Vaporisation as a possible mechanism of combustion instability

Dynamic response of an evaporating droplet to an acoustic field

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Presentation plan

- General approach
- Two-layer model
- N-layer model
- Conclusion and perspectives
Introduction

High frequency combustion instability in liquid rocket engines is a result of coupling between combustion processes and the chamber acoustics. Droplet evaporation is one possible driving mechanism of combustion instability.

In order to investigate this possibility, we consider an evaporating droplet submitted to an acoustic field. The objective of the study is to determine droplet response.
From the Rayleigh criterion to the response factor approach

[Source: Delplanque & Sirignano 1996]

“The Rayleigh criterion states that an initially small pressure perturbation will grow if the considered process adds energy in phase (or with a small enough phase lag) with pressure”

“To quantify this criterion in the case of droplet vaporization studies, a response factor was defined “:

\[
N = \iint_{V,t} q'(V,t) p'(V,t) dt \, dV / \iint_{V,t} (p'(V,t))^2 \, dt \, dV
\]

\[
p' = \left( p - \bar{p} \right) / \bar{p} \quad \text{Pressure relative perturbation}
\]

\[
q' = \left( q - \bar{q} \right) / \bar{q} \quad \text{Heat or mass relative perturbation}
\]
Analysis of frequency response characteristics

Response factor:

\[ N = \frac{\iint q'(V,t)p'(V,t)dt\,dV}{\iint (p'(V,t))^2\,dV\,dV} \]

For sinusoidal oscillations with the same period

For \( N > 0 \) --> destabilising effect of vaporisation

For \( N < 0 \) --> stabilising effect
The mean droplet of Heidmann and Wieber

A vaporizing drop of radius $r_S$, continuously supplied by a steady flow rate $\overline{M}$.

**Linearized theory:**

- Linearized equations are written for small perturbations
- Equations in $\hat{f}$ are deduced

\[
f' = \frac{f - \overline{f}}{\overline{f}}
\]

\[
f' = \hat{f} e^{i \omega t}
\]

\[
N = Re\left(\frac{\hat{M}}{\hat{\rho}}\right)
\]

\[
\theta = Arg\left(\frac{\hat{M}}{\hat{\rho}}\right)
\]
Extension of the Heidmann model with infinite conductivity

The Heidmann model makes the hypothesis of an infinite thermal conductivity
- in the steady-state regime
- in the unsteady regime too (uniform temperature inside the droplet)

--> this may have an influence on the droplet response

We made an extension of Heidmann model:
- Keeping the hypothesis of mean droplet at stabilized regime
- The objective is to evaluate more accurately the response factor and the cut-off frequency, taking into account the thermal wave inside the droplet in the unsteady regime
- As a first step, a two-layer model was developed
- We used the Spalding approach for the steady regime in the gaseous phase
Two-layer model for droplet transient heating

Temperature profiles inside the liquid droplet for three models
a) Conductive heat exchange (Law & Sirignano, 1977)
b) Infinite thermal conductivity (cf. Chin & Lefebvre, 1985)
c) Two-layer model (Present work): an alternative to a) and b)

N.B. : Heidmann model follows hypothesis b)
Two-layer model: a discrete system with two temperatures to describe the droplet evolution

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\[ M_L c_L \frac{dT_L}{dt} = Q_{SL} \]

Heat fluxes and energy equation of the liquid phase
Summary of the two-layer model

We suppose an ideal mixture of perfect gases and the quasi-steady hypothesis in the gaseous phase.

\[ M = \frac{4}{3} \pi r_s^3 \rho_L, \quad M_L = \frac{4}{3} \pi r_L^3 \rho_L, \quad M_S = \frac{4}{3} \pi (r_s^3 - r_L^3) \rho_L \]

\[ \dot{M} = 2 \pi \frac{k}{c_p} r_s \text{Nu} \ln(1 + B_T), \quad \dot{M} = 2 \pi \rho D r_s \text{Sh} \ln(1 + B_M) \]

\[ Q_{SL} = 2 \pi k_L r_L \text{Nu}_L (T_S - T_L), \quad Q_L = \dot{M} \frac{B_T'}{B_T} - 1 \]

\[ B_T = \frac{c_p (T_\infty - T_S)}{\ell + Q_L/\dot{M}}, \quad B_T' = \frac{c_p (T_\infty - T_S)}{\ell} \quad \ell: \text{latent heat} \]

At rest: \text{Nu}=2, \text{Nu}_L=2, \quad \text{Sh}=2.

Here: \( B_T \) (and \( B_M \)) are not constant. \quad \text{If Le=1, } B_T = B_M
Summary of the two-layer model

\[ \beta = \frac{3k Nu}{2\rho_L c_L r_s^2}, \quad \beta_S = \frac{\beta}{1-\varepsilon}, \quad \beta_L = \frac{3k_L Nu_L}{2\rho_L c_L r_s^2}, \quad \beta' = \frac{\varepsilon \beta_L}{1-\varepsilon}, \quad \varepsilon = \left( \frac{r_L}{r_s} \right)^3 \]

\[ \beta, \beta_S, \beta_L, \beta' \] heat exchange coefficients

Final energy equations:

\[
\begin{cases}
\frac{dT_L}{dt} = \beta_L (T_S - T_L) \\
\frac{dT_S}{dt} = -\beta' (T_S - T_L) + \beta_S \frac{\ln(1+B_T)}{B_T} (T_\infty - T_S) - \beta_S \frac{\ell}{c_p} \ln(1+B_T)
\end{cases}
\]
Summary of the two-layer model - linearisation of the equations

\[
\tau_v \frac{dM'}{dt} = -\dot{M}', \quad \dot{M}' = \frac{1}{3} M' + \alpha \left( p'_L - p'_C \right), \quad p'_L = \bar{b} T'_S, \quad p_L = \exp \left( a - \frac{b}{T_S - c} \right)
\]

assuming \( \bar{T}_S = \bar{T}_L \) (droplet in stabilised regime)

\[
\begin{align*}
\frac{dT'_L}{dt} &= \beta_L \left( T'_S - T'_L \right) , \quad \lambda \tau_v \left[ \varepsilon \frac{dT'_L}{dt} + \left( 1 - \varepsilon \right) \frac{dT'_S}{dt} \right] + \mu T'_S - \bar{a} p'_C = 0 \\
\end{align*}
\]

with \( \lambda = \frac{c_L \bar{T}_S}{\ell} \), \( \mu = \frac{\bar{T}_S}{T_C - \bar{T}_S} - \frac{2c}{\bar{T}_S - c} + \bar{b} \phi \), \( \bar{a} = \frac{\bar{T}_C}{T_C - \bar{T}_S} \frac{\gamma - 1}{\gamma} + \phi \),

\[
\varphi = \left( \frac{B_M}{Le} \left( 1 + B_M \right)^{\frac{1}{Le} - 1} \right) \left[ \bar{Y}_{AC} \bar{Y}_{FS} \bar{Y}_{AS} \frac{M_A}{M_F \bar{X}_{FS} + M_A \bar{X}_{AS}} + \left( M_A - M_F \right) \bar{Y}_{FS} \right] \left( M_A \right) - 3 \bar{b} - \mu \right) / \lambda, \quad B = 3\mu / \lambda, \quad \varepsilon = \left( r_L / r_S \right)^3
\]

\[
\Lambda = 3 \beta_L \tau_v, \quad \Lambda \varepsilon^3 = \theta, \quad \tau = \frac{3k_L c_p}{k c_L \ln(1 + B_T)} = \frac{9k_L \tau_v}{\rho_L c_L \bar{T}_S^2} = \frac{\tau_v}{\bar{T}_T}
\]

\( A, B, \varepsilon, \Lambda, \theta \) are constant coefficients ; \( \Lambda \) and \( \varepsilon \) are not independent only depends on propellant and surrounding conditions.
Summary of the two-layer model - complex transfer function

\[ Z = \frac{1}{\alpha} \frac{\hat{M}}{\hat{p}_C} = \frac{i u (\Lambda + i u)(A - i u) - \varepsilon u^2}{1 + i u (\Lambda + i u)(B + i u) + \varepsilon u^2} \]

\[ u = 3 \omega \bar{\tau}_v, \quad \varepsilon = \left( \frac{r_L}{r_S} \right)^3, \quad \Lambda = 3 \beta_L \bar{\tau}_v \]

Evolution of coefficients A and B versus temperature
Summary of the two-layer model - response factor

\[
\frac{N}{\alpha} = \text{Re}(Z) = \frac{u^2 \{ \Lambda^2 (AB + A + B) + u^2 [AB + A + B - \Lambda^2 - \varepsilon (A + B + \Lambda B - \Lambda A)] - u^4 (1 - \varepsilon)^2 \}}{(1 + u^2)(\Lambda B - (1 - \varepsilon)u^2)^2 + u^2 (\Lambda + B)^2}
\]

with \( \alpha = \frac{\bar{B}_M}{(1 + \bar{B}_M) \text{Ln}(1 + \bar{B}_M)} \frac{\bar{Y}_{AC} \bar{Y}_{FS}}{\bar{Y}_{AS}(\bar{Y}_{FS} - \bar{Y}_{FC})} \frac{M_A}{M_F \bar{X}_{FS} + M_A \bar{X}_{AS}} \)

Cut-off reduced frequency:

\[
u_c^2 = \frac{AB + A + B - \Lambda^2 - \varepsilon (A + B + \Lambda B - \Lambda A) + \sqrt{\Delta}}{2(1 - \varepsilon)^2}
\]

with \( \Delta = [AB + A + B - \Lambda^2 - \varepsilon (A + B + \Lambda B - \Lambda A)]^2 + 4(1 - \varepsilon)^2 (A + B + \Lambda B) \Lambda^2 \)

Limiting case of: \( \varepsilon = 0, \Lambda = \infty \) (analogous to Heidmann model hypotheses)

\[
Z = \frac{1}{\alpha} \frac{\hat{M}}{\hat{p}_c} = \frac{i u}{1 + i u} \frac{A - i u}{B + i u}
\]

\[
\frac{N}{\alpha} = \text{Re}(Z) = \frac{u^2 [AB + A + B - u^2]}{(1 + u^2)(B^2 + u^2)}
\]

\[ \rightarrow u_c^2 = AB + A + B \]
Two-layer model - Response factor as a function of reduced frequency for several couples ($\varepsilon$, $\Lambda$)

These results show the importance of taking into account internal thermal gradients for stability prediction.
The present model recovers exactly the Heidmann results (green curve coincides with $\varepsilon = 0.001$; $\Lambda = 5.65$). The Heidmann hypothesis leads to an underestimation of the amplification zone.
Two-layer model - Sensitivity of the cut-off frequency to the internal conductivity of the droplet

\[ k_0 = 0.11 \]

N.B. : \( k = k_0 \) corresponds to the case \( \text{O2/H2O} \); \( T = 3500 \text{ K} \); \( \text{Yac} = 0.9 \); \( P = 10\text{bar} \)
N-layer model - Principle and objectives

The model consists in refining the approach of the two-layer model, which permitted to take into account in a very simplified way the dynamic effect of internal thermal gradient. We hope so to have a better representation of the thermal internal profile.

We had two goals:
• to use for practical application a n-layer model with a limited number of layers (typically n < 10), in order to have a better precision as with a 2-layer model
• to deduce from the limit of the behavior of the n-layer model (of equal volumes) for great values of n (or n→infinity) the value of ε of the 2-layer model which would give the correct dynamic response
N-layer model - Hypotheses and formulation

Hypotheses:
- n layers of same volume and homogeneous temperatures
- Same Nusselt number for each layer

Matrix formulation

Tool used: Maple
N-layer model - General equations

\[ v_{j,n} = \frac{\nu}{n} = 4 \pi \frac{r_s^3}{n} \]

\[ r_{j,n} = \left( \frac{j+1}{n} \right)^\frac{1}{3} r_s \text{ for } 0 \leq j \leq n-1 \]

\[ \frac{\frac{\dot{M}c_L}{n}}{\frac{dT_0}{dt}} = 2\pi k_L Nu_L r_s \left( \frac{1}{n} \right)^\frac{1}{3} (T_1 - T_0) \]

\[ \beta_{0,n} = \frac{3k_L Nu_L}{2 \rho_L c_L r_{0,n}^2} = \frac{3k_L Nu_L n^3}{2 \rho_L c_L r_s^2} \]

\[ \Lambda_{0,n} = 3 \beta_{0,n} \bar{\tau}_v ; \quad \Lambda_{0,n} \left( \frac{1}{n} \right)^\frac{2}{3} = \theta = \frac{\bar{\tau}_v}{\bar{\tau}_T} \text{ (independent of n)} \]

\[ \frac{dT_j}{dt} = \beta_{0,n} \left[ (j+1)^\frac{1}{3} T_{j+1} - \left( (j+1)^\frac{1}{3} + j^\frac{1}{3} \right) T_j + j^\frac{1}{3} T_{j-1} \right] \text{ for } 1 \leq j < n-1 \]

\[ \frac{1}{n} \frac{d}{dt} \left( \sum_{j=0}^{n-1} T_j \right) = \frac{\dot{M}l}{Mc_L} \left( \frac{\tilde{B}_T}{B_T} - 1 \right) \]

Remark: the case \( n = 2 \) corresponds to the two-layer model with \( \epsilon = 0.5 \)
N-layer model - Equations for small perturbations

\[ \frac{dT'_0}{dt} = \beta_{0,n} \left( T'_1 - T'_0 \right) \]

\[ \frac{dT'_j}{dt} = \beta_{0,n} \left[ (j+1)^{\frac{1}{3}} T'_{j+1} - \left( (j+1)^{\frac{1}{3}} + j^{\frac{1}{3}} \right) T'_j + j^{\frac{1}{3}} T'_{j-1} \right] \text{ for } 1 \leq j < n-1 \]

\[ \frac{1}{n} \frac{d}{dt} \left( \sum_{j=0}^{n-1} T'_j \right) = \frac{l}{c_L T_s \bar{T}_v} \left( \frac{T_s}{T_c - T_s} T'_s - \frac{T_c}{T_c - T_s} T'_c + B_{T0} + l' \right) \]

Formulated so:

\[ \frac{\lambda \bar{T}_v}{n} \frac{d}{dt} \left( \sum_{j=0}^{n-1} T'_j \right) + \mu T'_s - \bar{\alpha} p'_c = 0 \]

\[ \bar{T}_v \frac{dM'}{dt} = -\dot{M'} \quad \dot{M'} = \frac{1}{3} M' + \alpha \left( p'_L - p'_c \right) \quad p'_L = \bar{b} T'_s \]
N-layer model – Final system

\[
\begin{align*}
    f' &= f e^{i \omega t} \\
    \left\{ \begin{array}{l}
    \frac{1 + iu}{iu} \hat{M} = \alpha (\tilde{b} \hat{T}_s - \hat{p}_c) \\
    \frac{iu}{n} \sum_{j=0}^{n-1} \hat{T}_j + B \hat{T}_s - \frac{3 \bar{a}}{\lambda} \hat{p}_c = 0 \\
    iu \hat{T}_0 = \Lambda_{0,n} \left( \hat{T}_1 - \hat{T}_0 \right) \\
    iu \hat{T}_j = \Lambda_{0,n} \left[ (j+1)^\frac{1}{3} \hat{T}_{j+1} - \left( (j+1)^\frac{1}{3} + j^\frac{1}{3} \right) \hat{T}_j + j^\frac{1}{3} \hat{T}_{j-1} \right] \quad \text{for} \ 1 \leq j \leq n-2
\end{array} \right.
\end{align*}
\]

\[
\begin{align*}
    u &= 3 \alpha \bar{\tau}_v \\
    \Lambda_{0,n} &= 3 \beta_{0,n} \bar{\tau}_v \\
    \beta_{0,n} &= \frac{3k_L Nu_L}{2 \rho_L c_s r_{0,n}^2} \\
    \Lambda_{0,n} \left( \frac{1}{n} \right)^\frac{2}{3} &= \theta = \frac{\bar{\tau}_v}{\bar{\tau}_T} \\
    r_{0,n} &= \frac{r_s^{\frac{1}{3}}}{n^3} \\
    B &= \frac{3 \mu}{\lambda} \\
    A &= \frac{3 \bar{a} \bar{b}}{\lambda} - B
\end{align*}
\]
**N-layer model - Dynamical response to a pressure perturbation**

\[
Z_n = \frac{1}{\alpha \hat{p}_c} \frac{\hat{M}}{\hat{M}} = \frac{iu}{1+iu} \frac{A - iu \frac{\sigma_n}{n}}{B + iu \frac{\sigma_n}{n}}
\]

with \( \sigma_n = \sigma_n(\Lambda_{0,n},u) = \left( \hat{T}_0 + \hat{T}_1 + \ldots + \hat{T}_s \right) / \hat{T}_s \)

\[
u = 3\omega \tau_v \\
B = \frac{3\mu}{\lambda} \\
A = \frac{3\bar{a}b}{\lambda} - B
\]

**Remark: two layers**: \( \varepsilon = \frac{1}{2} \rightarrow \)

\[
Z_2 = \frac{1}{\alpha \hat{p}_c} \frac{\hat{M}}{\hat{M}} = \frac{iu}{1+iu} \frac{\left( \Lambda_{0,2} + iu \right) \left( A - iu \right) - \frac{1}{2} u^2}{\left( \Lambda_{0,2} + iu \right) \left( B + iu \right) + \frac{1}{2} u^2}
\]
**N-layer model - Matrix formulation**

\[ \sigma_n = 1 + (n-1) \frac{1}{3} \sum_{n-1}^{T} M_{n-1}^{-1} e_{n-1} \]

\[
\begin{pmatrix}
\left( \frac{iu}{\Lambda_{0,n}} + 1 \right) & -1 & 0 & \ldots & 0 \\
-1 & \left( \frac{iu}{\Lambda_{0,n}} + \frac{1}{3} + 1 \right) & -\frac{1}{3} & 0 & \ldots & 0 \\
0 & -\frac{1}{3} & \left( \frac{iu}{\Lambda_{0,n}} + \frac{1}{3} + \frac{2}{3} \right) & -\frac{1}{3} & 0 & 0 \\
& & & & & \ddots & \vdots \\
0 & \ldots & 0 & -(n-3)\frac{1}{3} & \left( \frac{iu}{\Lambda_{0,n}} + (n-2)\frac{1}{3} + (n-3)\frac{1}{3} \right) & -(n-2)\frac{1}{3} \\
0 & \ldots & 0 & -(n-2)\frac{1}{3} & \left( \frac{iu}{\Lambda_{0,n}} + (n-1)\frac{1}{3} + (n-2)\frac{1}{3} \right) & \left( \frac{iu}{\Lambda_{0,n}} + (n-2)\frac{1}{3} \right)
\end{pmatrix}
\]

with \( M_{n-1} = \)

\[
\sum_{n-1}^{T} = \begin{bmatrix} 1 & 1 & \ldots & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} e_{n-1}^{T} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \ldots & 0 & 1 \end{bmatrix}
\]
**N-layer model - Results and conclusion**

- The cut-off frequency is very sensitive to the number of layers for low values of thermal conductivity.
- The convergence is too slow to obtain the limit with the tool we used and the limit is not easy to determine by analytical method (matrix model).
- The model may be used with a limited number of layers (<15).
Calibration of 2 layer-model using n-layer model, as a function of conductivity

- The 2-layer model gives results coherent with the 14-layer model for usual of artificially augmented values of conductivity
  - For low conductivities a value of $\varepsilon$ near 1 is suitable
  - For high conductivities a value of $\varepsilon$ near 0 is suitable
- For “medium” values of conductivity the 2-layer model underestimates the cut-off frequency
- N.B. : This analysis has to be extended to other physical conditions
**Conclusion**

The two-layer model shows the influence of the internal thermal gradient on the amplification zone.

This model needs a calibration of the $\varepsilon$ parameter in order to represent correctly the internal thermal gradient.

The n-layer model would allow to perform this calibration but the result will depend of the number of layers; however it can be used in situations where the two-layer model is not suitable.

First investigations make us to envisage the establishment of a continuous model in the small perturbation range, which would allow a validation of the n-layer model and a precise calibration of the two-layer model.
Perspectives

- Establish a continuous thermal conduction model in order to
  • validate the results of the n-layer model
  • calibrate more precisely the two-layer model

- Extend the analysis to other propellants and more realistic conditions, especially to high pressures.