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► **To cite this version:**

Flavien Mabilat. COMBINATORIAL DESCRIPTION OF THE PRINCIPAL CONGRUENCE SUBGROUP $\Gamma(2)$ IN $SL(2, \mathbb{Z})$. 2020. hal-02359509v2

HAL Id: hal-02359509

<https://hal.science/hal-02359509v2>

Preprint submitted on 8 Jan 2020 (v2), last revised 18 Feb 2022 (v3)

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COMBINATORIAL DESCRIPTION OF THE PRINCIPAL CONGRUENCE SUBGROUP $\Gamma(2)$ IN $SL(2, \mathbb{Z})$

FLAVIEN MABILAT

ABSTRACT. We characterize sequences of positive integers (c_1, c_2, \dots, c_n) for which the (2×2) -matrix $\begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix}$ belongs to the principal congruence subgroup of level 2 in $SL(2, \mathbb{Z})$. The answer is given in terms of dissections of a convex n -gon into a mixture of triangles and quadrilaterals.

1. INTRODUCTION

The classical modular group

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

and its quotient by the center, $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\{\pm \text{Id}\}$, play a central role in several classical areas, such as the theory of continued fractions, hyperbolic geometry, and the theory of modular forms. The group $SL(2, \mathbb{Z})$ naturally acts on the upper half-plane, and perhaps the most remarkable fact about it is that the quotient by this action is the moduli space of elliptic curves (this fact explains the name “modular group” due to Klein). The structure of the modular group and its subgroups was thoroughly studied; see [17]. An important class of subgroups are called “principal congruence subgroups of level N ” $\Gamma(N)$ defined as follows

$$\Gamma(N) = \{A \in SL(2, \mathbb{Z}) \mid A = \text{Id} \pmod{N}\},$$

where N is a positive integer.

This note is about a relation of the modular group to combinatorics. The idea is based on the fact that every element of $SL(2, \mathbb{Z})$ has a (canonical) presentation by a sequence of *positive integers*. This has been known for a long time (cf. [18]), but started to be exploited only very recently; see [16, 14]. One uses a general principle that positive integers must count some (geometric, combinatorial, etc.) objects.

Our approach is based on the work of Coxeter [7] and Conway-Coxeter [6]. Coxeter and Conway used the notion of *frieze pattern* to characterize sequences of positive integers (c_1, \dots, c_n) such that

$$(1.1) \quad \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix} = -\text{Id},$$

and that satisfy an extra condition of *total positivity*, formulated as the positivity of the entries of Coxeter’s *frieze pattern*. All positive solutions of Equation (1.1) were classified in [16]. For a detailed explanation of the total positivity property; see [16, 14] (and also [5]).

Our goal is twofold. We give a short overview of the combinatorial approach to the modular group, that we believe should be better known. We prove a new theorem that gives a combinatorial description of $\Gamma(2)$, the principal congruence subgroup of level 2.

2. SEQUENCES OF POSITIVE INTEGERS

The group $\mathrm{SL}(2, \mathbb{Z})$ has two generators whose standard choice is

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

These generators satisfy the relations: $S^2 = (TS)^3 = -\mathrm{Id}$, and this is a complete set of relations in $\mathrm{SL}(2, \mathbb{Z})$. This classical fact can be found in many textbooks, for a particularly elementary proof; see [1]. It readily implies that every element A of $\mathrm{SL}(2, \mathbb{Z})$ can be written, for some positive integer n , in the form

$$(2.1) \quad A = \pm T^{c_1} S T^{c_2} S \cdots T^{c_n} S = \pm \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix},$$

where c_1, \dots, c_n are positive integers; see [17, 18, 16] and explanation below. We will use the notation

$$M(c_1, \dots, c_n) := \begin{pmatrix} c_1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} c_2 & -1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} c_n & -1 \\ 1 & 0 \end{pmatrix},$$

For the generators one easily checks

$$T = -M(2, 1, 1), \quad T^{-1} = -M(1, 1, 2, 1), \quad \text{and} \quad S = -M(1, 1, 2, 1, 1).$$

A decomposition $A = \pm M(c_1, \dots, c_n)$ with each c_i positive can then be obtained for every chosen A by concatenation of any expression of A in terms of the generators.

The decomposition $A = \pm M(c_1, \dots, c_n)$ is obviously not unique (however a canonical, shortest expression was suggested in [14]). The first natural problem is thus to consider the equation

$$(2.2) \quad M(c_1, \dots, c_n) = \pm \mathrm{Id},$$

and look for a combinatorial description of the sequences of positive integer solutions. In other words, this problem is to give an explicit combinatorial description of relations in $\mathrm{PSL}(2, \mathbb{Z})$. This problem was studied in [6, 16]; see also [2, 14, 10] and Section 4 below. It turns out that equation (2.2) is related to triangulations of n -gons and to more sophisticated polygon dissections.

3. THE MAIN RESULT OF THIS PAPER

We will generalize the equation (2.2) and describe the sequences of positive integers (c_1, \dots, c_n) for which

$$(3.1) \quad M(c_1, \dots, c_n) \in \Gamma(2),$$

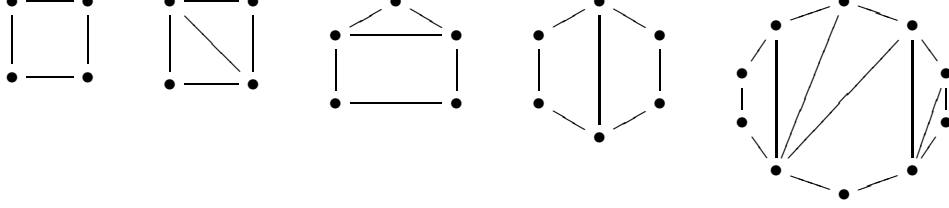
where $\Gamma(2)$ is the principal congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ of level 2, see the introduction..

Similarly to the case of equation (2.2), the property to be a solution of equation (3.1) is cyclically invariant (i.e., if an n -tuple (c_1, \dots, c_n) a solution of equation (3.1), then $(c_n, c_1, \dots, c_{n-1})$ is also a solution). It is thus often convenient to consider instead of an n -tuple (c_1, \dots, c_n) an n -periodic infinite sequence $(c_i)_{i \in \mathbb{Z}}$.

The solutions of equation (3.1) can be formulated in terms of polygon dissections.

Definition 3.1. We call a (3|4)-*dissection* a partition of a convex n -gon into sub-polygons by pairwise non-crossing diagonals, such that every subpolygon is a triangle or a quadrilateral.

Example 3.2. Let us give here simple examples:



For an integer a , we denote $\bar{a} := a + 2\mathbb{Z}$ the projection of a to $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$; if a is odd then $\bar{a} = \bar{1}$, if a is even then $\bar{a} = \bar{0}$. The following notion is analogous to that of [6].

Definition 3.3. The quiddity of a (3|4)-dissection of a convex n -gon is the (cyclically ordered) sequence $(\bar{c}_1, \dots, \bar{c}_n)$ of elements of $\mathbb{Z} \rightarrow \mathbb{Z}/2\mathbb{Z}$, such that for every i

$$\bar{c}_i = \begin{cases} \bar{1}, & \text{if the number of triangles adjacent to the } i\text{-th vertex is odd;} \\ \bar{0}, & \text{if the number of triangles adjacent to the } i\text{-th vertex is even.} \end{cases}$$

Example 3.4. The quiddities $(\bar{c}_1, \dots, \bar{c}_n)$ of the (3|4)-dissections of Example 3.2 are as follows.

- (a) For the first two examples, $(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = (\bar{0}, \bar{0}, \bar{0}, \bar{0})$, $(\bar{c}_1, \bar{c}_2, \bar{c}_3, \bar{c}_4) = (\bar{0}, \bar{1}, \bar{0}, \bar{1})$, and these are the only quiddities for $n = 4$.
- (b) For the remaining examples one has $(\bar{c}_1, \dots, \bar{c}_5) = (\bar{1}, \bar{1}, \bar{1}, \bar{0}, \bar{0})$, $(\bar{c}_1, \dots, \bar{c}_6) = (\bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{0})$, and $(\bar{c}_1, \dots, \bar{c}_{10}) = (\bar{0}, \bar{1}, \bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{0}, \bar{1}, \bar{0}, \bar{0})$, respectively.

The following statement is our main result. It gives a combinatorial characteristic of solutions of equation (3.1) for $n \geq 3$. Note that the product of elements of $SL(2, \mathbb{Z})$ commutes with the projection of the entries of matrices to $\mathbb{Z}/2\mathbb{Z}$. This allows one to make all the computations in $SL(2, \mathbb{Z}/2\mathbb{Z})$.

Theorem 1. (i) Every quiddity of a (3|4)-dissection of an n -gon is a solution of equation (3.1).

(ii) Every solution of equation (3.1) with $n \geq 3$ is a quiddity of a (3|4)-dissection of an n -gon.

This statement is proved in Section 5.

Let us mention that the number of solutions of equation (3.1), for a fixed n , can be deduced from the main result of [13] and is given by the Jacobsthal sequence (A001045 in OEIS [15]).

4. RELATIONS IN $PSL(2, \mathbb{Z})$ AND POLYGON DISSECTIONS

We give a brief overview of the theorems of Conway and Coxeter [6] (see also [2, 10]) and Ovsienko [16]. The first one relates equation (2.2) to one of the most classical notion of combinatorics, namely that of triangulation of an n -gon, while the second describes all positive integer solutions of equation (2.2) in terms of polygon dissections. This overview will allow us to compare equation (2.2) and equation (3.1). It also makes the presentation complete.

4.1. Triangulations and friezes. Fix a triangulation of a convex n -gon. Following [6], we call a *quiddity* of the triangulation the sequence of positive integers (c_1, \dots, c_n) , where c_i is equal to the number of triangles adjacent to the i -th vertex of the n -gon.

The theorem of Conway and Coxeter can be formulated in the following way (cf. [16], Corollary 2.3).

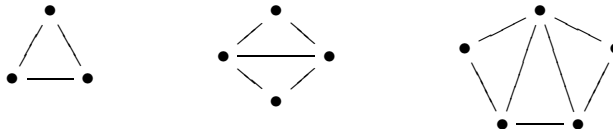
Theorem 2 ([6]). (i) The quiddity of a triangulation of an n -gon is a solution of $M(c_1, \dots, c_n) = -\text{Id}$.

(ii) Every solution (c_1, \dots, c_n) of equation (2.2), satisfying the condition

$$(4.1) \quad c_1 + c_2 + \dots + c_n = 3n - 6,$$

is the quiddity of a triangulation of an n -gon.

The simplest examples, with $n = 3, 4, 5$, are



It is easy to see that the corresponding quiddities $(c_1, c_2, c_3) = (1, 1, 1)$, $(c_1, c_2, c_3, c_4) = (1, 2, 1, 2)$ and $(c_1, c_2, c_3, c_4, c_5) = (1, 3, 1, 2, 2)$ are, indeed, solutions of $M(c_1, \dots, c_n) = -\text{Id}$.

The original formulation of Theorem 2 uses the beautiful notion of Coxeter's *frieze pattern*. Let us recall that a frieze pattern is an array of $(n - 1)$ infinite rows of positive integers with the rows 1 and $n - 1$ consisting in 1's. Every elementary 2×2 "diamond"

$$\begin{array}{ccc} & b & \\ a & & d \\ & c & \end{array},$$

of the frieze must satisfy the unimodular rule $ad - bc = 1$. Coxeter proved in [7] that the row 2 (and $n - 2$) is an n -periodic sequence satisfying equation (1.1). The Conway-Coxeter theorem [6] identifies Coxeter's friezes with triangulations.

Let us give here an example of a frieze pattern for $n = 5$.

$$\begin{array}{cccccc} \dots & 1 & 1 & 1 & 1 & 1 \\ & 1 & 3 & 1 & 2 & 2 & \dots \\ \dots & 2 & 2 & 1 & 3 & 1 \\ & 1 & 1 & 1 & 1 & 1 & \dots \end{array}$$

The 5-periodic sequence $(1, 3, 1, 2, 2)$ is the quiddity of a triangulation of a pentagon.

For a survey on frieze patterns; see [12]. Variants of frieze patterns involving other types of polygon dissections can be found in [4, 11, 9]. Links with frieze patterns and presentations of $\text{SL}(2, \mathbb{Z})$ also appeared in [3].

Remark 4.1. Let us mention that (4.1) turns out to be equivalent to the condition of total positivity; see [16], Corollary 2.3, and can be formulated in more standard terms of continued fractions and total positive (2×2) -matrices; see [14]. In terms used by Coxeter, this total positivity means that every entry of the frieze pattern is positive.

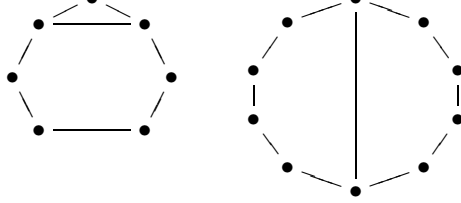
4.2. Complete solution of equation (2.2). For $n \geq 6$, there exist many solutions of equation (2.2) that cannot be obtained from triangulations of n -gons. The complete solution of equation (2.2) was given in [16] and led to the following notion of "3d-dissection".

Definition 4.2. (i) A *3d-dissection* is a partition of a convex n -gon into sub-polygons by means of pairwise non-crossing diagonals, such that the number of vertices of every sub-polygon is a multiple of 3.

(ii) The *quiddity* of a 3d-dissection of an n -gon is the (cyclically ordered) n -tuple of positive integers (c_1, \dots, c_n) such that c_i is the number of sub-polygons adjacent to the i -th vertex of the n -gon.

Theorem 3 ([16], Theorem 1). *Every quiddity of a 3d-dissection of an n -gon is a solution of equation (2.2). Conversely, every solution of equation (2.2) is a quiddity of a 3d-dissection of an n -gon.*

The simplest examples of $3d$ -dissections, that are not triangulations, are:



and the corresponding quiddities are: $(2, 1, 2, 1, 1, 1, 1)$ and $(2, 1, 1, 1, 1, 2, 1, 1, 1, 1)$.

4.3. The idea of the proof. The proof of Theorem 3 is inductive. The main idea uses the following “local surgery” operations.

$$\begin{aligned} (\alpha) \quad & (c_1, \dots, c_i, c_{i+1}, \dots, c_n) \mapsto (c_1, \dots, c_i + 1, 1, c_{i+1} + 1, \dots, c_n), \\ (\beta) \quad & (c_1, \dots, c_{i-1}, c_i, c_{i+1}, \dots, c_n) \mapsto (c_1, \dots, c_{i-1}, c'_i, 1, 1, c''_i, c_{i+1}, \dots, c_n), \end{aligned}$$

where $c'_i + c''_i = c_i + 1$. One readily checks that the matrix $M(c_1, \dots, c_n)$ is invariant under the operations of type (α) and changes the sign under the operations of type (β) .

A simple lemma then states that a sequence of positive integers (c_1, \dots, c_n) satisfying equation (2.2) always has some entries $c_i = 1$; cf. [6, 16]. This allows one to construct any solution of equation (2.2) from the simplest solution $(1, 1, 1)$.

The inductive step in the proof is based on the observation that the above surgery operations have a combinatorial meaning. Given a dissection of an n -gon, the operation (α) consists in gluing an additional triangle on the edge $(i, i + 1)$, while the operation (β) changes a $3k$ -gon adjacent to the i -th vertex in the dissection into a $(3k + 3)$ -gon; see [16].

5. PROOF OF THEOREM 1

Our proof of Theorem 1 is quite similar to that of Theorem 3. We give an inductive procedure of construction of all the solutions of equation (3.1).

5.1. Local surgery. Consider the following two families of “local surgery” operations for sequences of elements of $\mathbb{Z}/2\mathbb{Z}$.

(a) Operations of the first family insert $\bar{1}$ into the sequence $(\bar{c}_1, \bar{c}_2, \dots, \bar{c}_n)$:

$$(\bar{c}_1, \dots, \bar{c}_i, \bar{c}_{i+1}, \dots, \bar{c}_n) \mapsto (\bar{c}_1, \dots, \bar{c}_i + \bar{1}, \bar{1}, \bar{c}_{i+1} + \bar{1}, \dots, \bar{c}_n).$$

(b) Operations from the second family insert $\bar{0}$ in two consecutive positions:

$$(\bar{c}_1, \dots, \bar{c}_i, \bar{c}_{i+1}, \dots, \bar{c}_n) \mapsto (\bar{c}_1, \dots, \bar{c}_i, \bar{0}, \bar{0}, \bar{c}_{i+1}, \dots, \bar{c}_n).$$

Within the cyclic ordering, the operations (a) and (b) are defined for all $1 \leq i \leq n$. Every operation (a) transforms a sequence of n elements of $\mathbb{Z}/2\mathbb{Z}$ into a sequence of $(n + 1)$ elements of $\mathbb{Z}/2\mathbb{Z}$, and every operation (b) transforms a sequence of n elements of $\mathbb{Z}/2\mathbb{Z}$ into a sequence of $(n + 2)$ elements of $\mathbb{Z}/2\mathbb{Z}$.

The following statement means that equation (3.1) is invariant under the operations (a) and (b).

Lemma 5.1. *One has in the group $SL(2, \mathbb{Z}/2\mathbb{Z})$*

$$\begin{aligned} M(\bar{c}_1, \dots, \bar{c}_{n+1}) &= M(\bar{c}_1, \dots, \bar{c}_i + \bar{1}, \bar{1}, \bar{c}_{i+1} + \bar{1}, \dots, \bar{c}_n), \\ M(\bar{c}_1, \dots, \bar{c}_{n+1}) &= -M(\bar{c}_1, \dots, \bar{c}_i, \bar{0}, \bar{0}, \bar{c}_{i+1}, \dots, \bar{c}_n). \end{aligned}$$

Proof. An operation of type (a) replaces the matrix $\begin{pmatrix} \overline{c_i} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_{i+1}} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix}$ by

$$\begin{aligned} \begin{pmatrix} \overline{c_i + 1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_{i+1} + 1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} &= \begin{pmatrix} \overline{c_i} & \overline{c_i + 1} \\ \overline{1} & \overline{1} \end{pmatrix} \begin{pmatrix} \overline{c_{i+1} + 1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \\ &= \begin{pmatrix} \overline{c_i c_{i+1} + 1} & \overline{c_i} \\ \overline{c_{i+1}} & \overline{1} \end{pmatrix} \\ &= \begin{pmatrix} \overline{c_i} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_{i+1}} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix}. \end{aligned}$$

Therefore, $M(\overline{c_1}, \dots, \overline{c_{n+1}}) = M(\overline{c_1}, \dots, \overline{c_i} + \overline{1}, \overline{1}, \overline{c_{i+1}} + \overline{1}, \dots, \overline{c_n})$, as an element of $\text{SL}(2, \mathbb{Z}/2\mathbb{Z})$. An operation of type (b) adds $-Id$ in the product defining $M_n(\overline{c_1}, \dots, \overline{c_n})$. \square

5.2. The special cases $n = 2$ and $n = 3$. The equation (3.1) has no solution if $n = 1$. Consider now the simplest cases $n = 2$ and $n = 3$.

Lemma 5.2. (i) For $n = 2$, $(\overline{c_1}, \overline{c_2})$ is a solution of equation (3.1) if and only if $(\overline{c_1}, \overline{c_2}) = (\overline{0}, \overline{0})$.
(ii) For $n = 3$, $(\overline{c_1}, \overline{c_2}, \overline{c_3})$ is a solution of equation (3.1) if and only if $(\overline{c_1}, \overline{c_2}, \overline{c_3}) = (\overline{1}, \overline{1}, \overline{1})$.

Proof. Part(i). This follows from $\begin{pmatrix} \overline{c_1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_2} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{c_1 c_2 + 1} & \overline{c_1} \\ \overline{c_2} & \overline{1} \end{pmatrix}$.

Part (ii). This follows from

$$\begin{aligned} \begin{pmatrix} \overline{c_1} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_2} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \begin{pmatrix} \overline{c_3} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} &= \begin{pmatrix} \overline{c_1 c_2 + 1} & \overline{c_1} \\ \overline{c_2} & \overline{1} \end{pmatrix} \begin{pmatrix} \overline{c_3} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix} \\ &= \begin{pmatrix} \overline{c_1 c_2 c_3 + c_1 + c_3} & \overline{c_1 c_2 + 1} \\ \overline{c_2 c_3 + 1} & \overline{c_2} \end{pmatrix}. \end{aligned}$$

Hence the result. \square

5.3. Inductive construction. We now give an inductive procedure for construction of all the solutions of equation (3.1) starting from the simplest case $n = 2$ and the corresponding solution $(\overline{0}, \overline{0})$.

Proposition 5.3. An n -tuple $(\overline{c_1}, \dots, \overline{c_n})$ is a solution of equation (3.1) if and only if the sequence $(\overline{c_1}, \dots, \overline{c_n})$ can be obtained from $(\overline{0}, \overline{0})$ by applying the operations (a) and (b) in any order.

Proof. The “if” part follows from Lemma 5.1.

Conversely, given a solution $(\overline{c_1}, \dots, \overline{c_n})$ of equation (3.1), one has the following two possibilities.

A) One has $\overline{c_i} = \overline{0}$ for all i . Then n is even and $(\overline{c_1}, \dots, \overline{c_n})$ is obtained from $(\overline{0}, \overline{0})$ by a sequence of $\frac{n-2}{2}$ operations of type (b).

B) $\overline{c_i} = \overline{1}$ for some i , $1 \leq i \leq n$. Then the inverse of the operation of type (a) centered at i can be applied to $(\overline{c_1}, \dots, \overline{c_n})$. This results in an $(n-1)$ -tuple $(\overline{c_1}, \dots, \overline{c_{i-1}} + \overline{1}, \overline{c_{i+1}} + \overline{1}, \dots, \overline{c_n})$. The same computation as in the proof of Lemma 5.1 implies that this $(n-1)$ -tuple is a solution of equation (3.1). We conclude by induction assumption. \square

Let us mention that there exists an analog of Proposition 5.3 in the case of non-negative integer solutions of (1.1), see [8, Thm 3.1].

5.4. End of the proof of Theorem 1. We will need the following combinatorial interpretation of operation (a) and (b). Let $(\overline{c}_1, \dots, \overline{c}_n)$ be a sequence corresponding to a (3|4)-dissection of a convex n -gon, then the result of either operation is again a sequence corresponding to a (3|4)-dissection of a convex $(n + 1)$ -gon, or $(n + 2)$ -gon, respectively.

- (i) The operation (a) glues to a (3|4)-dissection a triangle on the segment $(i, i + 1)$.
- (ii) The operation (b) glues a quadrilateral on the segment $(i, i + 1)$.

We are ready to complete the proof of Theorem 2.

Part (i). The induction basis consists of two cases, $n = 3$ and $n = 4$. For $n = 3$, the quiddity of a (3|4)-dissection of a triangle is $(\overline{1}, \overline{1}, \overline{1})$ which is a solution of equation (3.1). For $n = 4$, the quiddity of a (3|4)-dissection of a quadrilateral is $(\overline{1}, \overline{0}, \overline{1}, \overline{0})$ (quadrilateral cut into two triangles) and $(\overline{0}, \overline{0}, \overline{0}, \overline{0})$ (quadrilateral alone) and it follows from Lemma 5.1 that they are solutions of equation (3.1).

Assume that an n -tuple $(\overline{c}_1, \dots, \overline{c}_n)$ is the quiddity of a (3|4)-dissection of a convex n -gon. Every (3|4)-dissection has (at least one) exterior triangle (such a triangle is sometimes called ‘‘an ear’’ in the literature), or quadrilateral. Cutting this exterior piece, one obtains either an $(n - 1)$ -tuple or an $(n - 2)$ -tuple of elements of $(\mathbb{Z}/2\mathbb{Z})$ which is the quiddity of a (3|4)-dissection of a convex $(n - 1)$ -gon or a convex $(n - 2)$ -gon. The result then follows from Lemma 5.1 and the induction assumption.

Part (ii). For $n = 3$, a triple $(\overline{c}_1, \overline{c}_2, \overline{c}_3)$ is a solution of equation (3.1) if and only if $(\overline{c}_1, \overline{c}_2, \overline{c}_3) = (\overline{1}, \overline{1}, \overline{1})$, which corresponds to a triangle. Similarly to Lemma 5.2, one shows the following, for $n = 4$, the (cyclically ordered) solutions are $(\overline{1}, \overline{0}, \overline{1}, \overline{0})$ and $(\overline{0}, \overline{0}, \overline{0}, \overline{0})$ already considered in Example 3.2.

Assume that a sequence $(\overline{c}_1, \dots, \overline{c}_n)$ is a solution of equation (3.1), and let us show that it is the quiddity of a (3|4)-dissection of a convex n -gon. By Proposition 5.3, this sequence is obtained from $(\overline{0}, \overline{0})$ by a sequence of the surgery operations (a) and (b).

If $\overline{c}_i = \overline{1}$ for some i , where $0 \leq i \leq n$, then, by induction assumption, the sequence

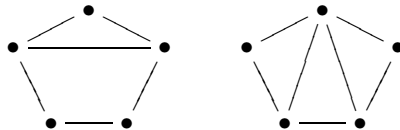
$$(\overline{c}_1, \dots, \overline{c_{i-1} + \overline{1}}, \overline{c_{i+1} + \overline{1}}, \dots, \overline{c}_n)$$

is the quiddity of a (3|4)-dissection of a convex $(n - 1)$ -gon. Therefore, $(\overline{c}_1, \dots, \overline{c}_n)$ is the quiddity of a (3|4)-dissection of a convex n -gon, obtained from this (3|4)-dissection by the gluing of a triangle.

If $\overline{c}_i = \overline{c_{i+1}} = \overline{0}$, then the sequence is of the form $(\overline{c}_1, \dots, \overline{c_{i-1}}, \overline{0}, \overline{0}, \overline{c_{i+2}}, \dots, \overline{c}_n)$. By induction assumption, $(\overline{c}_1, \dots, \overline{c_{i-1}}, \overline{c_{i+2}}, \dots, \overline{c}_n)$ is the sequence associated to a (3|4)-dissection of a convex $(n - 2)$ -gon. Therefore, $(\overline{c}_1, \dots, \overline{c}_n)$ is the quiddity of a (3|4)-dissection of a convex n -gon, obtained from this (3|4)-dissection by the gluing of a quadrilateral.

Theorem 1 is proved.

Remark 5.4. Part (ii) of Theorem 1 can be strengthened. Let (c_1, \dots, c_n) be a solution of equation (3.1). Assume that at least one element \overline{c}_i of $\mathbb{Z}/2\mathbb{Z}$ different from $\overline{0}$ (i.e., that not all of c_i are even). It turns out that, under this assumption, $(\overline{c}_1, \dots, \overline{c}_n)$ is the quiddity of a triangulation of a convex n -gon. For example, the two following (3|4)-dissections have the same quiddity $(\overline{1}, \overline{1}, \overline{1}, \overline{0}, \overline{0})$.



The proof of this strengthened statement is very similar to that of Theorem 1, Part (ii). It uses the following idea: if an $(n - 1)$ -tuple, obtained by applying the operation inverse to (a) centered at i to an n -tuple $(\overline{c}_1, \dots, \overline{c}_n)$, contains only $\overline{0}$, then the $(n - 1)$ -tuple obtained by the operation inverse to (a) and centered at $i + 1$ to the n -tuple $(\overline{c}_1, \dots, \overline{c}_n)$ contains an element different from $\overline{0}$.

Remark 5.5. Let us mention that Part (i) of Theorem 1 can be deduced from the combinatorial model and results of [9, Thm 7.3]. One can also deduce from this model that every quiddity $(\overline{c}_1, \dots, \overline{c}_n)$ coming from a (3|4)-dissection can be lift to an integer solution (c_1, \dots, c_n) of Equation (1.1).

We end the note with the following concluding remark.

The equation (3.1) naturally extends to arbitrary principal congruence subgroup $\Gamma(N)$ in $\mathrm{SL}(2, \mathbb{Z})$, and it would be interesting to find combinatorial description of the set of solutions in the general situation.

Acknowledgments. We are grateful to Valentin Ovsienko for the statement of the problem and guidance of this work. We are also grateful to Michael Cuntz and Thorsten Holm for valuable comments.

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