  $c_i$  positive can then be obtained for every chosen  $A$  by concatenation of any expression of  $A$  in terms of the generators.

The decomposition  $A = \pm M(c_1, \dots, c_n)$  is obviously not unique (even though a canonical, shortest expression was suggested in [15]). The first natural problem is thus to consider the equation

$$M(c_1, \dots, c_n) = \pm \text{Id}, \quad (3)$$

and look for a combinatorial description of the sequences of positive integer solutions. In other words, this problem is to give an explicit combinatorial description of relations in  $PSL(2, \mathbb{Z})$ . This problem was studied in [6], [7], [17]; see also [2], [15], [11] and Section 4 below. It turns out that equation (3) is related to triangulations of  $n$ -gons and to more sophisticated polygon dissections.

### 3 The main result of this paper

We will generalize equation (3) and describe the sequences of positive integers  $(c_1, \dots, c_n)$  for which

$$M(c_1, \dots, c_n) \in \Gamma(2), \quad (4)$$

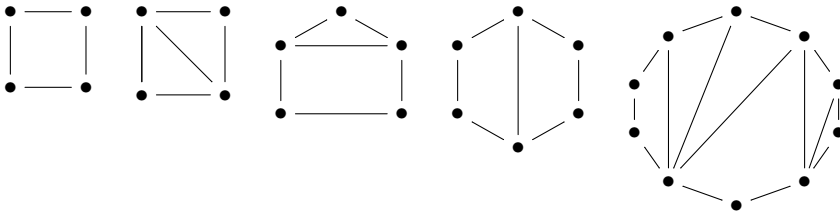
where  $\Gamma(2)$  is the principal congruence subgroup of  $SL(2, \mathbb{Z})$  of level 2, see the introduction.

Similarly to the case of equation (3), the property of being a solution of equation (4) is cyclically invariant (i.e., if an  $n$ -tuple  $(c_1, \dots, c_n)$  is a solution of equation (4), then  $(c_n, c_1, \dots, c_{n-1})$  is also a solution). It is thus often convenient to consider, instead of an  $n$ -tuple  $(c_1, \dots, c_n)$ , an  $n$ -periodic infinite sequence  $(c_i)_{i \in \mathbb{Z}}$ .

The solutions of equation (4) can be formulated in terms of polygon dissections.

**Definition 1.** Let a  $(3|4)$ -dissection be any partition of a convex  $n$ -gon into subpolygons by pairwise non-crossing diagonals, such that every subpolygon is a triangle or a quadrilateral.

**Example 1.** Let us give here simple examples:



For any integer  $a$ , we denote by  $\bar{a} := a + 2\mathbb{Z}$  the image of  $a$  under the projection  $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$ . If  $a$  is odd, then  $\bar{a} = \bar{1}$ ; if  $a$  is even, then  $\bar{a} = \bar{0}$ . The following notion is analogous to that of [6] and [7].

**Definition 2.** The *quiddity* of a  $(3|4)$ -dissection of a convex  $n$ -gon is the (cyclically ordered) sequence  $(\bar{c}_1, \dots, \bar{c}_n)$  of elements of  $\mathbb{Z}/2\mathbb{Z}$ , such that for every  $i$

$$\bar{c}_i = \begin{cases} \bar{1}, & \text{if the number of triangles adjacent to the } i\text{th vertex is odd;} \\ \bar{0}, & \text{if the number of triangles adjacent to the } i\text{th vertex is even.} \end{cases}$$

**Example 2.** The quiddities  $(\bar{c}_1, \dots, \bar{c}_n)$  of the  $(3|4)$ -dissections of Example 1 are as follows:

(a) For the first two examples,

$$(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = (\bar{0}, \bar{0}, \bar{0}, \bar{0}) \quad \text{and} \quad (\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = (\bar{0}, \bar{1}, \bar{0}, \bar{1}).$$

These are the only quiddities for  $n = 4$ .

(b) For the remaining examples one has, respectively,

$$\begin{aligned} (\overline{c}_1, \dots, \overline{c}_5) &= (\overline{1}, \overline{1}, \overline{1}, \overline{0}, \overline{0}), & (\overline{c}_1, \dots, \overline{c}_6) &= (\overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}) \\ \text{and } (\overline{c}_1, \dots, \overline{c}_{10}) &= (\overline{0}, \overline{1}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{0}, \overline{1}, \overline{0}, \overline{0}). \end{aligned}$$

The following statement is our main result. It gives a combinatorial characterization of the solutions of equation (4) for  $n \geq 3$ . Note that the product of elements of  $\text{SL}(2, \mathbb{Z})$  commutes with the projection of the entries of matrices to  $\mathbb{Z}/2\mathbb{Z}$ . This allows one to make all the computations in  $\text{SL}(2, \mathbb{Z}/2\mathbb{Z})$ .

**Theorem 1.**

- (i) Every quiddity of a (3|4)-dissection of an  $n$ -gon is a solution of equation (4).
- (ii) Every solution of equation (4) with  $n \geq 3$  is a quiddity of a (3|4)-dissection of an  $n$ -gon.

This statement is proved in Section 5.

Let us mention that the number of solutions of equation (4), for a fixed  $n$ , can be deduced from the main result of [14] and is given by the Jacobsthal sequence (A001045 in OEIS [16]).

## 4 Relations in $\text{PSL}(2, \mathbb{Z})$ and polygon dissections

We give a brief overview of the theorems of Conway and Coxeter [6], [7] (see also [2],[11]) and Ovsienko [17]. The first one relates equation (3) to one of the most classical notion of combinatorics, namely that of triangulation of an  $n$ -gon, while the second describes all positive integer solutions of equation (3) in terms of polygon dissections. This overview will allow us to compare equation (3) and equation (4). It also makes the presentation complete.

### 4.1 Triangulations and friezes

Fix a triangulation of a convex  $n$ -gon. Following [6] and [7], we call a *quiddity* of the triangulation the sequence of positive integers  $(c_1, \dots, c_n)$ , where  $c_i$  is equal to the number of triangles adjacent to the  $i$ th vertex of the  $n$ -gon.

The theorem of Conway and Coxeter can be formulated in the following way (cf. [17, Corollary 2.3]).

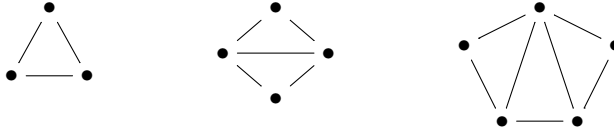
**Theorem 2 ([6] and [7]).**

- (i) The quiddity of a triangulation of an  $n$ -gon is a solution of the equation  $M(c_1, \dots, c_n) = -\text{Id}$ .
- (ii) Every solution  $(c_1, \dots, c_n)$  of equation (3) satisfying the condition

$$c_1 + c_2 + \dots + c_n = 3n - 6 \tag{5}$$

is the quiddity of a triangulation of an  $n$ -gon.

The simplest examples, with  $n = 3, 4, 5$ , are



It is easy to see that their corresponding quiddities, namely  $(c_1, c_2, c_3) = (1, 1, 1)$ ,  $(c_1, c_2, c_3, c_4) = (1, 2, 1, 2)$  and  $(c_1, c_2, c_3, c_4, c_5) = (1, 3, 1, 2, 2)$ , are, indeed, solutions of the equation  $M(c_1, \dots, c_n) = -\text{Id}$ .

The original formulation of Theorem 2 uses the beautiful notion of Coxeter's *frieze*. Let us recall that a frieze is an array of  $(n - 1)$  infinite rows of positive integers with the rows 1 and  $n - 1$  consisting in 1's. Every elementary  $2 \times 2$  "diamond"

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array},$$

of the frieze must satisfy the unimodular rule  $ad - bc = 1$ . Coxeter proved in [8] that the row 2 (and  $n - 2$ ) is an  $n$ -periodic sequence satisfying equation (1). The Conway-Coxeter theorem of [6] and [7] identifies Coxeter's friezes with triangulations.

Let us give here an example of a frieze for  $n = 5$ :

$$\begin{array}{cccccc} \dots & 1 & 1 & 1 & 1 & 1 \\ & 1 & 3 & 1 & 2 & 2 & \dots \\ \dots & 2 & 2 & 1 & 3 & 1 \\ & 1 & 1 & 1 & 1 & 1 & \dots \end{array}$$

The 5-periodic sequence  $(1, 3, 1, 2, 2)$  is the quiddity of a triangulation of a pentagon.

For a survey on friezes, see [13]. Variants of friezes involving other types of polygon dissections can be found in [4], [12], [10]. Links of friezes to presentations of  $SL(2, \mathbb{Z})$  also appeared in [3].

**Remark 1.** Let us mention that (5) turns out to be equivalent to the condition of total positivity, see [17, Corollary 2.3]; and it can be formulated in more standard terms of continued fractions and total positive  $(2 \times 2)$ -matrices, see [15]. In terms used by Coxeter, this total positivity means that every entry of the frieze is positive.

## 4.2 Complete solution of equation (3)

For  $n \geq 6$ , there exist many solutions of equation (3) that cannot be obtained from triangulations of  $n$ -gons. The complete solution of equation (3) was given in [17] and led to the following notion of "3d-dissection".

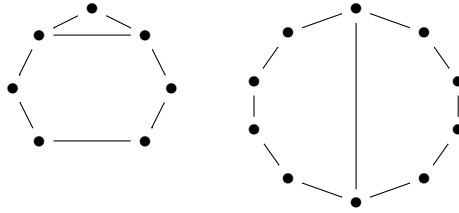
### Definition 3.

- (i) A *3d-dissection* is a partition of a convex  $n$ -gon into sub-polygons by means of pairwise non-crossing diagonals, such that the number of vertices of every sub-polygon is a multiple of 3.

- (ii) The *quiddity* of a 3d-dissection of an  $n$ -gon is the (cyclically ordered)  $n$ -tuple of positive integers  $(c_1, \dots, c_n)$  such that  $c_i$  is the number of sub-polygons adjacent to the  $i$ th vertex of the  $n$ -gon.

**Theorem 3 ([17], Theorem 1).** *Every quiddity of a 3d-dissection of an  $n$ -gon is a solution of equation (3). Conversely, every solution of equation (3) is a quiddity of a 3d-dissection of an  $n$ -gon.*

The simplest examples of 3d-dissections that are not triangulations are



and the corresponding quiddities are:  $(2, 1, 2, 1, 1, 1, 1)$  and  $(2, 1, 1, 1, 1, 2, 1, 1, 1, 1)$ .

### 4.3 Idea of the proof

The proof of Theorem 3 is inductive. The main idea uses the following “local surgery” operations:

$$(\alpha) (c_1, \dots, c_i, c_{i+1}, \dots, c_n) \mapsto (c_1, \dots, c_i + 1, 1, c_{i+1} + 1, \dots, c_n),$$

$$(\beta) (c_1, \dots, c_{i-1}, c_i, c_{i+1}, \dots, c_n) \mapsto (c_1, \dots, c_{i-1}, c'_i, 1, 1, c''_i, c_{i+1}, \dots, c_n),$$

where  $c'_i + c''_i = c_i + 1$ . One readily checks that the matrix  $M(c_1, \dots, c_n)$  is invariant under the operations of type  $(\alpha)$  and changes the sign under the operations of type  $(\beta)$ .

A simple lemma then states that a sequence of positive integers  $(c_1, \dots, c_n)$  satisfying equation (3) always has some entries  $c_i = 1$ ; cf. [6], [7], [17]. This allows one to construct any solution of equation (3) from the simplest solution  $(1, 1, 1)$ .

The inductive step in the proof is based on the observation that the above surgery operations have a combinatorial meaning. Given a dissection of an  $n$ -gon, the operation  $(\alpha)$  consists in gluing an additional triangle on the edge  $(i, i+1)$ , while the operation  $(\beta)$  changes a  $3k$ -gon adjacent to the  $i$ th vertex in the dissection into a  $(3k + 3)$ -gon; see [17].

## 5 Proof of Theorem 1

Our proof of Theorem 1 is quite similar to that of Theorem 3. We give an inductive procedure of construction of all the solutions of equation (4).

### 5.1 Local surgery

Consider the following two families of “local surgery” operations for sequences of elements of  $\mathbb{Z}/2\mathbb{Z}$ :

- (a) Operations of the first family insert  $\bar{1}$  into the sequence  $(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)$ :

$$(\bar{c}_1, \dots, \bar{c}_i, \bar{c}_{i+1}, \dots, \bar{c}_n) \mapsto (\bar{c}_1, \dots, \bar{c}_i + \bar{1}, \bar{1}, \bar{c}_{i+1} + \bar{1}, \dots, \bar{c}_n).$$

- (b) Operations from the second family insert two copies of  $\bar{0}$  between two consecutive positions:

$$(\bar{c}_1, \dots, \bar{c}_i, \bar{c}_{i+1}, \dots, \bar{c}_n) \mapsto (\bar{c}_1, \dots, \bar{c}_i, \bar{0}, \bar{0}, \bar{c}_{i+1}, \dots, \bar{c}_n).$$

Within the cyclic ordering, the operations (a) and (b) are defined for all  $1 \leq i \leq n$ . Every operation (a) transforms a sequence of  $n$  elements of  $\mathbb{Z}/2\mathbb{Z}$  into a sequence of  $n + 1$  elements of  $\mathbb{Z}/2\mathbb{Z}$ , and every operation (b) transforms a sequence of  $n$  elements of  $\mathbb{Z}/2\mathbb{Z}$  into a sequence of  $n + 2$  elements of  $\mathbb{Z}/2\mathbb{Z}$ .

The following statement means that equation (4) is invariant under the operations (a) and (b).

**Lemma 1.** *One has, in the group  $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$ ,*

$$\begin{aligned} M(\bar{c}_1, \dots, \bar{c}_{n+1}) &= M(\bar{c}_1, \dots, \bar{c}_i + \bar{1}, \bar{1}, \bar{c}_{i+1} + \bar{1}, \dots, \bar{c}_n), \\ M(\bar{c}_1, \dots, \bar{c}_{n+1}) &= -M(\bar{c}_1, \dots, \bar{c}_i, \bar{0}, \bar{0}, \bar{c}_{i+1}, \dots, \bar{c}_n). \end{aligned}$$

*Proof.* An operation of type (a) replaces the matrix  $\begin{pmatrix} \bar{c}_i & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{c}_{i+1} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}$  by

$$\begin{aligned} \begin{pmatrix} \bar{c}_i + \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{c}_{i+1} + \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix} &= \begin{pmatrix} \bar{c}_i & \bar{c}_i + \bar{1} \\ \bar{1} & \bar{1} \end{pmatrix} \begin{pmatrix} \bar{c}_{i+1} + \bar{1} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix} \\ &= \begin{pmatrix} \bar{c}_i \bar{c}_{i+1} + \bar{1} & \bar{c}_i \\ \bar{c}_{i+1} & \bar{1} \end{pmatrix} \\ &= \begin{pmatrix} \bar{c}_i & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix} \begin{pmatrix} \bar{c}_{i+1} & \bar{1} \\ \bar{1} & \bar{0} \end{pmatrix}. \end{aligned}$$

Therefore,  $M(\bar{c}_1, \dots, \bar{c}_{n+1}) = M(\bar{c}_1, \dots, \bar{c}_i + \bar{1}, \bar{1}, \bar{c}_{i+1} + \bar{1}, \dots, \bar{c}_n)$ , as an element of  $\mathrm{SL}(2, \mathbb{Z}/2\mathbb{Z})$ . An operation of type (b) adds  $-\mathrm{Id}$  in the product that defines  $M_n(\bar{c}_1, \dots, \bar{c}_n)$ .  $\square$

### 5.2 The special cases $n = 2$ and $n = 3$

The equation (4) has no solution if  $n = 1$ . Consider now the simplest cases  $n = 2$  and  $n = 3$ .

**Lemma 2.**

- (i) *For  $n = 2$ , a pair  $(\bar{c}_1, \bar{c}_2)$  is a solution of equation (4) if and only if*

$$(\bar{c}_1, \bar{c}_2) = (\bar{0}, \bar{0}).$$



(ii) For  $n = 3$ , the triple  $(\overline{c_1}, \overline{c_2}, \overline{c_3})$  is a solution of equation (4) if and only if

$$(\overline{c_1}, \overline{c_2}, \overline{c_3}) = (\overline{1}, \overline{1}, \overline{1}).$$

*Proof.* Part(i). This follows from the equality

$$\begin{pmatrix} \overline{c_1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_2} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{c_1 c_2 + 1} & \overline{c_1} \\ \overline{c_2} & \overline{1} \end{pmatrix}.$$

Part (ii). This follows from the equality

$$\begin{aligned} \begin{pmatrix} \overline{c_1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_2} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_3} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} &= \begin{pmatrix} \overline{c_1 c_2 + 1} & \overline{c_1} \\ \overline{c_2} & \overline{1} \end{pmatrix} \begin{pmatrix} \overline{c_3} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \\ &= \begin{pmatrix} \overline{c_1 c_2 c_3 + c_1 + c_3} & \overline{c_1 c_2 + 1} \\ \overline{c_2 c_3 + 1} & \overline{c_2} \end{pmatrix}. \end{aligned}$$

Hence the result.  $\square$

### 5.3 Inductive construction

We now give an inductive procedure for the construction of all the solutions of equation (4) starting from the simplest case  $n = 2$  and the corresponding solution  $(\overline{0}, \overline{0})$ .

**Proposition 1.** *An  $n$ -tuple  $(\overline{c_1}, \dots, \overline{c_n})$  is a solution of equation (4) if and only if the sequence  $(\overline{c_1}, \dots, \overline{c_n})$  can be obtained from  $(\overline{0}, \overline{0})$  by applying the operations (a) and (b) in any order.*

*Proof.* The “if” part follows from Lemma 1.

Conversely, given a solution  $(\overline{c_1}, \dots, \overline{c_n})$  of equation (4), one has the following two possibilities.

- A) One has  $\overline{c_i} = \overline{0}$  for all  $i$ . Then,  $n$  is even and  $(\overline{c_1}, \dots, \overline{c_n})$  is obtained from  $(\overline{0}, \overline{0})$  by a sequence of  $\frac{n-2}{2}$  operations of type (b).
- B)  $\overline{c_i} = \overline{1}$  for some  $i$ ,  $1 \leq i \leq n$ . Then, the inverse of the operation of type (a) centered at  $i$  can be applied to  $(\overline{c_1}, \dots, \overline{c_n})$ . This results in an  $(n-1)$ -tuple  $(\overline{c_1}, \dots, \overline{c_{i-1} + 1}, \overline{c_{i+1} + 1}, \dots, \overline{c_n})$ . The same computation as in the proof of Lemma 1 implies that this  $(n-1)$ -tuple is a solution of equation (4). We conclude by the induction assumption.  $\square$

Let us mention that there exists an analog of Proposition 1 in the case of non-negative integer solutions of (1), see [9, Thm 3.1].

### 5.4 End of the proof of Theorem 1

We will need the following combinatorial interpretation of operations (a) and (b). Let  $(\overline{c_1}, \dots, \overline{c_n})$  be a sequence corresponding to a (3|4)-dissection of a convex  $n$ -gon, then the result of either operation is again a sequence corresponding to a (3|4)-dissection of a convex  $(n+1)$ -gon or  $(n+2)$ -gon, respectively.

- (i) To a (3|4)-dissection, operation (a) glues a triangle on the segment  $(i, i + 1)$ .
- (ii) Operation (b) glues a quadrilateral on the segment  $(i, i + 1)$ .

We are ready to complete the proof of Theorem 1.

Part (i). The induction basis consists of two cases,  $n = 3$  and  $n = 4$ . For  $n = 3$ , the quiddity of a (3|4)-dissection of any triangle is  $(\bar{1}, \bar{1}, \bar{1})$  which is a solution of equation (4). For  $n = 4$ , the quiddity of a (3|4)-dissection of any quadrilateral is  $(\bar{1}, \bar{0}, \bar{1}, \bar{0})$  (quadrilateral cut into two triangles) or  $(\bar{0}, \bar{0}, \bar{0}, \bar{0})$  (quadrilateral alone) and it follows from Lemma 1 that they are solutions of equation (4).

Assume that an  $n$ -tuple  $(\bar{c}_1, \dots, \bar{c}_n)$  is the quiddity of a (3|4)-dissection of a convex  $n$ -gon. Every (3|4)-dissection has (at least) one exterior triangle (such a triangle is sometimes called an “ear” in the literature), or quadrilateral. Cutting this exterior piece, one obtains either an  $(n - 1)$ -tuple or an  $(n - 2)$ -tuple of elements of  $\mathbb{Z}/2\mathbb{Z}$  which is the quiddity of a (3|4)-dissection of a convex  $(n - 1)$ -gon or a convex  $(n - 2)$ -gon. The result then follows from Lemma 1 and the induction assumption.

Part (ii). For  $n = 3$ , a triple  $(\bar{c}_1, \bar{c}_2, \bar{c}_3)$  is a solution of equation (4) if and only if  $(\bar{c}_1, \bar{c}_2, \bar{c}_3) = (\bar{1}, \bar{1}, \bar{1})$ , which corresponds to a triangle. Similarly to Lemma 2, one shows the following: for  $n = 4$ , the (cyclically ordered) solutions are  $(\bar{1}, \bar{0}, \bar{1}, \bar{0})$  and  $(\bar{0}, \bar{0}, \bar{0}, \bar{0})$ , already considered in Example 1.

Assume that a sequence  $(\bar{c}_1, \dots, \bar{c}_n)$  is a solution of equation (4), and let us show that it is the quiddity of a (3|4)-dissection of a convex  $n$ -gon. By Proposition 1, this sequence is obtained from  $(\bar{0}, \bar{0})$  by a sequence of the surgery operations (a) and (b).

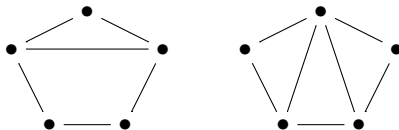
If  $\bar{c}_i = \bar{1}$  for some  $i$ , where  $0 \leq i \leq n$ , then, by the induction assumption, the sequence

$$(\bar{c}_1, \dots, \overline{c_{i-1} + 1}, \overline{c_{i+1} + 1}, \dots, \bar{c}_n)$$

is the quiddity of a (3|4)-dissection of a convex  $(n - 1)$ -gon. Therefore,  $(\bar{c}_1, \dots, \bar{c}_n)$  is the quiddity of a (3|4)-dissection of a convex  $n$ -gon, obtained from this (3|4)-dissection by the gluing of a triangle.

If  $\bar{c}_i = \bar{c}_{i+1} = \bar{0}$ , then the sequence is of the form  $(\bar{c}_1, \dots, \bar{c}_{i-1}, \bar{0}, \bar{0}, \bar{c}_{i+2}, \dots, \bar{c}_n)$ . By the induction assumption,  $(\bar{c}_1, \dots, \bar{c}_{i-1}, \bar{c}_{i+2}, \dots, \bar{c}_n)$  is the sequence associated to a (3|4)-dissection of a convex  $(n - 2)$ -gon. Therefore  $(\bar{c}_1, \dots, \bar{c}_n)$  is the quiddity of a (3|4)-dissection of a convex  $n$ -gon, obtained from this (3|4)-dissection by the gluing of a quadrilateral. Theorem 1 is proved.  $\square$

**Remark 2.** Part (ii) of Theorem 1 can be strengthened. Let  $(c_1, \dots, c_n)$  be a solution of equation (4). Assume that at least one element  $\bar{c}_i$  of  $\mathbb{Z}/2\mathbb{Z}$  is different from  $\bar{0}$  (i.e., not all of  $c_i$  are even). It turns out that, under this assumption,  $(\bar{c}_1, \dots, \bar{c}_n)$  is the quiddity of a triangulation of a convex  $n$ -gon. For example, the following two (3|4)-dissections have the same quiddity  $(\bar{1}, \bar{1}, \bar{1}, \bar{0}, \bar{0})$ .



The proof of this strengthened statement is very similar to that of Theorem 1, Part (ii). It uses the following idea: if an  $(n - 1)$ -tuple, obtained by applying the operation inverse to (a) centered at  $i$  to an  $n$ -tuple  $(\bar{c}_1, \dots, \bar{c}_n)$ , contains only  $\bar{0}$ , then the  $(n - 1)$ -tuple obtained by applying the operation inverse to (a) and centered at  $i + 1$  to the  $n$ -tuple  $(\bar{c}_1, \dots, \bar{c}_n)$  contains an element different from  $\bar{0}$ .

**Remark 3.** Let us mention that Part (i) of Theorem 1 can be deduced from the combinatorial model and results of [10, Thm 7.3]. One can also deduce from this model that every quiddity  $(\bar{c}_1, \dots, \bar{c}_n)$  coming from a  $(3|4)$ -dissection can be lifted to an integer solution  $(c_1, \dots, c_n)$  of equation (1).

**Concluding remark.** Equation (4) naturally extends to arbitrary principal congruence subgroups  $\Gamma(N)$  in  $\mathrm{SL}(2, \mathbb{Z})$ , and it would be interesting to find a combinatorial description of the set of solutions in the general situation.

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