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ANALYTIC SMOOTHING EFFECT FOR THE NONLINEAR LANDAU EQUATION OF MAXWELLIAN MOLECULES

Y. MORIMOTO & C.-J. XU

ABSTRACT. We consider the Cauchy problem of the nonlinear Landau equation of Maxwellian molecules, under the perturbation frame work to global equilibrium. We show that if $H_x^r(L_v^2)$, $r > 3/2$ norm of the initial perturbation is small enough, then the Cauchy problem of the nonlinear Landau equation admits a unique global solution which becomes analytic with respect to both position x and velocity v variables for any time $t > 0$. This is the first result of analytic smoothing effect for the spatially inhomogeneous nonlinear kinetic equation. The method used here is microlocal analysis and energy estimates. The key point is adopting a time integral weight associated with the kinetic transport operator.

1. INTRODUCTION

We consider the Cauchy problem for the spatially inhomogeneous Landau equation, a kinetic model from plasma physics that describes the evolution of a particle density $f(t, x, v) \geq 0$ with position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$ at time t . It reads

$$(1.1) \quad \begin{cases} \partial_t f + v \cdot \nabla_x f = Q_L(f, f), \\ f|_{t=0} = f_0, \end{cases}$$

where the term $Q_L(f, f)$ corresponds to the Landau collision operator associated to the bilinear operator

$$Q_L(g, f) = \nabla_v \cdot \left(\int_{\mathbb{R}^3} a(v - v_*) (g(v_*) (\nabla_v f)(v) - (\nabla_v g)(v_*) f(v)) dv_* \right).$$

Here $a = (a_{i,j})_{1 \leq i, j \leq 3}$ stands for the non-negative symmetric matrix

$$(1.2) \quad a(v) = |v|^\gamma (|v|^2 \mathbf{I} - v \otimes v) = |v|^{\gamma+2} \mathbb{P}_{v^\perp} \in M_3(\mathbb{R}), \quad -3 \leq \gamma < +\infty,$$

where \mathbb{P}_{v^\perp} is the orthogonal projection on v^\perp .

The Landau equation with $\gamma = -3$ was first proposed in 1936 [24] by the Russian theoretical physicist Lev Davidovitch Landau, as a transport equation for a system of charged particles, where the long range of the Coulomb interactions makes it impossible to use the normal Boltzmann equation. It soon became (in combination with the Vlasov equation) the most important mathematical kinetic model in the theory of collisional plasma.

The ‘‘generalized’’ Landau equation with $\gamma > -3$ was independently introduced by several authors (see e.g. [7, 15]). It plays a role of a model of the Boltzmann

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equation for various interactions including inverse power law potential ρ^{-n+1} , $n > 2$. This equation can be obtained as a limit of the Boltzmann equation when grazing collisions prevails (see [35, 36] for a detailed study of the limiting process and further references on the subjects). Though the Landau equation has no relation to physics in the non-Coulomb case, from various mathematical points of view, it has been intensively studied by mathematicians in last two decades, because it is a simple approximation of the non-cutoff Boltzmann equation (see, e.g. [1, 25]). We refer the reader to the surveys [27, 36, 10] and recent papers [9, 8, 13], as well as to the references therein for matters related to the derivation and basic results for that equation.

In this article, we focus our attention on the analytic smoothing effect of a solution for the Cauchy problem (1.1). More specifically, we study the Landau equation with Maxwellian molecules in *a close to equilibrium* framework. The Maxwellian molecules corresponds to the case when the parameter $\gamma = 0$ in the cross section (1.2). We consider the fluctuation

$$f = \mu + \sqrt{\mu}g,$$

around the Maxwellian equilibrium distribution

$$(1.3) \quad \mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

This distribution is a stationary solution for the Landau equation since it only depends on the velocity variable v and the fact that $Q_L(\mu, \mu) = 0$. We consider the linearized Landau operator around this equilibrium distribution given by

$$\mathcal{L}g = -\mu^{-1/2}Q_L(\mu, \mu^{1/2}g) - \mu^{-1/2}Q_L(\mu^{1/2}g, \mu).$$

The Cauchy problem for the Landau equation (1.1) is then reduced to the one for the fluctuation

$$(1.4) \quad \begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g), \\ g|_{t=0} = g_0, \end{cases}$$

with

$$\Gamma(g, f) = \mu^{-1/2}Q_L(\sqrt{\mu}g, \sqrt{\mu}f).$$

To state our main result, we define the Sobolev space $H_x^r(L_v^2)$ for $r \geq 0$ by

$$H_x^r(L_v^2) = \{u \in L^2(\mathbb{R}_{x,v}^6); \langle D_x \rangle^r u \in L^2(\mathbb{R}_{x,v}^6)\},$$

with $\langle D_x \rangle = \sqrt{1 - \Delta_x}$ and $D_x = -i\partial_x$. Let $\mathcal{A}(\mathbb{R}^n)$ denote the analytic function space on \mathbb{R}^n .

Theorem 1.1. *Let $r > 3/2$. There exists a small constant $\epsilon_0 > 0$ such that for all $g_0 \in H_x^r(L_v^2)$ satisfying*

$$\|g_0\|_{H_x^r(L_v^2)} \leq \epsilon_0,$$

the Cauchy problem (1.4) admits a unique global solution such that

$$g(t) \in \mathcal{A}(\mathbb{R}_{x,v}^6), \quad \forall t > 0.$$

Furthermore, there exists a $c_0 > 0$ such that,

$$(1.5) \quad e^{c_0\{\tilde{t}^2(-\Delta_x)^{1/2} + \tilde{t}(-\Delta_v)^{1/2}\}} g(t) \in L^\infty([0, +\infty), H_x^r(L_v^2)),$$

where $\tilde{t} = \min\{1, t\}$ for $t \geq 0$.

This article concerns the existence of a solution to the Cauchy problem for the Landau equation, as well as the smoothing properties of that solution, which is a topic studied in many previous works, as stated above. Among them we refer [35, 17, 18, 16] concerning the existence result, [32, 28] about higher regularity such as (ultra-)analyticity, in the spatially homogeneous case, while in the spatially inhomogeneous case, [5] concerning renormalized solution with defect measure, [22, 33, 14, 13, 12, 19] in a close to equilibrium setting, and recent regularity results by [11, 20, 23] under boundedness conditions on the mass, energy, entropy densities. Also we want to mention the related works on the non cut-off Boltzmann equation, e.g., the papers by [1, 30, 29, 21, 2, 3, 4, 26, 31, 6].

The rest of paper is arranged as follows: In Section 2 we recall the exact expression of the linear operator given in [25], introduce a version of the exponential weight used in the previous work [32], and show the estimate for the linear term with the exponential weight. In Section 3 we give an explicit form of the nonlinear term by means of creation and annihilation operators and spherical derivatives. In Section 4, we use this to show a trilinear estimate for the inner product of the nonlinear term and test function with exponential weight. Section 5 is devoted to complete the proof of the main theorem by constructing the time local solution with analytic smoothing property and by combining this and the known global existence result.

2. FOURIER ANALYSIS OF LINEAR LANDAU OPERATOR

With μ defined in (1.3), the linearized Landau operator \mathcal{L} is defined by

$$\mathcal{L}g = -\mu^{-1/2}Q_L(\mu, \mu^{1/2}g) - \mu^{-1/2}Q_L(\mu^{1/2}g, \mu),$$

is an unbounded symmetric operator on $L^2(\mathbb{R}_v^3)$. In the case of Maxwellian molecules, the linearized Landau operator may be computed explicitly (see e.g. [25], Proposition 1), and we have

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2,$$

with

$$\mathcal{L}_1 = 2\left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\right) - \Delta_{\mathbb{S}^2},$$

$$\begin{aligned} \mathcal{L}_2 &= \left[\Delta_{\mathbb{S}^2} - 2\left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\right)\right]\mathbb{P}_1 \\ &\quad + \left[-\Delta_{\mathbb{S}^2} - 2\left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2}\right)\right]\mathbb{P}_2 \end{aligned}$$

where

$$\Delta_{\mathbb{S}^2} = \frac{1}{2} \sum_{\substack{1 \leq j, k \leq 3 \\ j \neq k}} L_{k,j}^2, \quad L_{k,j} = v_j \partial_{v_k} - v_k \partial_{v_j},$$

stands for the Laplace-Beltrami operator on the unit sphere \mathbb{S}^2 , and \mathbb{P}_k denotes the orthogonal projection onto the Hermite basis $\mathcal{E}_k = \text{Span}\{\Phi_\alpha\}_{\alpha \in \mathbb{N}^3, |\alpha|=k}$. Here $\Phi_0(v) = \mu^{1/2}(v)$,

$$\Phi_\alpha = \frac{1}{\sqrt{\alpha!}} a_{+,1}^{\alpha_1} a_{+,2}^{\alpha_2} a_{+,3}^{\alpha_3} \Phi_0, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \quad \alpha! = \alpha_1! \alpha_2! \alpha_3!,$$

with

$$a_{\pm,j} = \frac{v_j}{2} \mp \frac{\partial}{\partial v_j}, \quad 1 \leq j \leq 3.$$

We have then,

$$\begin{aligned} (\mathcal{L}_1 g, g)_{L^2(\mathbb{R}_v^3)} &= 2 \sum_{j=1}^3 \left(\|\partial_{v_j} g\|_{L^2(\mathbb{R}_v^3)}^2 + \frac{1}{4} \|v_j g\|_{L^2(\mathbb{R}_v^3)}^2 \right) \\ &\quad + \frac{1}{2} \sum_{\substack{1 \leq j,k \leq 3 \\ j \neq k}} \|L_{k,j} g\|_{L^2(\mathbb{R}_v^3)}^2 - 3 \|g\|_{L^2(\mathbb{R}_v^3)}^2. \end{aligned}$$

The operators \mathcal{L}_2 is bounded in $L^2(\mathbb{R}_v^3)$. Putting

$$\begin{aligned} \|g\|_v^2 &= 2 \sum_{j=1}^3 \left(\|\partial_{v_j} g\|_{L^2(\mathbb{R}_v^3)}^2 + \frac{1}{4} \|v_j g\|_{L^2(\mathbb{R}_v^3)}^2 \right) + \frac{1}{2} \sum_{\substack{1 \leq j,k \leq 3 \\ j \neq k}} \|L_{k,j} g\|_{L^2(\mathbb{R}_v^3)}^2 \\ \|g\|_{r,0}^2 &= 2 \sum_{j=1}^3 \left(\|\partial_{v_j} g\|_{H_x^r(L_v^2)}^2 + \frac{1}{4} \|v_j g\|_{H_x^r(L_v^2)}^2 \right) + \frac{1}{2} \sum_{\substack{1 \leq j,k \leq 3 \\ j \neq k}} \|L_{k,j} g\|_{H_x^r(L_v^2)}^2, \end{aligned}$$

we have the following coercive estimates; there exists $C > 0$ such that for all $g \in \mathcal{S}(\mathbb{R}^3)$,

$$\|g\|_v^2 \leq (\mathcal{L} g, g)_{L^2(\mathbb{R}_v^3)} + C \|g\|_{L^2(\mathbb{R}_v^3)}^2.$$

Ultra-analytic smoothing effect of Kolmogorov equation. We recall the Cauchy problem of Kolmogorov equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f - \Delta_v f = 0, \\ f(0, x, v) = f_0(x, v) \in L^2(\mathbb{R}_{x,v}^{2n}). \end{cases}$$

Using the Fourier transform $(x, v) \leftrightarrow (\eta, \xi)$ we have

$$\hat{f}(t, \eta, \xi) = e^{-\int_0^t |\xi + \rho\eta|^2 d\rho} \hat{f}_0(\eta, \xi + t\eta)$$

because $(\partial_t - \eta \cdot \nabla_\xi + |\xi|^2) \hat{f} = 0$. Using the following Ukai inequality (2.1) with $\alpha = 2$,

$$\exists c > 0, \quad c(t|\xi|^2 + t^3|\eta|^2) \leq \int_0^t |\xi + \rho\eta|^2 d\rho.$$

Then we have

$$e^{c(-t\Delta_v - t^3\Delta_x)} f(t, \cdot, \cdot) \in L^2(\mathbb{R}_{x,v}^{2n}),$$

which shows the ultra-analytic smoothing effect of (x, v) variables for any $t > 0$.

Lemma 2.1. *For any $\alpha > 0$, there exists $c_\alpha > 0$ such that*

$$(2.1) \quad \int_0^t (1 + |\xi + \rho\eta|^2)^{\alpha/2} d\rho \geq c_\alpha t \{1 + |\xi|^2 + t^2|\eta|^2\}^{\alpha/2},$$

for all $t > 0$.

Remark 2.2. *The following simple proof is due to Seiji Ukai. On the other hand, there exists $C_\alpha > 0$ such that*

$$(2.2) \quad \int_0^t (1 + |\xi + \rho\eta|^2)^{\alpha/2} d\rho \leq C_\alpha t \{1 + |\xi|^2 + t^2|\eta|^2\}^{\alpha/2}, \quad \forall t > 0.$$

Proof. Put $\rho = t\tau$, $\tilde{\eta} = -t\eta$, then the estimate (2.1) is equivalent to

$$\int_0^1 \langle \xi - \tau\tilde{\eta} \rangle^\alpha d\tau \geq c_\alpha \{1 + |\xi|^2 + |\tilde{\eta}|^2\}^{\alpha/2},$$

with the notation $\langle \cdot \rangle = \sqrt{1 + |\cdot|^2}$. Since the case $\tilde{\eta} = 0$ is trivial, we assume $\tilde{\eta} \neq 0$. Notice

$$2 \int_0^1 \langle \xi - \tau\tilde{\eta} \rangle^\alpha d\tau \geq 1 + \int_0^1 |\xi - \tau\tilde{\eta}|^\alpha d\tau.$$

If $|\xi| \geq |\tilde{\eta}|$

$$\begin{aligned} \int_0^1 |\xi - \tau\tilde{\eta}|^\alpha d\tau &\geq |\xi|^\alpha \int_0^1 \left(1 - \tau \frac{|\tilde{\eta}|}{|\xi|}\right)^\alpha d\tau \geq |\xi|^\alpha \int_0^1 (1 - \tau)^\alpha d\tau \\ &= \frac{|\xi|^\alpha}{\alpha + 1} \geq \frac{1}{2^\alpha(\alpha + 1)} (|\xi|^2 + |\tilde{\eta}|^2)^{\alpha/2}. \end{aligned}$$

If $|\xi| < |\tilde{\eta}|$

$$\begin{aligned} \int_0^1 |\xi - \tau\tilde{\eta}|^\alpha d\tau &\geq |\tilde{\eta}|^\alpha \int_0^1 \left|\tau - \frac{|\xi|}{|\tilde{\eta}|}\right|^\alpha d\tau \\ &= |\tilde{\eta}|^\alpha \left\{ \int_0^{|\xi|/|\tilde{\eta}|} \left(\frac{|\xi|}{|\tilde{\eta}|} - \tau\right)^\alpha d\tau + \int_{|\xi|/|\tilde{\eta}|}^1 \left(\tau - \frac{|\xi|}{|\tilde{\eta}|}\right)^\alpha d\tau \right\} \\ &\geq \frac{|\tilde{\eta}|^\alpha}{\alpha + 1} \min_{0 \leq \theta \leq 1} (\theta^{\alpha+1} + (1 - \theta)^{\alpha+1}) = \frac{|\tilde{\eta}|^\alpha}{2^\alpha(\alpha + 1)} \\ &\geq \frac{1}{2^{2\alpha}(\alpha + 1)} (|\xi|^2 + |\tilde{\eta}|^2)^{\alpha/2}. \end{aligned}$$

We finally get (2.1). \square

We set

$$\Psi(t, \eta, \xi) = c_0 \int_0^t \langle \xi + \rho\eta \rangle d\rho = c_0 \int_0^t \langle \xi + (t - \rho)\eta \rangle d\rho,$$

for a sufficiently small $c_0 > 0$ which will be chosen later on. We have that

$$(\partial_t - \eta \cdot \nabla_\xi) \Psi = c_0 \langle \xi \rangle.$$

Then we can use (2.1) with $\alpha = 1$ to estimate Ψ .

To study the Gevery (and analytic) regularity of kinetic equations, exponential type weights were used in [30, 32] (see also [6]). Now we set

$$F_{\delta, \delta'}(t, \eta, \xi) = \frac{e^\Psi}{(1 + \delta e^\Psi)(1 + \delta' \Psi)^r}$$

for $0 < \delta \leq 1$, $r > 3/2$, $0 < r\delta' \leq 1$. Without the confusion, we use the same notation $F_{\delta, \delta'}$ for the symbol $F_{\delta, \delta'}(t, \eta, \xi)$ and also the pseudo-differential operator $F_{\delta, \delta'}(t, D_x, D_v)$. If A is a first order differential operator of (t, η, ξ) variables, then we have

$$(2.3) \quad AF_{\delta, \delta'} = \left(\frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right) (A\Psi)F_{\delta, \delta'},$$

and

$$\left| \frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right| \leq 1, \text{ since } 0 \leq a, b \leq 1 \text{ implies } |a - b| \leq 1.$$

We study now the apriori estimate of solution $g \in L^\infty(]0, T[, H_x^r(L_v^2))$ of the equation

$$\partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(g, g).$$

Since in this case,

$$\begin{aligned} v \cdot \nabla_x g &\in L^\infty(]0, T[, H_x^{r-1}(L_{-1}^2(\mathbb{R}_v^3))), \\ \mathcal{L}g, \Gamma(g, g) &\in L^\infty(]0, T[, H_x^r(H_{-2}^{-2}(\mathbb{R}_v^3))), \end{aligned}$$

we will take

$$\tilde{g} = F_{\delta, \delta'}(t, D_x, D_v) \langle \delta' v \rangle^{-4} F_{\delta, \delta'}(t, D_x, D_v) g \in L^\infty(]0, T[, H_4^{2r}(\mathbb{R}_{x,v}^6))$$

as test function, where $H_\ell^m(\mathbb{R}_{x,v}^6)$ is the weighted Sobolev space of order ℓ with respect to v variable. Taking the $H_x^r(L_v^2)$ inner product for $r > \frac{3}{2}$, we get,

$$\begin{aligned} &\left((\partial_t + v \cdot \nabla_x)g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} + \left(\mathcal{L}g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &= \left(\Gamma(g, g), F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)}. \end{aligned}$$

For the first term, we have

Proposition 2.3. *There exist $C_1, C_2 > 0$ independent of δ, δ' such that, for $0 < t \leq 1$,*

$$\begin{aligned} &\left((\partial_t + v \cdot \nabla_x)g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &\geq \frac{1}{2} \frac{d}{dt} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 - c_0 C_1 \|\langle D_v \rangle \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 \\ &\quad - C_2 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}. \end{aligned}$$

Proof. Using the Plancherel formula, we have

$$\begin{aligned} &\left((\partial_t + v \cdot \nabla_x)g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &= \frac{1}{(2\pi)^6} \left(\langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} (\partial_t - \eta \cdot \nabla_\xi) \hat{g}, \langle \eta \rangle^{2r} \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}(t, \eta, \xi) \right)_{L_{\eta, \xi}^2} \\ &= \frac{1}{(2\pi)^6} \left((\partial_t - \eta \cdot \nabla_\xi) \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}, \langle \eta \rangle^{2r} \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}(t, \eta, \xi) \right)_{L_{\eta, \xi}^2} \\ &\quad + \frac{1}{(2\pi)^6} \left([\langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'}, (\partial_t - \eta \cdot \nabla_\xi)] \hat{g}, \langle \eta \rangle^{2r} \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}(t, \eta, \xi) \right)_{L_{\eta, \xi}^2}, \end{aligned}$$

and

$$\begin{aligned} &\left((\partial_t - \eta \cdot \nabla_\xi) \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}, \langle \eta \rangle^{2r} \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}(t, \eta, \xi) \right)_{L_{\eta, \xi}^2} \\ &= \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^6} |\langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'} \hat{g}(t, \eta, \xi)|^2 \langle \eta \rangle^{2r} \frac{d\eta d\xi}{(2\pi)^6}. \end{aligned}$$

We study now the commutators term. Since $(\partial_t - \eta \cdot \nabla_\xi) \Psi = c_0 \langle \xi \rangle$, we have

$$-[F_{\delta, \delta'}, (\partial_t - \eta \cdot \nabla_\xi)] = (\partial_t - \eta \cdot \nabla_\xi) F_{\delta, \delta'} = c_0 \langle \xi \rangle \left(\frac{1}{1 + \delta e^\Psi} - \frac{r \delta'}{1 + \delta' \Psi} \right) F_{\delta, \delta'},$$

and

$$\begin{aligned} & [\langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'}, (\partial_t - \eta \cdot \nabla_\xi)] = \langle \delta' D_\xi \rangle^{-2} [F_{\delta, \delta'}, (\partial_t - \eta \cdot \nabla_\xi)] \\ & = -c_0 \left(\langle \delta' D_\xi \rangle^{-2} \langle \xi \rangle \left(\frac{1}{1 + \delta e^\Psi} - \frac{r \delta'}{1 + \delta' \Psi} \right) \langle \delta' D_\xi \rangle^2 \right) \langle \delta' D_\xi \rangle^{-2} F_{\delta, \delta'}. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \langle \delta' D_\xi \rangle^{-2} \left(\frac{\langle \xi \rangle}{1 + \delta e^\Psi} - \frac{r \delta' \langle \xi \rangle}{1 + \delta' \Psi} \right) \langle \delta' D_\xi \rangle^2 &= \langle \xi \rangle \left(\frac{1}{1 + \delta e^\Psi} - \frac{r \delta'}{1 + \delta' \Psi} \right) \\ &+ \langle \delta' D_\xi \rangle^{-2} \left[\langle \xi \rangle \left(\frac{1}{1 + \delta e^\Psi} - \frac{r \delta'}{1 + \delta' \Psi} \right), \langle \delta' D_\xi \rangle^2 \right], \end{aligned}$$

and

$$\begin{aligned} & \langle \delta' D_\xi \rangle^{-2} \left[\langle \xi \rangle \left(\frac{1}{1 + \delta e^\Psi} - \frac{r \delta'}{1 + \delta' \Psi} \right), \langle \delta' D_\xi \rangle^2 \right] \\ &= 2 \frac{(\delta')^2 D_\xi}{\langle \delta' D_\xi \rangle^2} \cdot \left(D_\xi \left(\frac{\langle \xi \rangle}{1 + \delta e^\Psi} - \frac{r \delta' \langle \xi \rangle}{1 + \delta' \Psi} \right) \right) \\ &- \langle \delta' D_\xi \rangle^{-2} (\delta')^2 \left(D_\xi^2 \left(\frac{\langle \xi \rangle}{1 + \delta e^\Psi} - \frac{r \delta' \langle \xi \rangle}{1 + \delta' \Psi} \right) \right). \end{aligned}$$

Now using

$$(2.4) \quad |\partial_{\xi_j} \Psi| = c_0 \left| \int_0^t \frac{\xi_j - \rho \eta_j}{\langle \xi - \rho \eta \rangle} d\rho \right| \leq c_0 t, \quad |\partial_\xi^\alpha \Psi| \leq C_\alpha c_0 t \text{ for } |\alpha| \geq 2,$$

we can complete the proof of Proposition 2.3. \square

We study now terms concerning linear operators.

Proposition 2.4. *If $0 < c_0 \ll 1$, then there exists a $C > 0$ independent of $\delta, \delta' > 0$ such that for any $0 < t \leq 1$ we have*

$$\begin{aligned} & \left(\mathcal{L}_1 g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ & \geq \frac{1}{2} \| \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \|_{r,0}^2 - C \| \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \|_{H_x^r(L_v^2)}^2. \end{aligned}$$

Proof. Note that

$$\mathcal{L}_1 = 2 \sum_{j=1}^3 \left(D_{v_j}^2 + \frac{v_j^2}{4} \right) - 3 - \frac{1}{2} \sum_{\substack{1 \leq j, k \leq 3 \\ j \neq k}} L_{k,j}^2.$$

Then we have firstly

$$\begin{aligned} & \left(D_{v_j}^2 g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} = \left(D_{v_j}^2 F_{\delta, \delta'} g, \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &= \left(\langle \delta' v \rangle^{-2} D_{v_j}^2 \langle \delta' v \rangle^2 \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &= \left(D_{v_j}^2 \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &+ \left(\langle \delta' v \rangle^{-2} [D_{v_j}^2, \langle \delta' v \rangle^2] \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)}. \end{aligned}$$

Since

$$[D_{v_j}^2, \langle \delta' v \rangle^2] = -2\delta'^2 - i4\delta'^2 v_j D_{v_j},$$

we have

$$\begin{aligned}
& \left| \left(\langle \delta' v \rangle^{-2} [D_{v_j}^2, \langle \delta' v \rangle^2] \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \right| \\
& \leq 2\delta'^2 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 \\
& \quad + 4\delta' \|D_{v_j} \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \\
& \leq 10\delta'^2 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 + \frac{1}{2} \|D_{v_j} \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2,
\end{aligned}$$

which gives

$$\begin{aligned}
(2.5) \quad \left(D_{v_j}^2 g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} & \geq \frac{1}{2} \|D_{v_j} \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 \\
& \quad - 10\delta'^2 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2.
\end{aligned}$$

Now, for the second terms, we write

$$\begin{aligned}
\left(v_j^2 g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} & = \left(v_j^2 \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\
& \quad + \left(\langle \delta' v \rangle^{-2} [F_{\delta, \delta'}, v_j^2] g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)},
\end{aligned}$$

and note that

$$\begin{aligned}
(2.6) \quad [F_{\delta, \delta'}, v_j^2] & = 2(D_{\xi_j} F_{\delta, \delta'})(t, D_x, D_v) v_j - (D_{\xi_j}^2 F_{\delta, \delta'})(t, D_x, D_v) \\
& = 2v_j (D_{\xi_j} F_{\delta, \delta'})(t, D_x, D_v) + (D_{\xi_j}^2 F_{\delta, \delta'})(t, D_x, D_v).
\end{aligned}$$

By using (2.3), we have

$$\begin{aligned}
(D_{\xi_j} F_{\delta, \delta'})(t, D_x, D_v) & = \left(\frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right) (D_{\xi_j} \Psi) F_{\delta, \delta'} \\
(2.7) \quad & := B_{j, \delta, \delta'}(t, D_x, D_v) F_{\delta, \delta'}
\end{aligned}$$

and moreover

$$\begin{aligned}
(D_{\xi_j}^2 F_{\delta, \delta'})(t, D_x, D_v) & = \left((D_{\xi_j} B_{j, \delta, \delta'})(t, D_x, D_v) + B_{j, \delta, \delta'}(t, D_x, D_v)^2 \right) F_{\delta, \delta'} \\
(2.8) \quad & := \tilde{B}_{j, \delta, \delta'}(t, D_x, D_v) F_{\delta, \delta'}.
\end{aligned}$$

Consequently

$$\begin{aligned}
& \left| \left(\langle \delta' v \rangle^{-2} [F_{\delta, \delta'}, v_j^2] g, \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \right| \\
& \leq 2 \|\langle \delta' v \rangle^{-2} B_{j, \delta, \delta'} \langle \delta' v \rangle^2 \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \|v_j \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \\
& \quad + \|\langle \delta' v \rangle^{-2} \tilde{B}_{j, \delta, \delta'} \langle \delta' v \rangle^2 \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}.
\end{aligned}$$

Since it follows from (2.4) that $\langle \delta' v \rangle^{-2} B_{\delta, \delta'} \langle \delta' v \rangle^2 = B_{\delta, \delta'} - \delta'^2 \langle \delta' v \rangle^{-2} [B_{\delta, \delta'}, |v|^2]$ is an $L^2(\mathbb{R}_{x,v}^6)$ bounded operator with a constant factor $c_0 t$ and the same fact holds for $\tilde{B}_{\delta, \delta'}$, we have that for $0 < t \leq 1$,

$$\begin{aligned}
(2.9) \quad \left(v_j^2 g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} & \\
& \geq \frac{1}{2} \|v_j \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 - C_1 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2.
\end{aligned}$$

Finally, for the last terms, let $\hat{L}_{k,j} = -\xi_k \partial_{\xi_j} + \xi_j \partial_{\xi_k}$. Then we have

$$(2.10) \quad \hat{L}_{k,j} F_{\delta,\delta'}(t, \eta, \xi) = \left(\frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right) F_{\delta,\delta'}(-\xi_k \partial_{\xi_j} \Psi + \xi_j \partial_{\xi_k} \Psi),$$

and hence, in view of $r\delta' \leq 1$,

$$\begin{aligned} & \left(-L_{k,j}^2 g, F_{\delta,\delta'} \langle \delta' v \rangle^{-4} F_{\delta,\delta'} g \right)_{H_x^r(L_v^2)} \\ &= \int_{\mathbb{R}^6} \left\{ \left(\hat{L}_{k,j} - \left(\frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right) (-\xi_k \partial_{\xi_j} \Psi + \xi_j \partial_{\xi_k} \Psi) \right) F_{\delta,\delta'} \hat{g} \right\} \langle \eta \rangle^{2r} \\ & \times \frac{\left\{ \left(\hat{L}_{k,j} + \left(\frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right) (-\xi_k \partial_{\xi_j} \Psi + \xi_j \partial_{\xi_k} \Psi) \right) \langle \delta' D_\xi \rangle^{-4} F_{\delta,\delta'} \hat{g} \right\} \frac{d\eta d\xi}{(2\pi)^6}}{\left\{ \left(\hat{L}_{k,j} + \left(\frac{1}{1 + \delta e^\Psi} - \frac{r\delta'}{1 + \delta' \Psi} \right) (-\xi_k \partial_{\xi_j} \Psi + \xi_j \partial_{\xi_k} \Psi) \right) \langle \delta' D_\xi \rangle^{-4} F_{\delta,\delta'} \hat{g} \right\} \frac{d\eta d\xi}{(2\pi)^6}} \\ & \geq \|L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2 \\ & \quad - c_0^2 t^2 C_2 \left(\|\partial_{v_j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2 + \|\partial_{v_k} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2 \right) \\ & \quad - c_0 t C_3 \|L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)} \\ & \quad \quad \times \left(\|\partial_{v_j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)} + \|\partial_{v_k} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)} \right) \\ & \quad - c_0^2 t^2 C_4 \|\langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2, \end{aligned}$$

where we have used $[\hat{L}_{k,j}, \langle \delta' D_\xi \rangle] = 0$, because $\langle \delta' D_\xi \rangle$ is radial.

Therefore, for $0 < t \leq 1$ and $0 < c_0 \ll 1$ we get

$$(2.11) \quad \begin{aligned} & \left(-L_{k,j}^2 g, F_{\delta,\delta'} \langle \delta' v \rangle^{-4} F_{\delta,\delta'} g \right)_{H_x^r(L_v^2)} \\ & \geq \frac{1}{2} \|L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2 \\ & \quad - \frac{1}{8} \left(\|\partial_{v_j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2 + \|\partial_{v_k} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2 \right) \\ & \quad - C_5 \|\langle \delta' v \rangle^{-2} F_{\delta,\delta'} g\|_{H_x^r(L_v^2)}^2. \end{aligned}$$

Combing the estimates (2.5),(2.9) and (2.11), we finish the proof of Proposition 2.4. \square

We recall the notations: $\Phi_0(v) = \mu^{1/2}(v)$,

$$\Phi_\alpha = \frac{1}{\sqrt{\alpha!}} a_{+,1}^{\alpha_1} a_{+,2}^{\alpha_2} a_{+,3}^{\alpha_3} \Phi_0, \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{N}^3, \quad \alpha! = \alpha_1! \alpha_2! \alpha_3!,$$

with

$$a_{\pm,j} = \frac{v_j}{2} \mp \frac{\partial}{\partial v_j}, \quad 1 \leq j \leq 3.$$

$$\Phi_{e_k} = v_k \Phi_0, \quad \Phi_{2e_k} = \frac{1}{\sqrt{2}} (v_k^2 - 1) \Phi_0, \quad \Phi_{e_j + e_k} = v_j v_k \Phi_0 \quad (j \neq k),$$

where (e_1, e_2, e_3) stands for the canonical basis of \mathbb{R}^3 .

Proposition 2.5. *For $0 < t \leq 1$, there exists $C_6 > 0$ independent of $0 < \delta \leq 1$, $0 < \delta' < r^{-1}$ and $0 < c_0 \ll 1$ such that we have*

$$\begin{aligned} & \left| \left(\mathcal{L}_2 g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} h \right)_{H_x^r(L_v^2)} \right| \\ & \leq C_6 \| \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \|_{H_x^r(L_v^2)} \| \langle \delta' v \rangle^{-2} F_{\delta, \delta'} h \|_{H_x^r(L_v^2)}. \end{aligned}$$

Proof. First we recall

$$\begin{aligned} \mathcal{L}_2 g &= \left[\Delta_{\mathbb{S}^2} - 2 \left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2} \right) \right] \mathbb{P}_1 g \\ &+ \left[-\Delta_{\mathbb{S}^2} - 2 \left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2} \right) \right] \mathbb{P}_2 g. \end{aligned}$$

Notice

$$\mathbb{P}_1 g = \sum_{k=1}^3 (g, \Phi_{e_k})_{L^2(\mathbb{R}_v^3)} \Phi_{e_k}, \quad \mathbb{P}_2 g = \sum_{|\alpha|=2} (g, \Phi_\alpha)_{L^2(\mathbb{R}_v^3)} \Phi_\alpha.$$

Then

$$\mathcal{L}_2 g = \sum_{k=1}^3 (g, \Phi_{e_k})_{L^2(\mathbb{R}_v^3)} \tilde{\Phi}_{e_k} + \sum_{|\alpha|=2} (g, \Phi_\alpha)_{L^2(\mathbb{R}_v^3)} \tilde{\Phi}_\alpha$$

with

$$\begin{aligned} \tilde{\Phi}_{e_k} &= \left[\Delta_{\mathbb{S}^2} - 2 \left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2} \right) \right] \Phi_{e_k} = p_{e_k}(v) e^{-\frac{|v|^2}{4}}, \\ \tilde{\Phi}_2 &= \left[-\Delta_{\mathbb{S}^2} - 2 \left(-\Delta_v + \frac{|v|^2}{4} - \frac{3}{2} \right) \right] \Phi_\alpha = p_\alpha(v) e^{-\frac{|v|^2}{4}}, \end{aligned}$$

where $p_*(v)$ are the polynomial of v -variables of 3 or 4 degrees. We then study one of terms, where $\tilde{\Phi}$ denotes Φ_{e_k}, Φ_α ,

$$\begin{aligned} & \left((g, \Phi)_{L^2(\mathbb{R}_v^3)} \tilde{\Phi}, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} h \right)_{H_x^r(L_v^2)} \\ &= \int_{\eta, \xi} (\hat{g}(t, \eta, \cdot), \mathcal{F}_{v^*}(\Phi))_{L^2(\mathbb{R}_{\xi^*}^3)} \mathcal{F}_v(\tilde{\Phi})(\xi) \overline{\langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'} \hat{h}(t, \eta, \xi) \langle \eta \rangle^{2r}} \frac{d\eta d\xi}{(2\pi)^6} \\ &= \int_{\eta, \xi, \xi^*} \hat{g}(t, \eta, \xi^*) \overline{\mathcal{F}_{v^*}(\Phi)(\xi^*)} \mathcal{F}_v(\tilde{\Phi})(\xi) \overline{\langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'} \hat{h}(t, \eta, \xi) \langle \eta \rangle^{2r}} \frac{d\eta d\xi d\xi^*}{(2\pi)^9} \\ &= \int_{\eta, \xi, \xi^*} (\langle \delta' D_{\xi^*} \rangle^{-2} F_{\delta, \delta'}(t, \eta, \xi^*) \hat{g}(t, \eta, \xi^*)) \overline{(\langle \delta' D_{\xi^*} \rangle^2 \mathcal{F}_{v^*}(\Phi)(\xi^*))} \mathcal{F}_v(\tilde{\Phi})(\xi) \\ &\quad \times \frac{F_{\delta, \delta'}(t, \eta, \xi)}{F_{\delta, \delta'}(t, \eta, \xi^*)} \overline{\langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'}(t, \eta, \xi) \hat{h}(t, \eta, \xi) \langle \eta \rangle^{2r}} \frac{d\eta d\xi d\xi^*}{(2\pi)^9} \\ &+ 2\delta'^2 \int_{\eta, \xi, \xi^*} (\langle \delta' D_{\xi^*} \rangle^{-2} F_{\delta, \delta'}(t, \eta, \xi^*) \hat{g}(t, \eta, \xi^*)) \overline{(D_{\xi^*} \mathcal{F}_{v^*}(\Phi)(\xi^*))} \mathcal{F}_v(\tilde{\Phi})(\xi) \\ &\quad \times \left(D_{\xi^*} \frac{F_{\delta, \delta'}(t, \eta, \xi)}{F_{\delta, \delta'}(t, \eta, \xi^*)} \right) \overline{\langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'}(t, \eta, \xi) \hat{h}(t, \eta, \xi) \langle \eta \rangle^{2r}} \frac{d\eta d\xi d\xi^*}{(2\pi)^9} \\ &+ \delta'^2 \int_{\eta, \xi, \xi^*} (\langle \delta' D_{\xi^*} \rangle^{-2} F_{\delta, \delta'}(t, \eta, \xi^*) \hat{g}(t, \eta, \xi^*)) \overline{(\mathcal{F}_{v^*}(\Phi)(\xi^*))} \mathcal{F}_v(\tilde{\Phi})(\xi) \\ &\quad \times \left(D_{\xi^*}^2 \frac{F_{\delta, \delta'}(t, \eta, \xi)}{F_{\delta, \delta'}(t, \eta, \xi^*)} \right) \overline{\langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'}(t, \eta, \xi) \hat{h}(t, \eta, \xi) \langle \eta \rangle^{2r}} \frac{d\eta d\xi d\xi^*}{(2\pi)^9}. \end{aligned}$$

On the other hand, we have

$$\frac{F_{\delta, \delta'}(t, \eta, \xi)}{F_{\delta, \delta'}(t, \eta, \xi^*)} = e^{c_0 \int_0^t (\langle \xi + \rho \eta \rangle - \langle \xi^* + \rho \eta \rangle) d\rho} \frac{1 + \delta e^{c_0 \int_0^t \langle \xi^* + \rho \eta \rangle d\rho}}{1 + \delta e^{c_0 \int_0^t \langle \xi + \rho \eta \rangle d\rho}} \times \left(\frac{1 + \delta' c_0 \int_0^t \langle \xi^* + \rho \eta \rangle d\rho}{1 + \delta' c_0 \int_0^t \langle \xi + \rho \eta \rangle d\rho} \right)^r.$$

Since

$$(2.12) \quad \begin{aligned} \langle \xi + \rho \eta \rangle &= |(1, \xi + \rho \eta)| \leq |(1, \xi^* + \rho \eta)| + |(0, \xi - \xi^*)| \\ &\leq \langle \xi^* + \rho \eta \rangle + |\xi| + |\xi^*|, \quad \forall \xi, \xi^* \in \mathbb{R}^3, \end{aligned}$$

we have, by using (2.3) and (2.4), for $0 < t \leq 1$, $1 \leq p \leq 2$,

$$\left| D_{\xi^*}^p \frac{F_{\delta, \delta'}(t, \eta, \xi)}{F_{\delta, \delta'}(t, \eta, \xi^*)} \right| \leq C e^{2c_0 t (|\xi| + |\xi^*|)} (1 + |\xi|^r + |\xi^*|^r).$$

Because, for $0 \leq p \leq 2$

$$|\xi|^r e^{2c_0 t |\xi|} D_{\xi}^p \mathcal{F}_v(\tilde{\Phi})(\xi) \in L^2(\mathbb{R}_{\xi}^3),$$

we proved that, for $0 < t \leq 1$,

$$\begin{aligned} &\left| \left((g, \Phi)_{L^2(\mathbb{R}_v^3)} \tilde{\Phi}, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} h \right)_{H_x^r(L_v^2)} \right| \\ &\leq C \int_{\eta} \|\langle \delta' D_{\xi^*} \rangle^{-2} F_{\delta, \delta'} \hat{g}(t, \eta, \cdot)\|_{L_{\xi^*}^2} \|\langle \delta' D_{\xi} \rangle^{-4} F_{\delta, \delta'} \hat{h}(t, \eta, \cdot)\|_{L_{\xi}^2} \langle \eta \rangle^{2r} \frac{d\eta}{(2\pi)^9} \\ &\leq C \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} h\|_{H_x^r(L_v^2)}. \end{aligned}$$

□

Remark 2.6. Compared to (2.12), there is no constant $C > 0$ such that

$$\langle \xi + \rho \eta \rangle^\alpha - \langle \xi^* + \rho \eta \rangle^\alpha \leq C (|\xi|^\alpha + |\xi^*|^\alpha), \quad \forall \xi, \xi^* \in \mathbb{R}^3,$$

if $\alpha > 1$. By this reason we do not seek for the ultra-analytic smoothing effect in the present paper.

In conclusion, for the linear operators, we get that there exists $C_7 > 1$ independent of $0 < \delta \leq 1$, $0 < \delta' < r^{-1}$ and $0 < c_0 \ll 1$ such that, for $0 < t \leq 1$

$$(2.13) \quad \begin{aligned} &\left((\partial_t + v \cdot \nabla_x + \mathcal{L})g, F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g \right)_{H_x^r(L_v^2)} \\ &\geq \frac{1}{2} \frac{d}{dt} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 + \frac{1}{4} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{r,0}^2 \\ &\quad - C_7 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2. \end{aligned}$$

3. DECOMPOSITION OF NONLINEAR OPERATORS

We compute now the nonlinear term $\Gamma(f, g)$,

Proposition 3.1.

$$(3.1) \quad (\Gamma(f, g), h)_{L^2(\mathbb{R}_v^3)} = D_1 + D_2 + D_3 + D_4 + D_5 + D_6 + D_7,$$

with

$$\begin{aligned}
D_1 &= \sqrt{2} \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_{2e_j})_{L^2(\mathbb{R}_v^3)} (a_{+,i}g, a_{-,i}h)_{L^2(\mathbb{R}_v^3)}, \\
D_2 &= - \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_0)_{L^2(\mathbb{R}_v^3)} (a_{-,i}g, a_{-,i}h)_{L^2(\mathbb{R}_v^3)}, \\
D_3 &= - \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_{e_i+e_j})_{L^2(\mathbb{R}_v^3)} (a_{+,j}g, a_{-,i}h)_{L^2(\mathbb{R}_v^3)}, \\
D_4 &= \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_{e_i})_{L^2(\mathbb{R}_v^3)} (g, a_{-,i}h)_{L^2(\mathbb{R}_v^3)}, \\
D_5 &= -\frac{1}{2} \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_0)_{L^2(\mathbb{R}_v^3)} (L_{i,j}g, L_{i,j}h)_{L^2(\mathbb{R}_v^3)}, \\
D_6 &= \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_{e_j})_{L^2(\mathbb{R}_v^3)} (L_{i,j}g, a_{-,i}h)_{L^2(\mathbb{R}_v^3)}, \\
D_7 &= - \sum_{\substack{1 \leq i, j \leq 3 \\ i \neq j}} (f, \Phi_{e_j})_{L^2(\mathbb{R}_v^3)} (a_{+,i}g, L_{i,j}h)_{L^2(\mathbb{R}_v^3)}.
\end{aligned}$$

where the creation (resp. annihilation) operator $a_{+,j}$ (resp. $a_{-,j}$) is given by

$$a_{\pm,j} = \frac{v_j}{2} \mp \frac{\partial}{\partial v_j} \quad \text{and} \quad L_{k,j} = v_j \partial_{v_k} - v_k \partial_{v_j}.$$

Proof. We begin by computing explicitly the bilinear term $\Gamma(g, f)$. Notice that for all $f, g \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned}
\Gamma(f, g) &= \mu^{-1/2}(v) \sum_{1 \leq k, j \leq 3} \partial_{v_k} \left(\int_{\mathbb{R}^3} a_{k,j}(v-v_*) \mu^{1/2}(v_*) f(v_*) \partial_{v_j} (\mu^{1/2}(v) g(v)) dv_* \right) \\
&\quad - \mu^{-1/2}(v) \sum_{1 \leq k, j \leq 3} \partial_{v_k} \left(\int_{\mathbb{R}^3} a_{i,j}(v-v_*) \partial_{v_j^*} (\mu^{1/2}(v_*) f(v_*)) \mu^{1/2}(v) g(v) dv_* \right),
\end{aligned}$$

that is,

$$\begin{aligned}
\Gamma(f, g) &= \sum_{1 \leq k, j \leq 3} \left(\partial_{v_k} - \frac{v_k}{2} \right) \left(\int_{\mathbb{R}^3} a_{k,j}(v-v_*) \mu^{1/2}(v_*) f(v_*) \left(\partial_j g(v) - \frac{v_j}{2} g(v) \right) dv_* \right) \\
&\quad - \sum_{1 \leq k, j \leq 3} \left(\partial_{v_k} - \frac{v_k}{2} \right) \left(\int_{\mathbb{R}^3} a_{k,j}(v-v_*) \mu^{1/2}(v_*) \left(\partial_j f(v_*) - \frac{v_j^*}{2} f(v_*) \right) g(v) dv_* \right).
\end{aligned}$$

It follows that for all $f, g, h \in \mathcal{S}(\mathbb{R}_v^3)$,

$$\begin{aligned} & \langle \Gamma(f, g), h \rangle_{L^2(\mathbb{R}_v^3)} \\ &= \sum_{1 \leq k, j \leq 3} \int_{\mathbb{R}^6} a_{k,j}(v - v_*) \mu^{1/2}(v_*) f(v_*) \left(\partial_j g(v) - \frac{v_j}{2} g(v) \right) \\ & \quad \times \overline{\left(-\partial_k h(v) - \frac{v_k}{2} h(v) \right)} dv_* dv \\ & - \sum_{1 \leq k, j \leq 3} \int_{\mathbb{R}^6} a_{k,j}(v - v_*) \mu^{1/2}(v_*) \left(\partial_j f(v_*) - \frac{v_j^*}{2} f(v_*) \right) g(v) \\ & \quad \times \overline{\left(-\partial_k h(v) - \frac{v_k}{2} h(v) \right)} dv_* dv. \end{aligned}$$

Integrating by parts, we obtain that

$$\begin{aligned} & \int_{\mathbb{R}^3} a_{k,j}(v - v_*) \mu^{1/2}(v_*) \left(\partial_j f(v_*) - \frac{v_j^*}{2} f(v_*) \right) dv_* \\ &= - \int_{\mathbb{R}^3} (\partial_{v_j^*} a_{k,j}(v - v_*)) \mu^{1/2}(v_*) f(v_*) dv_*, \end{aligned}$$

and this implies that

$$\begin{aligned} & \langle \Gamma(f, g), h \rangle_{L^2(\mathbb{R}_v^3)} = \sum_{1 \leq k, j \leq 3} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \\ & \times \left\langle a_{k,j}(v - v_*) \left(\partial_j g(v) - \frac{v_j}{2} g(v) \right) + \partial_{v_j^*} (a_{k,j}(v - v_*)) g(v), -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*. \end{aligned}$$

Since for the Maxwellian case,

$$a_{k,j}(z) = \delta_{kj} |z|^2 - z_k z_j, \quad 1 \leq k, j \leq 3.$$

We have that

$$\begin{aligned} & \langle \Gamma(f, g), h \rangle_{L^2(\mathbb{R}_v^3)} \\ &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \\ & \times \left\langle (v_j^2 - 2v_j v_j^* + (v_j^*)^2) \left(\partial_k g(v) - \frac{v_k}{2} g(v) \right), -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_* \\ & \quad + \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \\ & \times \left\langle -(v_k - v_k^*) (v_j - v_j^*) \left(\partial_j g(v) - \frac{v_j}{2} g(v) \right) + (v_k - v_k^*) g(v), \right. \\ & \quad \left. -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*. \end{aligned}$$

We obtain that

$$\langle \Gamma(g, f), h \rangle_{L^2(\mathbb{R}_v^3)} = A_0 + A_1 + A_2 + A_3 + A_4 + A_5 + A_6 + A_7,$$

with

$$\begin{aligned}
A_0 &= - \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} v_k^* v_j^* \mu(v_*)^{1/2} f(v_*) \\
&\quad \times \left\langle \partial_j g(v) - \frac{v_j}{2} g(v), -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*, \\
A_1 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} (v_j^*)^2 \mu(v_*)^{1/2} f(v_*) \\
&\quad \times \left\langle \partial_k g(v) - \frac{v_k}{2} g(v), -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*, \\
A_2 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \left\langle v_k g(v), -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*, \\
A_3 &= - \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} v_k^* \mu(v_*)^{1/2} f(v_*) \left\langle g(v), -\partial_k h(v) - \frac{v_k}{2} h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*, \\
A_4 &= \frac{1}{4} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \\
&\quad \times \left\langle (-v_k^2 v_j^2 + v_k^2 v_j v_j^* + v_k v_j^2 v_k^* + v_k^2 v_j^2 - 2v_k^2 v_j v_j^*) g(v), h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_* = 0,
\end{aligned}$$

since $\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} (v_k v_j^2 v_k^* - v_k^2 v_j v_j^*) = 0$. We have also

$$\begin{aligned}
(3.2) \quad A_5 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \\
&\quad \times \left\langle (v_j^2 - 2v_j v_j^*) \partial_k g(v) + (-v_k v_j + v_k v_j^* + v_j v_k^*) \partial_j g(v), -\partial_k h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*,
\end{aligned}$$

and

$$\begin{aligned}
A_6 &= -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \left\langle v_k \partial_j g(v), (-v_k v_j + v_k v_j^* + v_j v_k^*) h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
&\quad + \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \left\langle (-v_k v_j + v_k v_j^* + v_j v_k^*) g(v), v_j \partial_k h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*, \\
A_7 &= -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \left\langle v_j \partial_k g(v), (v_k v_j - 2v_k v_j^*) h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
&\quad + \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \left\langle (v_k v_j - 2v_k v_j^*) g(v), v_j \partial_k h(v) \right\rangle_{L^2(\mathbb{R}_v^3)} dv_*.
\end{aligned}$$

It follows from (3.2) that

$$\begin{aligned}
A_5 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle v_j \partial_k g(v) - v_k \partial_j g(v), -v_j \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
&+ \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle -2v_j v_j^* \partial_k g(v) + (v_k v_j^* + v_j v_k^*) \partial_j g(v), -\partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_*.
\end{aligned}$$

In the sequel, we shall use several times the (obvious) formula $\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \alpha_{k, j} = \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \alpha_{j, k}$. We notice that

$$\begin{aligned}
&\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle v_j \partial_k g(v) - v_k \partial_j g(v), -v_j \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&= \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle v_j \partial_k g(v) - v_k \partial_j g(v), -v_j \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&\quad + \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle v_k \partial_j g(v) - v_j \partial_k g(v), -v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&= -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle v_j \partial_k g(v) - v_k \partial_j g(v), v_j \partial_k h(v) - v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)},
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle -2v_j v_j^* \partial_k g(v) + (v_k v_j^* + v_j v_k^*) \partial_j g(v), -\partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} [\langle v_j \partial_k g(v), v_j^* \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} + \langle v_j^* \partial_k g(v), v_j \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)}] \\
&\quad - \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle v_k \partial_j g(v), v_j^* \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&\quad\quad - \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle v_j^* \partial_k g(v), v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} [\langle v_j \partial_k g(v) - v_k \partial_j g(v), v_j^* \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} \\
&\quad\quad + \langle v_j^* \partial_k g(v), v_j \partial_k h(v) - v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)}].
\end{aligned}$$

This implies that

$$\begin{aligned}
A_5 = & -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle v_j \partial_k g(v) - v_k \partial_j g(v), \\
& \qquad \qquad \qquad v_j \partial_k h(v) - v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
& + \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} v_j^* \mu(v_*)^{1/2} f(v_*) \left[\langle v_j \partial_k g(v) - v_k \partial_j g(v), \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} \right. \\
& \qquad \qquad \qquad \left. + \langle \partial_k g(v), v_j \partial_k h(v) - v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)} \right] dv_*.
\end{aligned}$$

On the other hand, we may write that

$$\begin{aligned}
A_6 + A_7 = & -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle v_k \partial_j g(v), \\
& \qquad \qquad \qquad (-v_k v_j + v_k v_j^* + v_j v_k^* + v_j - 2v_j v_k^*) h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
& + \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle (-v_k v_j + v_k v_j^* + v_j v_k^* + v_k v_j - 2v_k v_j^*) g(v), \\
& \qquad \qquad \qquad v_j \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_*.
\end{aligned}$$

It follows that

$$\begin{aligned}
A_6 + A_7 = & -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle v_k \partial_j g(v), (v_k v_j^* - v_j v_k^*) h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
& + \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle (v_j v_k^* - v_k v_j^*) g(v), v_j \partial_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_*.
\end{aligned}$$

This implies that

$$\begin{aligned}
A_6 + A_7 = & -\frac{1}{4} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle v_k \partial_j g(v) - v_j \partial_k g(v), \\
& \qquad \qquad \qquad (v_k v_j^* - v_j v_k^*) h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_* \\
& + \frac{1}{4} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} \mu(v_*)^{1/2} f(v_*) \langle (v_j v_k^* - v_k v_j^*) g(v), v_j \partial_k h(v) - v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_*,
\end{aligned}$$

that is,

$$\begin{aligned} A_6 + A_7 &= \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} v_j^* \mu(v_*)^{1/2} f(v_*) \langle v_j \partial_k g(v) - v_k \partial_j g(v), v_k h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_* \\ &\quad + \frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \int_{\mathbb{R}^3} v_k^* \mu(v_*)^{1/2} f(v_*) \langle v_j g(v), v_j \partial_k h(v) - v_k \partial_j h(v) \rangle_{L^2(\mathbb{R}_v^3)} dv_*. \end{aligned}$$

We obtain that

$$\langle \Gamma(f, g), h \rangle_{L^2(\mathbb{R}_v^3)} = E_1 + E_2 + E_3 + E_4 + E_5 + E_6 + E_7,$$

with

$$\begin{aligned} E_1 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, v_j^2 \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle \partial_k g - \frac{v_k}{2} g, -\partial_k h - \frac{v_k}{2} h \right\rangle_{L^2(\mathbb{R}_v^3)}, \\ E_2 &= - \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, v_k v_j \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle \partial_j g - \frac{v_j}{2} g, -\partial_k h - \frac{v_k}{2} h \right\rangle_{L^2(\mathbb{R}_v^3)}, \\ E_3 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle v_k g, -\partial_k h - \frac{v_k}{2} h \right\rangle_{L^2(\mathbb{R}_v^3)}, \\ E_4 &= - \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, v_k \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle g, -\partial_k h - \frac{v_k}{2} h \right\rangle_{L^2(\mathbb{R}_v^3)}, \\ E_5 &= -\frac{1}{2} \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle v_j \partial_k g - v_k \partial_j g, v_j \partial_k h - v_k \partial_j h \right\rangle_{L^2(\mathbb{R}_v^3)}, \\ E_6 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, v_j \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle v_j \partial_k g - v_k \partial_j g, \partial_k h + \frac{v_k}{2} h \right\rangle_{L^2(\mathbb{R}_v^3)}, \\ E_7 &= \sum_{\substack{1 \leq k, j \leq 3 \\ k \neq j}} \langle f, v_j \mu^{1/2} \rangle_{L^2(\mathbb{R}_v^3)} \left\langle \partial_k g - \frac{v_k}{2} g, v_j \partial_k h - v_k \partial_j h \right\rangle_{L^2(\mathbb{R}_v^3)}. \end{aligned}$$

This is exactly (3.1). \square

4. TRILINEAR ESTIMATES WITH EXPONENTIAL WEIGHTS

Proposition 4.1. *Let $0 < t \leq 1$ and $0 < c_0 \ll 1$. If $r > \frac{3}{2}$, then there exists $C_0 > 0$ independent of δ, δ', c_0 such that we have for suitable functions f, g, h ,*

$$\begin{aligned} (4.1) \quad & \left| \langle \Gamma(f, g), F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} h \rangle_{H_x^r(L_v^2)} \right| \\ & \leq C_0 \|F_{\delta, 0} f\|_{H_x^r(L_v^2)} \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{r, 0} + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)} \right) \\ & \quad \times \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} h\|_{r, 0} + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} h\|_{H_x^r(L_v^2)} \right). \end{aligned}$$

We consider the nonlinear term $(\Gamma(f, g), F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} h)_{H_x(L_v^2)}$. For instance we estimate a term of D_5 , that is,

$$\begin{aligned}
& \int_{\mathbb{R}_x^3} (f, \Phi_0)_{L^2(\mathbb{R}_v^3)} (L_{k,j} g, \langle D_x \rangle^{2r} L_{k,j} F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} h)_{L^2(\mathbb{R}_v^3)} dx \\
&= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} \hat{f}(\eta - \tilde{\eta}, \xi_*) \overline{\hat{\Phi}_0(\xi_*)} \frac{d\xi_*}{(2\pi)^3} \hat{L}_{k,j} \hat{g}(\tilde{\eta}, \xi) d\tilde{\eta} \right) \\
&\quad \times \langle \eta \rangle^{2r} \overline{\hat{L}_{k,j} F_{\delta, \delta'} \langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'} \hat{h}(\eta, \xi)} \frac{d\eta d\xi}{(2\pi)^6} \\
&= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} \hat{f}(\eta - \tilde{\eta}, \xi_*) \overline{\hat{\Phi}_0(\xi_*)} \frac{d\xi_*}{(2\pi)^3} \hat{L}_{k,j} \hat{g}(\tilde{\eta}, \xi) d\tilde{\eta} \right) \\
&\quad \times \langle \eta \rangle^{2r} \overline{F_{\delta, \delta'} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'} \hat{h}(\eta, \xi)} \frac{d\eta d\xi}{(2\pi)^6} \\
&\quad + \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} \hat{f}(\eta - \tilde{\eta}, \xi_*) \overline{\hat{\Phi}_0(\xi_*)} \frac{d\xi_*}{(2\pi)^3} \hat{L}_{k,j} \hat{g}(\tilde{\eta}, \xi) d\tilde{\eta} \right) \\
&\quad \times \langle \eta \rangle^{2r} \overline{[\hat{L}_{k,j}, F_{\delta, \delta'}] \langle \delta' D_\xi \rangle^{-4} F_{\delta, \delta'} \hat{h}(\eta, \xi)} \frac{d\eta d\xi}{(2\pi)^6}, \\
&:= \Gamma_1 + \Gamma_2.
\end{aligned}$$

We consider

$$F_{\delta, \delta'}(t, \eta, \xi) \langle \eta \rangle^r = \frac{e^\Psi}{(1 + \delta e^\Psi)} \left(\frac{\langle \eta \rangle}{(1 + \delta' \Psi)} \right)^r := F_{\delta, 0} G_{\delta'}.$$

By means of $\langle W + V \rangle \leq \langle W \rangle + \langle V \rangle$, we have

$$\begin{aligned}
\int_0^t \langle \xi - \rho \eta \rangle d\rho &\leq \int_0^t \langle \xi + \xi_* - \rho(\eta - \tilde{\eta}) - \rho \tilde{\eta} \rangle d\rho + \int_0^t \langle \xi_* \rangle d\rho \\
&\leq \int_0^t \langle \xi_* - \rho(\eta - \tilde{\eta}) \rangle d\rho + \int_0^t \langle \xi - \rho \tilde{\eta} \rangle d\rho + t \langle \xi_* \rangle.
\end{aligned}$$

Noting that $\tilde{F}_\delta(X) = e^X / (1 + \delta e^X)$ is an increasing function and that

$$\tilde{F}_\delta(X + Y) \leq 3\tilde{F}_\delta(X)\tilde{F}_\delta(Y),$$

we have

$$F_{\delta, 0}(t, \eta, \xi) \leq 9F_{\delta, 0}(t, \eta - \tilde{\eta}, \xi_*) F_{\delta, 0}(t, \tilde{\eta}, \xi) e^{c_0 t \langle \xi_* \rangle},$$

Since $\Psi(t, \eta, \xi) \sim c_0 t(1 + |\xi|^2 + t^2 |\eta|^2)^{1/2}$ and $(1 + Y)/(a + bY)$ for any constants $a \geq b > 0$ is increasing in Y , there exists a constant $C > 0$ independent of $\delta' > 0$ and $(t, \xi), \eta, \tilde{\eta}$ such that

$$G_{\delta'}(t, \eta, \xi) \leq C \frac{\langle \eta - \tilde{\eta} \rangle^r + \langle \tilde{\eta} \rangle^r}{(1 + \delta' \Psi(t, \tilde{\eta}, \xi))^r}.$$

Consequently, for another constant $C_7 > 0$ independent of $\delta' > 0$ and $(t, \xi), \eta, \tilde{\eta}$ we have

$$\begin{aligned}
(4.2) \quad & F_{\delta, \delta'}(t, \eta, \xi) \langle \eta \rangle^r \\
& \leq C_7 (\langle \eta - \tilde{\eta} \rangle^r + \langle \tilde{\eta} \rangle^r) F_{\delta, 0}(t, \eta - \tilde{\eta}, \xi_*) F_{\delta, \delta'}(t, \tilde{\eta}, \xi) e^{c_0 t \langle \xi_* \rangle}.
\end{aligned}$$

$$\begin{aligned}
\Gamma_1 &= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} \hat{f}(\eta - \tilde{\eta}, \xi_*) \overline{\hat{\Phi}_0(\xi_*)} \frac{d\xi_*}{(2\pi)^3} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} \hat{g}(\tilde{\eta}, \xi) d\tilde{\eta} \right) \\
&\quad \times \langle \eta \rangle^{2r} \overline{\langle \delta' D_\xi \rangle^2 F_{\delta,\delta'} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-4} F_{\delta,\delta'} \hat{h}(\eta, \xi)} \frac{d\eta d\xi}{(2\pi)^6} \\
&= \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} \hat{f}(\eta - \tilde{\eta}, \xi_*) \overline{\hat{\Phi}_0(\xi_*)} \frac{d\xi_*}{(2\pi)^3} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} \hat{g}(\tilde{\eta}, \xi) d\tilde{\eta} \right) \\
&\quad \times \langle \eta \rangle^{2r} \overline{F_{\delta,\delta'} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} F_{\delta,\delta'} \hat{h}(\eta, \xi)} \frac{d\eta d\xi}{(2\pi)^6} \\
&+ \int_{\mathbb{R}^6} \left(\int_{\mathbb{R}^3} \hat{f}(\eta - \tilde{\eta}, \xi_*) \overline{\hat{\Phi}_0(\xi_*)} \frac{d\xi_*}{(2\pi)^3} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} \hat{g}(\tilde{\eta}, \xi) d\tilde{\eta} \right) \\
&\quad \times \langle \eta \rangle^{2r} \overline{[\langle \delta' D_\xi \rangle^2, F_{\delta,\delta'}] \langle \delta' D_\xi \rangle^{-2} \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} F_{\delta,\delta'} \hat{h}(\eta, \xi)} \frac{d\eta d\xi}{(2\pi)^6} \\
&:= \Gamma_{1,1} + \Gamma_{1,2},
\end{aligned}$$

where we have used again $[\hat{L}_{k,j}, \langle \delta' D_\xi \rangle] = 0$. By means of (4.2) and Young inequality, we get

$$\begin{aligned}
|\Gamma_{1,1}| &\lesssim \| \langle \cdot \rangle^r F_{\delta,0}(\cdot, \xi_*) \hat{f}(\cdot, \xi_*) \|_{L^2(\mathbb{R}^6)} \| e^{c_0 t \langle \xi_* \rangle - |\xi_*|^2/4} \|_{L^2(\mathbb{R}_{\xi_*}^3)} \\
&\quad \times \| F_{\delta,\delta'}(\cdot, \xi) \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} \hat{g}(\cdot, \xi) \|_{L_x^2 L_v^1} \| L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} h \|_{H_x^{3/2+\epsilon} L_v^2} \\
&\quad + \| F_{\delta,0}(\cdot, \xi_*) \hat{f}(\cdot, \xi_*) \|_{L_{\xi_*}^2 L^1} \| e^{c_0 t \langle \xi_* \rangle - |\xi_*|^2/4} \|_{L^2(\mathbb{R}_{\xi_*}^3)} \\
&\quad \times \| \langle \cdot \rangle^r F_{\delta,\delta'}(\cdot, \xi) \hat{L}_{k,j} \langle \delta' D_\xi \rangle^{-2} \hat{g}(\cdot, \xi) \|_{L^2(\mathbb{R}^6)} \| L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} h \|_{H_x^r(L_v^2)} \\
&\lesssim \| F_{\delta,0} f \|_{H_x^r(L_v^2)} \| F_{\delta,\delta'} \langle \delta' v \rangle^{-2} L_{k,j} g \|_{H_x^r(L_v^2)} \| L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} h \|_{H_x^r(L_v^2)}.
\end{aligned}$$

By the same calculus as in (2.6), we note that

$$\begin{aligned}
F_{\delta,\delta'}(t, \eta, D_v) &= \langle \delta' v \rangle^{-2} F_{\delta,\delta'}(t, \eta, D_v) \langle \delta' v \rangle^2 \\
&+ 2\delta'^2 \sum_{j=1}^3 v_j \langle \delta' v \rangle^{-2} (D_{\xi_j} F_{\delta,\delta'})(t, \eta, D_v) - \delta'^2 \langle \delta' v \rangle^{-2} \sum_{j=1}^3 (D_{\xi_j}^2 F_{\delta,\delta'})(t, \eta, D_v).
\end{aligned}$$

By means of (2.3) and (2.4), there exists a constant $C_8 > 0$ independent of $t, \eta, \delta, \delta' > 0$ such that for $H(v) \in L_v^2(\mathbb{R}^3)$ we have

$$\begin{aligned}
\| F_{\delta,\delta'}(t, \eta, D_v) H \|_{L_v^2} &\leq \| \langle \delta' v \rangle^{-2} F_{\delta,\delta'}(t, \eta, D_v) \langle \delta' v \rangle^2 H \|_{L_v^2} \\
&\quad + C_8 \delta' c_0 \| F_{\delta,\delta'}(t, \eta, D_v) H \|_{L_v^2},
\end{aligned}$$

and hence

$$\| F_{\delta,\delta'}(t, \eta, D_v) H \|_{L_v^2} \leq 2 \| \langle \delta' v \rangle^{-2} F_{\delta,\delta'} \langle \delta' v \rangle^2 H \|_{L_v^2}$$

if $c_0 C_8 < 1/2$. Consequently, we obtain

$$\begin{aligned}
\| F_{\delta,\delta'} \langle \delta' v \rangle^{-2} L_{k,j} g \|_{H_x^r(L_v^2)} &\leq 2 \| \langle \delta' v \rangle^{-2} F_{\delta,\delta'} L_{k,j} g \|_{H_x^r(L_v^2)} \\
&\leq 2 \| L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g \|_{H_x^r(L_v^2)} + 2 \| \langle \delta' v \rangle^{-2} [L_{k,j}, F_{\delta,\delta'}] g \|_{H_x^r(L_v^2)} \\
&\lesssim \| L_{k,j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g \|_{H_x^r(L_v^2)} + \| D_{v_k} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g \|_{H_x^r(L_v^2)} \\
&\quad + \| D_{v_j} \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g \|_{H_x^r(L_v^2)} + \| \langle \delta' v \rangle^{-2} F_{\delta,\delta'} g \|_{H_x^r(L_v^2)}.
\end{aligned}$$

Here we have used (2.10) at the third inequality. As a consequence, we obtain

$$\begin{aligned} \|\Gamma_{1,1}\| &\lesssim \|F_{\delta,0}f\|_{H_x^r(L_v^2)} \left(\|\langle \delta'v \rangle^{-2} F_{\delta,\delta'}g\|_{r,0} + \|\langle \delta'v \rangle^{-2} F_{\delta,\delta'}g\|_{H_x^r(L_v^2)} \right) \\ &\quad \times \left(\|\langle \delta'v \rangle^{-2} F_{\delta,\delta'}h\|_{r,0} + \|\langle \delta'v \rangle^{-2} F_{\delta,\delta'}h\|_{H_x^r(L_v^2)} \right). \end{aligned}$$

Since it follows from the almost same calculation as in (2.6), (2.7) and (2.8) that

$$\begin{aligned} &[\langle \delta' D_\xi \rangle^2, F_{\delta,\delta'}] \langle \delta' D_\xi \rangle^{-2} \\ &= \sum_{j=1}^3 \left((D_{\xi_j} F_{\delta,\delta'}) (2\delta'^2 D_{\xi_j} \langle \delta' D_\xi \rangle^{-2}) + (D_{\xi_j}^2 F_{\delta,\delta'}) \delta'^2 \langle \delta' D_\xi \rangle^{-2} \right) \\ &= F_{\delta,\delta'}(t, \eta, \xi) \sum_{j=1}^3 \left(B_{j,\delta,\delta'}(t, \eta, \xi) (2\delta'^2 D_{\xi_j} \langle \delta' D_\xi \rangle^{-2}) + \tilde{B}_{j,\delta,\delta'}(t, \eta, \xi) \delta'^2 \langle \delta' D_\xi \rangle^{-2} \right), \end{aligned}$$

and since the last factor is a bounded operator, the estimation for $\Gamma_{1,2}$ is similar to the one for $\Gamma_{1,1}$,

As for the estimation of Γ_2 , we recall (2.10). Then

$$\begin{aligned} [\hat{L}_{k,j}, F_{\delta,\delta'}] \langle \delta' D_\xi \rangle^{-2} &= i F_{\delta,\delta'} \left(B_{k,\delta,\delta'} \xi_j - B_{j,\delta,\delta'} \xi_k \right) \langle \delta' D_\xi \rangle^{-2} \\ &= i F_{\delta,\delta'} \left(B_{k,\delta,\delta'} \langle \delta' D_\xi \rangle^{-2} \left(\xi_j - 2i \frac{\delta'^2 D_{\xi_j}}{\langle \delta' D_\xi \rangle^2} \right) - B_{j,\delta,\delta'} \langle \delta' D_\xi \rangle^{-2} \left(\xi_k - 2i \frac{\delta'^2 D_{\xi_k}}{\langle \delta' D_\xi \rangle^2} \right) \right). \end{aligned}$$

Writing

$$F_{\delta,\delta'} B_{k,\delta,\delta'} \langle \delta' D_\xi \rangle^{-2} = F_{\delta,\delta'} \langle \delta' D_\xi \rangle^{-2} \left(B_{k,\delta,\delta'} + [\langle \delta' D_\xi \rangle^2, B_{k,\delta,\delta'}] \langle \delta' D_\xi \rangle^{-2} \right),$$

we see that the estimation for Γ_2 is quite similar to the one for Γ_1 , because there exist bounded operators $R_k, R_j, R_{j,k}$ such that

$$[\hat{L}_{k,j}, F_{\delta,\delta'}] \langle \delta' D_\xi \rangle^{-2} = F_{\delta,\delta'} \langle \delta' D_\xi \rangle^{-2} \left(R_k \xi_j + R_j \xi_k + R_{j,k} \right).$$

Thus we have a desired estimate for D_5 . For the other terms D_j , we remark that $[\hat{a}_{\pm j}, \langle \delta' D_\xi \rangle] \neq 0$, but it is bounded, whence we also can estimate them by the same procedure. We omit the detail.

5. END OF PROOF OF MAIN THEOREM

In the first subsection we show the local existence of solution in $[0, 1]$ and its analytic smoothing effect. We remark the stability and uniqueness of this local solution in the next subsection. In the last subsection we complete the proof of the main theorem by using the global existence theorem given by Guo[22].

5.1. Existence of analytic time-local solution.

Lemma 5.1 (local existence for a linear equation). *Let $r > 3/2$ and $0 < c_0 \ll 1$. Assume that $0 < \delta \leq 1$. Then there exist $\epsilon_0 > 0$ and $C_9 > 1$ independent of δ such that for any $0 < T \leq 1$, $g_0 \in H_x^r(L_v^2)$, $f \in L^\infty([0, T]; H_x^r(L_v^2))$ satisfying*

$$(5.1) \quad \|F_{\delta,0}f\|_{L^\infty([0,T]; H_x^r(L_v^2))} \leq \epsilon_0,$$

the Cauchy problem

$$(5.2) \quad \begin{cases} \partial_t g + v \cdot \nabla_x g + \mathcal{L}g = \Gamma(f, g), \\ g|_{t=0} = g_0, \end{cases}$$

admits a weak solution $g \in L^\infty([0, T]; H_x^r(L_v^2))$ satisfying

$$(5.3) \quad \|F_{\delta,0}g\|_{L^\infty([0,T];H_x^r(L_v^2))}^2 + \int_0^T \|F_{\delta,0}g(s)\|_{r,0}^2 ds \leq C_9 \|g_0\|_{H_x^r(L_v^2)}^2.$$

Proof. Consider

$$\mathcal{Q} = -\partial_t + (v \cdot \nabla_x + \mathcal{L} - \Gamma(f, \cdot))^*,$$

where the adjoint operator $(\cdot)^*$ is taken with respect to the scalar product in $H_x^r(L_v^2)$. Then, by using (2.13) and (4.1) with $\delta, \delta', c_0 \rightarrow 0$ we see that for all $h \in C^\infty([0, T], \mathcal{S}(\mathbb{R}_{x,v}^6))$, with $h(T) = 0$ and $0 \leq t \leq T$,

$$\begin{aligned} \operatorname{Re}(h(t), \mathcal{Q}h(t))_{H_x^r(L_v^2)} &= -\frac{1}{2} \frac{d}{dt} (\|h\|_{H_x^r(L_v^2)}^2) \\ &\quad + \operatorname{Re}(v \cdot \nabla_x h, h)_{H_x^r(L_v^2)} + \operatorname{Re}(\mathcal{L}h, h)_{H_x^r(L_v^2)} - \operatorname{Re}(\Gamma(f, h), h)_{H_x^r(L_v^2)} \\ &\geq -\frac{1}{2} \frac{d}{dt} (\|h\|_{H_x^r(L_v^2)}^2) + \frac{1}{4} \|h(t)\|_{r,0}^2 - C_8 \|h(t)\|_{H_x^r(L_v^2)}^2 \\ &\quad - C_0 \|f(t)\|_{H_x^r(L_v^2)} \|h(t)\|_{r,0}^2, \end{aligned}$$

because \mathcal{L} is a selfadjoint operator and $\operatorname{Re}(v \cdot \nabla_x h, h)_{H_x^r(L_v^2)} = 0$. Since (5.1) implies $\|f\|_{L^\infty([0,T];H_x^r(L_v^2))} \leq 2\epsilon_0$, we have

$$\begin{aligned} -\frac{d}{dt} (e^{2C_8 t} \|h(t)\|_{H_x^r(L_v^2)}^2) + \frac{1}{4} e^{2C_8 t} \|h(t)\|_{r,0}^2 \\ \leq 2e^{2C_8 t} \|h(t)\|_{H_x^r(L_v^2)} \|\mathcal{Q}h(t)\|_{H_x^r(L_v^2)}, \end{aligned}$$

if $16\epsilon_0 C_0 < 1$. Since $h(T) = 0$, for all $t \in [0, T]$ we have

$$\begin{aligned} \|h(t)\|_{H_x^r(L_v^2)}^2 + \frac{1}{4} \int_t^T \|h(\tau)\|_{r,0}^2 d\tau \\ \leq 2 \int_t^T e^{2C_8(\tau-t)} \|h(\tau)\|_{H_x^r(L_v^2)} \|\mathcal{Q}h(\tau)\|_{H_x^r(L_v^2)} d\tau \\ \leq 2e^{2C_8 T} \|h\|_{L^\infty([0,T];H_x^r(L_v^2))} \|\mathcal{Q}h\|_{L^1([0,T],H_x^r(L_v^2))}, \text{ so that} \end{aligned}$$

$$(5.4) \quad \|h\|_{L^\infty([0,T];H_x^r(L_v^2))} \leq 2e^{2C_8 T} \|\mathcal{Q}h\|_{L^1([0,T],H_x^r(L_v^2))}.$$

We consider the vector subspace

$$\begin{aligned} \mathbb{W} &= \{w = \mathcal{Q}h : h \in C^\infty([0, T], \mathcal{S}(\mathbb{R}_{x,v}^6)), h(T) = 0\} \\ &\subset L^1([0, T], H_x^r(L_v^2)). \end{aligned}$$

This inclusion holds because it follows from Proposition 3.1 that for $g \in H_x^r(L_v^2)$

$$|(\Gamma(f, \cdot)^* h, g)_{H_x^r(L_v^2)}| = |(h, \Gamma(f, g))_{H_x^r(L_v^2)}| \lesssim \|f\|_{H_x^r(L_v^2)} \|g\|_{H_x^r(L_v^2)} \langle v \rangle^2 h \|_{H_x^r(H_v^2)},$$

and hence, for all $t \in [0, T]$,

$$\|\Gamma(f, \cdot)^* h\|_{H_x^r(L_v^2)} \lesssim \|f\|_{H_x^r(L_v^2)} \langle v \rangle^2 h \|_{H_x^r(H_v^2)}.$$

Since $g_0 \in H_x^r(L_v^2)$, we define the linear functional

$$\begin{aligned} \mathcal{G} : \quad \mathbb{W} &\longrightarrow \mathbb{C} \\ w = \mathcal{Q}h &\mapsto (g_0, h(0))_{H_x^r(L_v^2)} \end{aligned}$$

where $h \in C^\infty([0, T], \mathcal{S}(\mathbb{R}_{x,v}^6))$, with $h(T) = 0$. According to (5.4), the operator \mathcal{Q} is injective. The linear functional \mathcal{G} is therefore well-defined. It follows from (5.4) that \mathcal{G} is a continuous linear form on $(\mathbb{W}, \|\cdot\|_{L^1([0,T]; H_x^r(L_v^2))})$,

$$\begin{aligned} |\mathcal{G}(w)| &\leq \|g_0\|_{H_x^r(L_v^2)} \|h(0)\|_{H_x^r(L_v^2)} \\ &\leq 2e^{2C_8 T} \|g_0\|_{H_x^r(L_v^2)} \|\mathcal{Q}h\|_{L^1([0,T]; H_x^r(L_v^2))} \\ &= 2e^{2C_8 T} \|g_0\|_{H_x^r(L_v^2)} \|w\|_{L^1([0,T]; H_x^r(L_v^2))}. \end{aligned}$$

By using the Hahn-Banach theorem, \mathcal{G} may be extended as a continuous linear form on

$$L^1([0, T]; H_x^r(L_v^2)),$$

with a norm smaller than $2e^{2C_8 T} \|g_0\|_{H_x^r(L_v^2)}$. Hence there exists $g \in L^\infty([0, T]; H_x^r(L_v^2))$ satisfying

$$\|g\|_{L^\infty([0,T], H_x^r(L_v^2))} \leq 2e^{2C_8 T} \|g_0\|_{H_x^r(L_v^2)},$$

such that

$$\forall w \in L^1([0, T]; H_x^r(L_v^2)), \quad \mathcal{G}(w) = \int_0^T (g(t), w(t))_{H_x^r(L_v^2)} dt.$$

This implies that for all $h \in C_0^\infty((-\infty, T), \mathcal{S}(\mathbb{R}_{x,v}^6))$,

$$\begin{aligned} \mathcal{G}(\mathcal{Q}h) &= \int_0^T (g(t), \mathcal{Q}h(t))_{H_x^r(L_v^2)} dt \\ &= (g_0, h(0))_{H_x^r(L_v^2)}. \end{aligned}$$

This shows that $g \in L^\infty([0, T]; H_x^r(L_v^2))$ is a weak solution of the Cauchy problem (5.2).

It remains to show (5.3). Noting that $g \in L^\infty([0, T]; H_x^r(L_v^2))$ implies, for any $\delta > 0, \delta' > 0$,

$$\langle v \rangle (t^r \langle D_v \rangle^r + t^{2r} \langle D_x \rangle^r) \langle \delta' v \rangle^{-2} F_{\delta, \delta'} g \in L^\infty([0, T]; H_x^r(L_v^2)),$$

we multiply the first equation of (5.2) by $F_{\delta, \delta'} \langle \delta' v \rangle^{-4} F_{\delta, \delta'} g$ and take its $H_x^r(L_v^2)$ inner product. Then it follows from (2.13) and (4.1) that for $0 < t \leq T \leq 1$

$$\begin{aligned} (5.5) \quad &\frac{1}{2} \frac{d}{dt} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 + \frac{1}{4} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{r,0}^2 - C_8 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 \\ &\leq 2C_0 \|F_{\delta,0} f\|_{H_x^r(L_v^2)} \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{r,0}^2 + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{H_x^r(L_v^2)}^2 \right). \end{aligned}$$

Since ϵ_0 is chosen small enough that $16\epsilon_0 C_0 < 1$, we get

$$\begin{aligned} &\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g\|_{L^\infty([0,T]; H_x^r(L_v^2))}^2 + \frac{1}{4} \int_0^T \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g(s)\|_{r,0}^2 ds \\ &\leq 2e^{3C_8 T} \|\langle \delta' v \rangle^{-2} g_0\|_{H_x^r(L_v^2)}^2, \end{aligned}$$

which yields (5.3) with $C_9 = 8e^{3C_8}$, by letting $\delta' \rightarrow 0$. \square

Theorem 5.2 (analytic time-local solution). *Let $r > 3/2$. There exists an $\epsilon_1 > 0$ such that for all $g_0 \in H_x^r(L_v^2)$ satisfying*

$$\|g_0\|_{H_x^r(L_v^2)} \leq \epsilon_1,$$

the Cauchy problem (1.4) admits a solution such that

$$g(t) \in \mathcal{A}(\mathbb{R}_{x,v}^6), \quad 0 < \forall t \leq 1.$$

Furthermore, there exists a $0 < c_1 < 1$ such that,

$$e^{c_1\{t^2(-\Delta_x)^{1/2}+t(-\Delta_v)^{1/2}\}}g(t) \in L^\infty([0, 1], H_x^r(L_v^2)),$$

more precisely for any $0 < t \leq 1$

$$(5.6) \quad \|e^{c_1\{t^2(-\Delta_x)^{1/2}+t(-\Delta_v)^{1/2}\}}g(t)\|_{H_x^r(L_v^2)} \leq \sqrt{C_9}\|g_0\|_{H_x^r(L_v^2)}.$$

Proof. Consider the sequence of approximate solution defined by

$$(5.7) \quad \begin{cases} \partial_t g^{n+1} + v \cdot \nabla_x g^{n+1} + \mathcal{L}g^{n+1} = \Gamma(g^n, g^{n+1}). \\ g^{n+1}|_{t=0} = g_0, \end{cases}$$

with

$$g^0 = e^{-\Psi(t, D_x, D_v)}g_0.$$

We apply Lemma 5.1 with $f = g^n$ and $g = g^{n+1}$ by assuming $\sqrt{C_9}\epsilon_1 \leq \epsilon_0$. Then it follows from (5.3) with $T = 1$ that for $n \geq 1$

$$(5.8) \quad \|F_{\delta,0}g^n\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2 + \int_0^1 \|F_{\delta,0}g^n(s)\|_{r,0}^2 ds \leq C_9\|g_0\|_{H_x^r(L_v^2)}^2 \leq \epsilon_0^2$$

holds inductively because $\|F_{\delta,0}g^0\|_{H_x^r(L_v^2)} \leq \|g_0\|_{H_x^r(L_v^2)} \leq \epsilon_1 \leq \epsilon_0$. Setting $w^n = g^{n+1} - g^n$, from (5.7) we have

$$\partial_t w^n + v \cdot \nabla_x w^n + \mathcal{L}w^n = \Gamma(g^n, w^n) + \Gamma(w^{n-1}, g^n),$$

with $w^n|_{t=0} = 0$. Similar to the computation for (5.5), we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{H_x^r(L_v^2)}^2 + \frac{1}{4} \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{r,0}^2 - C_8 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{H_x^r(L_v^2)}^2 \right) \\ & \leq 2C_0 \|F_{\delta,0} g^n\|_{H_x^r(L_v^2)} \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{r,0}^2 + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{H_x^r(L_v^2)}^2 \right) \\ & \quad + 2C_0 \|F_{\delta,0} w^{n-1}\|_{H_x^r(L_v^2)} \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g^n\|_{r,0}^2 + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g^n\|_{H_x^r(L_v^2)}^2 \right)^{1/2} \\ & \quad \times \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{r,0}^2 + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{H_x^r(L_v^2)}^2 \right)^{1/2} \\ & \leq \left(2C_0 \|F_{\delta,0} g^n\|_{H_x^r(L_v^2)} + \frac{1}{16} \right) \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{r,0}^2 + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{H_x^r(L_v^2)}^2 \right) \\ & \quad + 32C_0^2 \|F_{\delta,0} w^{n-1}\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2 \\ & \quad \times \left(\|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g^n\|_{r,0}^2 + \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} g^n\|_{H_x^r(L_v^2)}^2 \right), \end{aligned}$$

which implies

$$\begin{aligned} & \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2 + \frac{1}{8} \int_0^1 \|\langle \delta' v \rangle^{-2} F_{\delta, \delta'} w^n(\tau)\|_{r,0}^2 d\tau \\ & \leq 64C_0^2 e^{3C_8} \left(\int_0^1 \|F_{\delta,0} g^n(\tau)\|_{r,0}^2 d\tau + \|F_{\delta,0} g^n\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2 \right) \\ & \quad \times \|F_{\delta,0} w^{n-1}\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2 \\ & \leq 64C_0^2 e^{3C_8} \times 2\epsilon_0^2 \|F_{\delta,0} w^{n-1}\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2, \end{aligned}$$

because of (5.8) and $16\epsilon_0 C_0 < 1$. Letting $\delta' \rightarrow 0$, we see that there exists a $0 < \lambda < 1$ such that

$$\|F_{\delta,0} w^n\|_{L^\infty([0,T]; H_x^r(L_v^2))}^2 \leq \lambda \|F_{\delta,0} w^{n-1}\|_{L^\infty([0,T]; H_x^r(L_v^2))}^2$$

if $\epsilon_0 > 0$ is small enough so that $128C_0^2e^{3C_8}\epsilon_0^2 \leq \lambda$. By taking $\delta \rightarrow 0$ we see that there exists a local solution $g \in L^\infty([0, T]; H_x^r(L_v^2))$ of the Cauchy problem (1.4) such that

$$\|e^\Psi(g^n - g)\|_{L^\infty([0,1]; H_x^r(L_v^2))} \rightarrow 0 \text{ as } n \rightarrow \infty,$$

and

$$\|e^\Psi g\|_{L^\infty([0,1]; H_x^r(L_v^2))}^2 + \int_0^1 \|e^\Psi g(s)\|_{r,0}^2 ds \leq C_9 \|g_0\|_{H_x^r(L_v^2)}^2 \leq \epsilon_0^2.$$

By means of Lemma 2.1, we get the desired estimate (5.6). \square

5.2. The stability and uniqueness of time-local solution.

Proposition 5.3 (Stability and uniqueness). *Let $r > 3/2, 0 < c_0 \ll 1$ and let $\epsilon_0 > 0$ satisfy $16C_0\epsilon_0 < 1$ for the constant C_0 in Proposition 4.1. Let $0 < T \leq 1$ and let $g_j(t) \in L^\infty([0, T]; H_x^r(L_v^2))$, $j = 1, 2$ be two solutions of the Cauchy problem (1.4) with initial data $g_{1,0}, g_{2,0} \in H_x^r(L_v^2)$, respectively. Assume that*

$$\|e^\Psi g_1\|_{L^\infty([0,T]; H_x^r(L_v^2))} \leq \epsilon_0,$$

and there exists $M > 0$, such that

$$\|e^\Psi g_2\|_{L^\infty([0,T]; H_x^r(L_v^2))}^2 + \int_0^T \|e^\Psi g_2(s)\|_{r,0}^2 ds \leq M.$$

Then there exists a $C_M > 1$ independent of c_0 such that

$$(5.9) \quad \|e^\Psi(g_1 - g_2)\|_{L^\infty([0,T]; H_x^r(L_v^2))}^2 + \int_0^T \|e^\Psi(g_1 - g_2)(s)\|_{r,0}^2 ds \leq C_M \|g_{0,1} - g_{0,2}\|_{H_x^r(L_v^2)}^2.$$

Remark 5.4. *It is easy to see that this proposition holds in the case where $c_0 = 0$, that is, e^Ψ is replaced by 1. Therefore the uniqueness of solutions belonging to $L^\infty([0, T]; H_x^r(L_v^2))$ holds under above conditions with $e^\Psi = 1$.*

Proof. Putting $w = g_1 - g_2$, we have

$$\partial_t w + v \cdot \nabla_x w + \mathcal{L}w = \Gamma(g_1, w) + \Gamma(w, g_2),$$

with an initial datum $w|_{t=0} = g_{0,1} - g_{0,2} \in H_x^r(L_v^2)$. Similar to the proof of Theorem 5.2, we obtain

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{H_x^r(L_v^2)}^2 + \frac{1}{4} \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{r,0}^2 - C_8 \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{H_x^r(L_v^2)}^2 \\ & \leq \left(2C_0 \|F_{\delta,0} g_1\|_{H_x^r(L_v^2)} + \frac{1}{16} \right) \left(\|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{r,0}^2 + \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{H_x^r(L_v^2)}^2 \right) \\ & \quad + 32C_0^2 \|F_{\delta,0} w\|_{H_x^r(L_v^2)}^2 \\ & \quad \times \left(\|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} g_2\|_{r,0}^2 + \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} g_2\|_{H_x^r(L_v^2)}^2 \right), \end{aligned}$$

which implies

$$\begin{aligned} & \frac{d}{dt} \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{H_x^r(L_v^2)}^2 + \frac{1}{8} \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{r,0}^2 - 3C_8 \|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} w\|_{H_x^r(L_v^2)}^2 \\ & \leq 64C_0^2 \|F_{\delta,0} w\|_{H_x^r(L_v^2)}^2 \left(\|\langle \delta'v \rangle^{-2} F_{\delta, \delta'} g_2\|_{r,0}^2 + M \right). \end{aligned}$$

By integrating from 0 to $t \in (0, T]$, we get

$$\begin{aligned} & \| \langle \delta' v \rangle^{-2} F_{\delta, \delta'} w(t) \|_{H_x^r(L_v^2)}^2 + \frac{1}{8} \int_0^t \| \langle \delta' v \rangle^{-2} F_{\delta, \delta'} w(\tau) \|_{r,0}^2 d\tau \\ & \leq \| w(0) \|_{H_x^r(L_v^2)}^2 + 64e^{3C_8 T_0} C_0^2 \int_0^t \| F_{\delta,0} w(\tau) \|_{H_x^r(L_v^2)}^2 \left(M + \| e^\Psi g_2(\tau) \|_{r,0}^2 \right) d\tau \end{aligned}$$

Let $\delta' \rightarrow 0$ and denote the right hand side by $Y_\delta(t)$. Then

$$Y'_\delta(t) \leq 64e^{3C_8 T_0} C_0^2 \left(M + \| e^\Psi g_2(t) \|_{r,0}^2 \right) Y_\delta(t), \quad a.e. \ t \in [0, T],$$

so that we obtain

$$Y_\delta(t) \leq \| w(0) \|_{H_x^r(L_v^2)}^2 e^{64e^{3C_8 T_0} C_0^2 (MT + \int_0^T \| e^\Psi g_2(s) \|_{r,0}^2 ds)}.$$

Finally, letting $\delta \rightarrow 0$ we obtain the desired estimate (5.9). \square

5.3. Time global solution and its analytic smoothing. To show the existence of a global solution and its analyticity smoothing, we refer the following theorem that was first proved by Guo[22] in the torus \mathbb{T}_x^3 case and was extended to the whole space \mathbb{R}_x^3 by Yang-Yu[34];

Theorem 5.5 ([22, 34]). *There exist some positive constants $\epsilon_2 > 0$, $C_{10} > 1$ such that if $g_0 \in H^8(\mathbb{R}_{x,v}^6)$ satisfies $\|g_0\|_{H_{x,v}^8} \leq \epsilon_2$ then the Cauchy problem (1.4) admits a unique global solution $g(t) \in L^\infty([0, \infty); H_{x,v}^8(\mathbb{R}^6))$ fulfilling*

$$\sup_{0 \leq t < \infty} \|g(t)\|_{H_{x,v}^8} \leq C_{10} \|g_0\|_{H_{x,v}^8}.$$

Assume that $0 < \epsilon_3 \leq (c_1)^8 / (8! C_9) \min\{\epsilon_1 / C_{10}, \epsilon_2\}$ and $\|g_0\|_{H_x^r(L_v^2)} \leq \epsilon_3$. Let $1 \leq \tau \leq 2$ and apply Theorem 5.2 with the initial time $t_0 = \tau - 1$, in view of $\sqrt{C_9} \epsilon_3 < \epsilon_1$. Then for any $\tau \in [1, 2]$ we have

$$\begin{aligned} & \| e^{c_1((-\Delta_x)^{1/2} + (-\Delta_v)^{1/2})} g(\tau) \|_{H_x^r(L_v^2)} \\ (5.10) \quad & = \| e^{c_1((\tau-t_0)^2(-\Delta_x)^{1/2} + (\tau-t_0)(-\Delta_v)^{1/2})} g(\tau) \|_{H_x^r(L_v^2)} \\ & \leq \sqrt{C_9} \|g(t_0)\|_{H_x^r(L_v^2)} \leq \sqrt{C_9} \epsilon_3, \end{aligned}$$

which implies the existence of local solution $g(t) \in L^\infty([0, 2]; H_x^r(L_v^2))$ satisfying

$$\sup_{[1,2]} \|g(t)\|_{H_{x,v}^8} \leq \frac{8!}{(c_1)^8} \sqrt{C_9} \epsilon_3 \leq \epsilon_2.$$

By using Theorem 5.5 with the initial time $t_1 = 1$, we obtain a global solution $g(t) \in L^\infty([1, \infty[; H_{x,v}^8(\mathbb{R}^6))$ satisfying

$$\sup_{[1, \infty[} \|g(t)\|_{H_x^r(L_v^2)} \leq \sup_{[1, \infty[} \|g(t)\|_{H_{x,v}^8} \leq \frac{8!}{(c_1)^8} \sqrt{C_9} C_{10} \epsilon_3 \leq \epsilon_1.$$

For $\tau > 2$, apply again Theorem 5.2 with the initial time $t_0 = \tau - 1$. Then we obtain

$$\sup_{\tau \geq 2} \| e^{c_1((-\Delta_x)^{1/2} + (-\Delta_v)^{1/2})} g(\tau) \|_{H_x^r(L_v^2)} \leq \sqrt{C_9} \sup_{\tau \geq 2} \|g(\tau - 1)\|_{H_x^r(L_v^2)} \leq \sqrt{C_9} \epsilon_1.$$

This together with (5.10) and Theorem 5.2 complete the proof of (1.5).

5.4. Global stability and uniqueness. It follows from Proposition 5.3 and the stability of global solutions in Theorem 5.5 that the stability of analytic global solutions holds. In fact, for any fixed $T > 2$, if $t \in [2, T]$ then

$$\begin{aligned} \|e^{c_1((-\Delta_x)^{1/2} + (-\Delta_v)^{1/2})}(g_1(t) - g_2(t))\|_{H_x^r(L_v^2)} &\leq \sqrt{C_M} \|g_1(t-1) - g_2(t-1)\|_{H_x^r(L_v^2)} \\ &\leq \sqrt{C_M} C_T \|g_1(1) - g_2(1)\|_{H_{x,v}^s} \\ &\leq \sqrt{C_M} \frac{8!C_T}{(c_1)^8} \|e^{c_1((-\Delta_x)^{1/2} + (-\Delta_v)^{1/2})}(g_1(1) - g_2(1))\|_{H_x^r(L_v^2)} \\ &\leq (C_M) \frac{8!C_T}{(c_1)^8} \|g_{0,1} - g_{0,2}\|_{H_x^r(L_v^2)}. \end{aligned}$$

The case $1 \leq t \leq 2$ is an easy consequence of Proposition 5.3. Thus the uniqueness of global solutions holds. Now the proof of Theorem 1.1 is complete.

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