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Global solutions to stochastic wave equations with superlinear coefficients

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Abstract

We prove existence and uniqueness of a random field solution $(u(t, x); (t, x) \in [0, T] \times \mathbb{R}^d)$ to a stochastic wave equation in dimensions $d = 1, 2, 3$ with diffusion and drift coefficients of the form $|z|(\ln_+(|z|))^a$ for some $a > 0$. The proof relies on a sharp analysis of moment estimates of time and space increments of the corresponding stochastic wave equation with globally Lipschitz coefficients. We give examples of spatially correlated Gaussian driving noises where the results apply.

Keywords: Stochastic wave equation, superlinear coefficients, global well-posedness
2010 MSC: Primary: 60H15, 60G60, Secondary: 35R60, 60G17

1. Introduction

In this paper, we study the stochastic wave equation in spatial dimension $d \in \{1, 2, 3\}$, with a multiplicative noise W ,

$$\begin{aligned} \frac{\partial^2}{\partial t^2} u(t, x) - \Delta_x u(t, x) &= b(u(t, x)) + \sigma(u(t, x)) \dot{W}(t, x), \quad (t, x) \in (0, T] \times \mathbb{R}^d, \\ u(0, x) &= u_0(x), \quad \frac{\partial}{\partial t} u(0, x) = v_0(x), \quad x \in \mathbb{R}^d. \end{aligned} \quad (1.1)$$

The choice of \dot{W} depends on the dimension d . First, we consider the case $d = 1$ with space-time white noise. Then, we consider the dimensions $d = 2, 3$ with a noise white in time and coloured in space. The initial conditions u_0 and v_0 are real-valued functions. The coefficients $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are locally Lipschitz functions such that, for $|z| \rightarrow \infty$,

$$|b(z)| \leq \theta_1 + \theta_2 |z| (\ln |z|)^\delta, \quad |\sigma(z)| \leq \sigma_1 + \sigma_2 |z| (\ln |z|)^a, \quad (1.2)$$

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where $\theta_i, \sigma_i \in \mathbb{R}_+, i = 1, 2, \theta_2, \sigma_2 > 0, \delta, a > 0$.

We are interested in studying conditions ensuring global existence of a random field solution to (1.1), that is, the existence of a stochastic process $(u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ satisfying

$$\begin{aligned} u(t, x) &= [G(t) * v_0](x) + \frac{\partial}{\partial t} [G(t) * u_0](x) + \int_0^t ds [G(s) * b(u(t-s, \cdot))](x) \\ &+ \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \sigma(u(s, y)) W(ds, dy), \quad \text{a.s.} \end{aligned} \quad (1.3)$$

either for all $(t, x) \in [0, T] \times \mathbb{R}^d$ or $(t, x) \in [0, T] \times D$, with $D \subset \mathbb{R}^d$ bounded. In (1.3), $G(t), t > 0$, is the fundamental solution to the wave operator, the notation “*” denotes the convolution in the space variable, and the stochastic integral is defined for example in [4].

It is a well-known phenomenon in PDEs that if the coefficients are superlinear, blow-up may occur (see for instance [12], and [23, Section X.13, p. 293]). For parabolic SPDEs, there is an extensive literature devoted to the study of blow-up phenomena. We refer the reader to [5] for a sample of references. There are however less results on stochastic wave equations. To the best of our knowledge, existence or absence of blow-up has been studied so far in the setting of functional-valued solutions, rather than for random field solutions, and mostly but not only, with strong conditions on the space covariance (see e.g. [3], [22], [18]). A quite general setting is considered in [17], where, existence (but not uniqueness) of functional-valued global solution is proved. The recent prepublication [11] is a new contribution to the problem.

Our research is motivated by [5], on the parabolic SPDE

$$\frac{\partial}{\partial t} u(t, x) - \frac{\partial^2}{\partial x^2} u(t, x) = b(u(t, x)) + \sigma(u(t, x)) \dot{W}(t, x), \quad (t, x) \in (0, T] \times (0, 1), \quad (1.4)$$

$u(0, x) = u_0(x), x \in [0, 1]$, with vanishing Dirichlet boundary conditions and locally Lipschitz coefficients such that, as $|z| \rightarrow \infty, |b(z)| = O(|z|(\ln |z|)), |\sigma(z)| = o(|z|(\ln |z|)^{1/4})$. One of the main results in [5] is the existence of a unique global random field solution to (1.4) on $\mathcal{C}(\mathbb{R}_+ \times [0, 1])$. This solution satisfies $\sup_{(t,x) \in [0,T] \times [0,1]} |u(t, x)| < \infty$, a.s., for any $T > 0$. If in equation (1.4), σ is constant and $|b(z)| \geq |z|(\ln |z|)^{1+\varepsilon}$ when $|z| \rightarrow \infty$, with ε arbitrarily close to zero, Bonder and Groisman [1] prove that blow-up occurs in finite time $t > 0$. The results in [5] imply that this condition on b is sharp.

The main results of this work are Theorem 3.5 and Theorem 4.13, relative to the two type of noises considered in the paper. Two scenarios are considered: (i) we restrict the spatial domain to a bounded set D ; (ii) the initial values have compact support and $b(0) = \sigma(0) = 0$. (see Section 6 for details). Loosely formulated, we prove:

If the initial conditions satisfy some Hölder continuity properties, the coefficients are such that (1.2) holds (see condition (Cs) in Section 3), and b dominates σ (see conditions (C1), (Cd) in Sections 3 and 4, respectively), then a global random field solution to (1.3) exists.

Our approach follows the L^∞ -method of [5], nevertheless, we do not use comparison theorems concerning monotony of coefficients since they do not hold for the wave equation. The main task consists in establishing qualitative sharp upper bounds on

$E \left(\sup_{(t,x) \in K} |u(t,x)|^p \right)$, for some range of values of p , when the coefficients are *globally Lipschitz* and K is compact subset of $\mathbb{R}_+ \times \mathbb{R}^d$. Such upper bounds depend on the value at the origin and the Lipschitz constants of the coefficients b and σ (see Propositions 3.4 and 4.12, and the notation (2.6)). These bounds are obtained from L^p -estimates of increments in time and in space of the process $(u(t,x))_{(t,x)}$ (see Propositions 3.3 and 4.11) via a version of Kolmogorov's theorem ([9, Theorem A.3.1]). Why is this important? Existence of solutions to equations with locally Lipschitz coefficients is often proved by transforming the coefficients into globally Lipschitz functions, using truncation. With a classical argument, involving an increasing sequence of stopping times $(\tau_N)_N$, if $\tau_N \uparrow \infty$ a.s., then existence of global solution follows. Let u_N denote the random field solution to (1.3) with truncated (by N) coefficients b^N, σ^N (see (3.25)). In our case, a sufficient condition for $\tau_N \uparrow \infty$ to hold (a.s.) is

$$E \left(\sup_{(t,x) \in K} |u_N(t,x)|^p \right) = o(N^p). \quad (1.5)$$

We prove (1.5) in the two scenarios described above, thereby deducing absence of blow-up.

In Section 3, we consider the case $d = 1$ and space-time white noise. The simplicity of this case allows to better highlight the approach. In Section 4, we deal with the case $d = 2, 3$. Since we are interested in random field solutions, in contrast with the case $d = 1$, we cannot take a space-time white noise. Instead, we consider a class of Gaussian noises white in time and coloured in space, for which a well developed stochastic integral theory exists (see e.g. [4], [9]). In comparison with Section 3, the arguments and computations are more difficult; they are inspired by the approach to sample path regularity of the random field solution of (1.3) for $d = 3$ given in [7] and [13]. Section 5 provides several examples of covariance densities where the results of the paper apply. Finally, in Section 6, we give the background on the two settings for the wave equation considered in the paper.

We end this introduction with some remarks. Addressing the question of critical growth of the coefficients for blow-up would be a natural and interesting continuation of this article. This is a plan for future work. Consider the case where b and σ are globally Lipschitz functions. From the first statement of Proposition 4.11 (see (4.72)), we deduce the existence of a version of the process $(u(t,x))_{(t,x)}$ with locally Hölder-continuous sample paths, jointly in (t,x) . Thus, for the class of spatial covariances considered in Section 4, this gives a unified approach to sample path regularity of the stochastic wave equation when $d = 2, 3$. Related results are in [19] for $d = 2$, and [7], [13] for $d = 3$.

Without much additional effort, the results of this paper can be extended to equation (1.3) with coefficients $b(t,x; u(t,x))$ and $\sigma(t,x; u(t,x))$.

2. Preliminaries and notations

We recall that for $d = 1, 2$ and for any fixed $t > 0$, the fundamental solution $G(t)$ to the partial differential operator $\frac{\partial^2}{\partial t^2} - \Delta_x$, is a function. More precisely,

$$G(t,x) = \begin{cases} \frac{1}{2} 1_{\{|x| < t\}}, & x \in \mathbb{R}, \\ \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x| < t\}}, & x \in \mathbb{R}^2, \end{cases} \quad (2.1)$$

while for $d = 3$,

$$G(t, dx) = \frac{1}{4\pi t} \sigma_t(dx), \quad x \in \mathbb{R}^3, \quad (2.2)$$

where $\sigma_t(dx)$ denotes the uniform surface measure on the sphere centred at zero and with radius t , (see e.g. [10, Ch. 5]).

Recall that, for any $d \geq 1$, the Fourier transform of $G(t, \cdot)$ is (see [27, p. 49])

$$\mathcal{F}G(t, \cdot)(\zeta) = \int_{\mathbb{R}^d} e^{-ix \cdot \zeta} G(t, dx) = \frac{\sin(t|\zeta|)}{|\zeta|}. \quad (2.3)$$

We will write $G(t, x - dy)$ to denote the translation by $-x$ of the measure $G(t, dy)$ in the distribution sense (see e.g. [24, p. 55]).

We will often write (1.3) in the compact form

$$u(t, x) = \sum_{i=0}^2 I_i(t, x), \quad t \in [0, T], \quad x \in \mathbb{R}^d, \quad (2.4)$$

where

$$\begin{aligned} I_0(t, x) &= [G(t) * v_0](x) + \frac{\partial}{\partial t} [G(t) * u_0](x), \\ I_1(t, x) &= \int_0^t ds [G(s) * b(u(t-s, \cdot))](x), \\ I_2(t, x) &= \int_0^t \int_{\mathbb{R}^d} G(t-s, x-dy) \sigma(u(s, y)) W(ds, dy). \end{aligned} \quad (2.5)$$

Notations

As mentioned in the introduction, we assume first that the coefficients of (1.3), b and σ , are globally Lipschitz continuous functions. Therefore, we have

$$|b(z)| \leq c(b) + L(b)|z|, \quad |\sigma(z)| \leq c(\sigma) + L(\sigma)|z|, \quad z \in \mathbb{R}, \quad (2.6)$$

with $c(b) = |b(0)|$, $c(\sigma) = |\sigma(0)|$ and $L(b)$, $L(\sigma)$, the Lipschitz constants of b and σ , respectively.

Let $\Phi : \Omega \times [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ be a jointly measurable random field. For fixed $\alpha > 0$, $p \in [2, \infty)$, we define the family of seminorms

$$\mathcal{N}_{\alpha, p}(\Phi) := \sup_{t \geq 0} \sup_{x \in \mathbb{R}^d} e^{-\alpha t} \|\Phi(t, x)\|_p, \quad (2.7)$$

where $\|\cdot\|_p$ denotes the norm in $L^p(\Omega)$.

For $\phi : \mathbb{R} \rightarrow \mathbb{R}$, set $\|\phi\|_\infty = \sup_{x \in \mathbb{R}} |\phi(x)|$ and, for $R \geq 0$, $\|\phi\|_{\infty, R} = \sup_{|x| \leq R} |\phi(x)|$. For $\gamma \in (0, 1)$, we define

$$\|\phi\|_\gamma = \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|^\gamma}. \quad (2.8)$$

Except if specified otherwise, $C, \bar{C}, \tilde{C}, c, \dots$ are positive and finite constants that may change throughout the paper, and $C(a), \bar{C}(a)$, etc., denote positive finite constants depending on the parameter a .

3. The stochastic wave equation in dimension one

In this section, we consider the stochastic wave equation (1.3) for $d = 1$, with a space-time white noise W and coefficients satisfying the superlinear growth condition (1.2). The study goes through several steps developed in the next subsections.

3.1. Qualitative moment estimates

We assume that the coefficients of (1.3), b and σ , are globally Lipschitz continuous functions therefore satisfying (2.6). We also suppose that $L(b)$ and $L(\sigma)$ are strictly positive. The goal is to obtain upper bounds on $\sup_{x \in \mathbb{R}} \|u(t, x)\|_p$ in terms of the constants $c(b)$, $c(\sigma)$, $L(b)$, $L(\sigma)$ for some range of values of p . This will be done using the approach of [14, Chapter 5] for the stochastic heat equation (see also [5]).

By the definition of $I_0(t, x)$ and G given in (2.5) and (2.1), respectively, we have

$$I_0(t, x) = \frac{1}{2} \int_{x-t}^{x+t} v_0(y) dy + \frac{1}{2} (u_0(x-t) + u_0(x+t)). \quad (3.1)$$

From this equality, we deduce

$$\sup_{x \in \mathbb{R}} |I_0(t, x)| \leq t \|v_0\|_\infty + \|u_0\|_\infty. \quad (3.2)$$

Clearly, if u_0, v_0 are bounded functions then $\sup_{x \in \mathbb{R}} |I_0(t, x)| < \infty$ for every $t \in [0, T]$.

Proposition 3.1. ([2, Proposition II.3]) *Assume that the function $(t, x) \mapsto I_0(t, x)$ is continuous and $\sup_{(t,x) \in [0,T] \times \mathbb{R}} |I_0(t, x)| < \infty$. Suppose that b and σ are globally Lipschitz continuous functions. Then (1.3) has a unique random field solution $(u(t, x); (t, x) \in [0, T] \times \mathbb{R})$. This solution satisfies*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}} \|u(t, x)\|_p < \infty, \quad \text{for any } p \in [1, \infty).$$

In the proof of the next proposition, the following facts will be used:

$$\sup_{t \geq 0} (t^k e^{-\alpha t}) = k^k (e\alpha)^{-k}, \quad k \in \mathbb{N}, \quad \sup_{t \geq 0} \int_0^t s e^{-\alpha s} ds = \alpha^{-2}, \quad \alpha > 0. \quad (3.3)$$

Proposition 3.2. *Let u_0 and v_0 be Borel functions satisfying $\|u_0\|_\infty + \|v_0\|_\infty < \infty$. Suppose that $L(b) \geq 8L(\sigma)^2$. Then, there exists a universal constant $C > 0$ such that, for any $p \in \left[2, \frac{L(b)}{4L(\sigma)^2}\right]$,*

$$\mathcal{N}_{2\sqrt{L(b)}, p}(u) \leq \mathcal{T}_0 + C \left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right], \quad (3.4)$$

where

$$\mathcal{T}_0 = \frac{e^{-1} \|v_0\|_\infty}{\sqrt{L(b)}} + 2 \|u_0\|_\infty. \quad (3.5)$$

Thus,

$$\sup_{x \in \mathbb{R}} E(|u(t, x)|^p) \leq e^{2pt\sqrt{L(b)}} \left\{ \mathcal{T}_0 + C \left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right] \right\}^p, \quad t \in [0, T]. \quad (3.6)$$

Proof. Fix $\alpha > 0$ and $p \in [2, +\infty)$. Using (3.2) and (3.3), we obtain

$$\mathcal{N}_{\alpha,p}(I_0) \leq \frac{e^{-1}}{\alpha} \|v_0\|_\infty + \|u_0\|_\infty. \quad (3.7)$$

Applying Minkowski's inequality, and then (2.6), we have

$$\begin{aligned} \|I_1(t, x)\|_p &\leq \int_0^t ds \int_{\mathbb{R}} dy G(t-s, x-y) \|b(u(s, y))\|_p \\ &\leq \int_0^t ds \int_{\mathbb{R}} dy G(t-s, x-y) [c(b) + L(b)\|u(s, y)\|_p]. \end{aligned}$$

Since $\int_{\mathbb{R}} G(t, x) dx = t$, using (3.3) we deduce

$$\begin{aligned} \mathcal{N}_{\alpha,p}(I_1) &\leq c(b) \sup_{t \geq 0} \left(\frac{t^2}{2} e^{-\alpha t} \right) + L(b) \mathcal{N}_{\alpha,p}(u) \sup_{t \geq 0} \int_0^t (s) e^{-\alpha(s)} ds \\ &\leq \frac{2e^{-2}}{\alpha^2} c(b) + \frac{1}{\alpha^2} L(b) \mathcal{N}_{\alpha,p}(u) \leq \frac{1}{\alpha^2} [c(b) + L(b) \mathcal{N}_{\alpha,p}(u)]. \end{aligned} \quad (3.8)$$

Applying first the version of Burkholder-Davies-Gundy's inequality given in [14, Theorem B1, p. 97], then Minkowski's inequality and (2.6), we obtain

$$\begin{aligned} \|I_2(t, x)\|_p^2 &\leq 4p \left\| \int_0^t \int_{\mathbb{R}} G^2(t-s, x-y) \sigma^2(u(s, y)) ds dy \right\|_{\frac{p}{2}} \\ &\leq 4p \int_0^t \int_{\mathbb{R}} G^2(t-s, x-y) \|\sigma^2(u(s, y))\|_{\frac{p}{2}} ds dy \\ &\leq 8p \left\{ \int_0^t ds \int_{\mathbb{R}} dy G^2(t-s, x-y) [c(\sigma)^2 + L(\sigma)^2 \|u(s, y)\|_p^2] \right\}. \end{aligned}$$

Since $G^2(t, x) = \frac{1}{2} G(t, x)$, using (3.3) we have

$$\begin{aligned} \mathcal{N}_{\alpha,p}(I_2) &\leq \sqrt{2p} c(\sigma) \sup_{t \geq 0} (te^{-\alpha t}) + \sqrt{8p} L(\sigma) \mathcal{N}_{\alpha,p}(u) \\ &\quad \times \left(\int_0^t ds \int_{\mathbb{R}} dy G^2(t-s, x-y) e^{-2\alpha(t-s)} \right)^{1/2} \\ &\leq \sqrt{2p} \frac{e^{-1}}{\alpha} c(\sigma) + \sqrt{8p} L(\sigma) \mathcal{N}_{\alpha,p}(u) \left(\int_0^t \frac{1}{2} s e^{-2\alpha s} ds \right)^{\frac{1}{2}} \\ &\leq \frac{\sqrt{p}}{\alpha} [c(\sigma) + L(\sigma) \mathcal{N}_{\alpha,p}(u)]. \end{aligned} \quad (3.9)$$

The inequalities (3.7), (3.8) and (3.9) imply

$$\mathcal{N}_{\alpha,p}(u) \leq \frac{e^{-1}}{\alpha} \|v_0\|_\infty + \|u_0\|_\infty + \frac{c(b)}{\alpha^2} + \frac{\sqrt{p}}{\alpha} c(\sigma) + 2 \max \left(\frac{L(b)}{\alpha^2}, \frac{\sqrt{p} L(\sigma)}{\alpha} \right) \mathcal{N}_{\alpha,p}(u). \quad (3.10)$$

Fix $\alpha^2 = 4L(b)$; since $L(b) \geq 8L(\sigma)^2$, the interval $\left[2, \frac{L(b)}{4L(\sigma)^2} \right]$ is nonempty. Since for any p in this interval we have $\sqrt{p} L(\sigma) \leq \frac{\sqrt{L(b)}}{2} = \frac{\alpha}{4}$, the choice of α implies

$\max\left(\frac{L(b)}{\alpha^2}, \frac{\sqrt{p}L(\sigma)}{\alpha}\right) = \frac{1}{4}$, and $\frac{\sqrt{p}}{\alpha} \leq \frac{1}{4L(\sigma)}$. Hence, from (3.10) we deduce (3.4). The estimate (3.6) is an immediate consequence of the definition of $\mathcal{N}_{\alpha,p}(u)$ for $\alpha = 2\sqrt{L(b)}$. \square

3.2. Uniform bounds on moments

In this section, we still assume that the coefficients of (1.3) are globally Lipschitz continuous functions. We prove an upper bound for

$$E\left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u(t, x)|^p\right), \quad (3.11)$$

for any $R > 0$, and for specific values of p that depend on the initial values u_0, v_0 , and the constants $c(b), c(\sigma), L(b), L(\sigma)$. This will be a consequence of the following proposition.

Proposition 3.3. *Let u_0 be locally Hölder continuous with exponent $\gamma_1 \in (0, 1]$, and v_0 be continuous. Set $\gamma = \gamma_1 \wedge \frac{1}{2}$, and fix $T, R \geq 0$. Then, for any $p \in [2, \infty)$, there exists a positive constant $C(p, T, R)$ such that, for any $t, \bar{t} \in [0, T]$, $x, \bar{x} \in [-R, R]$ and $\alpha > 0$,*

$$\frac{\|u(t, x) - u(\bar{t}, \bar{x})\|_p}{(|t - \bar{t}| + |x - \bar{x}|)^\gamma} \leq C(p, T, R) [\mathcal{M}_1 + \mathcal{M}_2 e^{\alpha T} \mathcal{N}_{\alpha,p}(u)], \quad (3.12)$$

where

$$\mathcal{M}_1 = \|u_0\|_{\gamma_1} + \|v_0\|_{\infty, R+T} + c(b) + \sqrt{p} c(\sigma), \quad \mathcal{M}_2 = L(b) + \sqrt{p} L(\sigma). \quad (3.13)$$

Moreover, if $L(b) \geq 8L(\sigma)^2$ then for any $p \in \left[2, \frac{L(b)}{4L(\sigma)^2}\right]$,

$$\frac{\|u(t, x) - u(\bar{t}, \bar{x})\|_p}{(|t - \bar{t}| + |x - \bar{x}|)^\gamma} \leq C(p, T, R) \left[\mathcal{M}_1 + \mathcal{M}_2 e^{2\sqrt{L(b)}T} \left(\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right) \right], \quad (3.14)$$

with \mathcal{T}_0 given in (3.5).

Proof. The function $V_0(z) = \int_0^z v_0(y) dy$ is continuously differentiable; hence,

$$\left| \int_{\bar{x}-\bar{t}}^{\bar{x}+\bar{t}} v_0(y) dy - \int_{x-t}^{x+t} v_0(y) dy \right| \leq 2 \|v_0\|_{\infty, R+T} (|x - \bar{x}| + |t - \bar{t}|).$$

Consequently, using the expression (3.1) and the γ_1 -Hölder continuity of u_0 , we obtain

$$|I_0(t, x) - I_0(\bar{t}, \bar{x})| \leq C(T, R) (\|u_0\|_{\gamma_1} + \|v_0\|_{\infty, R+T}) (|x - \bar{x}|^{\gamma_1} + |t - \bar{t}|^{\gamma_1}). \quad (3.15)$$

for some $C(T, R) > 0$.

In the next arguments, we will use the following inequalities, whose proofs are easy. For all $0 \leq \bar{t}, t \leq T$, $x, \bar{x} \in \mathbb{R}$, there exists a positive constant $C(T)$ such that

$$\begin{aligned} & \int_0^T ds \int_{\mathbb{R}} dy |G(t-s, x-y) - G(\bar{t}-s, \bar{x}-y)| \\ &= 2 \int_0^T ds \int_{\mathbb{R}} dy |G(t-s, x-y) - G(\bar{t}-s, \bar{x}-y)|^2 \leq C(T) (|t - \bar{t}| + |x - \bar{x}|). \end{aligned} \quad (3.16)$$

For any $\alpha > 0$, as in the proof of Proposition 3.2, Minkovski's inequality and (2.6) imply

$$\begin{aligned} \|I_1(t, x) - I_1(\bar{t}, \bar{x})\|_p &\leq \int_0^T ds \int_{\mathbb{R}} dy |G(t-s, x-y) - G(\bar{t}-s, \bar{x}-y)| \|b(u(s, y))\|_p \\ &\leq C(T) [c(b) + L(b)e^{\alpha T} \mathcal{N}_{\alpha, p}(u)] (|t - \bar{t}| + |x - \bar{x}|). \end{aligned} \quad (3.17)$$

Upper bounds of increments of I_2 are also obtained following the arguments in the proof of Proposition 3.2, based on the Burkholder-Davies-Gundy and Minkowski inequalities. More precisely,

$$\begin{aligned} \|I_2(t, x) - I_2(\bar{t}, \bar{x})\|_p^2 &\leq 4p \left\| \int_0^T ds \int_{\mathbb{R}} dy |G(t-s, x-y) - G(\bar{t}-s, \bar{x}-y)|^2 \sigma^2(u(s, y)) \right\|_{\frac{p}{2}} \\ &\leq 8p C(T) [c(\sigma)^2 + L(\sigma)^2 e^{2\alpha T} \mathcal{N}_{\alpha, p}(u)^2] (|t - \bar{t}| + |x - \bar{x}|), \end{aligned}$$

for any $\alpha > 0$. Consequently,

$$\|I_2(t, x) - I_2(\bar{t}, \bar{x})\|_p \leq 2\sqrt{2C(T)}\sqrt{p} [c(\sigma) + L(\sigma)e^{\alpha T} \mathcal{N}_{\alpha, p}(u)] (|t - \bar{t}| + |x - \bar{x}|)^{\frac{1}{2}}. \quad (3.18)$$

Let $\gamma = \gamma_1 \wedge \frac{1}{2}$; the inequalities (3.15), (3.17) and (3.18) imply (3.12).

Let $\alpha = 2\sqrt{L(b)}$ and $p \in [2, \frac{L(b)}{4L(\sigma)^2}]$; then (3.4) implies (3.14). The proof of the proposition is complete. \square

From Proposition 3.3, using Kolmogorov's continuity lemma (see [9, Theorem A.3.1] or [14, Theorem C-6]), we deduce the following.

Proposition 3.4. *Let the initial values u_0, v_0 be as in Proposition 3.3. Let $\gamma = \gamma_1 \wedge \frac{1}{2}$ and suppose that $L(b) > \frac{8}{\gamma} L(\sigma)^2$. Then u has a version, still denoted by u , which is locally Hölder continuous jointly in (t, x) with exponent $\eta \in (0, \gamma)$. Furthermore, given any $p \in (\frac{2}{\gamma}, \frac{L(b)}{4L(\sigma)^2}]$, there exists a constant $C(p, T, R)$ such that*

$$E\left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u(t, x)|^p\right) \leq 2^{p-1} \|u_0\|_{\infty, R}^p + C(p, T, R) \left[\mathcal{M}_1^p + \mathcal{M}_2^p \mathcal{M}_3^p e^{2pT\sqrt{L(b)}}\right], \quad (3.19)$$

where $\mathcal{M}_1, \mathcal{M}_2$ are defined in (3.13), and

$$\mathcal{M}_3 = \frac{e^{-1} \|v_0\|_{\infty, R}}{\sqrt{L(b)}} + 2\|u_0\|_{\infty, R} + C\left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)}\right],$$

with the universal constant C in the right-hand side of (3.4).

Proof. For any $s, t \in [0, T]$, $x, y \in [-R, R]$, set $\Delta(t, x; s, y) = |t - s|^\gamma + |x - y|^\gamma$. Proposition 3.3 implies

$$E(|u(t, x) - u(s, y)|^p) \leq K(\Delta(t, x; s, y))^p,$$

with

$$K := C(p, T, R) \left[\mathcal{M}_1^p + \mathcal{M}_2^p e^{\alpha p T} \mathcal{N}_{\alpha, p}(u)^p\right], \quad \alpha > 0. \quad (3.20)$$

Apply [9, Theorem A.3.1] with $k = 1$, $\alpha_1 = \alpha_2 = \gamma$, $I = [0, T]$, $J = [-R, R]$, $p \in (\frac{2}{\gamma}, \infty)$, to infer the existence of a version of u (that we still denote by u) with jointly Hölder continuous sample paths of exponent $\eta \in (0, \gamma)$. Moreover, since by (1.1), $C_1 := E\left(\sup_{|x| \leq R} |u(0, x)|^p\right) = \|u_0\|_{\infty, R}^p$, we deduce from [9, Equation(2.8.50)],

$$E\left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u(t, x)|^p\right) \leq 2^{p-1} \|u_0\|_{\infty, R}^p + C(p, T, R)K, \quad (3.21)$$

where K is defined in (3.20). Observe that K depends on α .

Choose $\alpha = 2\sqrt{L(b)}$. Then (3.21) and (3.20) yield

$$\begin{aligned} E\left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u(t, x)|^p\right) &\leq 2^{p-1} \|u_0\|_{\infty, R}^p \\ &+ C(p, T, R) \left[\mathcal{M}_1^p + \mathcal{M}_2^p e^{2pT\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)}, p}^p(u) \right]. \end{aligned} \quad (3.22)$$

Notice that, since $\gamma \leq 1/2$, the condition $L(b) > \frac{8}{\gamma} L(\sigma)^2$ implies that the hypotheses of Proposition 3.2 are satisfied. Hence, using (3.4) to upper estimate $\mathcal{N}_{2\sqrt{L(b)}, p}^p(u)$ on the right-hand side of (3.22), and since we are considering $|x| \leq R$, we obtain (3.19). \square

3.3. Existence and uniqueness of global solution

In this section, we consider the equation (1.3) with coefficients having superlinear growth and prove existence and uniqueness of a random field solution.

We introduce the following set of hypotheses.

(Cs) The functions $b, \sigma : \mathbb{R} \rightarrow \mathbb{R}$ are locally Lipschitz and such that as $|z_1|, |z_2| \rightarrow \infty$,

$$\begin{aligned} |b(z_1) - b(z_2)| &\leq \theta_2 |z_1 - z_2| [\ln_+(|z_1 - z_2|)]^\delta, \\ |\sigma(z_1) - \sigma(z_2)| &\leq \sigma_2 |z_1 - z_2| [\ln_+(|z_1 - z_2|)]^a, \end{aligned}$$

where $\theta_2, \sigma_2 \in (0, \infty)$, $\delta, a > 0$, and $\ln_+(z) = \ln(z \vee e)$ for $z \geq 0$.

(C1) The parameters δ, a in **(Cs)** satisfy one of the properties: (1) $\delta > 2a$; (2) $\delta = 2a$ and the constants θ_2 and σ_2 are such that $\theta_2 > \bar{\gamma}\sigma_2^2$, for some $\bar{\gamma} > 0$.

Notice that condition **(Cs)** implies (1.2), while **(C1)** says that b dominates σ . We define $\theta_1 := |b(0)|$ and $\sigma_1 := |\sigma(0)|$.

Theorem 3.5. *Assume that the initial condition u_0 is Hölder continuous with exponent γ_1 , and v_0 is continuous. Set $\gamma = \gamma_1 \wedge \frac{1}{2}$ and let the coefficients b and σ satisfy the conditions **(Cs)** and **(C1)** with $\delta < 2$ and $\bar{\gamma} = 8\gamma^{-1}$.*

1. For any $M > 0$, there exists a random field solution to (1.3) in $[-M, M]$, $(u(t, x), (t, x) \in [0, T] \times [-M, M])$. This solution is unique and satisfies

$$\sup_{(t, x) \in [0, T] \times [-M, M]} |u(t, x)| < \infty, \text{ a.s.} \quad (3.23)$$

2. Suppose that the initial conditions u_0, v_0 are functions with compact support included in $[-\rho, \rho]$, for some $\rho > 0$, and $b(0) = \sigma(0) = 0$. Then there exists a random field solution $(u(t, x), (t, x) \in [0, T] \times \mathbb{R})$ to (1.3). This solution is unique and satisfies

$$\sup_{(t, x) \in [0, T] \times [-(\rho+T), \rho+T]} |u(t, x)| < \infty, \text{ a.s.} \quad (3.24)$$

Proof. We start with some remarks. In statement 1. above, the notion of “*random field solution to (1.3) in $[-M, M]$ ”* is made rigorous in Section 6.2. According to Proposition 6.2, the support of the sample paths of this solution is included in $[0, T] \times [-(M+T), M+T]$.

By Proposition 6.1, the assumptions in statement 2. imply that the support of the sample paths of the solution $(u(t, x), (t, x) \in [0, T] \times \mathbb{R})$ is included in $[0, T] \times [-(\rho + T), \rho + T]$. Hence, (3.24) is equivalent to

$$\sup_{(t, x) \in [0, T] \times \mathbb{R}} |u(t, x)| < \infty, \text{ a.s.}$$

Solution for truncated Lipschitz continuous coefficients. For a locally Lipschitz function $g : \mathbb{R} \rightarrow \mathbb{R}$ and $N \geq 1$, we define a globally Lipschitz function g_N by

$$g_N(x) = g(x)1_{\{|x| \leq N\}} + g(N)1_{\{x > N\}} + g(-N)1_{\{x < -N\}}. \quad (3.25)$$

Using this definition for σ and b , we consider (1.3) with coefficients σ_N, b_N , and denote by $u_N := (u_N(t, x); (t, x) \in [0, T] \times \mathbb{R})$ its unique random field solution (see Proposition 3.1). From (Cs) we see that if $N \geq 2$, σ_N, b_N satisfy the conditions (2.6) with

$$c(b_N) = \theta_1, \quad c(\sigma_N) = \sigma_1, \quad L(b_N) = \theta_2(\ln(2N))^\delta, \quad L(\sigma_N) = \sigma_2(\ln(2N))^a. \quad (3.26)$$

Observe that, in the setting 2. of the Theorem, $\theta_1 = \sigma_1 = 0$.

Therefore, Proposition 3.3 applies; by Kolmogorov’s continuity criterion, there is a version of u_N with jointly Hölder continuous sample paths of exponent $\eta \in (0, \gamma)$ in both variables. In the sequel we will consider this version that we will still denote by u_N .

Bounds for L^p moments of u_N . Assume that condition (C1) (1) holds. Then, for N large enough, we have $L(b_N) > \frac{8}{\gamma} L(\sigma_N)^2$. On the other hand, if condition (C1) (2) is satisfied, then $L(b_N) > \frac{8}{\gamma} L(\sigma_N)^2$ holds for any $N \geq 2$. We can therefore apply Proposition 3.4 to see that for any $p \in \left(\frac{2}{\gamma}, \frac{\theta_2(\ln(2N))^\delta}{4\sigma_2^2(\ln(2N))^{2a}}\right]$, $R > 0$, N large enough (if necessary).

$$\begin{aligned} E \left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u_N(t, x)|^p \right) &\leq 2^{p-1} \|u_0\|_{\infty, R}^p \\ &\quad + C(p, T, R) \left[\mathcal{M}_1^p + \mathcal{M}_2^p(N) \mathcal{M}_3^p(N) e^{2pT\sqrt{L(b_N)}} \right], \end{aligned} \quad (3.27)$$

where

$$\begin{aligned} \mathcal{M}_1 &= \|u_0\|_{\gamma_1} + \|v_0\|_{\infty, R+T} + \theta_1 + \sqrt{p} \sigma_1, \quad \mathcal{M}_2(N) = L(b_N) + \sqrt{p} L(\sigma_N), \\ \mathcal{M}_3(N) &= \frac{e^{-1} \|v_0\|_{\infty, R}}{\sqrt{L(b_N)}} + 2\|u_0\|_{\infty, R} + C \left[\frac{\theta_1}{L(b_N)} + \frac{\sigma_1}{L(\sigma_N)} \right]. \end{aligned} \quad (3.28)$$

Existence and uniqueness of a global solution. Fix $R > 0$. For any $N \geq 2$, set

$$\tau_N := \inf \left\{ t > 0 : \sup_{|x| \leq R} |u_N(t, x)| \geq N \right\} \wedge T. \quad (3.29)$$

The uniqueness of the solution and the local property of stochastic integrals imply that $u_N(t, x) = u_{N+1}(t, x)$ a.s. for $t \leq \tau_N$. Hence, almost surely, $(\tau_N)_{N \geq 2}$ is an increasing sequence, bounded by T .

Assume that $\sup_N \tau_N = T$, a.s., and thus $\{t \leq \tau_N\} \uparrow \Omega$, a.s. On $\{t \leq \tau_N\}$, define $(u(t, x), (t, x) \in [0, T] \times \mathbb{R})$ by $u(t, x) = u_N(t, x)$; then $u(t, x) = u_M(t, x)$, for every $M \geq N$. The random variable $u(t, x)$ is well-defined and moreover, (1.3) holds for any (t, x) , a.s. Indeed, the definition of τ_N implies that on $\{t \leq \tau_N\}$,

$$\begin{aligned} u(t, x) = & I_0(t, x) + \int_0^t ds \int_{\mathbb{R}} dy G(t-s, x-y) b_N(u_N(s, y)) \\ & + \int_0^t \int_{\mathbb{R}} G(t-s, x-y) \sigma_N(u_N(s, y)) W(ds, dy). \end{aligned}$$

But on $\{t \leq \tau_N\}$, $b_N(u_N(s, y)) = b(u_N(s, y)) = b(u(s, y))$ and $\sigma_N(u_N(s, y)) = \sigma(u_N(s, y)) = \sigma(u(s, y))$. Since $\{t \leq \tau_N\} \uparrow \Omega$ a.s., we conclude that $(u(t, x), (t, x) \in [0, T] \times \mathbb{R})$ satisfies (1.3). Notice that, in this case, the stochastic integral in (1.3) is not defined in $L^2(\Omega)$, but using instead an extension defined a.s. (see e.g. [9]).

The last part of the proof is devoted to check that indeed, $\sup_N \tau_N = T$ a.s. This will follow from the property

$$\lim_{N \rightarrow \infty} P(\tau_N < T) = 0, \quad (3.30)$$

that we now establish. Let $C(p, T, R, N)$ denote the right-hand side of (3.27). To emphasise the terms that depend on N , we write

$$C(p, T, R, N) = C_1(p, T, R) + C_2(p, T, R, N), \quad (3.31)$$

with

$$\begin{aligned} C_1(p, T, R) &= 2^{p-1} \|u_0\|_{\infty, R}^p + C(p, T, R) \mathcal{M}_1^p, \\ C_2(p, T, R, N) &= C(p, T, R) \mathcal{M}_2^p(N) \mathcal{M}_3^p(N) e^{2pT\sqrt{L(b_N)}}. \end{aligned}$$

Fix $p \in \left(\frac{2}{\gamma}, \frac{\theta_2(\ln(2N))^\delta}{4\sigma_2^2(\ln(2N))^{2\alpha}}\right]$. Applying Chebychev's inequality and then (3.27), we have

$$\begin{aligned} P(\tau_N < T) &\leq P\left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u_N(t, x)| \geq N\right) \leq N^{-p} E\left(\sup_{t \in [0, T]} \sup_{|x| \leq R} |u_N(t, x)|^p\right) \\ &\leq N^{-p} C(p, T, R, N) = N^{-p} [C_1(p, T, R) + C_2(p, T, R, N)]. \end{aligned} \quad (3.32)$$

Assume that

$$C_2(p, T, R, N) = o(N^p). \quad (3.33)$$

Then, from (3.32), we clearly obtain (3.30).

For the proof of (3.33), we first write the expressions of $\mathcal{M}_2(N)$ and $\mathcal{M}_3(N)$ in (3.28), substituting $L(b_N)$ and $L(\sigma_N)$ by their respective values given in (3.26). Because of the property $\sup_{N \geq 2} \mathcal{M}_3(N) \leq C$, we obtain

$$C_2(p, T, R, N) = \tilde{C}_2(p, T, R) \exp\left(p \delta \ln[\ln(2N)] + 2pT\theta_2^{1/2}[\ln(2N)]^{\delta/2}\right).$$

Since $\delta < 2$, this implies (3.33).

Let $M > 0$ be as in Claim 1. From the above discussion, we deduce (3.23) by taking $R = M$. Similarly, Claim 2. is obtained by considering $R = \rho + T$.

The proof of the theorem is complete. \square

4. The stochastic wave equation in dimensions 2 and 3

The aim of this section is to discuss the same questions as in Section 3 when $d = 2, 3$, and the noise W is white in time and coloured in space. It is well-known that for dimensions $d \geq 2$, if W is a space-time white noise, the stochastic convolution in (1.3) fails to be a well-defined random variable in $L^2(\Omega)$, for almost any $(t, x) \in [0, T] \times \mathbb{R}^d$. This is the case even if σ is constant. However, we can still obtain a random field solution of (1.3) by taking a smoother noise in the spatial variable (see e.g. [28]). This leads to the introduction in the next subsection 4.1 of a new class of Gaussian noises.

4.1. Spatially homogeneous Gaussian noise and stochastic integrals

Let Λ be a non-negative definite distribution in $\mathcal{S}'(\mathbb{R}^d)$. By the Bochner-Schwartz theorem (see e.g. [24, Chap. VII, Theorem XVIII]), Λ is the Fourier transform of a non-negative, tempered, symmetric measure μ on \mathbb{R}^d called the spectral measure of Λ . In particular, Λ is also a tempered distribution.

On a complete probability space (Ω, \mathcal{A}, P) , we consider a Gaussian process $\{W(\varphi), \varphi \in \mathcal{C}_0(\mathbb{R}^{d+1})\}$, indexed by the set of Schwartz test functions, with mean zero and covariance

$$E(W(\varphi)W(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^d} \Lambda(dx) \left(\varphi(t) * \tilde{\psi}(t) \right) (x), \quad (4.1)$$

where “ $*$ ” denotes the convolution operator in the spatial variable and $\tilde{\psi}$ means reflection in the spatial variable too.

We will consider spatial covariances Λ satisfying the following hypothesis ([4]):

(h0) The spectral measure $\mu = \mathcal{F}^{-1}\Lambda$ is such that

$$\int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1 + |\zeta|^2} < \infty. \quad (4.2)$$

From (2.3), we see that this is equivalent to $\int_0^T dt \int_{\mathbb{R}^d} \mu(d\zeta) |\mathcal{F}G(t)(\zeta)|^2 < \infty$.

Consider a jointly measurable adapted process $Z = (Z(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ such that $\sup_{(t, x) \in [0, T] \times \mathbb{R}^d} E(|Z(t, x)|^p) < \infty$, for some $p \in [2, \infty)$, and assume **(h0)**. Then, the stochastic integral

$$((GZ) \cdot W)(t, x) := \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) Z(s, y) W(ds, dy)$$

is a well-defined random variable. Moreover, for any $x \in \mathbb{R}^d$, the process $((GZ) \cdot W)(t, x), t \in [0, T]$ is a martingale with respect to the natural filtration generated by W .

We will consider the particular class of covariances Λ described in **(h1)** below.

(h1) Λ is an absolutely continuous measure, $\Lambda(dx) = f(x)dx$, $f \geq 0$. Its spectral measure $\mu = \mathcal{F}^{-1}\Lambda$ is such that, for all signed measures Φ and Ψ with finite total variation,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Phi(dx) \Psi(dy) f(x-y) = C \int_{\mathbb{R}^d} \mu(d\zeta) \mathcal{F}\Phi(\zeta) \overline{\mathcal{F}\Psi(\zeta)}. \quad (4.3)$$

Remark 4.1. Assume $\int_{\mathbb{R}^d} \mu(d\zeta) [|\mathcal{F}(\|\Phi\|)(\zeta)|^2 + |\mathcal{F}(\|\Psi\|)(\zeta)|^2] < \infty$, where the notation $\|\cdot\|$ stands for the total variation. Suppose also that $f : \mathbb{R}^d \rightarrow [0, +\infty]$ is lower semicontinuous. Then if $\Phi = \Psi$, [16, Corollary 3.4] implies the validity of (4.3) with $C = (2\pi)^{-d}$. By a polarity argument, (4.3) can be extended to $\Phi \neq \Psi$ (see [15, p. 487]).

Assume **(h0)** and **(h1)**. Since for any $t > 0$, $G(t, dx)$ is a non-negative finite measure with compact support, using (4.3) with $\Phi = G(t, dx)$ and $\Psi = G(s, dy)$, $s, t > 0$, we have

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t, dx) G(s, dy) f(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mu(d\zeta) \mathcal{F}G(t, \cdot)(\zeta) \overline{\mathcal{F}G(s, \cdot)(\zeta)}. \quad (4.4)$$

In particular,

$$J(t) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t, dy) G(t, dx) f(x - y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mu(d\zeta) |\mathcal{F}G(t)(\zeta)|^2. \quad (4.5)$$

Using (2.3), we have $|\mathcal{F}G(t)(\zeta)|^2 \leq \frac{2}{1+|\zeta|^2} 1_{\{|\zeta| \geq 1\}} + t^2 1_{\{|\zeta| < 1\}} \leq \frac{2(1+t^2)}{1+|\zeta|^2}$ for $t > 0$. Hence,

$$J(t) \leq 2(1+t^2)C_\mu, \quad \text{where } C_\mu := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1+|\zeta|^2} < \infty, \quad (4.6)$$

which implies, $\sup_{t \in [0, T]} J(t) < \infty$.

Assuming **(h0)** and **(h1)**, the stochastic integral $((GZ) \cdot W)(t, x)$ satisfies the following sharp version of the Burkholder Davies Gundy inequality

$$\begin{aligned} \|((GZ) \cdot W)(t, x)\|_p^p &\leq (2\sqrt{p})^p \\ &\times E \left(\int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-dy) G(t-s, x-dz) f(y-z) Z(s, y) Z(s, z) \right)^{\frac{p}{2}} \end{aligned} \quad (4.7)$$

(see e.g. [9], [21]).

We end this section with a technical lemma related with the identity (4.3). For $d = 3$, with a different proof, the result can be found in [13, Lemma 6.5].

Lemma 4.2. Let $d \geq 1$, $t > 0$ and $G(t)$ be the fundamental solution of the wave operator on \mathbb{R}^d . Let φ, ψ be bounded Borel measurable functions defined on \mathbb{R}^d . Let Λ be a symmetric measure satisfying **(h1)**, with spectral measure $\mu = \mathcal{F}^{-1}\Lambda$ satisfying **(h0)**. Then, for any $s, t > 0$ and $z \in \mathbb{R}^d$, we have

$$\begin{aligned} &\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(x) G(t, dx) \psi(y) G(s, dy) f(x - y + z) \\ &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}(\varphi G(t))(\xi) \overline{\mathcal{F}(\psi G(s))(\xi)} e^{-iz \cdot \xi} \mu(d\xi). \end{aligned} \quad (4.8)$$

Proof. By applying the translation $\tau_z x = x + z$, the left-hand side of (4.8) equals

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(\tau_{-z}x) \tau_{-z}G(t, dx) \psi(y) G(s, dy) f(\tau_zx - y) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(w - y) \Phi(dw) \Psi(dy),$$

where $\Phi(dw) = \varphi(\tau_{-z}w) \tau_{-z}G(t, dw)$ and $\Psi = \psi(y) G(s, dy)$. We recall that $\tau_{-z}G(t, dw)$ stands for the translation of the measure $G(t, dw)$ by $-z$ in the distribution sense (see e.g. [24, p. 55]).

Because of the assumptions on φ and ψ , the measures $\Phi(dw)$ and $\Psi(dy)$ are signed measures with finite total variation. We can therefore apply (4.3) to deduce

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(z-y) \Phi(dw) \Psi(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}f(\xi) \mathcal{F}(\Phi)(\xi) \overline{\mathcal{F}(\Psi)(\xi)} d\xi.$$

Using the identities $\mathcal{F}(\Phi)(\xi) = \mathcal{F}(\tau_{-z}\varphi(\cdot)\tau_{-z}G(t, \cdot))(\xi) = e^{-i\xi \cdot z} \mathcal{F}(\varphi G(t, \cdot))(\xi)$, we obtain (4.8). \square

4.2. Qualitative moment estimates

We introduce a set of assumptions that ensure the existence and uniqueness of a random field solution to (1.3).

(he)

- (i) The functions b and σ are Lipschitz continuous;
- (ii) W is a spatially homogeneous noise as described in Section 4.1. Its covariance and spectral measures (Λ and μ , respectively) satisfy **(h0)** and **(h1)**.
- (iii) The initial values u_0, v_0 are such that the function $(t, x) \mapsto I_0(t, x)$ defined in (2.5) is continuous and

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |I_0(t, x)| < \infty. \quad (4.9)$$

Theorem 4.3. *Assume that **(he)** is satisfied. Then there exists a random field solution $(u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ to (1.3), and for any $p \in [1, \infty)$,*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p < \infty. \quad (4.10)$$

This solution is unique in the class of jointly measurable, adapted processes u satisfying (4.10) with $p = 2$.

In the case $u_0 = v_0 = 0$, this follows from [4, Theorem 13] applied to the wave operator. For non-null initial conditions, this follows from [6, Theorem 4.3].

In the sequel, as in Section 3, we will suppose that $L(b)$ and $L(\sigma)$ are strictly positive.

Proposition 4.4. *In addition to **(he)**, we assume that the initial values u_0, v_0 , satisfy the following conditions:*

1. for $d = 2$, u_0 is a bounded function of class C^1 with bounded partial derivatives; v_0 is continuous and bounded;
2. for $d = 3$, u_0 is a bounded function of class C^2 with bounded second order partial derivatives; v_0 is continuous and bounded.

*We also suppose that the covariance measure Λ satisfies **(h1)**, and the Lipschitz constants $L(b), L(\sigma)$ are such that $L(b) \geq (2^{12} 3^2 C_\mu^2 L(\sigma)^4) \vee \frac{1}{4}$, where C_μ is given in (4.6). Then, for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}\right]$ we have*

$$\mathcal{N}_{2\sqrt{L(b)}, p}(u) \leq C \left[\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right], \quad (4.11)$$

where C is a universal constant and

$$\mathcal{T}_0 = \begin{cases} \|u_0\|_\infty + \frac{1}{\sqrt{L(b)}} (\|\nabla u_0\|_\infty + \|v_0\|_\infty), & \text{if } d = 2, \\ \|u_0\|_\infty + \|\Delta u_0\|_\infty + \frac{1}{\sqrt{L(b)}} \|v_0\|_\infty, & \text{if } d = 3. \end{cases} \quad (4.12)$$

As a consequence, we deduce that for $t \in [0, T]$ and $p \in \left[2, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}\right]$,

$$\sup_{x \in \mathbb{R}^d} E(|u(t, x)|^p) \leq C^p e^{2pt\sqrt{L(b)}} \left[\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right]^p. \quad (4.13)$$

Proof. We will consider the contributions to $\mathcal{N}_{\alpha, p}$ of each of the terms $I_i(t, x)$ in (2.5).

Estimates of $\mathcal{N}_{\alpha, p}(I_0)$. Consider first the case $d = 2$. Using (1.11) and (1.12) from [19], we have for $t > 0$ and $x \in \mathbb{R}^2$

$$|[G(t) * v_0](x)| \leq t \|v_0\|_\infty, \quad \left| \frac{\partial}{\partial t} [G(t) * u_0](x) \right| \leq C \{ \|u_0\|_\infty + t \|\nabla u_0\|_\infty \}.$$

Since $\sup_{t \geq 0} (te^{-\alpha t}) = (e\alpha)^{-1}$, we deduce that for any $\alpha > 0$ and $p \in [2, \infty)$,

$$\mathcal{N}_{\alpha, p}(I_0) \leq C \left[\|u_0\|_\infty + \frac{e^{-1}}{\alpha} (\|v_0\|_\infty + \|\nabla u_0\|_\infty) \right]. \quad (4.14)$$

Let $d = 3$. Using (2.2) and $\int_{\mathbb{R}^3} G(t, dx) = t$, we obtain, for $t > 0$ and $x \in \mathbb{R}^3$,

$$|[G(t) * v_0](x)| = \left| \int_{|y|=t} v_0(x-y) G(t, dy) \right| \leq \|v_0\|_\infty \int_{|y|=t} G(t, dy) = t \|v_0\|_\infty.$$

By applying the formula $\frac{d}{dt} (G(t) * u_0) = \frac{1}{t} (G(t) * u_0) + \frac{1}{4\pi} \int_{\{|y| \leq 1\}} (\Delta u_0)(\cdot + ty) dy$ (see [25]), we have $\left| \frac{d}{dt} (G(t) * u_0)(x) \right| \leq \|u_0\|_\infty + \frac{1}{3} \|\Delta u_0\|_\infty$. Therefore,

$$\mathcal{N}_{\alpha, p}(I_0) \leq \frac{e^{-1}}{\alpha} \|v_0\|_\infty + \|u_0\|_\infty + \frac{1}{3} \|\Delta u_0\|_\infty. \quad (4.15)$$

Estimates of $\mathcal{N}_{\alpha, p}(I_1)$. Use the expression of $I_1(t, x)$ given in (2.5) and then Minkovski's inequality along with (2.6) to obtain

$$\begin{aligned} \|I_1(t, x)\|_p &\leq \int_0^t ds \int_{\mathbb{R}^d} G(t-s, dy) \left[c(b) + L(b) \|u(s, x-y)\|_p \right] \\ &= \frac{t^2}{2} c(b) + L(b) \int_0^t ds (t-s) \left(\sup_{x \in \mathbb{R}^d} \|u(s, x)\|_p \right). \end{aligned}$$

From the above estimates, an argument similar to that used to prove (3.8) implies

$$\begin{aligned} \mathcal{N}_{\alpha, p}(I_1) &\leq c(b) \sup_{t \geq 0} \left(\frac{t^2}{2} e^{-\alpha t} \right) \\ &\quad + L(b) \sup_{t \in [0, T]} \int_0^t ds (t-s) e^{-\alpha(t-s)} \left(\sup_{(s, x) \in [0, T] \times \mathbb{R}^d} e^{-\alpha s} \|u(s, x)\|_p \right) \\ &\leq \frac{2e^{-2}}{\alpha^2} c(b) + \frac{1}{\alpha^2} L(b) \mathcal{N}_{\alpha, p}(u). \end{aligned} \quad (4.16)$$

Estimates of $\mathcal{N}_{\alpha,p}(I_2)$ Applying (4.7) with $Z(s, y) := \sigma(u(s, y))$ and then Minkowski's inequality, we obtain

$$\begin{aligned} \|I_2(t, x)\|_p^2 &\leq 4p \left\{ E \left[\int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-dy) G(t-s, x-dz) f(y-z) \right. \right. \\ &\quad \left. \left. \times \sigma(u(s, y)) \sigma(u(s, z)) \right] \right\}^{\frac{2}{p}} \\ &\leq 4p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-dy) G(t-s, x-dz) f(y-z) \|\sigma(u(s, y)) \sigma(u(s, z))\|_{\frac{p}{2}} \\ &\leq 4p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-dy) G(t-s, x-dz) f(y-z) \|\sigma(u(s, y))\|_p \|\sigma(u(s, z))\|_p. \end{aligned}$$

Then, from (2.6) and the inequality $2ab \leq a^2 + b^2$ (valid for $a, b \in \mathbb{R}$), we deduce

$$\begin{aligned} \|I_2(t, x)\|_p^2 &\leq 4p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-dy) G(t-s, x-dz) f(y-z) \\ &\quad \times [c(\sigma) + L(\sigma) \|u(s, y)\|_p]^2 \\ &\leq 8p \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, x-dy) G(t-s, x-dz) f(y-z) \\ &\quad \times [c(\sigma)^2 + L(\sigma)^2 \|u(s, y)\|_p^2]. \end{aligned} \quad (4.17)$$

Using the notation introduced in (4.5), we can rewrite (4.17) as follows

$$\|I_2(t, x)\|_p^2 \leq 8p \left[c(\sigma)^2 \int_0^t ds J(t-s) + L(\sigma)^2 \int_0^t ds J(t-s) \sup_{y \in \mathbb{R}^d} \|u(s, y)\|_p^2 \right]. \quad (4.18)$$

From here, using the change of variables $s \mapsto t-s$, we have

$$\mathcal{N}_{\alpha,p}(I_2) \leq \sqrt{8p} \nu_1(\alpha) c(\sigma) + \sqrt{8p} \nu_2(\alpha) L(\sigma) \mathcal{N}_{\alpha,p}(u), \quad (4.19)$$

where the finite constants $\nu_1(\alpha)$ and $\nu_2(\alpha)$ are defined by

$$\nu_1(\alpha) := \sup_{t \in [0, T]} \left(e^{-2\alpha t} \int_0^t ds J(s) \right)^{\frac{1}{2}}, \quad \nu_2(\alpha) := \sup_{t \in [0, T]} \left(\int_0^t ds e^{-2\alpha s} J(s) \right)^{\frac{1}{2}}. \quad (4.20)$$

Thus, owing to (4.16), (4.19), we deduce

$$\begin{aligned} \mathcal{N}_{\alpha,p}(u) &\leq \mathcal{N}_{\alpha,p}(I_0) + \frac{2e^{-2}}{\alpha^2} c(b) + \sqrt{8p} c(\sigma) \nu_1(\alpha) \\ &\quad + 2 \max \left[\frac{L(b)}{\alpha^2}, \sqrt{8p} L(\sigma) \nu_2(\alpha) \right] \mathcal{N}_{\alpha,p}(u). \end{aligned} \quad (4.21)$$

Using (4.6) and the value of $\sup_{t \geq 0} (t^k e^{-\alpha t})$ for $k = 1, 3$, shown in (3.3), we see that

$$\nu_1(\alpha) \leq C_\mu^{\frac{1}{2}} \sup_{t \in [0, T]} \left(e^{-2\alpha t} \int_0^t 2(1+s^2) ds \right)^{\frac{1}{2}} \leq C_\mu^{\frac{1}{2}} \left(\frac{e^{-1}}{\alpha} + \frac{9}{4} \frac{e^{-3}}{\alpha^3} \right)^{\frac{1}{2}}. \quad (4.22)$$

Furthermore, using the inequality (4.6) and computing $\int_0^t s^2 e^{-\alpha s} ds$, we obtain

$$\nu_2(\alpha) \leq C_\mu^{\frac{1}{2}} \sup_{t \in [0, T]} \left(\int_0^t 2(1+s^2) e^{-2\alpha s} ds \right)^{\frac{1}{2}} \leq C_\mu^{\frac{1}{2}} \left(\frac{1}{\alpha} + \frac{1}{2\alpha^3} \right)^{\frac{1}{2}}. \quad (4.23)$$

Thus, (4.21)-(4.23) yield

$$\begin{aligned} \mathcal{N}_{\alpha, p}(u) &\leq \mathcal{N}_{\alpha, p}(I_0) + \frac{2e^{-2}}{\alpha^2} c(b) + \sqrt{8p} c(\sigma) C_\mu^{\frac{1}{2}} \left(\frac{e^{-1}}{\alpha} + \frac{9}{4} \frac{e^{-3}}{\alpha^3} \right)^{\frac{1}{2}} \\ &\quad + 2 \max \left[\frac{L(b)}{\alpha^2}, \sqrt{8p} L(\sigma) C_\mu^{\frac{1}{2}} \left(\frac{1}{\alpha} + \frac{1}{2\alpha^3} \right)^{\frac{1}{2}} \right] \mathcal{N}_{\alpha, p}(u). \end{aligned} \quad (4.24)$$

Choose $\alpha^2 = 4L(b)$. Since by assumption $L(b) \geq \frac{1}{4}$, we have $\alpha \geq 1$, which yields

$$\frac{e^{-1}}{\alpha} + \frac{9}{4} \frac{e^{-3}}{\alpha^3} \leq \frac{13}{8e} L(b)^{-\frac{1}{2}} \quad \text{and} \quad \frac{1}{\alpha} + \frac{1}{2\alpha^3} \leq \frac{3}{4} L(b)^{-\frac{1}{2}}.$$

Moreover, using once more the assumption $L(b) \geq [2^{12} 3^2 C_\mu^2 L(\sigma)^4] \vee \frac{1}{4}$, we see that for $\alpha^2 = 4L(b)$ and for any $p \in \left[2, \sqrt{L(b)} / (2^5 3 C_\mu L(\sigma)^2) \right]$,

$$\max \left[\frac{L(b)}{\alpha^2}, \sqrt{8p} L(\sigma) C_\mu^{\frac{1}{2}} \left(\frac{1}{\alpha} + \frac{1}{2\alpha^3} \right)^{\frac{1}{2}} \right] = \frac{1}{4}.$$

Hence, from (4.24), using the upper bound $p \leq \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2}$, (4.14) and (4.15), we deduce

$$\begin{aligned} \mathcal{N}_{2\sqrt{L(b)}, p}(u) &\leq 2\mathcal{N}_{2\sqrt{L(b)}, p}(I_0) + e^{-2} \frac{c(b)}{L(b)} + \left(\frac{13}{3e2^3} \right)^{\frac{1}{2}} \frac{c(\sigma)}{L(\sigma)} \\ &\leq C_1 \mathcal{T}_0 + C_2 \left[\frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right], \end{aligned}$$

with \mathcal{T}_0 defined in (4.12). This completes the proof of (4.11).

The inequality (4.13) follows from (4.11) using the definition of $\mathcal{N}_{\alpha, p}(u)$. \square

4.3. Uniform bounds on moments

In this section, we address the problems of Section 3.2 when $d = 2, 3$, and W is a noise white in time and coloured in space. The main task is to prove a result similar to Proposition 3.3 on moment estimates of increments in time and in space for the solution to equation (1.3) with globally Lipschitz coefficients.

Increments of $I_0(t, x)$ in time and space

Proposition 4.5. *Let $I_0(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$ be as in (2.5) and $R \geq 0$ be fixed.*

1. *Let $d = 2$. Assume that u_0 is \mathcal{C}^1 , ∇u_0 is Hölder continuous with exponent $\gamma_1 \in (0, 1]$, and v_0 is Hölder continuous with exponent $\gamma_2 \in (0, 1]$. Then, there exists a positive constant $C(T, R)$ such that, for any $t, \bar{t} \in [0, T]$, and any $x, \bar{x} \in B(0; R)$,*

$$\begin{aligned} |I_0(t, x) - I_0(\bar{t}, \bar{x})| &\leq C(T, R) (\|v_0\|_{\infty, R+T} + \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\nabla u_0\|_{\gamma_1}) \\ &\quad \times (|t - \bar{t}|^{\gamma_1 \wedge \gamma_2} + |x - \bar{x}|^{\gamma_1 \wedge \gamma_2}). \end{aligned} \quad (4.25)$$

2. Let $d = 3$. Assume that u_0 is C^2 , Δu_0 is Hölder continuous with exponent $\gamma_1 \in (0, 1]$, and v_0 is Hölder continuous with exponent $\gamma_2 \in (0, 1]$. Then, there exists a positive constant $C(T, R)$ such that, for any $t, \bar{t} \in [0, T]$, and any $x, \bar{x} \in B(0; R)$,

$$|I_0(t, x) - I_0(\bar{t}, \bar{x})| \leq C(T, R) [\|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1}] \times (|t - \bar{t}|^{\gamma_1 \wedge \gamma_2} + |x - \bar{x}|^{\gamma_1 \wedge \gamma_2}). \quad (4.26)$$

Proof. (1). Let $0 \leq t \leq \bar{t} \leq T$ and $x \in B(0; R)$ be fixed. The scaling property $G(t, dx) = tG(1, dx)$ for $t > 0$ implies

$$\begin{aligned} |[G(t) - G(\bar{t})] * v_0(x)| &= \left| \int_{\mathbb{R}^2} G(t, dy) \left[v_0(x - y) - \frac{\bar{t}}{t} v_0\left(x - \frac{\bar{t}}{t} y\right) \right] \right| \\ &\leq \frac{\bar{t}}{t} \int_{\mathbb{R}^2} G(t, dy) \left| v_0(x - y) - v_0\left(x - \frac{\bar{t}}{t} y\right) \right| + \left| 1 - \frac{\bar{t}}{t} \right| \int_{\mathbb{R}^2} G(t, dy) |v_0(x - y)| \\ &\leq T \|v_0\|_{\gamma_2} |t - \bar{t}|^{\gamma_2} + \|v_0\|_{\infty, R+T} |t - \bar{t}|. \end{aligned}$$

Consequently, $\sup_{|x| \leq R} |[G(t) - G(\bar{t})] * v_0(x)| \leq C(T) (\|v_0\|_{\infty, R+T} + \|v_0\|_{\gamma_2}) |t - \bar{t}|^{\gamma_2}$.

According to the computations in [19, p. 812-813], we have

$$\sup_{|x| \leq R} \left| \frac{\partial}{\partial t} [G(t) * u_0(x) - G(\bar{t}) * u_0(x)] \right| \leq C(\|\nabla u_0\|_{\infty, R+T} |t - \bar{t}| + \|\nabla u_0\|_{\gamma_1}) |t - \bar{t}|^{\gamma_1}.$$

Thus,

$$\sup_{|x| \leq R} |I_0(t, x) - I_0(\bar{t}, x)| \leq C(T) (\|v_0\|_{\infty, R+T} + \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\nabla u_0\|_{\gamma_1}) |t - \bar{t}|^{\gamma_1 \wedge \gamma_2}. \quad (4.27)$$

Let now $0 \leq t \leq T$ and $x, \bar{x} \in B(0, R)$ be fixed; then

$$\begin{aligned} |(G(t * v_0)(x) - (G(t * v_0)(\bar{x}))| &\leq \int_{\mathbb{R}^2} G(t, y) |v_0(x - y) - v_0(\bar{x} - y)| dy \\ &\leq \|v_0\|_{\gamma_2} |x - \bar{x}|^{\gamma_2} \left(\int_{\mathbb{R}^2} G(t, y) dy \right) \leq T \|v_0\|_{\gamma_2} |x - \bar{x}|^{\gamma_2}. \end{aligned} \quad (4.28)$$

According to the computations in [19, p. 815-816], we have

$$\left| \frac{\partial}{\partial t} [G(t) * u_0(x) - G(t) * u_0(\bar{x})] \right| \leq C(\|\nabla u_0\|_{\infty, T+R} |x - \bar{x}| + \|\nabla u_0\|_{\gamma_1} |x - \bar{x}|^{\gamma_1})$$

for $t \in [0, T]$. Therefore,

$$\sup_{0 \leq t \leq T} |I_0(t, x) - I_0(t, \bar{x})| \leq C(T, R) (\|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, T+R} + \|\nabla u_0\|_{\gamma_1}) |x - \bar{x}|^{\gamma_1 \wedge \gamma_2}. \quad (4.29)$$

From the estimates (4.27)–(4.29), we deduce (4.25).

(2). Fix $0 \leq t \leq \bar{t} \leq T$ and $x \in B(0, R)$. According to [7, Lemma 4.9, p. 43], we have

$$\begin{aligned} \sup_{|x| \leq R} \left\| \frac{\partial}{\partial t} (G(\cdot) * u_0)(x) \right\|_{\gamma_1} &\leq C(\|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1}), \\ \sup_{|x| \leq R} \|(G(\cdot) * v_0)(x)\|_{\gamma_2} &\leq C\|v_0\|_{\gamma_2}, \end{aligned}$$

where $C > 0$ is a universal constant. Consequently,

$$\sup_{|x| \leq R} |I_0(t, x) - I_0(\bar{t}, \bar{x})| \leq C(T, R) (\|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1} + \|v_0\|_{\gamma_2}) |t - \bar{t}|^{\gamma_1 \wedge \gamma_2}. \quad (4.30)$$

Fix $0 \leq t \leq T$ and $x, \bar{x} \in B(0, R)$. Using the arguments in [13, p. 362] (see also [7, Chapter 4]), and the validity of the computations in (4.28) in dimension 3, we deduce

$$\sup_{0 \leq t \leq T} \left| \frac{\partial}{\partial t} [G(t) * u_0(x) - G(t) * u_0(\bar{x})] \right| \leq C (\|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1}) |x - \bar{x}|^{\gamma_1}.$$

Hence,

$$\sup_{0 \leq t \leq T} |I_0(t, x) - I_0(t, \bar{x})| \leq C(T, R) (\|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1} + \|v_0\|_{\gamma_2}) |x - \bar{x}|^{\gamma_1 \wedge \gamma_2}. \quad (4.31)$$

The proof of (4.26) is a consequence of (4.30) and (4.31). \square

Remark 4.6. *In comparison with the assumptions (1) and (2) of Proposition 4.4, in Proposition 4.5 we restrict the space variable to a bounded set and therefore, the boundedness hypotheses are satisfied.*

Increments of $I_1(t, x)$ in time and space

Proposition 4.7. *Let $I_1(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$ be as in (2.5).*

1. *Assume that the hypotheses (he) are satisfied. Then there exists a positive constant $C(T)$ depending on T such that for any $(t, x), (\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^d$ and for any $p \in [2, \infty)$,*

$$\begin{aligned} \|I_1(t, x) - I_1(\bar{t}, \bar{x})\|_p &\leq C(T) \left\{ |t - \bar{t}| \left[c(b) + L(b) \sup_{(t, x) \times \mathbb{R}^d} \|u(t, x)\|_p \right] \right. \\ &\quad \left. + L(b) \int_0^t ds \left(\sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p + \sup_{|z_1 - z_2| \leq |t - \bar{t}|} \|u(s, z_1) - u(s, z_2)\|_p \right) \right\}. \end{aligned} \quad (4.32)$$

2. *Assume the hypotheses of Proposition 4.4. Then there exists a positive constant $C(T)$ depending on T such that for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^{5/3} C_\mu L(\sigma)^2}\right]$ and any $(t, x), (\bar{t}, \bar{x}) \in [0, T] \times \mathbb{R}^d$,*

$$\begin{aligned} \|I_1(t, x) - I_1(\bar{t}, \bar{x})\|_p &\leq C(T) \left\{ |t - \bar{t}| \left[c(b) + L(b) e^{2T\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)}, p}(u) \right] \right. \\ &\quad \left. + L(b) \int_0^t ds \left(\sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p + \sup_{|z_1 - z_2| \leq |t - \bar{t}|} \|u(s, z_1) - u(s, z_2)\|_p \right) \right\}, \end{aligned} \quad (4.33)$$

with $\mathcal{N}_{2\sqrt{L(b)}, p}(u)$ satisfying (4.11).

Proof. 1. Fix $t \in [0, T]$ and $x, \bar{x} \in \mathbb{R}^d$. The Minkowski inequality, the Lipschitz continuity of b and the property $\int_{\mathbb{R}^d} G(t, dx) = t$ yield

$$\begin{aligned} \|I_1(t, x) - I_1(t, \bar{x})\|_p &\leq L(b) \int_0^t ds (t - s) \left[\sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p \right] \\ &\leq L(b) T \int_0^t ds \sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p. \end{aligned} \quad (4.34)$$

By the triangle inequality $\|I_1(\bar{t}, x) - I_1(t, x)\|_p \leq T_1(p; t, \bar{t}, x) + T_2(p; t, \bar{t}, x)$, for any $0 \leq t \leq \bar{t} \leq T$, where

$$\begin{aligned} T_1(p; t, \bar{t}, x) &= \left\| \int_0^t ds \int_{\mathbb{R}^d} [G(\bar{t} - s, dy) - G(t - s, dy)] b(u(s, x - y)) \right\|_p, \\ T_2(p; t, \bar{t}, x) &= \left\| \int_t^{\bar{t}} ds \int_{\mathbb{R}^d} G(\bar{t} - s, dy) b(u(s, x - y)) \right\|_p. \end{aligned}$$

By the scaling property of the fundamental solution $G(t)$, $T_1(p; t, \bar{t}, x)$ equals

$$\left\| \int_0^t ds \int_{\mathbb{R}^d} G(1, dz) [(t - s)b(u(s, x - (t - s)z)) - (\bar{t} - s)b(u(s, x - (\bar{t} - s)z))] \right\|_p.$$

Apply Minkowski's inequality and use the Lipschitz property of b and (2.6); this yields

$$\begin{aligned} T_1(p; t, \bar{t}, x) &\leq |t - \bar{t}| \int_0^t ds \int_{\mathbb{R}^d} G(1, dz) [c(b) + L(b)\|u(s, x - (\bar{t} - s)z)\|_p] \\ &\quad + L(b) \int_0^t ds \int_{\mathbb{R}^d} G(1, dz) (t - s) \|u(s, x - (t - s)z) - u(s, x - (\bar{t} - s)z)\|_p. \end{aligned} \quad (4.35)$$

Since the support of $G(1, dz)$ is included in the closed ball $\overline{B(0; 1)}$, we have

$$\begin{aligned} &\int_0^t ds (t - s) \int_{\mathbb{R}^d} G(1, dz) \|u(s, x - (t - s)z) - u(s, x - (\bar{t} - s)z)\|_p \\ &\leq T \int_0^t ds \int_{\mathbb{R}^d} G(1, dz) \sup_{|z_1 - z_2| \leq |t - \bar{t}|} \|u(s, z_1) - u(s, z_2)\|_p \\ &= T \int_0^t ds \sup_{|z_1 - z_2| \leq |t - \bar{t}|} \|u(s, z_1) - u(s, z_2)\|_p. \end{aligned}$$

The first term on the right-hand side of (4.35) is bounded from above by

$$|t - \bar{t}| \left\{ Tc(b) + L(b) \int_0^t ds \sup_{x \in \mathbb{R}^d} \|u(s, x)\|_p \right\}. \quad (4.36)$$

Thus,

$$\begin{aligned} T_1(p; t, \bar{t}, x) &\leq T \left(L(b) \int_0^t \sup_{|z_1 - z_2| \leq |t - \bar{t}|} \|u(s, z_1) - u(s, z_2)\|_p ds \right. \\ &\quad \left. + |t - \bar{t}| \left\{ c(b) + L(b) \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p \right\} \right). \end{aligned} \quad (4.37)$$

With similar arguments, we deduce the following upper bounds for $T_2(p; t, \bar{t}, x)$:

$$\begin{aligned} T_2(p; t, \bar{t}, x) &\leq c(b) \int_t^{\bar{t}} ds \int_{\mathbb{R}^d} G(\bar{t} - s, dy) + L(b) \int_t^{\bar{t}} ds (\bar{t} - s) \sup_{x \in \mathbb{R}^d} \|u(s, x)\|_p \\ &\leq c(b) \frac{(\bar{t} - t)^2}{2} + L(b) \frac{(\bar{t} - t)^2}{2} \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p. \end{aligned} \quad (4.38)$$

From (4.37) and (4.38), we obtain (4.32) for $x = \bar{x}$. Along with (4.34), we obtain (4.32).

2. This claim follows from the definition of (2.7) and Proposition 4.4. \square

Space increments of $I_2(t, x)$

While keeping assumption **(h1)**, we consider a strengthening of **(h0)**, denoted by **(h2)**. This is condition (c') in [13, p. 367] on the spectral measure μ .

(h2) There exists $\gamma \in (0, 1)$ such that the Fourier transform of the tempered measure $|\zeta|^{2\gamma}\mu(d\zeta)$ is a non-negative locally integrable function g_γ , and moreover,

$$\int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1 + |\zeta|^{2-2\gamma}} < \infty.$$

Set

$$C_\mu^{(\gamma)} := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\mu(d\zeta)}{1 + |\zeta|^{2-2\gamma}}. \quad (4.39)$$

Proposition 4.8. *Let $I_2(t, x)$, $(t, x) \in [0, T] \times \mathbb{R}^d$ be as in (2.5).*

1. *Assume that the hypotheses **(he)**, **(h1)** and **(h2)** are satisfied. Then, for any $p \in [2, \infty)$ and $t \in [0, T]$, there exists a positive constant C such that, for every $x, \bar{x} \in \mathbb{R}^d$, $t \in [0, T]$,*

$$\begin{aligned} \|I_2(t, x) - I_2(t, \bar{x})\|_p^2 &\leq Cp(1 + T^2)C_\mu L(\sigma)^2 \left(\int_0^t ds \sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2 \right) \\ &\quad + Cp(T + T^3)C_\mu^{(\gamma)} |x - \bar{x}|^{2\gamma} \left[c(\sigma) + L(\sigma) \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} \|u(s, y)\|_p \right]^2, \end{aligned} \quad (4.40)$$

where $C_\mu, C_\mu^{(\gamma)}$ are defined in (4.6), (4.39), respectively.

2. *Assume that the hypotheses of Proposition 4.4 hold. Then, for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^{5/3}C_\mu L(\sigma)^2}\right]$, there exists a positive constant C such that, for every $x, \bar{x} \in \mathbb{R}^d$, $t \in [0, T]$,*

$$\begin{aligned} \|I_2(t, x) - I_2(t, \bar{x})\|_p^2 &\leq Cp(1 + T^2)C_\mu L(\sigma)^2 \left(\int_0^t ds \sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2 \right) \\ &\quad + Cp(T + T^3)C_\mu^{(\gamma)} |x - \bar{x}|^{2\gamma} \left[c(\sigma) + L(\sigma)e^{2T\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)}, p}(u) \right]^2, \end{aligned} \quad (4.41)$$

where $\mathcal{N}_{2\sqrt{L(b)}, p}(u)$ satisfies (4.11).

Proof. To simplify the presentation, we will use the notation of [13, Theorem 3.1] that we recall below. For $s \in [0, T]$ and $x, \bar{x}, y, z \in \mathbb{R}^d$, set $\xi = x - \bar{x}$ and

$$\begin{aligned} \Sigma_x(s, y) &= \sigma(u(s, x - y)), \quad \Sigma_{x, \bar{x}}(s, y) = \sigma(u(s, x - y)) - \sigma(u(s, \bar{x} - y)), \\ h_1(s, y, z) &= f(y - z)\Sigma_{x, \bar{x}}(s, y)\Sigma_{x, \bar{x}}(s, z), \\ h_2(s, y, z) &= [f(y - z + \xi) - f(y - z)]\Sigma_x(s, z)\Sigma_{x, \bar{x}}(s, y), \quad h_3(s, y, z) = h_2(s, z, y), \\ h_4(s, y, z) &= [2f(y - z) - f(y - z + \xi) - f(y - z - \xi)]\Sigma_x(s, y)\Sigma_x(s, z). \end{aligned}$$

Fix $p \in [2, \infty)$ and apply the Burkholder-Davies-Gundy inequality to obtain

$$\|I_2(t, x) - I_2(t, \bar{x})\|_p^2 \leq 4p \sum_{i=1}^4 \|Q_i(t; x, \bar{x})\|_{\frac{p}{2}}, \quad (4.42)$$

where, using the transfer of increments strategy introduced in [7, p. 19] (see also in [13, p. 374]), we set

$$Q_i(t; x, \bar{x}) = \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, dy) G(t-s, dz) h_i(s, y, z), \quad i = 1, \dots, 4.$$

In [13, Theorem 3.2], upper bounds of the terms $\|Q_i(t; x, \bar{x})\|_{\frac{p}{2}}$ are established. We sketch here their proofs, paying attention to the value of the relevant constants and checking that the arguments also hold for $d = 2$.

Upper bound of $\|Q_1(t; x, \bar{x})\|_{\frac{p}{2}}$. Using Minkowski's inequality, then the Cauchy-Schwarz inequality, the Lipschitz property of σ and (4.6) we obtain

$$\begin{aligned} \|Q_1(t; x, \bar{x})\|_{\frac{p}{2}} &\leq L(\sigma)^2 \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, dy) G(t-s, dz) f(y-z) \\ &\quad \times \left[\sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2 \right] \\ &\leq 2L(\sigma)^2 (1 + T^2) C_\mu \int_0^t ds \sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2. \end{aligned} \quad (4.43)$$

For the study of the remaining terms $\|Q_i(t; x, \bar{x})\|_{\frac{p}{2}}$, $i = 2, 3, 4$, in order to be in the setting of Lemma 4.2, we use a truncation argument on the processes $\Sigma_x(s, y)$, $\Sigma_{x, \bar{x}}(s, y)$. For $k \geq 1$, set $\Sigma_x^k(s, y) = \Sigma_x(s, y) 1_{\{|\Sigma_x(s, y)| \leq k\}}$, $\Sigma_{x, \bar{x}}^k(s, y) = \Sigma_{x, \bar{x}}(s, y) 1_{\{|\Sigma_{x, \bar{x}}(s, y)| \leq k\}}$, and

$$Q_i^k(t; x, \bar{x}) = \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-s, dy) G(t-s, dz) h_i^k(s, y, z), \quad i = 2, 3, 4,$$

where each $h_i^k(s, y, z)$ is defined as $h_i(s, y, z)$ by replacing $\Sigma_x(s, y)$ and $\Sigma_{x, \bar{x}}(s, y)$ by $\Sigma_x^k(s, y)$ and $\Sigma_{x, \bar{x}}^k(s, y)$, respectively.

Upper bound for $\|Q_2^k(t; x, \bar{x})\|_{\frac{p}{2}}$. Apply Lemma 4.2 to the bounded functions $\varphi(z) = \Sigma_x^k(s, z)$ and $\psi(y) = \Sigma_{x, \bar{x}}^k(s, y)$. Then, up to the constant $(2\pi)^{-d}$, $Q_2^k(t; x, \bar{x})$ is equal to

$$\int_0^t ds \int_{\mathbb{R}^d} \overline{\mathcal{F}(\Sigma_x^k(s, \cdot) G(t-s, \cdot))}(\zeta) \mathcal{F}(\Sigma_{x, \bar{x}}^k(s, \cdot) G(t-s, \cdot))(\zeta) [e^{-i\xi \cdot \zeta} - 1] \mu(d\zeta),$$

where $\xi = x - \bar{x}$. Since for any $\gamma \in (0, 1]$, $|e^{-i\xi \cdot \zeta} - 1| \leq C|\xi|^\gamma |\zeta|^\gamma$, and $2\sqrt{ab} \leq (a+b)$ for $a, b \geq 0$, computations similar to those in [13, p. 368] imply

$$\|Q_2^k(t; x, \bar{x})\|_{\frac{p}{2}} \leq C \left(\|Q_2^{k,1}(t; x, \bar{x})\|_{\frac{p}{2}} + \|Q_2^{k,2}(t; x, \bar{x})\|_{\frac{p}{2}} \right), \quad (4.44)$$

where

$$\begin{aligned} Q_2^{k,1}(t; x, \bar{x}) &:= |\xi|^{2\gamma} \int_0^t ds \int_{\mathbb{R}^d} |\mathcal{F}(\Sigma_x^k(s, \cdot) G(t-s, \cdot))(\zeta)|^2 |\zeta|^{2\gamma} \mu(d\zeta), \\ Q_2^{k,2}(t; x, \bar{x}) &:= \int_0^t ds \int_{\mathbb{R}^d} |\mathcal{F}(\Sigma_{x, \bar{x}}^k(s, \cdot) G(t-s, \cdot))(\zeta)|^2 \mu(d\zeta). \end{aligned}$$

Set $J^{(\gamma)}(t) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mu(d\zeta) |\zeta|^{2\gamma} |\mathcal{F}G(t)(\zeta)|^2$. A minor change in the proof of (4.6) yields

$$J^{(\gamma)}(t) \leq 2(1+t^2)C_\mu^{(\gamma)} < \infty, \quad (4.45)$$

where $C_\mu^{(\gamma)}$ is defined in (4.39), and we have used the assumption **(h2)**.

Using the Plancherel identity, the Minkowski inequality with respect to the non-negative measure $[G(t-s, \cdot) * G(t-s, \cdot)](y) g_\gamma(y) dy ds$ and once more the Plancherel identity, since $\tilde{G}(s, \cdot)$ is symmetric, we deduce (as in [13, p. 369]),

$$\begin{aligned} \|Q_2^{k,1}(t; x, \bar{x})\|_{\frac{p}{2}} &\leq C|x - \bar{x}|^{2\gamma} \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|\Sigma_x^k(s, y)\|_p^2 \int_0^t ds \int_{\mathbb{R}^d} \mu(d\zeta) |\zeta|^{2\gamma} |\mathcal{F}G(t-s)(\zeta)|^2 \\ &\leq C|x - \bar{x}|^{2\gamma} T(1+T^2) C_\mu^{(\gamma)} \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|\Sigma_x^k(s, y)\|_p^2. \end{aligned}$$

From (2.6) we have

$$\sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|\Sigma_x(s, y)\|_p \leq c(\sigma) + L(\sigma) \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|u(s, y)\|_p.$$

Therefore,

$$\|Q_2^{k,1}(t; x, \bar{x})\|_{\frac{p}{2}} \leq C|x - \bar{x}|^{2\gamma} (T+T^3) C_\mu^{(\gamma)} \left[c(\sigma) + L(\sigma) \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|u(s, y)\|_p \right]^2. \quad (4.46)$$

With similar arguments, we obtain

$$\begin{aligned} \|Q_2^{k,2}(t; x, \bar{x})\|_{\frac{p}{2}} &\leq C \int_0^t ds \sup_{y \in \mathbb{R}^d} \|\Sigma_{x, \bar{x}}^k(s, y)\|_p^2 \int_{\mathbb{R}^d} |\mathcal{F}(G(t-s, \cdot))(\zeta)|^2 \mu(d\zeta) \\ &\leq C \int_0^t \sup_{y \in \mathbb{R}^d} \|\Sigma_{x, \bar{x}}^k(s, y)\|_p^2 J(t-s) ds \leq C(1+T^2) C_\mu \int_0^t \sup_{y \in \mathbb{R}^d} \|\Sigma_{x, \bar{x}}^k(s, y)\|_p^2 ds \\ &\leq C(1+T^2) C_\mu L(\sigma)^2 \int_0^t ds \sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2, \end{aligned} \quad (4.47)$$

where in the last inequality, we have used (4.6), the Lipschitz property of σ and the upper estimate $\|\Sigma_{x, \bar{x}}^k(s, y)\|_p \leq \|\Sigma_{x, \bar{x}}(s, y)\|_p$.

Summarising, (4.44), along with (4.46) and (4.47) imply

$$\begin{aligned} \|Q_2^k(t; x, \bar{x})\|_{\frac{p}{2}} &\leq C(T+T^3) C_\mu^{(\gamma)} |x - \bar{x}|^{2\gamma} \left[c(\sigma) + L(\sigma) \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|u(s, y)\|_p \right]^2 \\ &\quad + C(1+T^2) C_\mu L(\sigma)^2 \int_0^t ds \sup_{|z_1 - z_2| = |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2, \end{aligned} \quad (4.48)$$

for some universal positive constant C .

Notice that, since $|e^{-i\xi \cdot \zeta} - 1| = |e^{i\xi \cdot \zeta} - 1|$, swapping y and z we deduce that (4.48) also holds for $\|Q_3^k(t; x, \bar{x})\|_{\frac{p}{2}}$.

Upper bound of $\|Q_4^k(t; x, \bar{x})\|_{\frac{p}{2}}$. Applying Lemma 4.2 with $\varphi = \psi = \Sigma_x^k(s, \cdot)$ and then Plancherel's identity, we obtain

$$\begin{aligned} |Q_4^k(t; x, \bar{x})| &\leq \frac{1}{(2\pi)^d} \int_0^t ds \int_{\mathbb{R}^d} dy |2 - e^{-i\xi \cdot \zeta} - e^{i\xi \cdot \zeta}| |\mathcal{F}(\varphi G(t-s, \cdot))|^2 \mu(d\zeta) \\ &\leq C|\xi|^{2\gamma} \int_0^t ds \int_{\mathbb{R}^d} dy g_\gamma(y) [(\Sigma_x^k(s, \cdot)G(t-s, \cdot)) * (\widetilde{\Sigma_x^k(s, \cdot)G(t-s, \cdot)})](y), \end{aligned}$$

where in the last inequality, we have used that for $\gamma \in (0, 1]$, $|1 - \cos(\xi \cdot \zeta)| \leq C(|\xi||\zeta|)^{2\gamma}$.

Consider the non-negative measure $g_\gamma(y)[G(t-s, \cdot) * G(t-s, \cdot)](y)ds dy$. The Minkowski inequality with respect to this measure, the Plancherel identity and (4.45) yield

$$\begin{aligned} \|Q_4^k(t; x, \bar{x})\|_{\frac{p}{2}} &\leq C|\xi|^{2\gamma} \int_0^t ds \int_{\mathbb{R}^d} dy g_\gamma(y) [G(t-s, \cdot) * G(t-s, \cdot)](y) \\ &\quad \times \sup_{y, z \in \mathbb{R}^d} \|\Sigma_x^k(s, y)\Sigma_x^k(s, y+z)\|_{\frac{p}{2}} \\ &\leq C|\xi|^{2\gamma} \int_0^t ds \sup_{y \in \mathbb{R}^d} \|\Sigma_x(s, y)\|_p^2 2(1+T^2)C_\mu^{(\gamma)}. \end{aligned}$$

Thus, an argument similar to that proving (4.46) implies

$$\|Q_4^k(t; x, \bar{x})\|_{\frac{p}{2}} \leq C(T+T^3)C_\mu^{(\gamma)} |x - \bar{x}|^{2\gamma} \left[c(\sigma) + L(\sigma) \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \|u(s, y)\|_p \right]^2. \quad (4.49)$$

The upper estimates (4.42), (4.43), (4.48) and (4.49) conclude the proof of (4.40).

The statement in part 2 is an immediate consequence of the definition of $\mathcal{N}_{2\sqrt{L(b)}, p}^{(\gamma)}(u)$ and Proposition 4.4. The proof of the proposition is complete. \square

From Propositions 4.4-4.8, we derive estimates on space increments of the random field (1.3) for $d = 2, 3$. Later on, they will be used to deduce estimates on time increments of $I_2(t, x)$. For its further use, set

$$K_0(u_0, v_0) = \begin{cases} \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\nabla u_0\|_{\gamma_1}, & d = 2, \\ \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1}, & d = 3. \end{cases} \quad (4.50)$$

Proposition 4.9. *We are assuming the following.*

1. The initial value functions u_0 and v_0 satisfy the conditions of Proposition 4.5 with some Hölder exponents $\gamma_1, \gamma_2 \in (0, 1]$.
2. The coefficients σ and b are globally Lipschitz continuous functions;
3. The covariance measure Λ of the noise W satisfies **(h1)**, and the corresponding spectral measure μ satisfies **(h2)**.

(i) Fix $T, R > 0$. Then, for any $p \in [2, \infty)$ and $\alpha > 0$, there exist positive constants $c_1(T, R)$, $c_2(T)$ and $c_3(T)$ such that if

$$\begin{aligned} C_1 &:= c_1(T, R) K_0(u_0, v_0), \\ C_2 &:= c_2(T) (pC_\mu^{(\gamma)})^{\frac{1}{2}} \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right], \\ C_3 &:= c_3(T) [L(b)^2 + pC_\mu L(\sigma)^2], \end{aligned} \quad (4.51)$$

with C_μ and $C_\mu^{(\gamma)}$ defined in (4.6) and (4.39), then for any $t \in [0, T]$, and $x, \bar{x} \in B(0; R)$,

$$\sup_{|z_1 - z_2| \leq |x - \bar{x}|} \|u(t, z_1) - u(t, z_2)\|_p^2 \leq \exp(TC_3) (C_1^2 |x - \bar{x}|^{2(\gamma_1 \wedge \gamma_2)} + C_2^2 |x - \bar{x}|^{2\gamma}). \quad (4.52)$$

Consequently,

$$\sup_{t \in [0, T]} \sup_{|z_1 - z_2| \leq |x - \bar{x}|} \|u(t, z_1) - u(t, z_2)\|_p \leq \tilde{C} |x - \bar{x}|^{\nu_1}, \quad (4.53)$$

with $\nu_1 = \min(\gamma, \gamma_1, \gamma_2)$, and $\tilde{C} = C(R) (C_1 + C_2) \exp(TC_3/2)$.

(ii) Suppose furthermore that the Lipschitz constants $L(b)$, $L(\sigma)$ and are such that $L(b) \geq (2^{12} 3^2 C_\mu^2 L(\sigma)^4) \vee \frac{1}{4}$. Then, for $p \in [2, \sqrt{L(b)} / (2^5 3 C_\mu L(\sigma)^2)]$ we have

$$C_2 \leq c_2(T) (pC_\mu^{(\gamma)})^{\frac{1}{2}} \left[c(\sigma) + L(\sigma) e^{2T\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)}, p}(u) \right],$$

with $\mathcal{N}_{2\sqrt{L(b)}, p}(u)$ satisfying (4.11).

Proof. (i). We first prove

$$\begin{aligned} \sup_{|z_1 - z_2| \leq |x - \bar{x}|} \|u(t, z_1) - u(t, z_2)\|_p^2 &\leq C_1^2 |x - \bar{x}|^{2(\gamma_1 \wedge \gamma_2)} + C_2^2 |x - \bar{x}|^{2\gamma} \\ &\quad + C_3 \int_0^t ds \sup_{|z_1 - z_2| \leq |x - \bar{x}|} \|u(s, z_1) - u(s, z_2)\|_p^2. \end{aligned} \quad (4.54)$$

Indeed, using (2.4) and (2.5), the first term on the right-hand side comes from (4.29) and (4.31). The second one comes from the last term on the right-hand side of (4.40). Finally, the very last term is obtained by the sum of the upper bound (4.34) and the first term on the right-hand side of (4.40).

Apply Gronwall's lemma to the function $t \mapsto \sup_{|z_1 - z_2| \leq |x - \bar{x}|} \|u(t, z_1) - u(t, z_2)\|_p^2$ to obtain (4.52), and then (4.53).

The claim (ii) follows from the definition of $\mathcal{N}_{2\sqrt{L(b)}, p}(u)$ and Proposition 4.4. \square

Time increments of $I_2(t, x)$

In order to deduce L^p -estimates of increments in time of the stochastic integral term $I_2(t, x)$, additional assumptions on the covariance of the noise are needed.

(h3) The spectral measure μ is such that there exists $\nu > 0$ and $C > 0$ for which

$$\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \mu(d\zeta) \leq Ct^\nu, \text{ for any } t \in [0, T]. \quad (4.55)$$

(h4) The covariance density function f satisfies the following conditions:

1. There exists $b > 0$ and $C > 0$ such that for any $h \in [0, T]$,

$$\int_0^T ds s \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) |f(s(y+z) + h(y+z)) - f(s(y+z) + hz)| \leq Ch^b. \quad (4.56)$$

2. There exists $\bar{b} > 0$ and $C > 0$ such that for any $h \in [0, T]$,

$$\begin{aligned} & \int_0^T ds s^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) \\ & \times |f(s(y+z) + h(y+z)) - f(s(y+z) + hy) - f(s(y+z) + hz) + f(s(y+z))| \\ & \leq Ch^{\bar{b}}. \end{aligned} \quad (4.57)$$

According to (4.5)–(4.6), the left-hand side of (4.55) is a function of t uniformly bounded over bounded intervals. Assumption **(h3)** provides a growth rate for this function.

Up to scalings, the assumption **(h4)** is on estimates of one and two-dimensional increments of the covariance density in a L^2 -type norm. We shall give in Section 5 examples where these conditions are satisfied.

Proposition 4.10. *Assume that the hypotheses (1)–(3) of Proposition 4.9 hold. Suppose also that the hypotheses **(h3)** and **(h4)** on the covariance of the noise are satisfied. Then there exists a constant $C(T, \nu)$ such that for any $p \in [2, \infty)$, $t, \bar{t} \in [0, T]$ and $x \in \mathbb{R}^d$,*

$$\begin{aligned} & \|I_2(t, x) - I_2(\bar{t}, x)\|_p^2 \leq C(T, \nu) p \left(C_\mu L(\sigma)^2 \tilde{C}^2 |t - \bar{t}|^{2\nu_1} \right. \\ & + \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right]^2 \{ |t - \bar{t}|^{1+\nu} + |t - \bar{t}|^{\min(b+1, \bar{b}, \tilde{\alpha})} \} \\ & + L(\sigma) \tilde{C} \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right] |t - \bar{t}|^{\nu_1 + \min(b, 1)} \Big), \end{aligned} \quad (4.58)$$

where $\nu_1 = \min(\gamma, \gamma_1, \gamma_2)$, \tilde{C} is defined in Proposition 4.9, $\tilde{\alpha} = (1 + \nu) \wedge 2$ if $\nu \neq 1$ and $\tilde{\alpha} < 2$ if $\nu = 1$.

If, as in Proposition 4.4, the Lipschitz constants $L(b)$, $L(\sigma)$ are such that $\frac{L(b)}{L(\sigma)} \geq (2^{12} 3^2 C_\mu^2 L(\sigma)^4) \vee \frac{1}{4}$, where C_μ is given in (4.6), then there exists a constant $\overline{C}(\nu, T)$ such that for $p \in \left[2, \frac{\sqrt{L(b)}}{2^5 3 C_\mu L(\sigma)^2} \right]$, $t, \bar{t} \in [0, T]$ and $x \in \mathbb{R}^d$,

$$\begin{aligned} & \|I_2(t, x) - I_2(\bar{t}, x)\|_p^2 \leq \overline{C}(\nu, T) p \left(C(T) C_\mu L(\sigma)^2 \tilde{C}^2 |t - \bar{t}|^{2\nu_1} \right. \\ & + \left[c(\sigma) + L(\sigma) e^{2T\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)}, p}(u) \right]^2 \{ |t - \bar{t}|^{1+\nu} + |t - \bar{t}|^{\min(b+1, \bar{b}, \tilde{\alpha})} \} \\ & + L(\sigma) \tilde{C} \left[c(\sigma) + L(\sigma) e^{2T\sqrt{L(b)}} \mathcal{N}_{2\sqrt{L(b)}, p}(u) \right] |t - \bar{t}|^{\nu_1 + \min(b, 1)} \Big), \end{aligned} \quad (4.59)$$

with $\mathcal{N}_{2\sqrt{L(b)}, p}(u)$ satisfying (4.11).

Proof. For $0 \leq t \leq \bar{t} \leq T$ and $x \in \mathbb{R}^d$, set

$$I_{2,1}(t, \bar{t}; x) = \int_t^{\bar{t}} \int_{\mathbb{R}^d} G(\bar{t} - s, x - dy) \sigma(u(s, y)) W(ds, dy).$$

By applying Burkholder-Davis-Gundy's inequality, and then Minkowski's inequality,

$$\begin{aligned} \|I_{2,1}(t, \bar{t}; x)\|_p^2 &\leq 4p \int_t^{\bar{t}} ds J(\bar{t} - s) \sup_{y \in \mathbb{R}^d} \|\sigma(u(s, y))\|_p^2 \\ &\leq p C |t - \bar{t}|^{1+\nu} \left[c(\sigma) + L(\sigma) \sup_{(s, y) \in [0, T] \times \mathbb{R}^d} \|u(s, y)\|_p \right]^2, \end{aligned}$$

where J is defined in (4.5), and the last upper estimate is deduced from **(h3)** (see (4.55)).

Let

$$I_{2,2}(t, \bar{t}; x) = \int_0^t \int_{\mathbb{R}^d} [G(\bar{t} - s, x - dy) - G(t - s, x - dy)] \sigma(u(s, y)) W(ds, dy). \quad (4.60)$$

We study the L_p -norm of this term following the proof of [13, Theorem 4.1]. This uses the transfer of increments trick introduced in [7, Section 3.2]. Applying the Burkholder-Davies-Gundy inequality, we obtain $\|I_{2,2}(t, \bar{t}; x)\|_p^2 \leq 4p \sum_{i=1}^4 \|R_i(t, \bar{t}; x)\|_{\frac{p}{2}}$, where, letting $h := \bar{t} - t$ and $\Theta_{t,x}(s, y) = \sigma(u(t - s, x - y))$, we set

$$\begin{aligned} R_1(t, \bar{t}; x) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) (s+h)^2 f((s+h)y - (s+h)z) \\ &\quad \times [\Theta_{t,x}(s, (s+h)y) - \Theta_{t,x}(s, sy)] [\Theta_{t,x}(s, (s+h)z) - \Theta_{t,x}(s, sz)], \\ R_2(t, \bar{t}; x) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) \\ &\quad \times [(s+h)^2 f((s+h)y - (s+h)z) - s(s+h) f(sy - (s+h)z)] \\ &\quad \times [\Theta_{t,x}(s, (s+h)z) - \Theta_{t,x}(s, sz)] \Theta_{t,x}(s, sy), \\ R_3(t, \bar{t}; x) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) \\ &\quad \times [(s+h)^2 f((s+h)y - (s+h)z) - s(s+h) f((s+h)y - sz)] \\ &\quad \times [\Theta_{t,x}(s, (s+h)y) - \Theta_{t,x}(s, sy)] \Theta_{t,x}(s, sz), \\ R_4(t, \bar{t}; x) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) \\ &\quad \times [(s+h)^2 f((s+h)y - (s+h)z) - s(s+h) f(sy - (s+h)z) \\ &\quad \quad - s(s+h) f((s+h)y - sz) + s^2 f(sy - sz)] \\ &\quad \times \Theta_{t,x}(s, sy) \Theta_{t,x}(s, sz). \end{aligned}$$

Notice that the linear growth and Lipschitz continuity assumptions on σ imply that for any $p \in [2, \infty)$, every $s, t \in [0, T]$ and $x, y, z \in \mathbb{R}^d$,

$$\sup_{0 \leq s \leq t \leq T; (x, y) \in \mathbb{R}^d} \|\Theta_{t,x}(s, y)\|_p \leq c(\sigma) + L(\sigma) \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p, \quad (4.61)$$

and

$$\begin{aligned} \sup_{0 \leq s \leq t \leq T; (x, y, z) \in \mathbb{R}^d, |y-z| \leq r} \|\Theta_{t,x}(s, y) - \Theta_{t,x}(s, z)\|_p &\leq L(\sigma) \sup_{t \in [0, T], |y-z| \leq r} \|u(t, y) - u(t, z)\|_p \\ &\leq L(\sigma) \tilde{C} r^{\nu_1}, \end{aligned} \quad (4.62)$$

where the last inequality follows from (4.53).

Upper bound of $\|R_1(t, \bar{t}; x)\|_{\frac{p}{2}}$. Apply the Minkowski and Cauchy-Schwarz inequalities. Then, using (4.62) we obtain

$$\begin{aligned} \|R_1(t, \bar{t}; x)\|_{\frac{p}{2}} &\leq L(\sigma)^2 \tilde{C}^2 |t - \bar{t}|^{2\nu_1} \\ &\quad \times \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) (s+h)^2 f((s+h)y - (s+h)z). \end{aligned} \quad (4.63)$$

Consider the change of variables $((s+h)y, (s+h)z) \mapsto (y, z)$; using the scaling property, (4.5) and (4.6), we deduce

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) (s+h)^2 f((s+h)y - (s+h)z) \\ = (2\pi)^{-d} \int_{\mathbb{R}^d} |\mathcal{F}G(s+h)(\zeta)|^2 \mu(d\zeta) \leq 2(1 + (2T)^2) C_\mu. \end{aligned} \quad (4.64)$$

Hence, (4.63) and (4.64) imply

$$\|R_1(t, \bar{t}; x)\|_{\frac{p}{2}} \leq 2 [1 + (2T)^2] C_\mu L(\sigma)^2 \tilde{C}^2 |t - \bar{t}|^{2\nu_1}, \quad (4.65)$$

where \tilde{C} is defined in Proposition 4.9.

Upper bound of $\|R_2(t, \bar{t}; x)\|_{\frac{p}{2}}$ and $\|R_3(t, \bar{t}; x)\|_{\frac{p}{2}}$. We will only consider $\|R_2(t, \bar{t}; x)\|_{\frac{p}{2}}$, since $\|R_3(t, \bar{t}; x)\|_{\frac{p}{2}}$ is similar. Set

$$\begin{aligned} R_{2,1}(t, \bar{t}; x) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) \\ &\quad \times s(s+h) [f((s+h)y - (s+h)z) - f(sy - (s+h)z)] \\ &\quad \times [\Theta_{t,x}(s, (s+h)z) - \Theta_{t,x}(s, sz)] \Theta_{t,x}(s, sy), \end{aligned}$$

Apply the change of variable $z \mapsto -z$ along with the Minkowski and Cauchy-Schwarz inequalities to obtain

$$\begin{aligned} \|R_{2,1}(t, \bar{t}; x)\|_{\frac{p}{2}} &\leq \sup_{0 \leq s \leq t \leq T; (x, z_1, z_2) \in \mathbb{R}^d, |z_1 - z_2| \leq h} (\|\Theta_{t,x}(s, z_1) - \Theta_{t,x}(s, z_2)\|_p) \\ &\quad \times \sup_{0 \leq s \leq t \leq T; (x, y) \in \mathbb{R}^d} (\|\Theta_{t,x}(s, y)\|_p) \\ &\quad \times \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) s(s+h) |f((s(y+z) + h(y+z)) - f(s(y+z) + hz)| \\ &\leq CTL(\sigma) \tilde{C} |t - \bar{t}|^{\nu_1 + b} \left[c(\sigma) + L(\sigma) \sup_{(t, x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p \right], \end{aligned} \quad (4.66)$$

where we have used (4.61), (4.62) and assumption **(h4)** (see (4.56)).

Define

$$R_{2,2}(t, \bar{t}; x) = \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) h(s+h) f((s+h)y - (s+h)z) \\ \times [\Theta_{t,x}(s, (s+h)z) - \Theta_{t,x}(s, sz)] \Theta_{t,x}(s, sy),$$

A computation similar to that used to upper estimate $\|R_{2,1}(t, \bar{t}; x)\|_p$ implies

$$\|R_{2,2}(t, \bar{t}; x)\|_{\frac{p}{2}} \leq CL(\sigma) \tilde{C} |t - \bar{t}|^{\nu_1} \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right] \\ \times \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) h(s+h) |f((s+h)y - (s+h)z)|.$$

Using the change of variables $((s+h)y, (s+h)z) \mapsto (y, z)$, the scaling property, (4.5) and **(h3)**, we obtain

$$\int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) h(s+h) [f((s+h)y - (s+h)z)] \\ = (2\pi)^{-d} h \int_0^t \frac{ds}{s+h} \int_{\mathbb{R}^d} \mu(d\zeta) |\mathcal{F}G(s+h)(\zeta)|^2 \leq C h \int_0^t ds (s+h)^{\nu-1} \leq CT^\nu h.$$

Thus,

$$\|R_{2,2}(t, \bar{t}; x)\|_{\frac{p}{2}} \leq CT^\nu L(\sigma) \tilde{C} |t - \bar{t}|^{\nu_1+1} \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right]. \quad (4.67)$$

Since $R_2(t, \bar{t}; x) = R_{2,1}(t, \bar{t}; x) + R_{2,2}(t, \bar{t}; x)$, from (4.66) and (4.67), we deduce

$$\|R_2(t, \bar{t}; x)\|_{\frac{p}{2}} \leq C(T + T^\nu) L(\sigma) \tilde{C} |t - \bar{t}|^{\nu_1 + \min(b,1)} \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right]. \quad (4.68)$$

Upper bound of $\|R_4(t, \bar{t}; x)\|_{\frac{p}{2}}$. Using Minkowski's inequality and (4.61), we obtain

$$\|R_4(t, \bar{t}; x)\|_{\frac{p}{2}} \leq \left[c(\sigma) + L(\sigma) \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \|u(t, x)\|_p \right]^2 I(t, h),$$

where

$$I(t, h) = \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) \left| (s+h)^2 f((s+h)(y-z)) \right. \\ \left. - s(s+h) f(sy - (s+h)z) - s(s+h) f((s+h)y - sz) + s^2 f(s(y-z)) \right|.$$

Use the change of variable $z \mapsto -z$ to see that $I(t, h) = \sum_{j=1}^4 \tilde{I}_j(t, h)$, with

$$\begin{aligned}\tilde{I}_1(t, h) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) s^2 \\ &\quad \times |f((s+h)(y+z)) - f(sy + (s+h)z) - f((s+h)y + sz) + f(s(y+z))|, \\ \tilde{I}_2(t, h) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) sh \\ &\quad \times |f((s+h)y + (s+h)z) - f(sy - (s+h)z)|, \\ \tilde{I}_3(t, h) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) sh \\ &\quad \times |f((s+h)y + (s+h)z) - f((s+h)y + sz)|, \\ \tilde{I}_4(t, h) &= \int_0^t ds \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(1, dy) G(1, dz) h^2 |f((s+h)y + (s+h)z)|.\end{aligned}$$

The hypothesis **(h4)** implies $\tilde{I}_1(t, h) \leq C h^{\bar{b}}$ and $\tilde{I}_2(t, h) + \tilde{I}_3(t, h) \leq C h^{b+1}$, (see (4.57) and (4.56), respectively). As for $\tilde{I}_4(t, h)$, we apply the change of variables $((s+h)y, (s+h)z) \mapsto (y, z)$, the scaling property, (4.5) and **(h3)**; this yields

$$\tilde{I}_4(t, h) = \int_0^t \frac{ds}{(s+h)^2} h^2 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(s+h, dy) G(s+h, dz) f(y-z) \leq Ch^2 \int_0^t (s+h)^{\nu-2} ds.$$

For $h \in (0, T]$ and $\varepsilon > 0$ arbitrarily small, this yields that, up to some multiplicative constant, $\tilde{I}_4(t, h)$ is upper estimated by $h^2 T^{\nu-1}$ if $\nu > 1$ (respectively by $h^{\nu+1}$ if $\nu < 1$, and by $T^\varepsilon h^{2-\varepsilon}$ if $\nu = 1$). Summarising the estimates above, we obtain

$$\begin{aligned}\|I_2(t, x) - I_2(\bar{t}, x)\|_p^2 &\leq 2 (\|I_{2,1}(t, \bar{t}; x)\|_p^2 + \|I_{2,2}(t, \bar{t}, x)\|_p^2) \\ &\leq Cp \left([c(\sigma) + L(\sigma) \sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p]^2 |t - \bar{t}|^{1+\nu} + C_\mu L(\sigma)^2 \tilde{C}^2 |t - \bar{t}|^{2\nu_1} \right. \\ &\quad \left. + (T + T^\nu) L(\sigma) \tilde{C} [c(\sigma) + L(\sigma) \sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p] |t - \bar{t}|^{\nu_1 + \min(b, 1)} \right. \\ &\quad \left. + \tilde{C}(\nu, T) [c(\sigma) + L(\sigma) \sup_{(t,x) \in [0, T] \times \mathbb{R}^d} \|u(t, x)\|_p]^2 |t - \bar{t}|^{\min(b+1, \bar{b}, \tilde{\alpha})} \right), \quad (4.69)\end{aligned}$$

where $\tilde{\alpha} = (1 + \nu) \wedge 2$ if $\nu \neq 1$, while $\tilde{\alpha} < 2$ if $\nu = 1$, and $\tilde{C}(\nu, T)$ is a positive constant. This completes the proof of (4.58).

From (4.58), using Proposition 4.4, we deduce (4.59). This concludes the proof. \square

From Propositions 4.5–4.10 we deduce Theorem 4.11 below, which is the main ingredient towards obtaining uniform bounds on moments.

The following constants ν_1 and ν_2 will be used in the next Theorem:

$$\nu_1 = \min(\gamma, \gamma_1, \gamma_2), \quad \nu_2 = \min \left(\nu_1, \frac{1}{2}[\nu_1 + \min(b, 1)], \frac{1+\nu}{2}, \frac{b+1}{2}, \frac{\bar{b}}{2}, \frac{\tilde{\alpha}}{2} \right). \quad (4.70)$$

We recall that γ_1, γ_2 , are the Hölder exponents of the initial values (see Proposition 4.5), γ is the parameter in the assumption **(h2)**, ν is defined in **(h3)**, b and \bar{b} in **(h4)**, and $\bar{\alpha}$ in the last part of the proof of Proposition 4.10.

Let

$$\bar{K}_0(u_0, v_0) = \begin{cases} \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\nabla u_0\|_{\gamma_1} + \|v_0\|_{\infty, R+T}, & d = 2, \\ \|v_0\|_{\gamma_2} + \|\nabla u_0\|_{\infty, R+T} + \|\Delta u_0\|_{\gamma_1}, & d = 3. \end{cases} \quad (4.71)$$

Comparing this definition with (4.50), we see that $K_0(u_0, v_0) \leq \bar{K}_0(u_0, v_0)$.

Theorem 4.11. *Suppose that the hypotheses (1)–(3) of Proposition 4.9 hold, and that the conditions **(h3)** and **(h4)** on the covariance of the noise are satisfied. Fix $T, R > 0$. Then the following holds.*

1. *For any $p \in [2, \infty)$, there exists a constant $C(p, T, R)$ such that, for any $t, \bar{t} \in [0, T]$, $x, \bar{x} \in B(0; R)$ and $\alpha > 0$,*

$$\frac{\|u(t, x) - u(\bar{t}, \bar{x})\|_p}{|x - \bar{x}|^{\nu_1} + |t - \bar{t}|^{\nu_2}} \leq C(p, T, R) \left[\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 e^{T\alpha} \mathcal{N}_{\alpha, p}(u) \right], \quad (4.72)$$

where

$$\begin{aligned} \mathcal{M}_1 &= \bar{K}_0(u_0, v_0) \left\{ 1 + \left[L(b) + \sqrt{p}(1 + \sqrt{C_\mu}) L(\sigma) \right] \exp\left(\frac{TC_3}{2}\right) \right\}, \\ \mathcal{M}_2 &= c(b) + \sqrt{p}c(\sigma) \left\{ 1 + (C_\mu^{(\gamma)})^{1/2} \left[L(b) + \sqrt{p}(1 + \sqrt{C_\mu}) L(\sigma) \right] \exp\left(\frac{TC_3}{2}\right) \right\}, \\ \mathcal{M}_3 &= \left[L(b) + \sqrt{p}(1 + \sqrt{C_\mu}) L(\sigma) \right] \left\{ 1 + (pC_\mu^{(\gamma)})^{1/2} L(\sigma) \exp\left(\frac{TC_3}{2}\right) \right\}, \end{aligned} \quad (4.73)$$

with $\bar{K}_0(u_0, v_0)$ and C_3 given in (4.71) and (4.51), respectively.

2. *Suppose further that $L(b) \geq (2^{12}3^2 C_\mu^2 L(\sigma)^4) \vee \frac{1}{4}$. Then, for any $p \in \left[2, \frac{\sqrt{L(b)}}{2^{53} C_\mu L(\sigma)^2}\right]$ and \mathcal{T}_0 defined in (4.12), we have*

$$\begin{aligned} \frac{\|u(t, x) - u(\bar{t}, \bar{x})\|_p}{|x - \bar{x}|^{\nu_1} + |t - \bar{t}|^{\nu_2}} &\leq C(p, T, R) \\ &\times \left[\mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 e^{2T\sqrt{L(b)}} \left(\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right) \right]. \end{aligned} \quad (4.74)$$

Proof. Fix $x \in B(0; R)$ and consider the time increment $\|u(t, x) - u(\bar{t}, x)\|_p$, with $t, \bar{t} \in [0, T]$. Using the estimates (4.27), (4.30) for the increments of I_0 in dimension $d = 2, 3$, respectively, then (4.37), (4.38) and the definition of (2.7) for the increments of I_1 , and finally (4.59) for the increments of I_2 , we obtain

$$\begin{aligned} \|u(t, x) - u(\bar{t}, x)\|_p &\leq C(T, R) \left\{ \bar{K}_0(u_0, v_0) |t - \bar{t}|^{\min(\gamma_1, \gamma_2)} \right. \\ &\quad + [c(b) + L(b)e^{T\alpha} \mathcal{N}_{\alpha, p}(u)] |t - \bar{t}| \\ &\quad + \tilde{C} \left(\left[L(b) + \sqrt{p}\sqrt{C_\mu} L(\sigma) \right] |t - \bar{t}|^{\nu_1} + \sqrt{p} L(\sigma) |t - \bar{t}|^{\frac{1}{2}[\nu_1 + \min(b, 1)]} \right) \\ &\quad + \sqrt{p} [c(\sigma) + L(\sigma)e^{T\alpha} \mathcal{N}_{\alpha, p}(u)] \\ &\quad \left. \times \left[|t - \bar{t}|^{\frac{1}{2}[\nu_1 + \min(b, 1)]} + |t - \bar{t}|^{\frac{1+\nu}{2}} + |t - \bar{t}|^{\frac{1}{2} \min(b+1, \bar{b}, \bar{\alpha})} \right] \right\}, \end{aligned} \quad (4.75)$$

where the constant \tilde{C} is the same as in (4.53). Here we have applied the inequality $\sqrt{AB} \leq \frac{1}{2}(A+B)$ to the product of constants $A := \tilde{C}L(\sigma)$ and $B := \left[c(\sigma) + L(\sigma)e^{T\alpha}\mathcal{N}_{\alpha,p}(u) \right]$ in the last line of (4.59) (with α instead of $2\sqrt{L(b)}$).

Since by (4.53) we have $\sup_{t \in [0, T]} \|u(t, x) - u(t, \bar{x})\|_p \leq \tilde{C}|x - \bar{x}|^{\nu_1}$ for $x, \bar{x} \in B(0; R)$, we deduce that the L^p norm or space-time increment $\|u(t, x) - u(\bar{t}, \bar{x})\|_p$ is bounded from above by the sum of the left-hand side of (4.75) and $\tilde{C}|x - \bar{x}|^{\nu_1}$. Using the definition of \tilde{C} and grouping terms, we obtain the inequality (4.72).

Part 2. follows from Proposition 4.4 (see (4.11)). This concludes the proof. \square

With an approach similar to that used in Section 3, from part 2. of Theorem 4.11 and the Kolmogorov continuity lemma ([9, Theorem A.3.1]), we deduce the uniform L^p moment estimates stated in the next Proposition. They are essential in the proof of existence and uniqueness of a global random field solution to (1.3). Set

$$\mathcal{K}(c(b), c(\sigma), L(b), L(\sigma)) = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 e^{2T\sqrt{L(b)}} \left(\mathcal{T}_0 + \frac{c(b)}{L(b)} + \frac{c(\sigma)}{L(\sigma)} \right), \quad (4.76)$$

where \mathcal{M}_j , $j = 1, 2, 3$ and \mathcal{T}_0 are defined in (4.73) and (4.12), respectively. Observe that, up to a constant factor, $\mathcal{K}(c(b), c(\sigma), L(b), L(\sigma))$ equals the right-hand side of (4.74).

Proposition 4.12. *Suppose that the hypotheses (1)–(3) of Proposition 4.9 hold, and the hypotheses (h3) and (h4) on the covariance of the noise are satisfied. Let ν_1 and ν_2 be the parameters defined in (4.70). Suppose that the Lipschitz coefficients $L(b)$ and $L(\sigma)$ satisfy $L(b) \geq (2^{12}3^2 C_\mu^2 L(\sigma)^4) \vee \frac{1}{4}$ and*

$$\frac{\sqrt{L(b)}}{2^{53} C_\mu L(\sigma)^2} > \frac{1}{\nu_1} + \frac{d}{\nu_2}, \quad d = 2, 3. \quad (4.77)$$

Fix $T, R > 0$. Then, for any $p \in \left(\frac{1}{\nu_1} + \frac{d}{\nu_2}, \sqrt{L(b)}/(2^{53} C_\mu L(\sigma)^2) \right]$, there exists positive constants C_1 and $C_2(p, T, R)$ such that

$$E \left(\sup_{(t,x) \in [0, T] \times B(0; R)} |u(t, x)|^p \right) \leq 2^{p-1} C_1 + C_2(p, T, R) \mathcal{K}(c(b), c(\sigma), L(b), L(\sigma)), \quad (4.78)$$

with $\mathcal{K}(c(b), c(\sigma), L(b), L(\sigma))$ defined in (4.76).

The proof is analogous to that of Proposition 3.4; it is omitted.

4.4. Existence and uniqueness of a global solution

In this section, we consider the equation (1.3) in spatial dimensions $d = 2, 3$. We assume that the coefficients b and σ satisfy the hypothesis **(Cs)** of Section 3.3, thereby having superlinear growth. We also assume that b dominates σ , in the terms expressed by the condition **(Cd)** below.

(Cd) The parameters δ and a in (1.2) satisfy one of the properties:

- (1) $\delta > 4a$;
 - (2) $\delta = 4a$ and θ_2 and σ_2 are such that $\theta_2 > 2^{12}3^2 C_\mu^2 \sigma_2^4 \left(\frac{1}{\nu_1} + \frac{d}{\nu_2} \right)^2$, $d = 2, 3$,
- where C_μ is defined in (4.6) and ν_1, ν_2 are given in (4.70).

The next theorem is the main result of this section.

Theorem 4.13. *The hypothesis are as follows.*

(i) *The initial values u_0 and v_0 are functions satisfying the hypotheses of Proposition 4.5 with some Hölder exponents $\gamma_1, \gamma_2 \in (0, 1)$.*

(ii) *The coefficients b and σ satisfy **(Cs)** and **(Cd)** with $\delta < \frac{1}{2}$.*

(iii) *The covariance of the noise satisfies conditions **(h1)**, **(h2)**, **(h3)** and **(h4)**.*

1. *For any $M > 0$, there exists a random field solution to (1.3) in $B(0; M)$, $(u(t, x), (t, x) \in [0, T] \times B(0; M))$. This solution is unique and satisfies*

$$\sup_{(t,x) \in [0,T] \times B(0;M)} |u(t, x)| < \infty, \text{ a.s.} \quad (4.79)$$

2. *Suppose that the initial conditions u_0, v_0 are functions with compact support included in $B(0; \rho)$, for some $\rho > 0$, and $b(0) = \sigma(0) = 0$. Then there exists a random field solution $(u(t, x), (t, x) \in [0, T] \times \mathbb{R})$ to (1.3). This solution is unique and satisfies*

$$\sup_{(t,x) \in [0,T] \times B(0,\rho+T)} |u(t, x)| < \infty, \text{ a.s.} \quad (4.80)$$

Equivalently,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |u(t, x)| < \infty, \text{ a.s.} \quad (4.81)$$

Proof. We refer to Section 6 for details on the settings of the claims. The proof uses the same approach as in the proof of Theorem 3.5. First, for $g = b, \sigma$, we consider the truncated globally Lipschitz functions b_N, σ_N , defined in (3.25). The assumption **(Cs)** imply that (3.26) holds. Moreover, by **(Cd)**, we see that the Lipschitz coefficients $L(b_N), L(\sigma_N)$ satisfy the hypotheses of Proposition 4.12.

Let $u_N = (u_N(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ be the unique global random field solution to (1.3) with coefficients b_N, σ_N . Fix $R > 0$. Under the standing hypotheses, we can apply Proposition 4.12 to the stochastic process u_N to deduce that, for any $p \in \left[\frac{1}{\nu_1} + \frac{d}{\nu_2}, \frac{\sqrt{L(b_N)}}{2^5 3 C_\mu L(\sigma_N)^2} \right]$ (and N large enough if necessary), there exist positive constants C_1 and $C_2(p, T, R)$, not depending on N , such that

$$E \left(\sup_{(t,x) \in [0,T] \times B(0;R)} |u_N(t, x)|^p \right) \leq 2^{p-1} C_1 + C_2(p, T, R) \mathcal{K}(c(b_N), c(\sigma_N), L(b_N), L(\sigma_N)). \quad (4.82)$$

Here, $\mathcal{K}(c(b_N), c(\sigma_N), L(b_N), L(\sigma_N))$ is given by (4.76), with $c(b), c(\sigma), L(b), L(\sigma)$ replaced by $c(b_N), c(\sigma_N), L(b_N), L(\sigma_N)$. Recall that

$$c(b_N) = \theta_1, \quad c(\sigma_N) = \sigma_1, \quad L(b_N) = \theta_2 (\ln(2N))^\delta, \quad L(\sigma_N) = \sigma_2 (\ln(2N))^a,$$

(see (3.26)). Because of **(Cd)**, and since $\max(a, \delta) = \delta < \frac{1}{2}$, we have

$$\mathcal{K}(c(b_N), c(\sigma_N), L(b_N), L(\sigma_N)) = o(N^p). \quad (4.83)$$

Consider the sequence of increasing stopping times defined in (3.29). Using (4.83), we see that $\sup_N \tau_N = T$, a.s. By the standard localization argument (see the details of the proof of Theorem 3.5), we finish the proof. \square

5. Examples of covariance densities

In this section, we give three examples of covariances which satisfy the conditions **(h0)**–**(h4)** of section 4; for each of them, we identify the values of the parameters ν_1 , ν_2 in (4.70). For $d = 3$, the same examples are studied in [13]. We only sketch some arguments and refer the reader to the extended version of this article in arXiv ([20]) for details.

5.1. Riesz kernels. For $\beta \in (0, d)$, let $f_\beta : \mathbb{R}^d \rightarrow [0, +\infty]$ be defined by $f_\beta(x) = |x|^{-\beta}$ for $x \in \mathbb{R}^d \setminus \{0\}$, and $f_\beta(0) = +\infty$. Let Λ be the non-negative definite tempered distribution given by $\Lambda(dx) = f_\beta(x) dx$. According to [26, Chapter V], its spectral measure is $\mu_\beta(d\zeta) = c_{d,\beta} f_{d-\beta}(\zeta) d\zeta$, where $c_{d,\beta} = 2^{-\beta+d/2} \Gamma((d-\beta)/2) / \Gamma(\beta/2)$, where Γ denotes the Euler Gamma function.

Observe that the integral $\int_{\mathbb{R}^d} \frac{\mu_\beta(d\zeta)}{1+|\zeta|^2}$ converges if and only if $\beta \in (0, 2 \wedge d)$. In the sequel, we consider the dimensions $d = 2, 3$, and assume that $\beta \in (0, 2)$. Thus, μ_β satisfies the condition **(h0)**. Since f_β is a lower semicontinuous function, from Remark 4.1 we deduce that **(h1)** holds.

Fix $\gamma \in (0, 1)$; the integral $\int_{\mathbb{R}^d} \frac{\mu_\beta(d\zeta)}{1+|\zeta|^{2-2\gamma}}$ is finite if and only if $\gamma < (2-\beta)/2$. Since $|\zeta|^{2\gamma} \mu_\beta(d\zeta) = c_{d,\beta} |\zeta|^{-(d-\beta-2\gamma)} d\zeta$, and the Fourier transform of this measure is $g_\gamma(x) = \tilde{c}(\beta, d) |x|^{-(\beta+2\gamma)}$ (for some positive constant $\tilde{c}(\beta, d)$), if $\beta + 2\gamma < d$, the function $|\zeta|^{2\gamma} \mu_\beta(d\zeta)$ is locally integrable. Therefore, μ_β satisfies the condition **(h2)** for any $\gamma \in (0, (2-\beta)/2)$.

Apply the change of variable $\eta = t\zeta$ to deduce $\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \mu_\beta(d\zeta) = c_{d,\beta} t^{2-\beta} I_{d,\beta}$, where the integral $I_{d,\beta} := \int_{\mathbb{R}^d} \frac{\sin^2(|\eta|)}{|\eta|^{2+d-\beta}} d\eta$ is finite. Hence μ_β satisfies the condition **(h3)** with $\nu = 2 - \beta$ and $C := c_{d,\beta} I_{d,\beta}$.

The function f_β satisfies the conditions **(h4)** 1. and **(h4)** 2. for any $b \in (0, \min(2-\beta, 1))$ and $\bar{b} \in (0, 2-\beta)$, respectively. For $d = 3$, this is proven in [13, Proposition 5.3, p. 383-386], relying on [7, Lemma 2.6, p. 10] (see also Lemmas 6.4 and 6-5 in [7]). Going through the details of the proofs, we see that they can be extended to $d = 2$, thanks to Lemma 4.2 (see [20, Section 5] for further details).

Conclusion. Let $d = 2, 3$ and $\beta \in (0, 2)$. For spatially homogeneous Gaussian noises with covariance given by (4.1) with $\Lambda(dx) = f_\beta(x) dx$, the parameters in (4.70) are $\nu_1 = \nu_2 = \min(\gamma, \gamma_1, \gamma_2)$, with $\gamma < \frac{2-\beta}{2}$. Hence, (4.72) implies that almost all sample paths of the solution to (1.3) are locally Hölder continuous, jointly in (t, x) , with exponent $\theta \in (0, \min((2-\beta)/2, \gamma_1, \gamma_2))$. For $d = 3$, this is [7, Theorem 4.11, p. 48]. Moreover, the critical exponent $\min((2-\beta)/2, \gamma_1, \gamma_2)$ is sharp in both dimensions, $d = 2, 3$ (see [7], [8]).

5.2. Bessel kernels. For any $\kappa > 0$, the Bessel kernel is the function defined by $\tilde{f}_\kappa(x) = \int_0^\infty w^{\frac{\kappa-d-2}{2}} e^{-w} e^{-\frac{|x|^2}{4w}} dw$ if $x \in \mathbb{R}^d \setminus \{0\}$, $\tilde{f}_\kappa(0) = \infty$ if $0 < \kappa \leq d$, and $\tilde{f}_\kappa(0) = c(d, \kappa)$ if $\kappa > d$, where $0 < c(d, \kappa) < \infty$. Let Λ be the measure defined by $\Lambda(dx) = \tilde{f}_\kappa(x) dx$; its spectral measure is $\tilde{\mu}_\kappa(d\zeta) = C_{d,\kappa} (1 + |\zeta|^2)^{-\frac{\kappa}{2}} d\zeta$ (see [26, Chapter V]). In the sequel, we consider the case $d = 2, 3$, and we assume $\kappa > d - 2$.

Since \tilde{f}_κ is lower semicontinuous, the condition **(h1)** holds (see Remark 4.1). Fix $\gamma \geq 0$; the integral $\int_{\mathbb{R}^d} \frac{\tilde{\mu}_\kappa(d\zeta)}{1+|\zeta|^{2-2\gamma}}$ is finite if and only if $2\gamma < \kappa - d + 2$. Take $\gamma = 0$ to

deduce that **(h0)** holds. Furthermore, for $\gamma \in (0, \min(\frac{\kappa-d+2}{2}, 1))$, the constant $C_{\tilde{\mu}_\kappa}^{(\gamma)}$ defined in (4.39) is finite and therefore, **(h2)** holds.

To check **(h3)**, we fix $t > 0$ and write

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \tilde{\mu}_\kappa(d\zeta) &\leq \tilde{C}_{d,\kappa} \left(\int_0^1 t^2 r^{d-1} dr + \int_1^{\frac{T}{t}} t^2 r^{d-1-\kappa} dr + \int_{\frac{T}{t}}^\infty r^{d-1} r^{-2} r^{-\kappa} dr \right) \\ &\leq \tilde{C}_{d,\kappa} \left(\frac{t^2}{d} + \int_1^{\frac{T}{t}} t^2 r^{d-1-\kappa} dr + \frac{1}{\kappa-d+2} \left(\frac{t}{T} \right)^{\kappa-d+2} \right). \end{aligned} \quad (5.1)$$

Set $I(t) := \int_1^{\frac{T}{t}} t^2 r^{d-1-\kappa} dr$; we see that $I(t)$ is upper bounded by $\frac{t^2}{d-\kappa} \left(\frac{T}{t} \right)^{d-\kappa}$ if $d-2 < \kappa < d$ (respectively by $t^2 \ln \left(\frac{T}{t} \right)$ if $\kappa = d$, and by $\frac{t^2}{\kappa-d}$ if $d < \kappa$). Therefore, (5.1) implies

$$\int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \tilde{\mu}_\kappa(d\zeta) \leq C(d, \kappa, T) t^\nu, \quad t \in [0, T],$$

where $C(d, \kappa, T)$ is some positive constant and $\nu < \min(2, \kappa - d + 2)$. Hence, **(h3)** holds with $\nu < \min(2, \kappa - d + 2)$.

For $d = 3$, the validity of **(h4)** is proved in [13, Section 5.3]. Going through the proof, we see that they also hold for $d = 2$ (see the details in [20, Section 5]).

Conclusion. Let $d = 2, 3$, $\kappa > d - 2$. For spatially homogeneous Gaussian noises with covariance given by (4.1) with $\Lambda(dx) = \tilde{f}_\kappa(x) dx$, the parameters defined in (4.70) are $\nu_1 = \nu_2 = \min(\gamma, \gamma_1, \gamma_2)$, with $\gamma < \min(\frac{\kappa-d+2}{2}, 1)$. Thus, we deduce that almost all sample paths of the solution to (1.3) are locally Hölder continuous, jointly in (t, x) , with exponent $\theta \in (0, \min(\frac{\kappa-d+2}{2}, 1, \gamma_1, \gamma_2))$. When $d = 3$, we recover the results in [13, p. 393]. Whether this Hölder exponent is sharp seems to be an open question.

5.3. Fractional kernels Let $d = 2, 3$, and $H = (H_i)_{1 \leq i \leq d}$, with $H_i \in (1/2, 1)$. Let $\tilde{f}_H(x) = C_H \prod_{i=1}^d |x_i|^{2H_i-2}$, when $\prod_{i=1}^d x_i \neq 0$, where $C(H) = \prod_{i=1}^d H_i(2H_i - 1)$, and $\tilde{f}_H(x) = +\infty$, otherwise. In this section, we consider the non-negative definite tempered distribution $\Lambda(dx) = \tilde{f}_H(x) dx$. Its spectral measure is $\tilde{\mu}_H(\zeta) = C_H \prod_{i=1}^d |\zeta_i|^{1-2H_i} d\zeta$, where C_H is some positive constant.

Since the function \tilde{f}_H is lower semicontinuous, condition **(h1)** holds, by Remark 4.1. As proved in [20, p. 44], the condition **(h0)** is satisfied if $\sum_{i=1}^d H_i > d - 1$.

Set $\bar{\kappa} := \sum_{i=1}^d H_i - (d - 1) > 0$. The condition **(h2)** holds if $\gamma < \bar{\kappa}$ (see [20, p. 44]).

Let $t \in [0, T]$; then

$$\begin{aligned} \int_{\mathbb{R}^d} |\mathcal{F}G(t)(\zeta)|^2 \tilde{\mu}_H(d\zeta) &\leq C \left[\int_0^{t^{-1}} t^2 r^{2d-1-2\sum_{i=1}^d H_i} dr + \int_{t^{-1}}^\infty r^{d-1-2+d-2\sum_{i=1}^d H_i} dr \right] \\ &\leq C t^{2\bar{\kappa}}. \end{aligned}$$

Thus, the condition **(h3)** holds with $\nu = 2\bar{\kappa}$.

For $d = 3$, the validity of **(h4)** is proved in [13]. With minor changes, the arguments also hold for $d = 2$ (see [20, Section 5] for more details).

Conclusion. Let $d = 2, 3$, $H = (H_i)_{1 \leq i \leq d}$, with $H_1 \in (1/2, 1)$ and $\bar{\kappa} = \sum_{i=1}^d H_i - d + 1 > 0$. For spatially homogeneous Gaussian noises with covariance given by (4.1) with

$\Lambda(dx) = f_H(x) dx$, the parameters in (4.70) are

$$\nu_1 = \nu_2 = \min(\gamma_1, \gamma_2, \bar{\kappa}, \min(H_i - 1/2; i = 1, \dots, d)).$$

As a consequence, from (4.72) we deduce that almost all sample paths of the solution to (1.3) are locally Hölder continuous, jointly in (t, x) , with exponent

$$\theta \in (0, \min(\gamma_1, \gamma_2, \bar{\kappa}, \min(H_i - 1/2; i = 1, \dots, d))).$$

For $d = 3$, this is [13, Theorem 6.1]. In this case, following [13, Theorem 6.2], the critical exponent should be $\min(\gamma_1, \gamma_2, \bar{\kappa})$; therefore the above result is not optimal.

6. Appendix: some elements on stochastic wave equations

In this section, we give some basic elements relative to the stochastic wave equations considered in this article.

Throughout the section, $(u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ denotes the random field solution to the stochastic wave equation in each one of the settings:

- (s1) $d = 1$, W is a space-time white noise, and the conditions of Proposition 3.1 hold;
- (s2) $d = 2, 3$, W is a noise white in time and coloured in space, and the hypotheses of Proposition 4.3 are satisfied.

For any bounded Borel set $\mathcal{O} \subset \mathbb{R}^d$ and ε_0 , $\mathcal{O}^{(\varepsilon_0)}$ denotes the closed ε_0 -neighborhood of \mathcal{O} , that is,

$$\mathcal{O}^{(\varepsilon_0)} = \{z \in \mathbb{R}^d, d(x, \mathcal{O}) \leq \varepsilon_0\}.$$

6.1. Propagation of support

For the homogeneous (deterministic) wave equation there is a well-known *compact support property* saying that if the initial conditions have compact support then the solution in the classical sense also has compact support. The proposition below tells us that, under suitable conditions, this property extends to the stochastic wave equation in dimensions $d \in \{1, 2, 3\}$. As for the deterministic case, this relies on the fact that the support of the fundamental solution of the wave equation $G(t, \cdot)$, $t \in [0, T]$, is included in the closed ball $\overline{B(0; t)}$.

Proposition 6.1. *Consider the two cases (s1) and (s2) described above. Assume that*

- (i) *the initial conditions u_0, v_0 are functions with compact support $K \subset \mathbb{R}^d$;*
- (ii) *the coefficients b and σ satisfy $b(0) = \sigma(0) = 0$.*

Then, for any $t \in [0, T]$,

$$u(t, x) = 0, \quad \text{for any } x \notin K^{(t)}. \tag{6.1}$$

Hence, the support of the sample paths of the solution $(u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ is included in $[0, T] \times K^{(T)}$.

Proof. We will extend the arguments in [18, p. 925] in dimension $d = 2$ to any $d \in \{1, 2, 3\}$.

First, notice that since the mapping $[0, T] \ni t \mapsto K^{(t)}$ is increasing, the last statement is an immediate consequence of (6.1).

Next, we prove that (6.1) holds with $u(t, x)$, replaced by $u^n(t, x)$ —the n -th Picard iteration of u defined by

$$\begin{cases} u^0(t, x) &= I_0(t, x), \\ u^n(t, x) &= I_0(t, x) + \int_0^t ds \int_{\mathbb{R}^d} dy G(t-s, x-y) b(u^{n-1}(s, y)) \\ &\quad + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \sigma(u^{n-1}(s, y)) W(ds, dy), \quad n \geq 0. \end{cases} \quad (6.2)$$

Indeed, fix $t \in [0, T]$ and let $n = 0$. If $x \notin K^{(t)}$, we have $|x - y| > t$ for any $y \in K$. Therefore, the integrals defining $I_0(t, x)$ (see (2.5)) vanish, because of the above mentioned property on the support of $G(t, \cdot)$.

In the next induction step, we will make use of the following fact:

(PS) *Let $x \notin K^{(t)}$. Then for all $s \in [0, t]$ and all $y \in \mathbb{R}^d$ such that $|x - y| \leq t - s$, we have $y \notin K^{(s)}$.*

Indeed, if $x \notin K^{(t)}$ then for any $z \in \mathbb{R}^d$, $|z - x| > t$. By the triangle inequality, this implies $|z - y| \geq |z - x| - |x - y| > t - (t - s) = s$. Thus, $y \notin K^{(s)}$.

Assume that (6.1) holds with $u(t, x)$ replaced by $u^l(t, x)$, $l = 0, \dots, n-1$. We observe that the integrands in (6.2) (with $n := n-1$) vanish if $|x - y| > t - s$. If on the contrary, $|x - y| \leq t - s$, from **(PS)** and the induction assumption, we have that $u^{(n-1)}(s, y) = 0$. Hence, by assumption (ii), we have $b(u^{n-1}(s, y)) = \sigma(u^{n-1}(s, y)) = 0$, which implies (6.1) with $u(t, x)$ replaced by $u^n(t, x)$.

In Propositions 3.1 and 4.3 the random field solutions $(u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ are obtained as limits of the Picard iterations (6.2). Thus, the *propagation of support property* (6.1) holds. \square

6.2. The solution of the stochastic wave equation restricted to a bounded domain

Let $d \in \{1, 2, 3\}$ and $T > 0$. Following the setting of [7, Chapter 4], we consider a bounded domain $D \subset \mathbb{R}^d$ and the associated past light cone

$$D^{(T-t)} = \{z \in \mathbb{R}^d : d(z, D) \leq T - t\}, \quad t \in [0, T].$$

For any $(t, x) \in [0, T] \times \mathbb{R}^d$, set

$$\begin{aligned} u(t, x) 1_{D^{(T-t)}}(x) &= 1_{D^{(T-t)}}(x) \left([G(t) * v_0](x) + \frac{\partial}{\partial t} [G(t) * u_0](x) \right) \\ &\quad + 1_{D^{(T-t)}}(x) \int_0^t ds \int_{\mathbb{R}^d} dy G(t-s, x-y) b(u(s, y) 1_{D^{(T-s)}}(y)) \\ &\quad + 1_{D^{(T-t)}}(x) \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) \sigma(u(s, y) 1_{D^{(T-s)}}(y)) W(ds, dy). \end{aligned} \quad (6.3)$$

By the triangle inequality, if $x \in D^{(T-t)}$ and $x - y \in \overline{B(0; t-s)}$ then $y \in D^{(T-s)}$ for all $0 \leq s \leq t \leq T$ (compare with property **(PS)** in Section 6.1). Hence, (6.3) is

consistent in the following sense: when $x \notin D^{(T-t)}$, it is a trivial equation, while when $x \in D^{(T-t)}$, the stochastic process $(u(t, x)1_{D^{(T-t)}}(x), (t, x) \in [0, T] \times \mathbb{R}^d)$ satisfies (1.3) (with $(t, x) \in [0, T] \times D^{(T-t)}$). In [7], this process is called a solution to the wave equation “in D ”.

Observe that $[0, T] \ni t \mapsto D^{(T-t)}$ is decreasing and $\bigcap_{t \in [0, T]} D^{(T-t)} = D$.

Proposition 6.2. *Consider the cases (s1), (s2) described above.*

Let $u = (u(t, x), (t, x) \in [0, T] \times \mathbb{R}^d)$ be the respective random field solutions given in Propositions 3.1 and 4.3.

Let $D \subset \mathbb{R}^d$ be a bounded domain. Then, almost surely, $u(t, x) = u(t, x)1_{D^{(T-t)}}(x)$ for all $(t, x) \in [0, T] \times D$, where $(u(t, x)1_{D^{(T-t)}}(x), (t, x) \in [0, T] \times \mathbb{R}^d)$ is the random field solution to (6.3).

Therefore, the support of the sample paths of the random field $u = (u(t, x), (t, x) \in [0, T] \times D)$ is included in $[0, T] \times D^{(T)}$.

Proof. For $d = 3$ and covariance densities of the noise belonging to a class that include Riesz kernels, the existence and uniqueness of a random field solution to (6.3) is established in [7, Chapter 4] (see, in particular Proposition 4.3 and Theorem 4.6 there). With covariance densities satisfying the conditions (h0) and (h1) of Section 4.1, the results are still valid ([6], [13]). With similar but simpler arguments, those results can be extended to $d = 1$ (with space time white noise) and to $d = 2$ with covariance densities satisfying the conditions (h0) and (h1) of Section 4.1.

Because of the uniqueness of solution assertions, for all $(t, x) \in [0, T] \times D$, $u(t, x) = u(t, x)1_{D^{(T-t)}}(x)$ a.s. Actually, since the sample paths of both processes are continuous (and even locally Hölder continuous), the processes $(u(t, x), (t, x) \in [0, T] \times D)$ and $(u(t, x)1_{D^{(T-t)}}(x), (t, x) \in [0, T] \times D)$ are indistinguishable. \square

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