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THE TORSION IN SYMMETRIC POWERS ON CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

JONATHAN PFAFF AND JEAN RAIMBAULT

ABSTRACT. In this paper we prove that for a fixed neat principal congruence subgroup of a Bianchi group the order of the torsion part of its second cohomology group with coefficients in an integral lattice associated to the m -th symmetric power of the standard representation of $\mathrm{SL}_2(\mathbb{C})$ grows exponentially in m^2 . We give upper and lower bounds for the growth rate. Our result extends a result of W. Müller and S. Marshall, who proved the corresponding statement for closed arithmetic 3-manifolds, to the finite-volume case. We also prove a limit multiplicity formula for combinatorial Reidemeister torsions on higher dimensional hyperbolic manifolds.

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1. INTRODUCTION

1.1. **Torsion homology growth of arithmetic groups.** An arithmetic group is, roughly speaking, a group of the form $G(\mathbb{Z})$ where G is a semisimple \mathbb{Q} -subgroup of some GL_n . It is then a lattice (discrete subgroup with finite covolume) in the Lie group $G(\mathbb{R})$. If Γ is such a group, given a \mathbb{Q} -representation ρ of G on a vector space V there exists a lattice $\Lambda \subset V(\mathbb{R})$ which is preserved by $\rho(\Gamma)$. The cohomology groups $H^*(\Gamma; \Lambda)$ are then finitely generated Abelian groups and as such they split noncanonically as a direct sum

$$H^k(\Gamma; \Lambda) = H^k(\Gamma; \Lambda)_{free} \oplus H^k(\Gamma; \Lambda)_{tors}$$

where $H^k(\Gamma; \Lambda)_{free}$ is free abelian and $H^k(\Gamma; \Lambda)_{tors}$ is finite. The classes in the free summand can be interpreted as automorphic forms for the group G and have been extensively

studied both from the theoretical and the computational perspective. More recently the torsion subgroup and mod- p classes have attracted a lot of interest from number theorists, as it gained significance in a mod- p analogue of the Langlands programme (see for example the work of N. Bergeron–A. Venkatesh [BV13], F. Calegari–A. Venkatesh [CV12] and P. Scholze [Sch15]) and, perhaps less prominently, for links with the K -theory of number fields (see [CV12, 9.7] and V. Emery’s paper [Eme14]).

Our topic of interest in this paper lies purely on the side of the study of the groups $H^*(\Gamma; \Lambda)$, without any further number-theoretical considerations (though number theory will feature prominently in our arguments). More precisely, motivated in particular by the above it is interesting to study the growth rate of the size $|H^k(\Gamma; \Lambda)_{tors}|$ when either the group Γ or the representation ρ (and thus the Γ -module Λ) varies. The question of studying sequences of congruence lattices in a fixed Lie group $G(\mathbb{R})$ is the topic of the seminal work [BV13]. We note that this reference dealt only with uniform lattices and that further work on non-uniform lattices by the authors of the present paper is in [Pfa14] and [Rai13]. In [BV13] it is conjectured that, for a lattice $\Gamma \subset G(\mathbb{R})$ and a fixed Γ -module Λ as above and a sequence of congruence subgroups $\Gamma_m \subset \Gamma$,¹ the size of the group $H^k(\Gamma_m, \Lambda)$ is asymptotically exponential in the covolume of $\Gamma_m \backslash G(\mathbb{R})$, with a rate depending only on $G(\mathbb{R})$, if the following two conditions are satisfied:

- (1) $k = (d + 1)/2$ where d is the dimension of the symmetric space of $G(\mathbb{R})$;
- (2) $G(\mathbb{R})$ has “fundamental rank” equal to 1, for example $G(\mathbb{R}) \cong \mathrm{SO}(n, 1)$ or $G(\mathbb{R}) \cong \mathrm{SL}_3(\mathbb{R})$.

Otherwise they conjecture that the size of $H^k(\Gamma_n, \Lambda)$ is always subexponential in the covolume. They prove their conjecture when $G(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{C}) \times K$ (with K compact, so that $d = 3$ and the fundamental rank is 1) and ρ is a symmetric power of the adjoint representation. The general case is completely open at present, though they prove that exponential growth occurs in at least one of the groups $H^k(\Gamma_n; \Lambda)$ in the cases where it is expected (when $G(\mathbb{R})$ has fundamental rank one). Let us note that in this case $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})$ the conjecture is also of interest to topologists, see for example [Le14].

We will study what happens when we fix the arithmetic lattice Γ and vary the representation ρ . In this setting it is expected that a statement similar to the Bergeron–Venkatesh conjecture should hold, with covolume replaced by a function of the \mathbb{Z} -rank $\mathrm{rank}_{\mathbb{Z}}(\Lambda(m))$ (see e.g. [MP14b]). We only study the case where $G(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})$ and the sequence of representations $\rho_m = \mathrm{Sym}^m(\rho_1)$ of $G(\mathbb{R})$, where ρ_1 is the tautological representation of $G(\mathbb{R})$ on \mathbb{C}^2 . The most obvious \mathbb{Q} -forms of $\mathrm{SL}_2(\mathbb{C})$ which admit \mathbb{Q} -representations which become ρ_1 over the reals are the Weil restrictions of the groups SL_2/F where F is an imaginary quadratic field, that is $F = \mathbb{Q}(\sqrt{-D})$ for some square-free integer D . The associated arithmetic lattices are (up to commensurability) the so-called Bianchi groups $\Gamma_D = \mathrm{SL}_2(\mathcal{O}_D)$ where \mathcal{O}_D is the ring of integers in $\mathbb{Q}(\sqrt{-D})$. Note that they represent all commensurability classes of nonuniform arithmetic lattices in $\mathrm{SL}_2(\mathbb{C})$. Fixing D , it is obvious that the lattice $\mathcal{O}_D^2 \subset \mathbb{C}^2$ is stable under $\rho_1(\Gamma_D)$ and we are thus interested in the

¹In fact Bergeron and Venkatesh only consider exhaustive towers of such groups, but in view of [ABB⁺17, Section 8] it is natural to extend their conjecture to all congruence subgroups.

groups $H^k(\Gamma; \Lambda(m))_{tors}$ as m tends to infinity, where

$$\Lambda(m) = \text{Symm}^m(\mathcal{O}_D)$$

and $\Gamma \subset \Gamma_D$ is a fixed subgroup of finite index. We will restrict to the case where Γ is a principal congruence subgroup, that is

$$\Gamma = \Gamma(\mathfrak{a}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathcal{O}_D) : a, d \in 1 + \mathfrak{a}, b, c \in \mathfrak{a} \right\}.$$

Our main result is then the following.

Theorem A. *There exist constants $C_1(\Gamma_D) > 0$, $C_2(\Gamma_D) > 0$, which depend only on Γ_D such that for each non-zero ideal \mathfrak{a} of \mathcal{O}_D with $N(\mathfrak{a}) > C_1(\Gamma_D)$ one has*

$$(1.1) \quad \liminf_{m \rightarrow \infty} \left(\frac{\log |H^2(\Gamma(\mathfrak{a}); \Lambda(m))_{tors}|}{m^2} \right) \geq \frac{1}{4} \cdot \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \left(1 - \frac{C_1(\Gamma_D)}{N(\mathfrak{a})} \right) > 0$$

and

$$(1.2) \quad \limsup_{m \rightarrow \infty} \left(\frac{\log |H^2(\Gamma(\mathfrak{a}); \Lambda(m))_{tors}|}{m^2} \right) \leq \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \left(1 + \frac{C_2(\Gamma_D)}{N(\mathfrak{a})} \right).$$

Finally, we also have

$$(1.3) \quad |H^1(\Gamma(\mathfrak{a}); \Lambda(m))_{tors}| = O(m \log m),$$

as $m \rightarrow \infty$.

We were not able to prove the expected limit

$$(1.4) \quad \lim_{m \rightarrow \infty} \frac{\log |H^2(\Gamma(\mathfrak{a}); \Lambda(m))_{tors}|}{m^2} = \frac{1}{2} \cdot \frac{\text{vol}(X_{\mathfrak{a}})}{\pi}$$

but it is conceivable that various refinements of our arguments could be used to do this. The constants $C_1, C_2(\Gamma_D)$ can be made explicit, and if the class number of F is one and $\mathcal{O}_D^* = \{\pm 1\}$, we can take $C_1(\Gamma_D) = 4$ (this is the case exactly for $D = -7, -11, -19, -43, -67$ and -163 by the Heegner–Stark Theorem).

We end this introductory section with a few comments on the analogue of our result for a uniform Γ , which was studied by S. Marshall and W. Müller in [MM13]. The simplest case is then when G is the group of norm 1 elements in a quaternion algebra A over an imaginary quadratic field F , and Γ is (a torsion-free congruence subgroup in) the set of units in an order \mathcal{O} of A . The representation ρ_2 is the adjoint representation and it can be defined as a rational representation for G since it is isomorphic to the conjugation action of G on the subspace U of purely imaginary quaternions in $A \otimes_F \mathbb{C}$, and the lattice $\Lambda(2) = U \cap \mathcal{O}$ is stable under Γ as are its symmetric powers $\Lambda(2m) \subset \text{Symm}^{2m}(\mathbb{C}^2)$. Then Marshall and Müller prove that

$$\lim_{m \rightarrow \infty} \frac{\log |H^2(\Gamma; \Lambda(2m))_{tors}|}{m^2} = \frac{2 \text{vol}(X_{\mathfrak{a}})}{\pi}.$$

For higher-dimensional groups $G(\mathbb{R})$ there are results of the first author together with W. Müller [MP14b].

1.2. Sketch of proof. The statement (1.3) on H^1 is deduced from explicit computations, see Lemma 7.2. For H^2 the upper and lower bound are proven separately, and by very different methods. Since the proof for the upper bound is much shorter and more straightforward we will explain it first.

In general it is very easy to get rough bounds for the order of all torsion subgroups of $H^k(\Gamma; \Lambda(m))$ of the same order as in (1.2). We will first explain the idea for the similar case where one considers a sequence $\Gamma_m \subset \Gamma$ of finite index subgroups and want to get exponential bounds in $[\Gamma : \Gamma_m]$ for the torsion subgroup of $H^k(\Gamma_m; \Lambda)$. Let \tilde{X} be the symmetric space of $G(\mathbb{R})$ and $X = \Gamma \backslash \tilde{X}$, which is a $K(\Gamma, 1)$. There exists a local system \mathcal{L} of \mathbb{Z} -modules Λ on X such that $H^*(\Gamma; \Lambda) = H^*(X; \mathcal{L})$. There exists a triangulation T of X , and lifting it to $X_m = \Gamma_m \backslash \tilde{X}$ we get a triangulation T_m of X_m with at most $C[\Gamma : \Gamma_m]$ simplices in each dimension, and where each simplex is incident to at most C other simplices of the same dimension (for some constant C depending only on X). This means that the cohomology $H^*(\Gamma_m; \Lambda) = H^*(X_m; \mathcal{L})$ can be computed from the cochain complex $(C^*(T_m; \mathcal{L}), d_*)$ which satisfies

$$\text{rank}_{\mathbb{Z}}(C^k(T_m; \mathcal{L})) \ll \text{rank}_{\mathbb{Z}}(\Lambda) \cdot [\Gamma : \Gamma_m]$$

and the entries of each differential d_k are uniformly bounded (depending on coefficients of $\rho(\gamma)$ for γ in a finite subset of Γ) and at most C are nonzero in each column. So $H^k(\Gamma_m; \Lambda)$ embeds into $\mathbb{Z}^r / A\mathbb{Z}^s$ where $r, s \ll [\Gamma : \Gamma_m]$ and A is a matrix with at most C nonzero entries in each column, and by a well-known lemma in elementary linear algebra it follows that the torsion is at most $(C')^{[\Gamma : \Gamma_m]}$ (see Lemma 8.1). This method was used in [Sau16] in a different context.

When varying Λ things are different since entries in the differentials d_k can be arbitrarily large, but in our case where $\rho_m = \text{Symm}^m(\rho_1)$ they can be at most C^m (for some C depending on Γ). Since the rank of the $C^k(X; \Lambda(m))$ is linear in $\text{rank}_{\mathbb{Z}}(\Lambda(m)) = m$ it follows from the same lemma that the torsion in $H^k(\Gamma; \Lambda(m))$ is (roughly) at most $(C^m)^m = C^{m^2}$. To get the more precise estimate (1.2) we use a few more tricks.

Getting a lower bound for the torsion in H^2 is much more involved. In fact we, following the earlier work [BV13] and [MM13], prove the abundance of torsion classes via a somewhat indirect path. First we need a “limit multiplicity formula” for the analytic torsion $T_{X_0}(X; E_{\rho_m})$ (see Section 2 for the definition—here and later E_{ρ} is the flat vector bundle over X associated to the representation ρ) as in [BV13, Section 4]. This means that

$$\lim_{m \rightarrow +\infty} \left(\frac{\log T_{X_0}(X; E_{\rho_m})}{m^2} \right) = -\frac{\text{vol}(X)}{2\pi}.$$

In our context such a limit is proven to hold by the first author and W. Müller in [MP12]. Then we use an equality relating the analytic torsion to a Reidemeister torsion $\tau_{Eis}(X; E_{\rho_m})$ (see Section 4 and [Pfa17] for the definition), and following [BV13] we relate the latter to torsion homology. Bergeron and Venkatesh as well as Müller and Marshall use an extension of the classical Cheeger–Müller theorem due to Müller and J.-M. Bismut–W. Zhang, but for non-compact manifolds we need a recent result of the first author [Pfa17]. There are

actually some additional terms in the formula for cusped manifolds (see Proposition 2.3 below), and while a more thorough analysis might show that they disappear in the limit we use a simpler trick, which gives us only a lower bound on the exponential growth of Reidemeister torsion (Proposition 3.2). A similar result was proven by P. Menal-Ferrer and J. Porti in [MFP14], for a different normalisation of the Reidemeister torsion that we cannot use here and using very different methods.

Once we get this exponential growth it remains to extract the behaviour of the specific torsion subgroup $H^2(X; \Lambda(m))$ from that of $\tau_{Eis}(X; E_{\rho_m})$. This is the main new contribution of our work, and we will now detail a bit how this is done. The relation of Reidemeister torsion with cohomological torsion is given by:

$$(1.5) \quad \tau_{Eis}(X; E_{\rho_m}) = \frac{|H^1(\Gamma; \Lambda(m))_{tors}| \cdot R^2(X, \mathcal{L}(m))}{|H^2(\Gamma; \Lambda(m))_{tors}| \cdot R^1(X, \mathcal{L}(m))}$$

where the “regulator” terms $R^k(X; \mathcal{L}(m))$ are defined as the covolume of the lattice $H^k(X; \mathcal{L}(m)) \subset H^k(X; E_{\rho_m})$ where the right-hand space is identified with a space of harmonic forms with the L^2 -metric. For the Bianchi groups this is somewhat tricky since when X is noncompact there is no natural L^2 -metric on the space $H^k(X; E_{\rho_m})$. However, the theory of Eisenstein cohomology developed by G. Harder in [Har87] allows to “transport” the L^2 -metric from the harmonic forms on the boundary $\partial\bar{X}$ of the Borel–Serre compactification \bar{X} (see Section 4). This way of defining regulators is similar to that used in [CV12, Section 6.3].

In view of (1.5), to prove exponential growth of $H^2(\Gamma; \Lambda(m))_{tors}$ we would like to show that

$$\lim_{m \rightarrow +\infty} \left(\frac{\log R^k(X; \mathcal{L}(m))}{m^2} \right) = 0.$$

This only causes serious problems for R^1 , so let us explain more precisely how this regulator is defined and how to deal with it. When $m > 0$ there is an isomorphism $E : H^1(\partial\bar{X}; E_{\rho_m})_- \rightarrow H^1(X; E_{\rho_m})$ defined through Eisenstein series (where the right-hand side is the $(0, 1)$ -part of the cohomology in the Hodge decomposition, see (4.2) below). The problem is then that, while the map E is rational it is not integral. This means that we do not have $E(H^1(\partial\bar{X}; \mathcal{L}(m))_- \subset H^1(X; \mathcal{L}(m))$ but there exists a $N \in \mathbb{Z}_{>0}$ such that

$$E(H^1(\partial\bar{X}; \mathcal{L}(m))_- \subset N^{-1} \cdot H^1(X; \mathcal{L}(m)).$$

So to estimate $R^1(X; \mathcal{L}(m))$ we need to get an upper bound on N (it is easy to estimate the covolume of the integral lattice $H^1(\partial\bar{X}; \mathcal{L}(m))_-$ in $H^1(\partial\bar{X}; E_{\rho_m})_-$, see Lemma 7.1).

We will not deal with this problem completely in this paper (but it is very likely that this could be done using additional arguments from [CV12]). However, using the long exact sequence we can estimate R^1 in terms of the torsion in H^2 and an integer M such that $\iota^* E(H^1(\partial\bar{X}; \mathcal{L}(m))_- \subset M^{-1} H^1(\partial\bar{X}; \mathcal{L}(m))_-$ (where ι is the inclusion $\partial\bar{X} \subset \bar{X}$), see Lemma 4.1. This allows us to balance the two against each other, and thus instead of N we need only to control the smaller number M (at the expense of getting a less precise result, halving the lower bound). This is easier since we need only to deal with boundary

integrals and it boils down to evaluating the denominator of the “intertwining matrix” $\underline{\mathbf{C}} \in \text{Hom}(H^1(\partial\bar{X}; E_{\rho_m})_-, H^1(\partial\bar{X}; E_{\rho_m}))$ defined by

$$\iota^* E(\omega) = \omega + \underline{\mathbf{C}}\omega$$

(see also Section 6.6 in [CV12]). After classical computations summarised in (5.17) this in turn reduces to controlling a ratio of values of Hecke L -functions. To deal with this we use two papers of R. Damerell [Dam70, Dam71]: we have to give an effective version of one of his results and analyse [Dam70] to this effect in Appendix 6.10. In the end we obtain in Proposition 6.1 an estimate $M \leq (m!)^A$. This essentially finishes the proof, and the wrapping up is done in Proposition 7.3.

In this sketch we were not completely accurate. In particular, we will work with the self-dual lattices $\bar{\Lambda}(m) = \Lambda(m) \oplus \Lambda^\vee(m)$ to avoid some issues with duality, and relate $H^*(\Gamma; \bar{\Lambda}(m))$ with $H^*(\Gamma; \Lambda(m))$ using Proposition 7.4 at the end.

1.3. Some further comments. Theorem A also holds with the same proof for slightly more general rays of local systems. Namely, the finite dimensional irreducible representations of $\text{SL}_2(\mathbb{C})$ are parametrised as $\text{Symm}^{n_1} \otimes \overline{\text{Symm}}^{n_2}$, where $n_1, n_2 \in \mathbb{N}$ and where $\overline{\text{Symm}}^{n_2}$ is the complex conjugate of Symm^{n_2} . Each such representation space carries a canonical \mathbb{Z} -lattice preserved by the action of Γ_D . If we fix n_1 and n_2 with $n_1 \neq n_2$ and let $\rho_m(n_1, n_2)$ be the representation $\text{Symm}^{mn_1} \otimes \overline{\text{Symm}}^{mn_2}$, then the analog of Theorem A holds if we replace the factor m^2 by $m \dim \rho_m(n_1, n_2)$ which grows as m^3 if both n_1 and n_2 are not zero. However, we can by no means remove the assumption $n_1 \neq n_2$. In other words, the ray $\rho_m(1, 1) = \text{Symm}^m \otimes \overline{\text{Symm}}^m$, which is the ray carrying cuspidal cohomology, cannot be studied by our methods. For a fixed $\Gamma(\mathfrak{a})$, we can only show that the size of cohomological torsion with coefficients in the canonical lattice associated to $\rho_m(1, 1)$ grows at most as $m \text{rk} \rho_m(1, 1) = O(m^3)$, but we can say nothing about the existence of torsion along this ray, i.e. we cannot establish any bound from below. The reason is that here 0 belongs to the essential spectrum of the Laplacian in the middle dimension. Therefore, essentially none of the results on analytic torsion we use in our proof is currently available for $\rho_m(1, 1)$ and also the regulator would be more complicated. We remark that, as far as we know, even in the compact case no result for the growth of torsion along this particular ray has been obtained. For an investigation of the dimension of the (cuspidal) cohomology along this ray, we refer to [FGT10].

The results we use on limit multiplicity for analytic torsion and its relation with Reidemeister torsion are valid for hyperbolic manifolds in all odd dimensions, i.e. for lattices in the groups $\text{SO}(d, 1)$ for odd $d > 3$. However intertwining operators in higher dimensions are more difficult to deal with in groups other than SL_2 , which are of absolute rank > 1 . This implies that the L -functions one needs to deal with are no longer Hecke L -functions. We have no idea about how to estimate the absolute value and denominators of normalised L -values in this more general class of functions, and are in consequence unable to extend our result on cohomology growth to higher dimensions even in a weaker form similar to what was done for uniform lattices in [MP14b].

Using the first author's result relating analytic and Reidemeister torsion and his result with Müller on limit multiplicities for analytic torsion in sequences of manifolds [MP14a] we prove in Corollary 2.4 that in the case where $G = \mathrm{SO}(d, 1)$ for any odd $d \geq 1$, and Γ_m is a sequence of principal congruence subgroups in $G(\mathbb{Z})$, and $X_m = \Gamma_m \backslash \mathbb{H}^d$ we have:

$$\lim_{m \rightarrow +\infty} \left(\frac{\log \tau_{Eis}(X_m; E_\rho)}{\mathrm{vol}(X_m)} \right) = c \neq 0$$

For Bianchi groups a similar result is proven in [Rai13]. However even in this absolute rank 1 setting we are currently not able to deduce exponential growth for the torsion cohomology. This is because it is harder to deal with absolute norms of Hecke L -functions when allowing the characters to change at finite places rather than infinite ones as we do in this paper.

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2. THE REGULARIZED ANALYTIC TORSION FOR COVERINGS

In this section we shall review the definition of regularized traces and the regularized analytic torsion of hyperbolic manifolds X of finite volume. These objects are defined in terms of a fixed choice of truncation parameters on X and there are two different ways to perform such a truncation which are relevant in the present paper. Firstly, one can define a truncation of X via a fixed choice of Γ -cuspidal parabolic subgroups of G . Secondly, if X is a finite covering of a hyperbolic orbifold X_0 , then a choice of truncation parameters on X_0 gives a truncation on X in terms of which one can define another regularized analytic torsion. We shall compute the difference between the associate regularized analytic torsions explicitly. For more details we refer to [MP12], [MP14a].

2.1. Hyperbolic manifolds with cusps. We denote by $\mathrm{SO}^0(d, 1)$ the identity-component of the isometry group of the standard quadratic form of signature $(d, 1)$ on \mathbb{R}^{d+1} . Let $\mathrm{Spin}(d, 1)$ denote the universal covering of $\mathrm{SO}^0(d, 1)$. We let either $G := \mathrm{SO}^0(d, 1)$ or $G := \mathrm{Spin}(d, 1)$. We assume that d is odd and write $d = 2n + 1$. Let $K := \mathrm{SO}(d)$, if $G = \mathrm{SO}^0(d, 1)$ or $K := \mathrm{Spin}(d)$, if $G = \mathrm{Spin}(d, 1)$. We let \mathfrak{g} be the Lie algebra of G . Let θ denote the standard Cartan involution of \mathfrak{g} and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ denote the associated Cartan decomposition of \mathfrak{g} , where \mathfrak{k} is the Lie algebra of K . Let B be the Killing form. Then

$$(2.1) \quad \langle X, Y \rangle := -\frac{1}{2(d-1)} B(X, \theta Y)$$

is an inner product on \mathfrak{g} . Moreover, the globally symmetric space G/K , equipped with the G -invariant metric induced by the restriction of (2.1) to \mathfrak{p} is isometric to the d -dimensional real hyperbolic space \mathbb{H}^d . Let $\Gamma \subset G$ be a discrete, torsion-free subgroup. Then $X := \Gamma \backslash \mathbb{H}^d$, equipped with the push-down of the metric on \mathbb{H}^d , is a d -dimensional hyperbolic manifold.

We let P be a fixed parabolic subgroup of G with Langlands decomposition $P = M_P A_P N_P$ as in [MP12]. Let \mathfrak{a} denote the Lie algebra of A_P and $\exp : \mathfrak{a} \rightarrow A_P$ the exponential map. Then we fix a restricted root e_1 of \mathfrak{a} in \mathfrak{g} , let $H_1 \in \mathfrak{a}$ be such that $e_1(H_1) = 1$ and define $\iota_P : (0, \infty) \rightarrow A_P$ by $\iota_P(t) := \exp(\log t H_1)$. If P_1 is another parabolic subgroup of G , we fix $k_P \in K$ with $P_1 = k_{P_1} P k_{P_1}^{-1}$ and define $A_{P_1} := k_{P_1} A_P k_{P_1}^{-1}$, $M_{P_1} := k_{P_1} M_P k_{P_1}^{-1}$, $N_{P_1} := k_{P_1} N_P k_{P_1}^{-1}$. Moreover, for $t \in (0, \infty)$ we define $\iota_{P_1}(t) := k_{P_1} \iota_P(t) k_{P_1}^{-1} \in A_{P_1}$. For $Y > 0$ we let $A_{P_1}[Y] := \iota_{P_1}([Y, \infty))$.

A parabolic subgroup P_1 of G is called Γ -cuspidal if $\Gamma \cap N_{P_1}$ is a lattice in N_{P_1} . From now on, we assume that $\text{vol}(X)$ is finite and that Γ is neat in the sense of Borel, i.e. that $\Gamma \cap P_1 = \Gamma \cap N_{P_1}$ for each Γ -cuspidal P_1 . If P_1 is Γ -cuspidal, then for $Y > 0$ we put

$$F_{P_1, \Gamma}(Y) := (\Gamma \cap N_{P_1}) \backslash N_{P_1} \times A_{P_1}[Y] \cong (\Gamma \cap N_{P_1}) \backslash N_{P_1} \times [Y, \infty).$$

We equip $F_{P_1, \Gamma}(Y)$ with the metric $y^{-2} g_{N_{P_1}} + y^{-2} dy^2$ where $g_{N_{P_1}}$ is the push-down of the invariant metric on N_{P_1} induced by the innerer product (2.1) restricted to \mathfrak{n}_{P_1} .

2.2. Trace formula and analytic torsions. Let $\text{Rep}(G)$ denote the set of finite-dimensional irreducible representations of G . For $\rho \in \text{Rep}(G)$ the associated vector space V_ρ possesses a distinguished inner product $\langle \cdot, \cdot \rangle_\rho$ which is called admissible and which is unique up to scaling. We shall fix an admissible inner product on each V_ρ . If $\rho \in \text{Rep}(G)$, then the restriction of ρ to Γ induces a flat vector bundle $E_\rho := \tilde{X} \times_{\rho|_\Gamma} V_\rho$. This bundle is canonically isomorphic to the locally homogeneous bundle $E'_\rho := \Gamma \backslash G \times_{\rho|_K} V_\rho$ induced by the restriction of ρ to K . In particular, since $\rho|_K$ is a unitary representation on $(V_\rho, \langle \cdot, \cdot \rangle_\rho)$, the inner product $\langle \cdot, \cdot \rangle_\rho$ induces a smooth bundle metric on E'_ρ and therefore on E_ρ . For $p = 0, \dots, d$ let $\Delta_p(\rho)$ denote the flat Hodge Laplacian acting on the smooth E_ρ -valued p -forms of X . Since X is complete, $\Delta_p(\rho)$ with domain the smooth, compactly supported E_ρ -valued p -forms is essentially selfadjoint and its L^2 -closure shall be denoted by the same symbol. Let $e^{-t\Delta_p(\rho)}$ denote the heat semigroup of $\Delta_p(\rho)$ and let

$$K_X^{\rho, p}(t, x, y) \in C^\infty(X \times X; E_\rho \boxtimes E_\rho^*)$$

be the integral kernel of $e^{-t\Delta_p(\rho)}$.

We let \mathfrak{P}_Γ be a fixed set of representatives for the Γ -conjugacy classes of Γ -cuspidal parabolic subgroups of G . Then \mathfrak{P}_Γ is finite, and it is non-empty if and only if X is non-compact. Moreover, $\kappa(\Gamma) := |\mathfrak{P}_\Gamma|$ equals the number of cusps of X which from now on we assume to be nonzero. The choice of \mathfrak{P}_Γ and of a fixed base-point in \tilde{X} determine an exhaustion of X by smooth compact manifolds $X_{\mathfrak{P}_\Gamma}(Y)$ with boundary, $Y \gg 0$. This exhaustion depends on the choice of \mathfrak{P}_Γ . Then one can show that the integral of $K_X^{\rho, p}(t, x, x)$ over $X_{\mathfrak{P}_\Gamma}(Y)$ has an asymptotic expansion

$$(2.2) \quad \int_{X_{\mathfrak{P}_\Gamma}(Y)} \text{Tr} K_X^{\rho, p}(t, x, x) dx = \alpha_{-1}(t) \log Y + \alpha_0(t) + o(1),$$

as $Y \rightarrow \infty$, [MP12][section 5]. Now one can define the regularized trace $\text{Tr}_{\text{reg}; X; \mathfrak{P}_\Gamma} e^{-t\Delta_p(\rho)}$ of $e^{-t\Delta_p(\rho)}$ with respect to the choice of \mathfrak{P}_Γ by $\text{Tr}_{\text{reg}; X; \mathfrak{P}_\Gamma} e^{-t\Delta_p(\rho)} := \alpha_0(t)$, where $\alpha_0(t)$ is the constant term in the asymptotic expansion in (2.2).

From now on, we also assume that there is a hyperbolic orbifold $X_0 := \Gamma_0 \backslash \mathbb{H}^d$ such that X is a finite covering of X_0 . Let $\pi : X \rightarrow X_0$ denote the covering map. Then if a set of truncation parameters on X_0 , or in other words a set \mathfrak{P}_{Γ_0} of representatives of Γ_0 -cuspidal parabolic subgroups are fixed, one obtains truncation parameters on X by pulling back the truncation on X_0 via π . One can again show that there is an asymptotic expansion

$$(2.3) \quad \int_{\pi^{-1}X_{\mathfrak{P}_{\Gamma_0}}(Y)} \mathrm{Tr} K_X^{\rho,p}(t, x, x) dx = \tilde{\alpha}_{-1}(t) \log Y + \tilde{\alpha}_0(t) + o(1),$$

as $Y \rightarrow \infty$ and one can define the regularized trace with respect to the truncation parameters on X_0 as $\mathrm{Tr}_{\mathrm{reg}; X; X_0} e^{-t\Delta_p(\rho)} := \tilde{\alpha}_0(t)$. This regularized trace depends only on the choice of a set \mathfrak{P}_{Γ_0} of representatives of Γ_0 -cuspidal parabolic subgroups of G . Put

$$(2.4) \quad K_{X; \mathfrak{P}_{\Gamma}}(t, \rho) := \sum_p (-1)^p p \mathrm{Tr}_{\mathrm{reg}; \mathfrak{P}_{\Gamma}} e^{-t\Delta_p(\rho)}; \quad K_{X; X_0}(t, \rho) := \sum_p (-1)^p p \mathrm{Tr}_{\mathrm{reg}; X; X_0} e^{-t\Delta_p(\rho)}.$$

Now assume that ρ satisfies $\rho \neq \rho_\theta$. Then one defines the analytic torsion with respect to the two truncations of X by

$$(2.5) \quad \log T_{\mathfrak{P}_{\Gamma}}(\rho) := \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K_{X; \mathfrak{P}_{\Gamma}}(t, \rho) dt \right);$$

$$(2.6) \quad \log T_{X_0}(\rho) := \frac{1}{2} \frac{d}{ds} \Big|_{s=0} \left(\frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} K_{X; X_0}(t, \rho) dt \right).$$

Here the integrals converge absolutely and locally uniformly for $\Re(s) > d/2$ and are defined near $s = 0$ by analytic continuation, [MP12, section 7], [MP14a, section 9].

To compare the two analytic torsions in (2.5) and (2.6), we need to introduce some more notation. We fix $\mathfrak{P}_{\Gamma_0} = \{P_{0,1}, \dots, P_{0,\kappa(\Gamma_0)}\}$ and $\mathfrak{P}_{\Gamma} = \{P_1, \dots, P_{\kappa(\Gamma)}\}$. Then for each $P_j \in \mathfrak{P}_{\Gamma}$ there exists a unique $l(j) \in \{1, \dots, \kappa(\Gamma_0)\}$ and a $\gamma_j \in \Gamma_0$ such that $\gamma_j P_j \gamma_j^{-1} = P_{0,l(j)}$. Write

$$(2.7) \quad \gamma_j = n_{0,l(j)} \iota_{P_{0,l(j)}}(t_{P_j}) k_{0,l(j)},$$

$n_{0,l(j)} \in N_{P_{0,l(j)}}$, $t_{P_j} \in (0, \infty)$, $\iota_{P_{0,l(j)}}(t_{P_j}) \in A_{P_{0,l(j)}}$ as above, and $k_{0,l(j)} \in K$. Since $P_{0,l(j)}$ equals its normalizer, the projection of γ_j to $(\Gamma_0 \cap P_{0,l(j)}) \backslash \Gamma_0$ is unique. Moreover, since $P_{0,l(j)}$ is Γ_0 -cuspidal, one has $\Gamma_0 \cap P_{0,l(j)} = \Gamma_0 \cap N_{P_{0,l(j)}} M_{P_{0,l(j)}}$. Thus t_{P_j} depends only on \mathfrak{P}_{Γ_0} and P_j .

Now the analytic torsions $\log T_{\mathfrak{P}_{\Gamma}}(X; E_\rho)$ and $\log T_{X_0}(X; E_\rho)$ are compared in the following proposition. The representation $\sigma_{\rho,k}$ of M_P and the positive number $\lambda_{\rho,k}$ are defined respectively in [Pfa17, (6.5) and (6.6)].

Proposition 2.1. *One has*

$$\log T_{\mathfrak{P}_{\Gamma}}(X; E_\rho) = \log T_{X_0}(X; E_\rho) + \sum_{P_j \in \mathfrak{P}_{\Gamma}} \log(t_{P_j}) \left(\sum_{k=0}^n \frac{(-1)^k \dim(\sigma_{\rho,k}) \lambda_{\rho,k}}{2} \right).$$

Proof. This follows by an application of a theorem of Kostant [Kos61] on nilpotent Lie algebra cohomology. For $p \in \{0, \dots, d\}$ define a representation of K on $\Lambda^p \mathfrak{p}^* \otimes V(\rho)$ by $\nu_p(\rho) := \Lambda^p \text{Ad}^* \otimes \rho$. Let $\tilde{E}_{\nu_p(\rho)} := G \times_{\nu_p(\rho)} K$, which is a homogeneous vector bundle over $\mathbb{H}^d = G/K$. Let Ω be the Casimir element of \mathfrak{g} . Then $-\Omega + \rho(\Omega)$ induces canonically a Laplace-type operator $\tilde{\Delta}_p(\rho)$ which acts on the smooth sections of $\tilde{E}_{\nu_p(\rho)}$. The heat semigroup of $e^{-t\tilde{\Delta}_p(\rho)}$ is canonically represented by a smooth function $H_t^{p,\rho} : G \rightarrow \text{End}(\Lambda^p \mathfrak{p}^* \otimes V_\rho)$, [MP12, section 4, section 7]. Let $h_t^{p,\rho} := \text{Tr } H_t^{p,\rho}$ and put

$$k_t^{p,\rho} := \sum_p (-1)^p p h_t^{p,\rho}.$$

Then by the definition of the regularized traces, one has

$$(2.8) \quad \int_{X_{\mathfrak{P}_\Gamma(Y)}} \left(\sum_{\gamma \in \Gamma} k_t^{p,\rho}(x^{-1}\gamma x) \right) dx = \alpha_{-1}(t; \rho) \log Y + K_{X; \mathfrak{P}_\Gamma}(t, \rho) + o(1),$$

as $Y \rightarrow \infty$, resp.

$$(2.9) \quad \int_{\pi^{-1}(X_{\mathfrak{P}_{\Gamma_0}(Y)})} \left(\sum_{\gamma \in \Gamma} k_t^{p,\rho}(x^{-1}\gamma x) \right) dx = \tilde{\alpha}_{-1}(t; \rho) \log Y + K_{X; X_0}(t, \rho) + o(1),$$

as $Y \rightarrow \infty$, where we use the notation (2.4). On the other hand, for $k = 0, \dots, n$ let $h_t^{\sigma_{\rho,k}} \in C^\infty(G)$ be defined as in [MP12, (8.7)]. If we apply the same considerations as in [MP14a, section 6] to the functions $h_t^{\sigma_{\rho,k}}$, then combining [MP12, Proposition 8.2], (2.8) and (2.9) we obtain

$$K_{X; X_0}(t, \rho) = K_{X; \mathfrak{P}_\Gamma}(t, \rho) - \sum_{P_j \in \mathfrak{P}_\Gamma} \log t_{P_j} \left(\sum_{k=0}^n \frac{(-1)^k \dim(\sigma_{\rho,k}) e^{-t\lambda_{\rho,k}^2}}{\sqrt{4\pi t}} \right).$$

The Proposition follows by aking the Mellin transform of this expression. \square

Next, as in [Pfa17], for each $P_j \in \mathfrak{P}_\Gamma$ and for $Y > 0$ one can define the regularized analytic torsion $T(F_{P_j; \Gamma}(Y), \partial F_{P_j; \Gamma}(Y); E_\rho)$ of $F_{P_j; \Gamma}(Y)$ and the bundle $E_\rho|_{F_{P_j; \Gamma}(Y)}$, where one takes relative boundary conditions (that is, we restrict the Laplacian to smooth forms whose normal component at the boundary vanishes). For different Y , these torsions are compared by the following gluing formula.

Lemma 2.2. *Let $c(n) \in \mathbb{R}$ be as in [Pfa17, equation 15.10]. Then for $Y_1 > 0$ and $Y_2 > 0$ one has*

$$\begin{aligned} & \log T(F_{P_j; \Gamma}(Y_1), \partial F_{P_j; \Gamma}(Y_1); E_\rho) - c(n) \text{vol}(\partial F_{P_j; \Gamma}(Y_1)) \text{rk}(E_\rho) \\ &= \log T(F_{P_j; \Gamma}(Y_2), \partial F_{P_j; \Gamma}(Y_2); E_\rho) - c(n) \text{vol}(\partial F_{P_j; \Gamma}(Y_2)) \text{rk}(E_\rho) \\ & \quad + \sum_{k=0}^n (-1)^{k+1} \lambda_{\rho,k} \dim(\sigma_{\rho,k}) (\log(Y_2) - \log(Y_1)). \end{aligned}$$

Proof. This follows immediately from [Pfa17, Corollary 15.4, equation (15.11) and Corollary 16.2]. \square

2.3. Comparison of analytic and Reidemeister torsions. We let \overline{X} denote the Borel-Serre compactification of X and we let $\tau_{Eis}(X; E_\rho)$ be the Reidemeister torsion of \overline{X} with coefficients in E_ρ , defined as in [Pfa17, section 9]. For simplicity, we assume that Γ is normal in Γ_0 . Then for the torsion $T_{X_0}(X; E_\rho)$, the main result of [Pfa17] can be restated as follows.

Proposition 2.3. *For $l = 1, \dots, \kappa(\Gamma_0)$ let $a_l = |\{P_j \in \mathfrak{P}_\Gamma : \gamma_j P_j \gamma_j^{-1} = P_{0,l}\}|$ be the number of cusps of X lying over the l th cusp of X_0 . For the analytic torsion $T_{X_0}(X; E_\rho)$ we have:*

$$\begin{aligned} \log \tau_{Eis}(X; E_\rho) &= \log T_{X_0}(X; E_\rho) - \frac{1}{4} \sum_{k=0}^{d-1} (-1)^k \log |\lambda_{\rho,k}| \dim H^k(\partial \overline{X}; E_\rho) \\ &\quad - \sum_{l=1}^{\kappa(\Gamma_0)} a_l (\log T(F_{P_{0,l};\Gamma}(1), \partial F_{P_{0,l};\Gamma}(1); E_\rho) - c(n) \text{vol}(\partial F_{P_{0,l};\Gamma}(1)) \text{rk}(E_\rho)). \end{aligned}$$

Proof. By [Pfa17, Theorem 1.1] and by Proposition 2.1 we have

$$\begin{aligned} \log \tau_{Eis}(X; E_\rho) &= \log T_{X_0}(X; E_\rho) + \sum_{P_j \in \mathfrak{P}_\Gamma} \log(t_{P_j}) \left(\sum_{k=0}^n \frac{(-1)^k \dim(\sigma_{\rho,k}) \lambda_{\rho,k}}{2} \right) \\ &\quad - \sum_{P_j \in \mathfrak{P}_\Gamma} (\log T(F_{P_j;\Gamma}(1), \partial F_{P_j;\Gamma}(1); E_\rho) - c(n) \text{vol}(\partial F_{P_j;\Gamma}(1)) \text{rk}(E_\rho)) \\ &\quad - \frac{1}{4} \sum_{k=0}^{d-1} (-1)^k \log |\lambda_{\rho,k}| \dim H^k(\partial \overline{X}; E_\rho) \end{aligned}$$

where we recall that the regularized analytic torsion used in [Pfa17, Theorem 1.1] is the torsion denoted $T_{\mathfrak{P}_\Gamma}(X; E_\rho)$ here. Using that Γ is normal in Γ_0 , it easily follows from the definition of t_{P_j} that for each $P_j \in \mathfrak{P}_\Gamma$ one has a canonical isometry $\iota_{P_j; P_{0,l(j)}} : F_{P_j;\Gamma}(1) \cong F_{P_{0,l(j);\Gamma}(t_{P_j})}$. It is easy to see that also $\iota_{P_j; P_{0,l(j)}}^* \left(E_\rho|_{F_{P_{0,l(j);\Gamma}(t_{P_j})}} \right)$ is isometric to $E_\rho|_{F_{P_j;\Gamma}(1)}$. Thus we have

$$\begin{aligned} (2.10) \quad \log T(F_{P_j;\Gamma}(1), \partial F_{P_j;\Gamma}(1); E_\rho) - c(n) \text{vol}(\partial F_{P_j;\Gamma}(1)) \text{rk}(E_\rho) \\ = \log T(F_{P_{0,l(j);\Gamma}(t_{P_j})}, \partial F_{P_{0,l(j);\Gamma}(t_{P_j})}; E_\rho) - c(n) \text{vol}(\partial F_{P_{0,l(j);\Gamma}(t_{P_j})}) \text{rk}(E_\rho). \end{aligned}$$

Applying Lemma 2.2, the Proposition follows. \square

2.4. Limit multiplicity property for Reidemeister torsions. Although the main topic of this paper is the behaviour of cohomological torsion of congruence subgroups of Bianchi groups under a variation of the local system, we now state the following limit multiplicity formula for Reidemeister torsion in arithmetic hyperbolic congruence towers of arbitrary odd dimension, since the latter is an easy corollary. It is easy to generalise this result to more general arithmetic lattices and sequences of congruence subgroup but for convenience we restrict to the simplest case.

Corollary 2.4. *Let $G := \mathrm{SO}^0(d, 1)$, d odd, and for $q \in \mathbb{N}$ let $\Gamma(q) := \ker(G(\mathbb{Z}) \rightarrow G(\mathbb{Z}/q\mathbb{Z}))$ denote the principal congruence subgroup of level q . Let $X_q := \Gamma(q) \backslash \mathbb{H}^d$. Then for any $\rho \in \mathrm{Rep}(G)$ with $\rho \neq \rho_\theta$ one has*

$$\lim_{q \rightarrow \infty} \frac{\log \tau_{Eis}(X_q; E_\rho)}{\mathrm{vol}(X_q)} = t_{\mathbb{H}^d}^{(2)}(\rho),$$

where $t_{\mathbb{H}^d}^{(2)}(\rho)$ is the L^2 -invariant associated to ρ and \mathbb{H}^d which is defined as in [BV13], [MP14a] and which is never zero. The same holds for every sequence $X_{\mathbf{a}}$ of arithmetic hyperbolic 3-manifolds associated to principal congruence subgroups $\Gamma(\mathbf{a})$ of Bianchi groups if $N(\mathbf{a}) \rightarrow \infty$.

Proof. Let us give a rough idea of the proof before getting more formal. For analytic torsion the corresponding result is known, so we need to prove that in the sequence $\Gamma(q)$ the terms on the right-hand side in the equality of Proposition 2.3 are all $o([\Gamma : \Gamma(q)])$. This is easy since the number of cusps is an $o([\Gamma : \Gamma(q)])$ and the geometry of the cusps stays within a finite number of conformal classes of flat manifolds, with only mild rescaling in addition.

First we assume that $G = \mathrm{SO}^0(d, 1)$. Let $\Gamma_0 := G(\mathbb{Z})$ and $X_0 := \Gamma_0 \backslash \mathbb{H}^d$. By [MP14a, Corollary 1.3], for the analytic torsion $T_{X_0}(X_q; E_\rho)$ one has

$$(2.11) \quad \lim_{q \rightarrow \infty} \frac{\log T_{X_0}(X_q; E_\rho)}{\mathrm{vol}(X_q)} = t_{\mathbb{H}^d}^{(2)}(\rho),$$

as $q \rightarrow \infty$. Next it is well-known that for the number $\kappa(\Gamma(q))$ of cusps of $\Gamma(q)$ one has

$$(2.12) \quad \lim_{q \rightarrow \infty} \frac{\kappa(\Gamma(q))}{\mathrm{vol}(X_q)} = 0,$$

see for example [MP14a, Proposition 8.6]. For $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ we let $\Lambda_{\Gamma(q)}(P_{0,l}) := \log((\Gamma(q) \cap N_{P_{0,l}}))$, which is a lattice in $\mathfrak{n}_{P_{0,l}}$. By a result of Deitmar and Hoffmann [DH99, Lemma 4], for each $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$, there exists a finite set of lattices $\mathcal{L}_{P_{0,l}} = \{\Lambda_1(P_{0,l}), \dots, \Lambda_m(P_{0,l})\}$ in $\mathfrak{n}_{P_{0,l}}$ such that for each $q \in \mathbb{N}$ the lattice $\Lambda_{\Gamma(q)}(P_{0,l})$ arises by scaling one of the lattices $\Lambda_j(P_{0,l})$, $j = 1, \dots, m$, see [MP14a, Lemma 10.1]. For $\Lambda_j(P_{0,l}) \in \mathcal{L}_{P_{0,l}}$ we let $T_{\Lambda_j(P_{0,l})} := \Lambda_j(P_{0,l}) \backslash \mathfrak{n}_{P_{0,l}}$, equipped with the flat metric (2.1) restricted to $\mathfrak{n}_{P_{0,l}}$ which we shall denote by $g_{\Lambda_j(P_{0,l})}$. Then we let $F_{\Lambda_j(P_{0,l})}(1) := [1, \infty) \times \Lambda_j(P_{0,l})$, equipped with the metric $y^{-2}(dy^2 + g_{\Lambda_j(P_{0,l})})$, $y \in [1, \infty)$. If $\Lambda_{\Gamma(q)}(P_{0,l}) = \mu_{q;P_{0,l}} \Lambda_j(P_{0,l})$ with $\Lambda_j(P_{0,l}) \in \mathcal{L}_{P_{0,l}}$ and $\mu_{q;P_{0,l}} \in (0, \infty)$, then by Lemma 2.2 one has

$$\begin{aligned} & \log T \left(F_{P_{0,l};\Gamma(q)}(1), \partial F_{P_{0,l};\Gamma(q)}(1); E_\rho \right) - c(n) \mathrm{vol}(\partial F_{P_{0,l};\Gamma(q)}(1)) \mathrm{rk}(E_\rho) \\ &= \log T \left(F_{\Lambda_j(P_{0,l})}(1), \partial F_{\Lambda_j(P_{0,l})}(1); E_\rho \right) - c(n) \mathrm{vol}(\Lambda_j(P_{0,l})) \mathrm{rk}(E_\rho) \\ & \quad + \log \mu_{q;P_{0,l}} \sum_{k=0}^n (-1)^k \lambda_{\rho,k} \dim(\sigma_{\rho,k}). \end{aligned}$$

We can obviously estimate $\mu_{q;P_{0,l}} \leq C_1 [\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]$, where C_1 is a constant which is independent of q . Thus there exists a constant C_2 such that for all q one can

estimate

$$(2.13) \quad \left| \log T \left(F_{P_{0,l};\Gamma(q)}(1), \partial F_{P_{0,l};\Gamma(q)}(1); E_\rho \right) - c(n) \operatorname{vol}(\partial F_{P_{0,l};\Gamma(q)}(1)) \operatorname{rk}(E_\rho) \right| \leq C_2 \log[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}].$$

Recall that $a_l = |\{P_j \in \mathfrak{P}_{\Gamma(q)}, \gamma_j P_j \gamma_j^{-1} = P_{0,l}\}|$. For each $l \in \{1, \dots, \kappa(\Gamma_0)\}$ one has

$$\frac{a_l}{[\Gamma_0 : \Gamma(q)]} = \frac{|\Gamma(q) \backslash \Gamma_0 / (\Gamma_0 \cap P_{0,l})|}{[\Gamma_0 : \Gamma(q)]} \leq \frac{1}{[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]}.$$

Thus for all q one can estimate

$$(2.14) \quad \sum_{l=1}^{\kappa(\Gamma_0)} \frac{a_l}{[\Gamma_0 : \Gamma(q)]} \left| \log T \left(F_{P_{0,l};\Gamma(q)}(1), \partial F_{P_{0,l};\Gamma(q)}(1); E_\rho \right) - c(n) \operatorname{vol}(\partial F_{P_{0,l};\Gamma(q)}(1)) \operatorname{rk}(E_\rho) \right| \leq C_2 \sum_{l=1}^{\kappa(\Gamma_0)} \frac{\log[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]}{[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]}.$$

Since $\cap_q \Gamma(q) = \{1\}$, $[\Gamma_0 \cap N_{P_{0,l}} : \Gamma(q) \cap N_{P_{0,l}}]$ as $q \rightarrow \infty$ and thus the last term in (2.14) goes to zero as q tends to infinity. Since $\dim H^k(\partial \overline{X}_q) = O(\kappa(\Gamma_q))$, the corollary follows by applying Proposition 2.3, equation (2.11) and equation (2.12). For a sequence X_a associated to principal congruence subgroups of Bianchi groups, one argues in the same way. \square

3. ANALYTIC AND COMBINATORIAL TORSION FOR CONGRUENCE SUBGROUPS OF BIANCHI GROUPS

From now on, we let $G = \operatorname{Spin}(3, 1) = \operatorname{SL}_2(\mathbb{C})$, $K = \operatorname{Spin}(3) = \operatorname{SU}(2)$. We take P to be the standard parabolic subgroup of G consisting of all upper triangular matrices in G . Then the unipotent radical N_P of P is given by all upper triangular matrices whose diagonal entries are one. Moreover, we let M_P and A_P denote the subgroups of $\operatorname{SL}_2(\mathbb{C})$ defined by

$$(3.1) \quad M_P := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : \theta \in [0, 2\pi] \right\}; \quad A_P := \left\{ \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} : t \in (0, \infty) \right\}.$$

Then $P = M_P A_P N_P$. Let \mathfrak{n}_P denote the Lie algebra of N_P and let \mathfrak{a}_P denote the Lie algebra of A_P . For $k \in \mathbb{Z}$, $\lambda \in \mathbb{C}$ we let $\sigma_k : M_P \rightarrow \mathbb{C}$, $\xi_\lambda : A_P \rightarrow \mathbb{C}$ be defined by

$$\sigma_k \left(\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right) := e^{ik\theta}; \quad \xi_\lambda \left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \right) := t^{2\lambda}.$$

Then the assignment $\lambda \rightarrow \xi_\lambda$ is consistent with our earlier identification $\mathbb{C} \cong (\mathfrak{a}_P^*)_{\mathbb{C}}$. The group K_∞ acts on $\mathfrak{g}/\mathfrak{k}$ by Ad and we let K_∞ act on $\mathfrak{a}_P \oplus \mathfrak{n}_P$ by using the canonical identification $\mathfrak{g}/\mathfrak{k} \cong \mathfrak{a}_P \oplus \mathfrak{n}_P$.

Let $F := \mathbb{Q}(\sqrt{-D})$ be an imaginary quadratic number field and let d_F be its class number. Let \mathcal{O}_D denote the ring of integers of F . Let \mathcal{O}_D^* be the group of units of \mathcal{O}_D , i.e. $\mathcal{O}_D^* = \{\pm 1\}$ for $D \neq 1, 3$, $\mathcal{O}_D^* = \{\pm 1, \pm \sqrt{-1}\}$ for $D = 1$, $\mathcal{O}_D^* = \{\pm 1, \pm \frac{1 \pm \sqrt{-3}}{2}\}$ for $D = 3$.

Let $\Gamma_D := \mathrm{SL}_2(\mathcal{O}_D)$. The quotient $X_0 := \Gamma_D \backslash \mathbb{H}^3$ is a hyperbolic orbifold of finite volume, see for example [EGM98]. We have

$$(3.2) \quad \Gamma_D \cap N_P = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathcal{O}_D \right\}$$

We fix a set $\mathfrak{P}_{\Gamma_D} = \{P_{0,1}, \dots, P_{0,\kappa(\Gamma_D)}\}$ of representatives of Γ_D -cuspidal parabolic subgroups of G , where we require each parabolic to be F -rational and where we assume that $P_{0,1} = P$. Let $\mathbb{P}^1(F)$ be the one-dimensional projective space of F . As usual, we write ∞ for the element $[1, 0] \in \mathbb{P}^1(F)$. Then $\mathrm{SL}_2(F)$ acts transitively on $\mathbb{P}^1(F)$ and P is the stabilizer of ∞ . In particular for each subgroup Γ of Γ_D one has $\kappa(\Gamma) = |(\Gamma \backslash G/P)| = |(\Gamma \backslash \mathbb{P}^1(F))|$. Moreover, by [EGM98, Chapter 7.2, Theorem 2.4], one has $\kappa(\Gamma_D) = d_F$. Let $\{\eta_l : l = 1, \dots, \eta_{d_F}\}$ denote fixed representatives of $\Gamma_D \backslash \mathbb{P}^1(F)$ such that $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ is the stabilizer of η_l in $\mathrm{SL}_2(\mathbb{C})$ for each l .

For \mathfrak{a} a non-zero ideal of \mathcal{O}_D we let $\Gamma(\mathfrak{a})$ denote the principal congruence subgroup of Γ_D of level \mathfrak{a} . This group is normal in Γ_D ; moreover, for $N(\mathfrak{a})$ sufficiently large ($N(\mathfrak{a}) \geq 3$ in the case $\mathcal{O}_D^* = \pm 1$), the group $\Gamma(\mathfrak{a})$ is neat. We shall assume from now on that this is the case. Then $X_{\mathfrak{a}} := \Gamma(\mathfrak{a}) \backslash \mathbb{H}^3$ is a hyperbolic 3-manifold of finite volume. According to the previous section, for $\rho \in \mathrm{Rep}(G)$ with $\rho \neq \rho_{\theta}$, we can define the analytic torsion $\log T_{X_0}(X_{\mathfrak{a}}; E_{\rho})$ of $X_{\mathfrak{a}}$ with coefficients in E_{ρ} with respect to the choice of truncation parameters coming from the covering $\pi : X_{\mathfrak{a}} \rightarrow X_0$ and the choice of \mathfrak{P}_{Γ_D} .

We shall now simplify the formula in Proposition 2.1 a bit further for the specific manifolds $X_{\mathfrak{a}}$. Let \mathfrak{b} be a non-zero ideal of \mathcal{O}_D . Taking the identification (3.2), we shall regard \mathfrak{b} as a lattice in \mathfrak{n}_P . We denote this lattice by $\Lambda_P(\mathfrak{b})$ and we let $T_{\Lambda_P(\mathfrak{b})} := \exp(\Lambda_P(\mathfrak{b})) \backslash N_P$ denote the corresponding torus. As above, for $r > 0$ we let $F_{\Lambda_P(\mathfrak{b})}(r) := [r, \infty) \times T_{\Lambda_P(\mathfrak{b})}$ denote the corresponding cusp. We fix ideals \mathfrak{s}_l , $l = 1, \dots, d_F$ in \mathcal{O}_D which represent the class group of F . Then we have:

Proposition 3.1. *There exist $n_{1,\Gamma(\mathfrak{a})}, \dots, n_{d_F,\Gamma(\mathfrak{a})} \in \mathbb{N} \cup \{0\}$, with $n_{1,\Gamma(\mathfrak{a})} + \dots + n_{d_F,\Gamma(\mathfrak{a})} = d_F$, and there exist $\mu_{1,\Gamma(\mathfrak{a})}, \dots, \mu_{d_F,\Gamma(\mathfrak{a})} \in (0, \infty)$ such that*

$$\begin{aligned} \log \tau_{Eis}(X_{\mathfrak{a}}; E_{\rho}) &= \log T_{X_0}(X_{\mathfrak{a}}; E_{\rho}) - \sum_{l=1}^{d_F} \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*| N(\mathfrak{a})} \left(n_{l,\Gamma(\mathfrak{a})} \log T(F_{\Lambda_P(\mathfrak{s}_l)}(1), \partial F_{\Lambda_P(\mathfrak{s}_l)}(1); E_{\rho}) \right. \\ &\quad \left. - n_{l,\Gamma(\mathfrak{a})} c(1) \mathrm{vol}(\Lambda_P(\mathfrak{s}_l)) \mathrm{rk}(E_{\rho}) + (\lambda_{\rho,1} - \lambda_{\rho,0}) \mu_{l,\Gamma(\mathfrak{a})} \log \mu_{l,\Gamma(\mathfrak{a})} \right) \\ &\quad - \frac{\kappa(\Gamma(\mathfrak{a}))}{2} (\lambda_{\rho,0} - \lambda_{\rho,1}). \end{aligned}$$

Here the $n_{1,\Gamma(\mathfrak{a})}, \dots, n_{d_F,\Gamma(\mathfrak{a})} \in \mathbb{N} \cup \{0\}$ and the $\mu_{1,\Gamma(\mathfrak{a})}, \dots, \mu_{d_F,\Gamma(\mathfrak{a})} \in (0, \infty)$ depend on $\Gamma(\mathfrak{a})$, but not on the representation ρ .

Proof. Let $\{\eta_l : l = 1, \dots, \eta_{d_F}\}$ denote fixed representatives of $\Gamma_D \backslash \mathbb{P}^1(F)$ such that $P_{0,l} \in \mathfrak{P}_{\Gamma_0}$ is the stabilizer of η_l in $\mathrm{SL}_2(\mathbb{C})$ for each l . For each $l = 1, \dots, d_F$ we fix $B_{\eta_l} \in \mathrm{SL}_2(F)$ with $B_{\eta_l} \eta_l = \infty$. Then $P_{0,l} =: P_{\eta_l} = B_{\eta_l}^{-1} P B_{\eta_l}$. Let $(\Gamma_D)_{\eta_l} = \Gamma_D \cap P_{\eta_l}$ resp. $\Gamma(\mathfrak{a})_{\eta_l} =$

$\Gamma(\mathbf{a}) \cap P_{\eta_l}$ be the stabilizers of η_l in Γ_D resp. $\Gamma(\mathbf{a})$. One has

$$(3.3) \quad B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} = \left\{ J(B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} \cap N), J \in \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix}, \alpha \in \mathcal{O}_D^* \right\} \right\}.$$

In particular, one has $B_{\eta_l} \Gamma(\mathbf{a})_{\eta_l} B_{\eta_l}^{-1} \cap P = B_{\eta_l} \Gamma(\mathbf{a})_{\eta_l} B_{\eta_l}^{-1} \cap N$ for $N(\mathbf{a})$ sufficiently large.

Write $B_{\eta_l} = \begin{pmatrix} \alpha_l & \beta_l \\ \gamma_l & \delta_l \end{pmatrix} \in \mathrm{SL}_2(F)$ and let \mathbf{u}_l be the \mathcal{O}_D -module generated by γ_l and δ_l .

Then one has:

$$(B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1}) \cap N = \left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}, \omega \in \mathbf{u}_l^{-2} \right\}$$

and

$$(B_{\eta_l} \Gamma(\mathbf{a})_{\eta_l} B_{\eta_l}^{-1}) \cap N = \left\{ \begin{pmatrix} 1 & \omega' \\ 0 & 1 \end{pmatrix}; \omega' \in \mathbf{a} \mathbf{u}_l^{-2} \right\}$$

(the first equality is proved in [EGM98, Chapter 8.2, Lemma 2.2] and the second can be proved using the same arguments). Thus one has

$$[B_{\eta_l}(\Gamma_D)_{\eta_l} B_{\eta_l}^{-1} \cap N : B_{\eta_l} \Gamma(\mathbf{a})_{\eta_l} B_{\eta_l}^{-1} \cap N] = N(\mathbf{a}).$$

Thus by (3.3), for each $l = 1, \dots, d_F$ and $N(\mathbf{a})$ sufficiently large one has $[(\Gamma_D)_{\eta_l} : \Gamma(\mathbf{a})_{\eta_l}] = |\mathcal{O}_D^*| N(\mathbf{a})$ and so one gets

$$|\{P_j \in \mathfrak{P}_{\Gamma(\mathbf{a})}, \gamma_j P_j \gamma_j^{-1} = P_{0,l}\}| = \frac{[\Gamma_D : \Gamma(\mathbf{a})]}{|\mathcal{O}_D^*| N(\mathbf{a})}.$$

For each l there is a constant $\kappa_l > 0$ which depends only on B_{η_l} such that one has a canonical isometry $\iota : F_{P_{0,l}; \Gamma(\mathbf{a})}(1) \cong F_{\Lambda_P(\kappa_l \mathbf{a} \mathbf{u}_l^{-2})}(1)$ which induces an isometry $\iota^* : E_\rho|_{F_{\Lambda_P(\kappa_l \mathbf{a} \mathbf{u}_l^{-2})}(1)} \cong E_\rho|_{F_{P_{0,l}; \Gamma(\mathbf{a})}(1)}$. Next, there exists a unique map σ from $\{1, \dots, d_F\}$ into itself such that for each $l = 1, \dots, d_F$ there exists a $\tilde{\mu}_{l, \Gamma(\mathbf{a})} \in F^*$ with $\mathbf{a} \mathbf{u}_l^{-2} = \tilde{\mu}_{l, \Gamma(\mathbf{a})} \mathfrak{s}_{\sigma(l)}$. We let $\mu_{l, \Gamma(\mathbf{a})} := \kappa_l |\tilde{\mu}_{l, \Gamma(\mathbf{a})}|$. Then it follows that there is a canonical isometry $\iota : F_{P_{0,l}; \Gamma(\mathbf{a})}(1) \cong F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(\mu_{l, \Gamma(\mathbf{a})}^{-1})$ which extends to an isometry $\iota^* : E_\rho|_{F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(\mu_{l, \Gamma(\mathbf{a})}^{-1})} \cong E_\rho|_{F_{P_{0,l}; \Gamma(\mathbf{a})}(1)}$. Thus it follows from Lemma 2.2 that

$$\begin{aligned} & \log T(F_{P_{0,l}; \Gamma(\mathbf{a})}(1), \partial F_{P_{0,l}; \Gamma(\mathbf{a})}(1); E_\rho) - c(1) \mathrm{vol}(\partial F_{P_{0,l}; \Gamma(\mathbf{a})}(1)) \mathrm{rk}(E_\rho) \\ &= \log T\left(F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(1), \partial F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(1); E_\rho\right) - c(1) \mathrm{vol}\left(\partial F_{\Lambda_P(\mathfrak{s}_{\sigma(l)})}(1)\right) + \log \mu_{l, \Gamma(\mathbf{a})} (\lambda_{\rho,1} - \lambda_{\rho,0}). \end{aligned}$$

For each $l = 1, \dots, d_F$ we let $n_{l, \Gamma(\mathbf{a})} := |\{\sigma^{-1}(l)\}|$. Then, since $\dim(\sigma_{\rho,k}) = 1$ in the case of $G = \mathrm{SL}_2(\mathbb{C})$, the Proposition follows from Proposition 2.3. \square

We let ρ_1 denote the standard representation of $\mathrm{SL}_2(\mathbb{C})$ on \mathbb{C}^2 and for $m \in \mathbb{N}$ we consider the representation $\rho_m := \mathrm{Symm}^m \rho_1$ on $V(m) := \mathrm{Symm}^m \mathbb{C}^2$. We can now deduce the following result on the growth of the torsion $\tau_{Eis}(X_{\mathbf{a}}; E(\rho_m))$ if $m \rightarrow \infty$.

Proposition 3.2. *Let \mathfrak{a} and \mathfrak{a}_0 be two ideals in \mathcal{O}_D such that $\Gamma(\mathfrak{a})$ and $\Gamma(\mathfrak{a}_0)$ are neat and such that, in the notation of the previous proposition, one has $n_{l,\Gamma(\mathfrak{a})} = n_{l,\Gamma(\mathfrak{a}_0)}$ for each $l = 1, \dots, d_F$. Then one has*

$$\begin{aligned} & \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*|N(\mathfrak{a}_0)} \log \tau_{Eis}(X_{\mathfrak{a}}; E(\rho_m)) - \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*|N(\mathfrak{a})} \log \tau_{Eis}(X_{\mathfrak{a}_0}; E(\rho_m)) \\ &= -\frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)][\Gamma_D : \Gamma(\mathfrak{a})]}{2|\mathcal{O}_D^*|\pi} \left(\frac{1}{N(\mathfrak{a}_0)} - \frac{1}{N(\mathfrak{a})} \right) \text{vol}(\Gamma_D \backslash \mathbb{H}^3) m^2 + O(m \log m), \end{aligned}$$

as $m \rightarrow \infty$.

Proof. In the notation of [MP12], the representation ρ_m is the representation of highest weight $(m/2, m/2)$. If Γ is a neat finite index subgroup of Γ_D and if we let $X := \Gamma \backslash \mathbb{H}^3$, then specializing [MP12, Theorem 1.1] to the present situation, we obtain

$$\log T_{X_0}(X; E_{\rho_m}) = -\frac{1}{2\pi} \text{vol}(X) m^2 + O(m \log m),$$

as $m \rightarrow \infty$. Moreover, one has $\lambda_{\rho_m,0} = (m+1)/2$ and $\lambda_{\rho_m,1} = m/2$. Thus the proposition follows from the previous Proposition 3.1. \square

There is a constant $C'_1(\Gamma)$ such that $\Gamma(\mathfrak{a})$ is neat for all ideals \mathfrak{a} of \mathcal{O}_D with $N(\mathfrak{a}) \geq C'_1(\Gamma)$. If $\mathcal{O}_D^* = \{\pm 1\}$, one has $C'_1(\Gamma) = 3$. Next we remark that by the requirement $n_{1,\Gamma(\mathfrak{a})} + \dots + n_{d_F}(\mathfrak{a}) = d_F$ there is a finite set \mathcal{A} of ideals of \mathcal{O}_D such that $N(\mathfrak{a}_0) \geq C'_1(\Gamma)$ for each $\mathfrak{a}_0 \in \mathcal{A}$ and such that for each non-zero ideal \mathfrak{a} of \mathcal{O}_D with $N(\mathfrak{a}) \geq C'_1(\Gamma)$ there is an $\mathfrak{a}_0 \in \mathcal{A}$ such that $n_{l,\Gamma(\mathfrak{a})} = n_{l,\Gamma(\mathfrak{a}_0)}$ for each $l = 1, \dots, d_F$. We let $\tilde{C}_1(\Gamma) := \max\{N(\mathfrak{a}_0) : \mathfrak{a}_0 \in \mathcal{A}\}$ and we let $C_1(\Gamma) := \max\{\tilde{C}_1(\Gamma), C'_1(\Gamma)\}$.

4. TORSION IN COHOMOLOGY AND REIDEMEISTER TORSION

4.1. Local systems and cohomology. We keep the notation of the preceding section. Let $\Lambda(m) := \text{Symm}^m \mathcal{O}_D$. Then $\Lambda(m)$ is a lattice in $V(m)$ which is preserved by $\Gamma(D)$; we denote the representation of $\Gamma(D)$ on $\text{Aut}_{\mathbb{Z}}(\Lambda(m))$ by $\rho_{m;\mathbb{Z}}$. Let $\Lambda^\vee(m) := \text{Hom}_{\mathbb{Z}}(\Lambda(m), \mathbb{Z})$ denote the dual lattice of $\Lambda(m)$ and let $\check{\rho}_{m;\mathbb{Z}}$ denote the contragredient representations of $\Gamma(D)$ on Λ_m^\vee . We let $\bar{\rho}_{\mathbb{Z};m} := \rho_{\mathbb{Z};m} \oplus \check{\rho}_{\mathbb{Z};m}$ denote the corresponding integral representation of $\Gamma(D)$ on the lattice

$$\bar{\Lambda}(m) := \Lambda(m) \oplus \Lambda^\vee(m).$$

We let $\bar{\rho}_m$ denote the corresponding real representation of Γ on $\bar{V}(m) := V(m) \oplus V(m)^*$. Over \mathbb{R} , the representation ρ_m is self-dual, i.e. one has $\bar{\rho}_m \cong \rho_m \oplus \rho_m$. In particular, no irreducible summand of $\bar{\rho}_m$ is invariant under the Cartan involution θ .

Let Γ be any neat, finite index subgroup of $\Gamma(D)$. Then we may regard each $\bar{\Lambda}(m)$ as a Γ -module. Let $H^*(\Gamma, \bar{\Lambda}(m))$ be the cohomology groups of Γ with coefficients in $\bar{\Lambda}(m)$. These groups are finitely generated abelian groups and thus they admit a (non-canonical) decomposition as a direct sum

$$H^*(\Gamma, \bar{\Lambda}(m)) = H^*(\Gamma, \bar{\Lambda}(m))_{free} \oplus H^*(\Gamma, \bar{\Lambda}(m))_{tors},$$

where $H^*(\Gamma, \bar{\Lambda}(m))_{free}$ are free, finite-rank \mathbb{Z} -modules and $H^*(\Gamma, \bar{\Lambda}(m))_{tors}$ are finite abelian groups. Let X be the hyperbolic manifold $\Gamma \backslash \mathbb{H}^3$ and let \bar{X} denote its Borel-Serre compactification. The latter is a compact manifold with toric boundary which is homotopy equivalent to X . In particular, $\bar{\rho}_{\mathbb{Z}, m}$ defines an integral local system $\mathcal{L}(m)$ over \bar{X} . Since the universal covering of X resp. \bar{X} is contractible, one has a canonical isomorphism $H^*(\Gamma, \bar{\Lambda}(m)) \cong H^*(\bar{X}, \mathcal{L}(m))$. Let $E_{\bar{\rho}_m}$ denote the flat vector bundle over X resp. \bar{X} corresponding to $\bar{\rho}_m$ and let $H^*(\bar{X}; E_{\bar{\rho}_m})$ denote the singular cohomology groups of \bar{X} with coefficients in $E_{\bar{\rho}_m}$. Then $H^*(\Gamma, \bar{\Lambda}(m))_{free}$ is a lattice in $H^*(\bar{X}; E_{\bar{\rho}_m})$.

4.2. Eisenstein bases for the characteristic zero homology. We now recall the description of the canonical bases in the cohomology $H^*(\bar{X}; E_{\bar{\rho}_m})$ which are used to define the Reidemeister torsion $\tau_{Eis}(X; E_{\bar{\rho}_m})$. For more details, we refer [Pfa17, section 8], the notation of which we shall also use. Let $\partial\bar{X}$ denote the boundary of \bar{X} and $\iota : \partial\bar{X} \rightarrow X$ denote the inclusion map. The corresponding maps $\iota_k^* : H^k(\bar{X}; E_{\bar{\rho}_m}) \rightarrow H^k(\partial\bar{X}; E_{\bar{\rho}_m})$ in cohomology are injective for $k \in \{1, 2\}$, see [Pfa17, Lemma 8.3]. Thus the cohomology $H^*(\bar{X}; E_{\bar{\rho}_m})$ is completely described in terms of the Eisenstein cohomology described by Harder in [Har87], which we will explain below in our particular case.

For each $P_j \in \mathfrak{P}_\Gamma$ let $\mathcal{H}^k(\mathfrak{n}_{P_j}; \bar{V}(m))$ denote the harmonic forms of degree k in the Lie algebra cohomology complex of \mathfrak{n}_{P_j} with coefficients in $\bar{V}(m)$. We equip this space with the inner product induced by the restriction of the inner product (2.1) on \mathfrak{g} to \mathfrak{n}_{P_j} and the admissible inner product on $\bar{V}(m)$. Let $\sigma_{\bar{\rho}_m, 1}^- \in \hat{M}_P$ and $\lambda_{\bar{\rho}_m, 1}^- \in (-\infty, 0)$ resp. $\sigma_{\bar{\rho}_m, 2} \in \hat{M}_P$ and $\lambda_{\bar{\rho}_m, 2} \in (-\infty, 0)$ be defined as in [Pfa17, section 6]. In the present situation, we have $\sigma_{\bar{\rho}_m, 1}^- = \sigma_{-m-2}$ and $\lambda_{\bar{\rho}_m, 1}^- = -m/2$ resp. $\sigma_{\bar{\rho}_m, 2} = \sigma_{-m}$ and $\lambda_{\bar{\rho}_m, 2} = -(m+1)/2$. By the finite-dimensional Hodge theorem and a theorem of van Est one has a canonical isomorphism

$$(4.1) \quad H^k(\partial\bar{X}; E_{\bar{\rho}_m}) \cong \bigoplus_{P_j \in \mathfrak{P}_\Gamma} \mathcal{H}^k(\mathfrak{n}_{P_j}, \bar{V}(m)).$$

In degree 1, Kostant's theorem gives a splitting

$$\mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m)) = \mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m))_- \oplus \mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m))_+.$$

We define the positive and negative parts of the cohomology by:

$$(4.2) \quad H^1(\partial\bar{X}; E_{\bar{\rho}_m})_{\pm} := \bigoplus_{j=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_{P_j}, \bar{V}(m))_{\pm}.$$

An essential ingredient in the theory of Eisenstein series, which will be central to our argument as well, is the intertwining operator

$$\underline{\mathbf{C}}(\sigma_{-m-2}, m/2) : H^1(\partial\bar{X}; E_{\bar{\rho}_m})_- \rightarrow H^1(\partial\bar{X}; E_{\bar{\rho}_m})_+.$$

We will give other interpretations of this map in later sections, but for the moment we will only observe that it can be defined out of the following relation: for any $\Phi \in H^1(\partial\bar{X}; E_{\bar{\rho}_m})_-$ we have

$$\iota_1^* E(\Phi, m/2) = \Phi + \underline{\mathbf{C}}(\sigma_{-m-2}, m/2)\Phi.$$

Now let us describe the cohomology in degree 1. We have $\dim \mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{\pm} = 2 \dim \mathcal{H}^1(\mathfrak{n}_{P_j}; V(m))_{\pm} = 2$. For each $P_j \in \mathfrak{P}_{\Gamma}$ let $\Phi_{i,j}^1$ be an orthonormal basis of $\mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{-}$. Then the set

$$\mathcal{B}^1(\Gamma; \bar{\rho}_m) := \{E(\Phi_{i,j}^1, m/2), j = 1, \dots, \kappa(\Gamma), i = 1, \dots, \dim \mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{-}\}$$

forms a basis of $H^1(\bar{X}; E_{\bar{\rho}_m})$, where $E(\Phi_{i,j}^1, m/2)$ denotes again the Eisenstein series evaluated at $m/2$ as in [Pfa17, (7.3)], which is regular at this point by [Pfa17, Proposition 8.4]. As noted above for $\Phi_{i,j}^1 \in \mathcal{H}^1(\mathfrak{n}_{P_j}; \bar{V}(m))_{-}$ one has (pending the definition of the intertwining operators):

$$(4.3) \quad \iota_1^* E(\Phi_{i,j}^1, m/2) = \Phi_{i,j}^1 + \mathbf{C}(\sigma_{-m-2}, m/2) \Phi_{i,j}^1.$$

Things are simpler in degree 2, since we have $\dim \mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m)) = 2 \dim \mathcal{H}^2(\mathfrak{n}_{P_j}; V(m)) = 2$ and the map ι_2^* is an isomorphism. Let $\Phi_{i,j}^2$ be an orthonormal basis of $\mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m))$. Then the family

$$\mathcal{B}^2(\Gamma; \bar{\rho}_m) := \{E(\Phi_{i,j}^2, (m+1)/2) : j = 1, \dots, \kappa(\Gamma) : i = 1, \dots, \dim \mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m))\}$$

forms a basis of $H^2(\bar{X}; E_{\bar{\rho}_m})$. Here $E(\Phi_{i,j}^2, (m+1)/2)$ denotes the Eisenstein series associated to $\Phi_{i,j}^2$ evaluated at $(m+1)/2$ (which is regular at this point). For $\Phi_{i,j}^2 \in \mathcal{H}^2(\mathfrak{n}_{P_j}; \bar{V}(m))$ one has the simpler restriction formula:

$$(4.4) \quad \iota_2^* E(\Phi_{i,j}^2, (m+1)/2) = \Phi_{i,j}^2.$$

4.3. The Reidemeister torsion τ_{Eis} . We will use the expression for the Reidemeister torsion in terms of the ‘‘regulators’’ introduced in [BV13], which in our case are similar to those studied in [CV12, 6.3.2] and [Rai13]. We need some notation and terminology to define those. If L is a discrete subgroup (not necessarily a lattice) in an Euclidean space V then we denote by $\text{vol}(L)$ (with implicit reference to V) as the covolume of L is the subspace of V it spans, alternatively it equals the square root of the determinant of the Gram matrix of a \mathbb{Z} -basis of L (which also makes sense in Hermitian spaces).

By the definition of [Pfa17, section 9], the Reidemeister torsion $\tau_{Eis}(X; E_{\bar{\rho}_m})$ is taken using the bases in the cohomology defined in the previous subsection. We can also endow the spaces $H^k(X; \bar{V}(m))$ with an Euclidean structure by taking the bases $\mathcal{B}_k(\Gamma, \bar{\rho}_m)$ to be orthonormal. The size of the groups $H^*(\Gamma, \bar{\Lambda}(m))_{tors}$ is then related to the combinatorial torsion $\tau_{Eis}(X; E_{\bar{\rho}_m})$ in the following way: by [BV13, section 2.2] one has

$$(4.5) \quad \tau_{Eis}(X; E_{\bar{\rho}_m}) = \frac{|H^1(\Gamma; \bar{\Lambda}(m))_{tors}| \cdot \text{vol}(H^2(X, \bar{\Lambda}(m))_{free})}{|H^2(\Gamma; \bar{\Lambda}(m))_{tors}| \cdot \text{vol}(H^1(X, \bar{\Lambda}(m))_{free})}$$

In the notation of [BV13], the term in the second line of (4.5) is called the regulator.

4.4. Boundary denominators of Eisenstein classes. We denote by $H^*(\partial\bar{X}; \bar{\Lambda}(m))$ resp. $H^*(\partial\bar{X}; E(\bar{\rho}_m))$ the cohomology of $\partial\bar{X}$ with coefficients in $\mathcal{L}(m)$ resp. $E(\bar{\rho}_m)$ restricted to $\partial\bar{X}$. Again there is a (noncanonical) splitting of $H^*(\partial\bar{X}; \bar{\Lambda}(m))$ into a direct sum $H^*(\partial\bar{X}; \bar{\Lambda}(m))_{free} \oplus H^*(\partial\bar{X}; \bar{\Lambda}(m))_{tors}$ and $H^*(\partial\bar{X}; \bar{\Lambda}(m))_{free}$ is a lattice in $H^*(\partial\bar{X}; E_{\bar{\rho}_m})$. It is easy to see that the subgroups defined by

$$H^1(\partial\bar{X}; \bar{\Lambda}(m))_{\pm} := H^1(\partial\bar{X}; \bar{\Lambda}(m))_{free} \cap H^1(\partial\bar{X}; E_{\bar{\rho}_m})_{\pm}$$

are \mathbb{Z} -lattices in $H^1(\partial\bar{X}; E_{\bar{\rho}_m})_{\pm}$. This is a particular case of results in [Har87], see also [Rai13, Lemma 6.3] for a more hands-on proof in our case. Thus $H^1(\partial\bar{X}; \bar{\Lambda}(m))_{-} \oplus H^1(\partial\bar{X}; \bar{\Lambda}(m))_{+}$ is a \mathbb{Z} -sublattice of finite index in $H^1(\partial\bar{X}; \bar{\Lambda}(m))_{free}$. Moreover, with respect to the lattices $H^1(\partial\bar{X}; \bar{\Lambda}(m))_{\pm}$, the matrix $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ is \mathbb{Q} -rational. This is something that will be proven along with more precise results (see Proposition 6.1) but it is also proven by Harder in [Har87] (see Corollary 4.2.1 there).

As a consequence of the rationality of intertwining operators there exists $N \in \mathbb{N}$ such that $N \cdot \underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ defines a map

$$(4.6) \quad N \cdot \underline{\mathbf{C}}(\sigma_{-m-2}, m/2) : H^1(\partial\bar{X}; \bar{\Lambda}(m))_{-} \rightarrow H^1(\partial\bar{X}; \bar{\Lambda}(m))_{+}.$$

We let $d_{Eis, \mathbf{C}}(\Gamma, \bar{\rho}_m)$ denote the smallest $N \in \mathbb{N}$ such that (4.6) holds.

4.5. Inequalities for regulators. We will not deal with the denominator of Eisenstein cohomology classes which would be needed, but only with the denominator of the constant term matrix (equivalently that of its restriction to the boundary). We will be able to use this to establish exponential growth of the torsion part of H^2 thanks to the following crucial result which allows us to avoid a more detailed analysis.

Lemma 4.1. *We have*

$$\text{vol}(H^1(X, \bar{\Lambda}(m)))_{free} \leq |H^2(X, \bar{\Lambda}(m))_{tors}| \cdot (d_{Eis, \mathbf{C}}(\Gamma, \bar{\rho}_m))^{\kappa(\Gamma)} \cdot \text{vol}(H^1(\partial\bar{X}; \bar{\Lambda}(m))_{-}).$$

Proof. For convenience we will denote $H^1(X, \bar{\Lambda}(m))_{free}$ by $H^1(X)$ and similarly for other integral cohomology groups. The rough idea of the proof is to compare volumes between $H^1(X)$ and the lattice B of Eisenstein classes whose restriction to the boundary has an integral component on $H^1(\partial\bar{X}; \bar{V}(m))_{-}$: the quotient will depend on the default of integrality d_{Eis} of the intertwining operator, and the default of integrality of Eisenstein classes whose restriction is integral which is controlled by the torsion in $H^2(X)$ via a long exact sequence.

Let A be the submodule of $H^1(\partial\bar{X})$ defined by

$$A = (\iota_1^* H^1(X) \otimes \mathbb{Q}) \cap H^1(\partial\bar{X}).$$

Then the finite abelian group $A/\iota_1^* H^1(X)_{free}$ embeds into $H^1(\partial\bar{X})/\iota_1^* H^1(X)$, and from the long exact sequence in cohomology for the pair $(\bar{X}, \partial\bar{X})$ we see that the torsion subgroup of the latter embeds in $H^2(X, \partial\bar{X})_{tors}$. So we get that:

$$[A : \iota_1^* H^1(X)] \leq |H^2(\bar{X}, \partial\bar{X})_{tors}|.$$

It follows from Poincaré duality [Bro82, p. 222] that $H^2(\bar{X}, \partial\bar{X}) \cong H_1(\bar{X})$, and from the universal coefficient theorem that $H_1(\bar{X})_{tors} \cong H^2(X)_{tors}$. We will explain a bit more how to get this last isomorphism, as in [Pfa14]. Since $\bar{\Lambda}(m)$ is self-dual we have an isomorphism $C^*(\bar{X}) \cong \text{Hom}(C_*(\bar{X}), \mathbb{Z})$. By the universal coefficient theorem as stated in [Bro82, p. 8] we get an exact sequence :

$$0 \rightarrow \text{Ext}_{\mathbb{Z}}^1(H_1(\bar{X}), \mathbb{Z}) \rightarrow H^2(\bar{X}) \rightarrow \text{Hom}(H_2(\bar{X}), \mathbb{Z}) \rightarrow 0$$

which proves our claim since $\text{Ext}_{\mathbb{Z}}^1(M, \mathbb{Z}) = M_{tors}$ for any \mathbb{Z} -module M , and the term on the right is torsion-free. Together with the inequality above this implies that:

$$(4.7) \quad [A : \iota_1^* H^1(X)] \leq |H^2(X)_{tors}|.$$

Since restrictions of rational Eisenstein classes span $\iota_1^* H^1(X) \otimes \mathbb{Q}$ we get that A is a subgroup of finite index in:

$$B = \{\eta + \underline{\mathbf{C}}(\eta) : \eta \in H^1(\partial\bar{X})_-\}.$$

Moreover, by definition of d_{Eis} we have that $B \subset d_{Eis}^{-1} A$ and as $\kappa(\Gamma) = \dim(A \otimes \mathbb{Q})$ we deduce that

$$(4.8) \quad [B : A] \leq (d_{Eis})^{\kappa(\Gamma)}.$$

Now let π be the orthogonal projection of $H^1(\partial\bar{X}) \otimes \mathbb{R}$ onto $H^1(\partial\bar{X})_- \otimes \mathbb{R}$. Since $\pi(\iota_1^* E(\Phi, m/2)) = \Phi$ by (4.3), the map $\pi \circ \iota_1^*$ is by definition an isometry between $H^1(X) \otimes \mathbb{R}$ and $H^1(\partial\bar{X})_- \otimes \mathbb{R}$. Since $\pi(B) = H^1(\partial\bar{X})_-$ we deduce that

$$\begin{aligned} \text{vol}(H^1(X)) &= \text{vol} \pi(\iota_1^* H^1(X)) \\ &= [B : \iota_1^* H^1(X)] \cdot \text{vol} \pi(B) = [B : \iota_1^* H^1(X)] \cdot \text{vol}(H^1(\partial\bar{X})_-) \\ &= [B : A] \cdot [A : \iota_1^* H^1(X)] \cdot \text{vol}(H^1(\partial\bar{X})_-) \end{aligned}$$

and the lemma follows from the last line together with (4.8) and (4.7). \square

In degree 2 the situation is much simpler and we have complete control over the regulator.

Lemma 4.2. *We have :*

$$\text{vol}(H^2(\partial\bar{X}; \bar{\Lambda}(m))) \leq \text{vol}(H^2(X, \bar{\Lambda}(m)))_{free} \leq |H^1(X, \bar{\Lambda}(m))|_{tors} \cdot \text{vol}(H^2(\partial\bar{X}; \bar{\Lambda}(m))).$$

Proof. The map $\iota_2^* : H^2(X) \otimes \mathbb{R} \rightarrow H^2(\partial\bar{X}) \otimes \mathbb{R}$ is by definition an isometry and thus we need only to show that

$$[\iota_2^* H^2(X) : H^2(\partial\bar{X})] \leq |H^1(X)_{tors}|$$

which follows immediately from the long exact sequence of $(\bar{X}, \partial\bar{X})$. \square

5. THE ADELIC INTERTWINING OPERATORS

The main goal of this section and the next one is to establish an estimate of $d_{Eis, \mathbf{C}}(\Gamma, \bar{\rho}_m)$. For this we will use the computation of intertwining operators via integrals over adèle groups.

5.1. **Adélic setting.** We let $G := \mathrm{SL}_2$ as an algebraic group over F . Let \mathbb{A} denote the adèle ring of F and let \mathbb{A}_f be the finite adèles. For a linear algebraic group H defined over F let $H(\mathbb{A})$ denote its adelic points. For v a finite place we let F_v denote the completion of F at v , we let \mathcal{O}_v denote the integers in F_v and we let $\pi_v \in \mathcal{O}_v$ be a fixed uniformizer. We let Γ be a fixed neat congruence subgroup of $\mathrm{SL}_2(\mathcal{O}_D)$ of level $\mathfrak{a} = \prod_{v \text{ finite}} \mathfrak{p}_v^{n_v}$, where \mathfrak{p}_v is the prime ideal corresponding to v .

Let P be the parabolic subgroup of G consisting of upper triangular matrices and let T denote the set of diagonal matrices of determinant one. Let N_P denote the unipotent radical of P , that is the subgroup of upper triangular matrices with 1 as diagonal entries. We regard both P and N_P as algebraic groups over F . Then $P = TN_P$.

For notational convenience we shall write $K_\infty := \mathrm{SU}(2)$. Let

$$K_{max} := K_\infty \times \prod_{v \text{ finite}} \mathrm{SL}_2(\mathcal{O}_v).$$

Then one has the Iwasawa decomposition $G(\mathbb{A}) = P(\mathbb{A})K_{max}$. Let $K(\Gamma)_f \subset K_{max}$ be the compact subgroup of $G(\mathbb{A}_f)$ corresponding to Γ , meaning that $\Gamma = G(F) \cap K(\Gamma)_f$ where Γ and $G(F)$ are embedded diagonally into $G(\mathbb{A}_f)$.

Let $h = \kappa(\Gamma)$ and P_1, \dots, P_h be representatives of the Γ -conjugacy classes of parabolic subgroups in $G(F)$ (these are F -algebraic subgroups of G). Let $g_{P_i} \in G(F)$ such that $P_i := g_{P_i}^{-1} P g_{P_i}$, so that g_{P_1}, \dots, g_{P_h} are representatives of $P(F) \backslash G(F) / \Gamma$. We will assume that $P_1 = P$. In this section and the next one, we let N_{P_i} denote the unipotent radical of P_i regarded as an algebraic group over F and we shall denote by $N_{P_i, \infty}$ the corresponding real subgroup of $\mathrm{SL}_2(\mathbb{C})$.

We embed $G(F)$, $P(F)$ as well as the elements g_{P_1}, \dots, g_{P_h} diagonally into $G(\mathbb{A})$. Then we have a canonical isomorphism

$$(5.1) \quad \mathcal{I}_{\mathbb{A}} : P(F) \backslash G(\mathbb{A}) / K(\Gamma)_f \xrightarrow{\sim} \bigsqcup_{i=1}^h (\Gamma \cap N_{P_i, \infty}) \backslash \mathrm{SL}_2(\mathbb{C})$$

which is defined as follows. By the strong approximation theorem one has $G(\mathbb{A}) = G(F) \mathrm{SL}_2(\mathbb{C}) K(\Gamma)_f$. This implies that each $g \in G(\mathbb{A})$ can be written as

$$(5.2) \quad g = b g_{P_i} g_\infty k_f,$$

where $b \in P(F)$, $g_{P_i} \in \{g_{P_1}, \dots, g_{P_h}\}$ is uniquely determined and g_∞ is unique up to $\Gamma \cap P_i(F) = \Gamma \cap N_{P_i, \infty}$. Let $\pi : G(\mathbb{A}) \rightarrow P(F) \backslash G(\mathbb{A}) / K(\Gamma)_f$ denote the projection. Then according to (5.2), for $g \in G(\mathbb{A})$ we set $\mathcal{I}_{\mathbb{A}}(\pi(g)) := (\Gamma \cap N_{P_i, \infty}) g_\infty$, where $(\Gamma \cap N_{P_i, \infty}) g_\infty$ denotes the equivalence class of g_∞ in $(\Gamma \cap N_{P_i, \infty}) \backslash \mathrm{SL}_2(\mathbb{C})$. We also denote by

$$\mathcal{I}_{\mathbb{A}, P_i}^{-1} : (\Gamma \cap N_{P_i, \infty}) \backslash \mathrm{SL}_2(\mathbb{C}) \rightarrow P(F) \backslash G(\mathbb{A}) / K(\Gamma)_f$$

the maps induced by $\mathcal{I}_{\mathbb{A}}$ (each corresponding to the embedding of a cusp into the disjoint union (5.1)).

5.2. **Function spaces.** Let

$$W = (C^\infty(P(F)N(\mathbb{A})\backslash G(\mathbb{A})/K(\Gamma)_f) \otimes (\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m))^{K_\infty}.$$

The map $\mathcal{I}_\mathbb{A}$ induces an isomorphism

$$(5.3) \quad (\mathcal{I}_\mathbb{A})^* : \bigoplus_{i=1}^{\kappa(\Gamma)} (C^\infty(N_{P_i, \infty} \backslash \mathrm{SL}_2(\mathbb{C})) \otimes (\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m))^{K_\infty} \xrightarrow{\sim} W.$$

Here K_∞ acts on the C^∞ -spaces by right translation and on $(\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m)$ by $\mathrm{Ad}^* \otimes \rho(m)$. We shall denote this representation also by $\nu_1(\rho(m))$.

We regard the real subgroups M_P and A_P of $\mathrm{SL}_2(\mathbb{C})$ from (3.1) or more generally the real subgroups M_{P_i} and A_{P_i} for a parabolic P_i introduced above as subgroups of $G(\mathbb{A})$. For $\sigma \in \hat{M}_P$ with $[\nu_1(\rho(m)) : \sigma] \neq 0$ and $\lambda \in \mathbb{C}$ we let $W_{\bar{\rho}_m}(\sigma, \lambda)$ be the subspace of W defined by:

$$(5.4) \quad W_{\bar{\rho}_m}(\sigma, \lambda) = \{f \in W : \forall g \in G(\mathbb{A}), a \in A_P, t \in M_P, f(atg) = \xi_{\lambda+1}(a)\sigma^{-1}(t)f(g)\}.$$

If we view W as a space of (vector-valued) differential forms on the cusps of $X = \Gamma \backslash \mathbb{H}^3$ then the spaces $W_{\bar{\rho}_m}(\sigma, \lambda)$ correspond to the summands of the Mellin decomposition of compactly-supported elements of W (with a further refinement corresponding to the M_P -equivariance). See also [CV12, 6.6] for further explanations. The functions in $W_{\bar{\rho}_m}(\sigma, \lambda)$ are eigenforms for the Laplace operator (with eigenvalue depending on λ, ρ), and we will be interested in the harmonic ones which will serve to describe the Eisenstein cohomology classes.

Let $K_{P_i, \infty} := g_{P_i}^{-1} K_\infty g_{P_i}$. Then $K_{P_i, \infty}$ acts on $(\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m)$ by conjugating with g_{P_i} . We let $\mathcal{E}_{P_i}(\sigma, \lambda, \nu_1(\rho_m))$ be the space of all $f \in (C^\infty(N_{P_i, \infty} \backslash \mathrm{SL}_2(\mathbb{C})) \otimes (\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes \bar{V}(m))^{K_{P_i, \infty}}$ which additionally satisfy

$$f(a_{P_i} m_{P_i} g) = \xi_{P_i, \lambda+1}(a_{P_i}) \sigma_{P_i}(m_{P_i}^{-1}) f(g) \quad \forall g \in \mathrm{SL}_2(\mathbb{C}), \quad \forall a_{P_i} \in A_{P_i}, \quad \forall m_{P_i} \in M_{P_i}.$$

Here $\xi_{P_i, \lambda+1}$ and σ_{P_i} are the characters which arise from ξ_λ and σ by conjugating with g_{P_i} . If ν is a finite-dimensional representation of K on a complex vector space V , we let $V^{\sigma_{P_i}}$ denote the σ_{P_i} -isotypical component of the restriction of the representation ν to M_{P_i} . Then we have the following Lemma.

Lemma 5.1. *For $f \in W_{\bar{\rho}_m}(\sigma, \lambda)$ define $\mu_{\sigma, \lambda}(f) \in \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$ by*

$$\mu_{\sigma, \lambda}(f) := \sum_{i=1}^{\kappa(\Gamma)} (\mathcal{I}_{\mathbb{A}, P_i}^{-1})^* f(1) = \sum_{i=1}^{\kappa(\Gamma)} f(g_{P_i})$$

Then $\mu_{\sigma, \lambda}$ defines an isomorphism

$$\mu_{\sigma, \lambda} : W_{\bar{\rho}_m}(\sigma, \lambda) \cong \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$$

Proof. It is easy to see that each function $(\mathcal{I}_{\mathbb{A}, P_i}^{-1})^* f$ belongs to $\mathcal{E}_{P_i}(\sigma, \lambda, \nu_1(\rho_m))$ and is therefore determined by its value at 1, which moreover belongs to $((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$. On the other hand, for $\Phi_{P_i} \in ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{P_i}}$ we define $\Phi_{P_i; \lambda} \in \mathcal{E}_{P_i}(\sigma, \lambda, \nu_1(\rho_m))$ by

$$(5.5) \quad \Phi_{P_i; \lambda}(n_{P_i} a_{P_i} k) := \xi_{P_i; \lambda+1}(a_{P_i}) \nu_1(\rho_m)(k^{-1}) \Phi_{P_i}.$$

Let $\mu(\sigma, \lambda)^{-1}(\Phi_{P_i}) := \mathcal{I}_{\mathbb{A}}^*(\Phi_{P_i; \lambda})$. Then it immediately follows from the definitions that $\mu(\sigma, \lambda)$ and $\mu(\sigma, \lambda)^{-1}$ are inverse to each other. \square

We shall from now on identify $T(\mathbb{A})$ with the ring of ideles \mathbb{A}^* by sending $x \in \mathbb{A}^*$ to the diagonal matrix $\text{diag}(x, x^{-1})$. Let $U(\Gamma)_f := T(\mathbb{A}) \cap K(\Gamma)_f$. Let $\sigma = \sigma_k$, $k \in \mathbb{Z}$ and let $\lambda \in \mathbb{C}$. Then we combine σ^{-1} and λ to a character $\chi_{\infty, \sigma, \lambda}$ of $MA = T_{\infty} \cong \mathbb{C}^*$ by putting

$$(5.6) \quad \chi_{\infty, \sigma, \lambda}(z) := |z|^{2(\lambda+1)} \left(\frac{\bar{z}}{|z|} \right)^k.$$

We let $\mathcal{H}(\sigma, \lambda, K(\Gamma)_f)$ denote the set of all Hecke characters $\chi : F^* \backslash \mathbb{A}^* \rightarrow \mathbb{C}$ which are trivial on $U(\Gamma)_f$ and which satisfy $\chi_{\infty} = \chi_{\infty, \sigma, \lambda}$. If $|\cdot|_{\mathbb{A}}$ denotes the usual norm on the adèles, then each $\chi \in \mathcal{H}(\sigma, \lambda, K(\Gamma)_f)$ can be uniquely written as

$$(5.7) \quad \chi = |\cdot|_{\mathbb{A}}^{2(\lambda+1)} \chi_1,$$

where χ_1 is unitary. For $\chi \in \mathcal{H}(\sigma, \lambda, K(\Gamma)_f)$ with χ as in (5.7) we define

$$w_0 \chi = \chi^{-1} = |\cdot|_{\mathbb{A}}^{2(-\lambda+1)} \bar{\chi}_1 \in \mathcal{H}(w_0 \sigma, -\lambda, K(\Gamma)_f).$$

Since $T(\mathbb{A})$ normalizes $N(\mathbb{A})$ and T is abelian the group $T(\mathbb{A})$ acts on $W_{\bar{\rho}_m}(\sigma, \lambda)$ by left translations and thus we obtain a decomposition of $W_{\bar{\rho}_m}(\sigma, \lambda)$ into χ -isotypical subspaces:

$$(5.8) \quad W_{\bar{\rho}_m}(\sigma, \lambda) = \bigoplus_{\chi \in \mathcal{H}(\sigma, \lambda, K(\Gamma)_f)} W_{\bar{\rho}_m}(\sigma, \lambda)_{\chi}.$$

Let $t_{P_1}, \dots, t_{P_h} \in T(\mathbb{A}_f)$ denote fixed representatives of $T(F) \backslash T(\mathbb{A}_f) / U(\Gamma)_f$. Then for $f \in W_{\bar{\rho}_m}(\sigma, \lambda)$, its projection f_{χ} onto $W_{\bar{\rho}_m}(\sigma, \lambda)_{\chi}$ is given by

$$(5.9) \quad f_{\chi}(g) = \frac{1}{\kappa(\Gamma)} \sum_{i=1}^{\kappa(\Gamma)} \bar{\chi}(t_{P_i}^{-1}) f(t_{P_i} g).$$

5.3. Cohomology spaces as function spaces. Now we use the the notations of the previous sections for the various cohomology groups. We can canonically identify the Lie algebra \mathfrak{n}_{P_i} of $N_{P_i, \infty}$ with \mathfrak{n}_P . Then we have canonical embeddings

$$\bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_+ \hookrightarrow \bigoplus_{i=1}^{\kappa(\Gamma)} (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m); \quad \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_- \hookrightarrow \bigoplus_{i=1}^{\kappa(\Gamma)} (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m).$$

Moreover, an easy computation shows that in the present case these embeddings in fact give isomorphisms

$$\begin{aligned} \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_+ &\cong \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{m+2}} = (\mathbb{C}X_+ \otimes v_+)^2; \\ \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_- &\cong \bigoplus_{i=1}^{\kappa(\Gamma)} ((\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m))^{\sigma_{-m-2}} = (\mathbb{C}X_- \otimes v_-)^2 \end{aligned}$$

where X_{\pm} are the vectors of weight ± 2 for $\mathfrak{su}(2)$ in \mathfrak{p}^* and v_{\pm} the vectors of weight $\pm m$ for $\mathfrak{su}(2)$ in $V(m)$ (recall that $\bar{V}(m) \cong V(m)^2$). Thus together with Lemma 5.1, we obtain isomorphisms:

$$(5.10) \quad \mu_+(m) : W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2) \cong \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_+ \cong H^1(\partial\bar{X}; \bar{V}(m))_+$$

and

$$(5.11) \quad \mu_-(m) : W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2) \cong \bigoplus_{i=1}^{\kappa(\Gamma)} \mathcal{H}^1(\mathfrak{n}_P; \bar{V}(m))_- \cong H^1(\partial\bar{X}; \bar{V}(m))_-.$$

5.4. Integral formula for the intertwining operators. The various function spaces we consider are subspaces of the invariant subspace

$$(L^2(G(F)\backslash G(\mathbb{A})) \otimes \mathfrak{p}^* \otimes \bar{V}(m))^{K_{\infty}K_f(\Gamma)}.$$

This is naturally isomorphic the $\nu_1(\bar{\rho}_m)$ -isotypic component of $L^2(G(F)\backslash G(\mathbb{A}))$ and in the sequel we will view it as such, so that we implicitly interpret all vector-valued functions on various quotients of X as scalar-valued functions on $G(\mathbb{A})$ (we will use an explicit isomorphism when needed).

We shall denote the operator from $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ to $W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ induced by $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ and the isomorphisms $\mu_{\pm}(m)$ by $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$ too. The constant term of a differential form $\omega \in \Omega^*(T_i; \bar{V}(m))$ is defined by

$$\frac{1}{\text{vol}((\Gamma \cap N_i)\backslash N_i)} \int_{(\Gamma \cap N_i)\backslash N_i} n^* \omega dn.$$

It follows immediately from the fact that the group action $\mathbb{T}^2 \curvearrowright H^*(T_i; \bar{V}(m))$ is trivial that the de Rham class of a differential form is the same as that of its constant term. In turn, this implies that the intertwining operators are given by the constant term of Eisenstein series and it follows that for $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ and $g \in G(\mathbb{A})$ one has

$$\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)f(g) = \int_{N(\mathbb{A})} f(w_0 n g) dn, \quad w_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(see e.g. [Rai13, 2.2.4] or [CV12, 6.6.2], we use the same normalisation of Haar measure on $N(\mathbb{A})$ as in these references). With respect to the decompositions of $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ resp.

$W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ into Hecke-isotypical subspaces given in (5.8), the operator $\underline{C}(\sigma_{-m-2}, m/2)$ splits as

$$\underline{C}(\sigma_{-m-2}, m/2) = \bigoplus_{\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)} C(\chi),$$

where $C(\chi) : W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)_\chi \rightarrow W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)_{w_0\chi}$. Let $f \in W(\sigma_{-m-2}, m/2)_\chi$. For our later purposes we can assume that $f = f_\infty \otimes \bigotimes_{v \text{ finite}} f_v$. Then one has for $\text{Re}(s)$ large enough so that the product converges (in general we need analytic continuation of the global expression):

$$C(\chi)f = C_\infty(\sigma_{-m-2}, m/2)f_\infty \otimes \bigotimes_{v \text{ finite}} C_v(\chi_v)f_v.$$

where the operators $C_v(\chi_v)$ for v finite or infinite are defined by

$$(5.12) \quad C_v(\chi_v)f_v(g) = \int_{F_v} f_v \left(w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) dx$$

The integrals above can be computed more or less explicitly at each place, which we will do separately for finite and infinite places.

5.4.1. Finite places. Let us set some further notation. Recall that n_v is the v -valuation of the level of the congruence subgroups Γ . Let χ_1 be as in (5.7). For v a finite place of F we let $\chi_{1,v}$ be the local component of χ_1 at v . Then we say that χ_1 is unramified at v if $\chi_{1,v}$ is trivial on \mathcal{O}_v^\times . Otherwise we say that χ_1 is ramified at v . If χ_1 is unramified at v , then $\chi_{1,v}(\pi_v)$ does not depend on the choice of π_v (this happens in particular if $n_v = 0$). Let $q = |\mathcal{O}_v/\pi_v\mathcal{O}_v|$. For v a finite place, χ_1 unramified at v and $s \in \mathbb{C}$ the local L -factor $L_v(\chi_{1,v}, s)$ is defined by

$$(5.13) \quad L_v(\chi_{1,v}, s) := \frac{1}{1 - \chi_{1,v}(\pi_v)q^{-s}}.$$

Then the following Lemma holds.

Lemma 5.2. *Let $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ and v be a finite place of F . Suppose that f_v takes integral values on $\text{SL}_2(\mathcal{O}_v) = K_{\max,v}$. Then we have:*

(1) *If $n_v = 0$, one has*

$$C_v(\chi_v)f_v(1_v) = \frac{L_v(\chi_{1,v}, m+2)}{L_v(\chi_{1,v}, m+1)}f_v(1_v).$$

(2) *If $n_v > 0$ and χ is ramified at v , then for $k_v \in K_{\max,v}$ one has*

$$C_v(\chi_v)f_v(k_v) = I_v(k_v)f_v(k_v),$$

where $I_v(k_v) \in q^{-2n_v(m+2)}\overline{\mathbb{Z}}$.

(3) *If $n_v > 0$ and χ is unramified at v , then for $k_v \in K_{\max,v}$ one has*

$$C_v(\chi_v)f_v(k_v) = \frac{L_v(\chi_{1,v}, m+2)}{L_v(\chi_{1,v}, m+1)}I_v(k_v)f_v(k_v),$$

where $I_v(k_v) \in q^{-2n_v m}\overline{\mathbb{Z}}$.

Proof. For convenience we will denote $n = n_v$ and $s = 1 + m/2$ during the whole proof. We prove (i) first. For $x \in F_v, |x|_v > 1$ we have the Iwasawa decomposition

$$w_0 \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^{-1} & -1 \\ 0 & x \end{pmatrix} \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix}.$$

On the other hand for $g = nak$ we have $f_v(g) = \alpha(a)^s \chi_{1,v}(a) f(\text{Id})$. Hence we get

$$\begin{aligned} I_v &= \int_{|x|_v > 1} |x|_v^{-2s} \chi_{1,v}(x)^{-1} f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx + \int_{\mathcal{O}_v} f_v \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} dx \\ &= \left(\int_{|x|_v > 1} |x|_v^{-2s} \chi_{1,v}(x)^{-1} dx + \int_{\mathcal{O}_v} 1 dx \right) f_v(\text{Id}) \\ &= \left(\sum_{k \geq 1} (1 - q^{-1}) q^k \cdot \chi_{1,v}(\pi_v)^k q^{-2sk} + 1 \right) f_v(\text{Id}) \\ &= \left(1 + \frac{(\chi_{1,v}(\pi_v) q^{2s-1})^{-1} (1 - q^{-1})}{1 - \chi_{1,v}(\pi_v) q^{-2s+1}} \right) f_v(\text{Id}) = \frac{1 - \chi_{1,v}(\pi_v) q^{-2s}}{1 - \chi_{1,v}(\pi_v) q^{-2s+1}} \cdot f_v(\text{Id}). \end{aligned}$$

Now let us prove (ii). It obviously suffices to deal with $k_v = \text{Id}$. Recall from the preceding proof that :

$$(5.14) \quad I_v = \int_{|x|_v > 1} |x|_v^{-2s} \chi_{1,v}(x) f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx + \int_{\mathcal{O}_v} f_v \begin{pmatrix} 0 & -1 \\ 1 & x \end{pmatrix} dx.$$

The second term is a linear combination of integers (values of f_v) with coefficients in $q^{-n}\mathbb{Z}$ (the measure of a coset on which f_v is constant is equal to q^{-n}) and hence lies in $q^{-n}\overline{\mathbb{Z}}$. It remains to deal with the second term. Since χ_1 is ramified at v we have for any $k \in \mathbb{Z}$ the equality

$$\int_{|x|_v = q^k} \chi_{1,v}(x) dx = 0.$$

It follows that :

$$\begin{aligned} \int_{|x|_v > 1} |x|_v^{-2s} \chi_{1,v}(x) f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx &= \int_{1 < |x|_v \leq q^n} |x|_v^{-2s} \chi_{1,v}(x) f_v \begin{pmatrix} 1 & 0 \\ x^{-1} & 1 \end{pmatrix} dx \\ &= \sum_{a \in v\mathcal{O}_F/v^n} |a|_v^{2s-2} \chi_{1,v}(a) f_v \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} |a|^{-1} \int_{1+a^{-1}v^n\mathcal{O}_v} \chi_{1,v}(x) dx. \end{aligned}$$

Now there are two possibilities for the integral on the second line: either $\chi_{1,v}$ is trivial on $1 + a^{-1}v^n$, in which case the integral equals $q^{-n}|a|^{-1}$, or the integral vanishes. In either case it lies in $q^{-n}\mathbb{Z}$, hence the sum lies in $q^{-(2s-1)n}\overline{\mathbb{Z}}$.

The proof of (iii) is just a combination of those of (i) and (ii). The decomposition (5.14) is still valid, and the estimate for the denominator of the first factor done there is still

valid. For the first one we have :

$$\begin{aligned} \int_{|x|_v < 1} |x|_v^{2-2s} \chi_{1,v}(x) f \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} dx &= \sum_{a \in \mathcal{O}_F/v^n} f_v \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \int_{a+v^n\mathcal{O}_v} |x|_v^{2-2s} \chi_{1,v}(x) dx \\ &= q^{-n} \sum_{a \neq 0} \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \chi_{1,v}(a) |a|_v^{2s-2} + f_v(\text{Id}) \int_{|x|_v \leq q^{-n}} |x|_v \chi_{1,v}(x) dx \end{aligned}$$

(since χ_1 is not ramified at v it is constant on every coset $a + v^n\mathcal{O}_v$). The terms with $a \neq 0$ belong to $q_v^{-(2s-1)n} \overline{\mathbb{Z}}$, and the same computation as in the proof of (i) yields that the last term lies in $q^{-2sn} \frac{L_v(\chi_1, 2s)}{L_v(\chi_1, 2s-1)} \overline{\mathbb{Z}}$. \square

5.4.2. *Infinite place.* Finally, the term $C_\infty(\sigma_{-m-2}, m/2)(f_\infty)$, which is always a ratio of Γ -functions, can be described explicitly in the present case. There is $\Phi \in ((\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes V(\rho(m)))^\sigma$ such that $f_\infty = \Phi_{m/2}$. Moreover, there is an M_∞ -equivariant isomorphism

$$\nu_1(\rho_m)(w_0) : ((\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes V(\rho(m)))^\sigma \cong ((\mathfrak{n}_P \oplus \mathfrak{a}_P)^* \otimes V(\rho(m)))^{w_0\sigma}.$$

The representation $\nu_1(\rho_m)$ of K is not irreducible. However, if ν_{m+2} denotes the representation of K of highest weight $m+2$ in the canonical parametrization, then ν_{m+2} occurs with multiplicity one in $\nu_1(\rho_m)$ and belong to the ν_{m+2} -isotypical subspace. Thus we have

$$(5.15) \quad C_\infty(\sigma_{-m-2}, m/2)(\Phi_{m/2}) = c_{\nu_{m+2}}(\sigma_{-m-2} : m/2) \cdot (\nu_1(\rho_m)(w_0)\Phi)_{-m/2},$$

where $c_{\nu_{m+2}}(\sigma_{-m-2} : m/2) \in \mathbb{C}$ is the value of generalized Harish-Chandra c -function. In the present case, the latter is known explicitly. Namely, by [Coh74, Appendix 2], taking the different parametrizations into account, one has

$$(5.16) \quad c_{\nu_{m+2}}(\sigma_{-m-2} : m/2) = \frac{1}{\pi} \frac{1}{im + m + 2}.$$

5.5. **Final expression for $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$.** It follows from (5.16) and Lemma 5.2 that we have an expression of the following form for $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$:

$$(5.17) \quad \underline{\mathbf{C}}(\sigma_{-m-2}, m/2)\Phi = N^{-m} \frac{1}{\pi} \frac{L(\chi_1, m+2)}{L(\chi_1, m+1)} \Psi$$

where

- L is the Hecke L -function associated with the character χ_1 ;
- N is an integer depending on Γ (via the possible ramification of χ_1);
- Ψ is a function taking integral values on $K_{f, \max}$.

We will use this expression in the next section to study the integrality of the operator $\underline{\mathbf{C}}(\sigma_{-m-2}, m/2)$.

6. ESTIMATION OF THE DENOMINATOR OF THE C-MATRIX

We keep the notation of the preceding section. Our goal here is to prove the following estimate for the denominator of the intertwining matrices.

Proposition 6.1. *Let Γ be a (principal) congruence subgroup of Γ_D . Then there exists a constant $C_0(\Gamma)$ such that one can estimate*

$$\log |d_{Eis, \mathbb{C}}(\Gamma, \bar{\rho}_m)| \leq C_0(\Gamma) m \log(m)$$

for all $m \in \mathbb{N}$.

Using the maps $\mu_{\pm}(m)$ from (5.10) and (5.11) we obtain distinguished integral lattices $\mu_+^{-1}(m)(H^1(\partial\bar{X}; \bar{\Lambda}(m))_+)$ in the space $W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ and $\mu_-^{-1}(m)(H^1(\partial\bar{X}; \bar{\Lambda}(m))_-)$ in $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$. More generally, if $R \subset \mathbb{C}$ is a ring with $\mathbb{Z} \subset R$ we will say that $f \in W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ is defined over R if it is in the image of $\mu_+^{-1}(m)(H^1(\partial\bar{X}; \bar{\Lambda}(m))_+ \otimes_{\mathbb{Z}} R)$ and we make the corresponding definition for $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$.

6.1. Reduction to the isotypic case. The decomposition of $W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ with respect to Hecke characters given in (5.8) does not respect the \mathbb{Z} -structure on this space just introduced. In other words, if $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ is defined over \mathbb{Z} and if we decompose

$$(6.1) \quad f = \sum_{\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)} f_{\chi},$$

then we cannot expect the f_{χ} to be defined over \mathbb{Z} . However, we have the following Proposition which controls this defect.

Proposition 6.2. *There exists an algebraic integer $\alpha \in \bar{\mathbb{Z}}$ which depends on the group Γ but not on the representation $\rho(m)$ such that if $f \in W_{\bar{\rho}_m}(\sigma_{-m-2}, m/2)$ is defined over \mathbb{Z} then $\alpha^m f_{\chi}$ is defined over $\bar{\mathbb{Z}}$ for each f_{χ} in the decomposition (6.1).*

Proof. Recall that the character $\chi_{\infty, \sigma_{-m-2}, m/2}$ is the character

$$(6.2) \quad \chi_{\infty, m+2} : T(\mathbb{C}) \rightarrow \mathbb{C}, \quad \chi_{\infty, m+2} \left(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} \right) := z^{m+2}$$

on $T(\mathbb{C})$. Let $\mathcal{H}^1(\mathfrak{n}_P; \bar{\Lambda}(m))_-$ denote the integral lattice in corresponding to $H^1(\partial\bar{X}; \bar{\Lambda}(m))_-$. Without loss of generality, we may assume that $f = \mu_-^{-1}(\Phi)$, where $\Phi \in H^1(\mathfrak{n}_P; \bar{\Lambda}(m))_-$. We fix $\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)$. We have

$$\mu_- \left((\mu_-^{-1}(\Phi))_{\chi} \right) = \sum_{i=1}^{\kappa(\Gamma)} (\mu_-^{-1}(\Phi))_{\chi}(g_{P_i}).$$

On the other hand, by (5.9), for each i we have

$$(6.3) \quad (\mu_-^{-1}(\Phi))_{\chi}(g_{P_i}) = \frac{1}{\kappa(\Gamma)} \sum_{j=1}^{\kappa(\Gamma)} \chi(t_{P_j})(\mu_-^{-1}(\Phi))(t_{P_j}^{-1} g_{P_i})$$

We will now evaluate the denominators of the $\chi(t_{P_j})$ and of the $(\mu^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i})$ separately. Essentially, we will see that these values which depend on m have denominators which are $(m+2)$ th-powers of a finite number of algebraic integers. For the $\chi(t_{P_j})$ this follows from the fact that the finite part of the characters χ that we need to consider does not change on algebraic units, and for the $(\mu^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i})$ it follows from a simple computation using the fact that $\Phi \in W_{\bar{\rho}_m}(m/2, \sigma_{-(m+2)})$.

Each Hecke character $\chi : F^* \backslash \mathbb{A}^* \rightarrow \mathbb{C}$ which is trivial on $U(\Gamma)$ and which satisfies $\chi_\infty = \chi_{m+2, \infty}$ is of the form $\tilde{\chi}^{m+2}$, where $\tilde{\chi} : F^* \backslash \mathbb{A}^* \rightarrow \mathbb{C}$ is a character which is trivial on $U(\Gamma)$ and satisfies $\tilde{\chi}_\infty(z) = z$ for $z \in \mathbb{C}^*$. The set of such characters $\tilde{\chi}$ is finite. Thus it follows that there exists a $\beta \in \bar{\mathbb{Z}}$ such that:

$$(6.4) \quad \beta^{m+2} \chi(t_{P_j}) \in \bar{\mathbb{Z}}$$

for all $j = 1, \dots, \kappa(\Gamma)$ and all $\chi \in \mathcal{H}(\sigma_{-m-2}, m/2, K(\Gamma)_f)$.

For each i, j there exists a unique $l = l(i, j) \in \{1, \dots, h\}$ and a $g_\infty = g_\infty(i, j) \in \mathrm{SL}_2(\mathbb{C})$ such that

$$(6.5) \quad t_{P_j}^{-1}g_{P_i} = bg_{P_l}g_\infty k,$$

where $b \in P(F)$, $k \in K_f(\Gamma)$. We fix g_∞ satisfying (6.5). If $g_{P_i} \neq g_{P_i} = 1$, then, by definition one has $(\mu^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i}) = 0$. On the other hand, if $g_{P_i} = 1$, then by definition one has

$$(6.6) \quad (\mu^{-1}(\Phi))(t_{P_j}^{-1}g_{P_i}) = \Phi_{m/2}(g_\infty(i, j)),$$

where $\Phi_{m/2} = \Phi_{P, m/2} \in \mathcal{E}_P(\sigma_{-m-2}, m/2, \nu_1(\rho_m))$ is as in (5.5). Write the Iwasawa decomposition $g_\infty = p_\infty k_\infty$, where $p_\infty \in P_\infty$, $k_\infty \in K_\infty$. Then:

$$\Phi_{m/2}(g_\infty(i, j)) = \nu_1(\rho_m)(k_\infty(i, j))^{-1} \Phi_{m/2}(p_\infty(i, j)).$$

One has $\Phi_{m/2}(p_\infty) \in (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m)^{\sigma-m-2}$ and $\nu_1(\rho_m)(k_\infty)^{-1} \Phi_{m/2}(p_\infty) \in (\mathfrak{a}_P \oplus \mathfrak{n}_P)^* \otimes \bar{V}(m)^{\sigma-m-2}$. It is easy to see that this implies $k_\infty \in M_\infty$, i.e. $g_\infty \in P_\infty$. Moreover, together with (6.5) it follows that $g_\infty \in P(F)$. Thus one can write $g_\infty = t_\infty n_\infty$ with $t_\infty \in T(F)$ and $n_\infty \in N(F)$. We write $t_\infty = \mathrm{diag}(t_{i,j}, t_{i,j}^{-1})$ with $t_{i,j} \in F^*$. Then by (6.2) and by the definition of $\Phi_{m/2}$ we have $\Phi_{m/2}(g_\infty) = t_{i,j}^{m+2} \Phi$. Thus if $\alpha_{i,j} \in \mathcal{O}_F^*$ is the denominator of $t_{i,j}$, i.e. $\alpha_{i,j} t_{i,j} \in \mathcal{O}_F^*$ we get that

$$(6.7) \quad \alpha_{i,j}^{m+2} \Phi_{m/2}(g_\infty(i, j)) \in H^1(\mathfrak{n}_P, \bar{\Lambda}(m))_- \otimes_{\mathbb{Z}} \mathcal{O}_F.$$

Combining (6.3), (6.6), (6.7) and (6.4) we get the statement in the proposition. \square

6.2. Integrality of quotients of L-values. For a unitary Hecke character χ_1 recall the Euler product formula for the Hecke L-function:

$$L(\chi_1, s) = \prod_v L_v(\chi_1, v, s).$$

For a place v where χ does not ramify we defined the local factor L_v in (5.13); we take the convention that $L_v(\chi_1, s) := 1$ if χ_1 is ramified at v . The infinite product converges

absolutely for $\operatorname{Re}(s) > 1$ and admits a meromorphic continuation to \mathbb{C} . For $a \in \overline{\mathbb{Q}}$ we shall denote by $|a|_{\overline{\mathbb{Q}}/\mathbb{Q}}$ or simply $|a|$ its absolute norm given by

$$|a|_{\overline{\mathbb{Q}}/\mathbb{Q}} := |B|, \quad B = \prod_{b \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \cdot a} b \in \mathbb{Q},$$

where $\operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ is the absolute Galois group of \mathbb{Q} . The following proposition evaluates the denominator of the quotient of L -values appearing in the computation of the intertwining operators in Lemma 5.2. It is essentially contained in the work of Damerell [Dam70],[Dam71]: our argument consist in keeping track of norm estimates along the steps in the proof of the main theorem of [Dam70]. We learned of Damerell's theorem through the book [CV12], which makes use of it for purposes similar to our own in Section 6.7.

Proposition 6.3. *Let \mathfrak{J} be an ideal in \mathcal{O}_D . There is $A \in \mathbb{Z}_{>0}$ such that for all unitary Hecke characters χ whose conductor divides \mathfrak{J} and whose infinite part $\chi_\infty(z) = (z/|z|)^n = (\bar{z}/z)^{n/2}$ for an even integer n , and for any integer $s \in \{0, \dots, n/2\}$ there is an integer $a' \in \overline{\mathbb{Z}}$ such that $|a'| \leq (n!)^A$ and we have :*

$$(6.8) \quad 2^{n-2s+1} a' \frac{L(\chi, s)}{L(\chi, s-1)} \in \pi \overline{\mathbb{Z}}.$$

Proof. The arguments used in [Dam70] can be adapted to yield the following effective version of Theorem 1 in this reference; we will detail how to use Damerell's calculations to obtain this in the appendix A.

Lemma 6.4. *There is an $\Omega \in \mathbb{C}^\times$ (depending only on the field F) such that the following holds. Let n be an even integer, χ a unitary Hecke character of F with infinite part $\chi_\infty(z) = (\bar{z}/z)^{n/2}$ and s an integer in the range $\{0, \dots, n/2\}$. Then*

$$(6.9) \quad \pi^{n/2-s} L(\chi, s) / \Omega^n$$

is an algebraic number, whose degree over \mathbb{Q} is bounded by a constant depending only on F and the conductor \mathfrak{f} of χ and whose absolute norm is bounded by $C(n!)^A$ for positive integers C, A depending only on F .

Bounds for the denominators of special values of L -functions are also given by the work of Damerell. We will use the following statement to evaluate denominators of the L -part of the intertwining integrals: let n be an even integer, χ be a unitary Hecke character of F , with infinite part

$$\chi_\infty(z) = (\bar{a}/a)^{n/2}$$

and finite part χ_f of conductor \mathfrak{J} . Then [Dam71, Theorem 2] states that for the complex number $\Omega \in \mathbb{C}^\times$ appearing in Lemma 6.4 there is an algebraic integer $a \in \overline{\mathbb{Z}}$, whose absolute norm $|a| = |a|_{\overline{\mathbb{Q}}/\mathbb{Q}}$ is bounded independently of s, χ_∞ , such that for all integers $s \in \{0, \dots, n/2\}$ we have

$$(6.10) \quad 2^{n/2-s} a \cdot L(\chi, s) \in \Omega^n / \pi^{n/2-s} \overline{\mathbb{Z}}.$$

The proposition follows from this and the bound for the absolute norm of normalized L -values given in Lemma 6.4 (the transcendental factors cancel between the numerator and denominator). \square

6.3. Pairing and integrality. To be able to use Proposition 6.3 together with the formula 5.17 we need to relate the values of a function $\Phi \in W(\sigma_{-m-2}, m/2)$ with the integrals of the 1-form $\mu_-(m)\Phi$ against integral cycles in $H_1(\partial\bar{X}; \bar{V}(m))$, since the 1-form is integral if and only if these are integral.

First we describe the integral homology classes. Fix a component T_i of $\partial\bar{X}$ and identify it with $(\Gamma \cap N_i) \backslash N_i$. Let $\mathfrak{n}_{P_i, \mathbb{Z}} = \exp^{-1}(\Gamma \cap N_i)$; in this case since N_i is unipotent abelian this just means that $\Gamma \cap N_i = \{\text{Id} + Y : Y \in \mathfrak{n}_{P_i, \mathbb{Z}}\}$. Let $Y \in \mathfrak{n}_{P_i, \mathbb{Z}}$ and $n = 1 + Y$. Let $v \in \bar{\Lambda}(m)$ such that $v - \rho(n)v = 0$. Then the 1-chain $c_{Y,v}$ defined by:

$$c_{Y,v}(t) = (1 + tY)_* v, t \in [0, 1]$$

is an integral cycle, and the classes associated to these cycles span $H_1(T_i; \bar{\Lambda}(m))$.

Now let $\langle \cdot, \cdot \rangle$ be the $G(\mathbb{R})$ -invariant nondegenerate pairing on $\bar{V}(m)$ for which $\bar{\Lambda}(m)$ is self-dual. For an element $\Phi \in W$ and $Y \in \mathfrak{n}_{P_i}, g \in G(\mathbb{A})$ let $\Phi(g)(Y) \in \bar{V}(m)$ be defined by

$$\Phi(g)(Y) = \sum_{i=1}^n f_i(g) X_i^\vee(Y) v_i \text{ if } \Phi = \sum_{i=1}^n f_i \otimes X_i^\vee \otimes v_i \in C^\infty(G(\mathbb{A})) \otimes (\mathfrak{n}_{P_i} \oplus \mathfrak{a}_{P_i})^* \otimes \bar{V}(m).$$

The pairing on $H^1(T_i; \bar{V}(m)) \times H_1(T_i; \bar{V}(m))$ is given by

$$\mu_+(\Phi)(c_{Y,v}) = \int_0^1 \langle (1 + tY)_* v, (1 + tY)_* \Phi(1 + tY)(Y) \rangle dt$$

and by the invariance property of Φ we have that $t \mapsto \langle c(t), \exp(tY)_* \Phi(\exp(tY))(Y) \rangle$ is constant, equal to $\langle v, \Phi(\text{Id})(Y) \rangle$. Let $f_{Y,v} = \langle v, \Phi(\cdot)(Y) \rangle$. We see that the class $\mu_-(\Phi)$ is integral if and only if $f_{Y,v} \in L^2(G(F) \backslash G(\mathbb{A}))$ takes integral values on $K_f(\Gamma)$ for any given $Y \in \mathfrak{n}_{P_i, \mathbb{Z}}$ and $v \in \bar{\Lambda}(m)$ as above. Of course the same statement holds for $\Phi \in W(\sigma_{m+2}, m/2)$ with μ_- replaced by μ_+ .

6.4. Proof of Proposition 6.1. According to the previous paragraph the statement in the proposition reduces to the following claim: let K'_f be a compact-open subgroup in $G(\mathbb{A}_f)$ and $f \in W_{\bar{\rho}_m}(\sigma_{m+2}, -m/2)$ corresponding to a rational integral cohomology class (the space W being defined as above with $K_f(\Gamma)$ replaced by K'_f). Then we claim that there are $N, C, A \in \mathbb{Z}_{>0}$ depending only on K'_f such that we have

$$\mathbf{C}(s(m))f(\text{Id}) \in C^{-1} N^{-m} (m!)^A \mathbb{Z}f(\text{Id}).$$

To prove we note that it suffices to prove a similar result over $\bar{\mathbb{Z}}$, namely that for all f as above corresponding to a cohomology class with coefficients in $\bar{\Lambda}(m) \otimes \bar{\mathbb{Z}}$ we have

$$(6.11) \quad \mathbf{C}(s(m))f(\text{Id}) \in a^{-1} \mathbb{Z}f(\text{Id})$$

for an algebraic integer a with $|a|_{\overline{\mathbb{Q}}/\mathbb{Q}} \leq C(m!)^A$ (indeed, since we know a priori that if the right-hand side is defined over \mathbb{Q} the proposition follows by taking the product of Galois conjugates of each side).

Let us prove (6.11). First, it follows from Proposition 6.2 that it suffices to prove it for $f \in W_\chi$. By Lemma 5.2 and (5.17) we get that it suffices to prove that

$$\frac{1}{\pi} \cdot \frac{L(\chi, m)}{L(\chi, m-1)} \in b^{-1}\overline{\mathbb{Z}}$$

for an $a \in \overline{\mathbb{Z}}$ with absolute norm $|b| \leq (m!)^A$. This last statement follows from Proposition 6.3.

7. BOUNDING THE TORSION FROM BELOW

In this section we prove the estimate (1.1) (and also (1.3), which we actually need to prove the former) from our main Theorem A.

7.1. Regulators on the boundary. We have evaluated most terms from the estimates in Lemmas 4.1 and 4.2 but we still need to control the covolumes of integral homologies from the boundary components. This is elementary and we give estimates in the next lemma.

Lemma 7.1. *We have*

$$\log \text{vol}(H^1(\partial\overline{X}; \overline{\Lambda}(m))_-), \log \text{vol}(H^2(\partial\overline{X}, \overline{\Lambda}(m))) = O(m \log m)$$

as $m \rightarrow +\infty$.

Proof. We will prove this for cohomology with coefficients in $\Lambda(m)$. Since $\overline{\Lambda}(m)$ splits as $\Lambda(m) \oplus \Lambda(m)^\vee$ and $\Lambda(m)^\vee$ contains a sub-lattice of index at most $(m!)^c$ which is Γ -equivariantly isomorphic to $\Lambda(m)$ (see Lemma 7.7 below) it implies the statement for $\overline{\Lambda}(m)$. We will be a bit sketchy in our arguments below.

Since $H^1(\partial\overline{X}; \Lambda(m))_-$ splits as the direct sum $\bigoplus_{i=1}^{\kappa(\Gamma)} H^1(\Gamma \cap N_i; \Lambda(m))_-$ it is enough to estimate the volume of each lattice $H^1(\Gamma \cap N_i; \Lambda(m))_-$. We let ω be the cohomology class corresponding to $X_- \otimes v_-$ in the isomorphism (5.11) where v_- spans over \mathcal{O}_D the integral vectors in $V(m)$ of weight $-m$ for the Cartan subalgebra \mathfrak{a}_{P_i} corresponding to P_i . The class ω is not necessarily integral but we will see that it is rational and control its denominator. Since its norm is clearly bounded independently of m this will imply the estimate we want.

Let $i \in \{1, \dots, \kappa(\Gamma)\}$ and $X_1, X_2 \in \mathfrak{n}_{P_i}$ such that $\Gamma \cap P_i$ is generated by $1 + X_1, 1 + X_2$. Let v_l be a vector of weight $m - 2l$ for \mathfrak{a}_{P_i} in $V(m)$. There exists integral polynomials Q_k^l such that

$$\rho(1 + zX_1)v_l = v_l + \sum_{j=l+1}^m Q_k^l(z)v_k.$$

and using this equality it is possible to produce cycles θ_1, θ_2 which span $H_1(\Gamma \cap N_i, V(m))$ over \mathbb{Q} , which are not necessarily integral but whose denominators are controlled linearly by the coefficients of the Q_k^l appearing above. Moreover the integrals $\omega(\theta_1) = 0$ and $\omega(\theta_2)$

also have denominators controlled by these coefficients. Since all of these are essentially binomial coefficients, and thus at most exponential in m this finishes the proof.

In degree 2 the cohomology/homology for each component is 1-dimensional and the proof is much simpler and left to the reader. \square

7.2. Torsion in $H^*(\Gamma, \bar{\Lambda}(m))$. We first show directly that the order of the group $H^1(\Gamma; \bar{\Lambda}(m))_{tors}$ grows slower in m than our leading term. Since we work with a split algebraic group the proof of this is simpler than that of the corresponding statement in [MM13].

Lemma 7.2. *Let Γ be a congruence subgroup of Γ_D . Then*

$$\log |H^1(\Gamma; \bar{\Lambda}(m))_{tors}| = O(m \log m),$$

as $m \rightarrow \infty$.

Proof. One has $H^1(\Gamma; \bar{\Lambda}(m))_{tors} \cong H_0(\Gamma; \bar{\Lambda}(m))_{tors}$ by the universal coefficient theorem. Let $\Lambda^0(m)$ denote the submodule of $\Lambda(m)$ generated by all $(\rho_m(\gamma) - \text{Id})v$, where $v \in \Lambda(m)$ and $\gamma \in \Gamma$. Then $H_0(\Lambda(m)) = \Lambda(m)/\Lambda^0(m)$. There exists an $a \in \mathbb{N}$ such that for $n_a := \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\bar{n}_a := \begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}$ one has $n_a, \bar{n}_a \in \Gamma$. If we let X, Y denote the standard basis of \mathbb{C}^2 , then $X^m, X^{m-1}Y, \dots, Y^m$ is a basis of $\Lambda(m)$ and in this basis, $\rho_m(n_a) - \text{Id}$ is represented by an upper triangular nilpotent matrix. For $j > i$, the entry in the i -th row, j -th column of this matrix is given by $a^{j-i} \binom{j-1}{j-i}$. Thus it follows inductively that $(l+1)aX^{m-l}Y^l \in \Lambda^0(m)$ for $0 \leq l < m$. On the other hand, one has $(\rho_m(\bar{n}_a) - \text{Id})XY^{m-1} = aY^m$. Thus one has $|H_0(\Lambda(m))| \leq a^{m+1}m!$. i.e. $\log |H_0(\Lambda(m))| = O(m \log m)$ as $m \rightarrow \infty$. For $\Lambda^\vee(m)$ one can argue similarly. \square

The main result we prove here is the exponential growth of the torsion subgroup of $H^2(\Gamma, \bar{\Lambda}(m))$; in the remainder of this subsection we will show how all the work done in sections 4–6 implies the following result.

Proposition 7.3. *We have*

$$(7.1) \quad \liminf_{m \rightarrow +\infty} \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|}{m^2} \geq \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \left(1 - \frac{N(\mathfrak{a}_0)}{N(\mathfrak{a})} \right)$$

Proof. Let \mathcal{A} and $C_1(\Gamma)$ be as in the end of section 3. Let \mathfrak{a} be a non-zero ideal of \mathcal{O}_D with $N(\mathfrak{a}) > C_1(\Gamma)$ and let $\mathfrak{a}_0 \in \mathcal{A}$ such that $n_{l, \Gamma(\mathfrak{a})} = n_{l, \Gamma(\mathfrak{a}_0)}$ for each $l = 1, \dots, d_F$. For brevity we shall use the following notation in the remaining computations:

$$R^i(X, \bar{\mathcal{L}}(m)) = \text{vol} \left(H^i(\Gamma, \bar{\Lambda}(m)) \right).$$

With this notation (4.5) becomes

$$(7.2) \quad \begin{aligned} \log \tau_{Eis}(X; E_{\bar{\rho}_m}) &= \log R^2(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m)) + \log |H^1(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}| \\ &\quad - \log R^1(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m)) - \log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}|. \end{aligned}$$

By Lemmas 4.2 and 7.2 we have that

$$(7.3) \quad \left| \log \left(|H^1(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{tors}| \right) \right|, \left| \log R^2(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m)) \right| \ll m \log(m).$$

On the other hand, applying by Lemma 7.1, Proposition 6.1 to the right-hand side of the estimate in Lemma 4.1 we get that

$$\liminf_{m \rightarrow +\infty} \frac{\log R^1(X_{\mathfrak{a}}, \bar{\mathcal{L}}(m))}{m^2} \leq \liminf_{m \rightarrow +\infty} \frac{\log |H^2(\Gamma(\mathfrak{a}), \Lambda(m))_{tors}|}{m^2}$$

From (7.3) we get that in the expression (7.2) for the Reidemeister torsion $\tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}_m})$ all terms but $\log |H^2|$ and $\log R^1$ are $O(m \log(m))$ and using the preceding inequality we get that

$$(7.4) \quad \liminf_{m \rightarrow +\infty} \left(2 \frac{\log |H^2(\Gamma(\mathfrak{a}), \Lambda(m))_{tors}|}{m^2} \right) \geq \liminf_{m \rightarrow +\infty} \left(- \frac{\log \tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}_m})}{m^2} \right).$$

On the other hand, we also get from Lemmas 4.1 and 7.1 that

$$\liminf_{m \rightarrow +\infty} \left(\frac{-\log R^1(X_{\mathfrak{a}_0}, \bar{\mathcal{L}}(m))}{m^2} \right) \leq 0$$

from which and (7.3) (used for \mathfrak{a}_0 instead of \mathfrak{a}) it follows that

$$(7.5) \quad \liminf_{m \rightarrow +\infty} \left(\frac{\log \tau_{Eis}(X_{\mathfrak{a}_0}; E_{\bar{\rho}_m})}{m^2} \right) \leq 0.$$

Putting together (7.4) and (7.5) we get that

$$\begin{aligned} \liminf_{m \rightarrow +\infty} \left(2 \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*| \cdot N(\mathfrak{a}_0)} \cdot \frac{\log |H^2(\Gamma(\mathfrak{a}), \Lambda(m))_{tors}|}{m^2} \right) &\geq \\ \liminf_{m \rightarrow +\infty} \left(\frac{-[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*| \cdot N(\mathfrak{a}_0)} \cdot \frac{\log \tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}_m})}{m^2} + \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*| \cdot N(\mathfrak{a})} \cdot \frac{\log \tau_{Eis}(X_{\mathfrak{a}_0}; E_{\bar{\rho}_m})}{m^2} \right). \end{aligned}$$

Finally, the right-hand side above converges to

$$\begin{aligned} 2 \cdot \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)] \cdot [\Gamma_D : \Gamma(\mathfrak{a})]}{2|\mathcal{O}_D^*|\pi} \left(\frac{1}{N(\mathfrak{a}_0)} - \frac{1}{N(\mathfrak{a})} \right) \text{vol}(\Gamma_D \backslash \mathbb{H}^3) \\ = \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*|N(\mathfrak{a}_0)} \cdot \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \cdot \left(1 - \frac{N(\mathfrak{a}_0)}{N(\mathfrak{a})} \right) \end{aligned}$$

by Proposition 3.2 (which we are allowed to use for the representation $\rho_m \oplus \rho_m$ instead of ρ_m since the analytic or Reidemeister torsion of the former is the square of that of the latter). This finishes the proof of Proposition 7.3. \square

7.3. Independence from the lattice. Here we prove that $\log |H^2(\Gamma, \Lambda_m)_{tors}|$ does not depend on the choice of lattices $\Lambda_m \subset V(m)$ up to an error term of size $m \log(m)$.

Proposition 7.4. *Let Γ be a finite-index subgroup of the Bianchi group Γ_D . There is a constant C depending only on Γ such that for any $m \geq 1$ and any two Γ -invariant lattices Λ_1, Λ_2 in $V(m)$ we have*

$$\left| \log \left(\frac{|H^2(\Gamma, \Lambda_1)_{tors}|}{|H^2(\Gamma, \Lambda_2)_{tors}|} \right) \right| \leq Cm \log(m).$$

We will deduce the proposition from the two next lemmas.

Lemma 7.5. *Let Γ be a subgroup of a Bianchi group, ρ be a representation of $\mathrm{SL}_2(\mathbb{C})$ on a vector space V and Λ, Λ' two $\rho(\Gamma)$ -invariant lattices in V such that $M \cdot \Lambda \subset \Lambda' \subset \Lambda$ for some integer $M \in \mathbb{Z}_{>0}$. Let $\mathcal{L}, \mathcal{L}'$ be the local systems on $X = \Gamma \backslash \mathbb{H}^3$ induced by Λ, Λ' and E_ρ the Euclidean bundle on X induced by ρ . Then we have*

$$1 \leq \frac{R^1(X, \mathcal{L}')}{R^1(X, \mathcal{L})} \leq M^{\dim H^1(X, E_\rho)}.$$

Proof. Let $h = \dim H^1(X, E_\rho)$ and let $c_1, \dots, c_h \in Z^1(X, \mathcal{L})$ such that the cohomology classes $[c_1], \dots, [c_h]$ generate the free part of $H^1(X, \mathcal{L})$. Then each $M \cdot c_i$ belongs to $Z^1(X, \mathcal{L}')$ and together the $M \cdot [c_i]$ generate a finite-index subgroup of $H^1(X, \mathcal{L}')$. Thus we get

$$M \cdot H^1(X, \mathcal{L}) \subset H^1(X, \mathcal{L}')$$

and the inequality

$$R^1(X, \mathcal{L}') \leq [H^1(X, \mathcal{L}) : M \cdot H^1(X, \mathcal{L})] R^1(X, \mathcal{L}) = M^h R^1(X, \mathcal{L})$$

follows immediately. □

Lemma 7.6. *There is a constant $c \in \mathbb{Z}_{>0}$ depending on Γ such that if Λ_1, Λ_2 are two Γ -invariant lattices in $V(m)$ then there exists $a \in \mathbb{Q}$ such that*

$$a\Lambda_1 \subset \Lambda_2 \subset a(m!)^{-c}\Lambda_1.$$

Proof. Let $\langle \cdot, \cdot \rangle$ be the pairing on $V(m) = \mathrm{Sym}^m \mathbb{C}^2$ induced by the determinant on $\mathbb{C}^2 \times \mathbb{C}^2$ (which is hence nondegenerate and Γ -invariant) and let Λ'_1 be the $\langle \cdot, \cdot \rangle$ -dual lattice of Λ_1 in $V(m)$, that is

$$\Lambda'_1 = \{v \in V(m) : \forall u \in \Lambda_1, \langle u, v \rangle \in \mathbb{Z}\}.$$

Then

$$(7.6) \quad m!\Lambda'_1 \subset \Lambda_1 \subset (m!)^{-1}\Lambda'_1$$

as follows from the expression of $\langle \cdot, \cdot \rangle$ in coordinates (see for example [Ber08, 2.4]). Now let u be a primitive vector in Λ_2 which is a vector of maximal weight for the standard parabolic subgroup of $\mathrm{SL}_2(\mathbb{C})$ in $V(m)$ (i.e. a rational multiple of X^m); there exists an

$a \in \mathbb{Q}$ such that au is a primitive vector in Λ'_1 . Then $\Lambda_3 := \langle \Gamma \cdot au \rangle \subset \Lambda'_1$: indeed, for any $v \in \Lambda_1$ and $\gamma \in \Gamma$ we have

$$\langle v, \gamma \cdot au \rangle = \langle \gamma^{-1} \cdot v, au \rangle \in \mathbb{Z}.$$

From this and (7.6) we get that $\Lambda_3 \subset (m!)^{-1}\Lambda_1$. By arguments similar to those used in the proof of Lemma 7.2 (which we will detail after we explain how to conclude the proof from there), we also have that

$$(7.7) \quad a\Lambda_2 \subset N^{-m}(m!)^{-2}\Lambda_3$$

for some $N \in \mathbb{Z}_{>0}$ depending on Γ . It finally follows that

$$a\Lambda_2 \subset N^{-m}(m!)^{-3}\Lambda_1$$

which proves half the lemma; the second half also follows by a completely symmetric argument.

Let us explain how (7.7) is proved. For ease of notation we will suppose that $au = X^m$. We put $\Lambda_0 = \Lambda(m) = \mathcal{O}_D X^m \oplus \mathcal{O}_D X^{m-1}Y \oplus \dots \oplus \mathcal{O}_D Y^m$. Let $z \in \mathbb{Z}$ such that $n_z \in \Gamma$ and $k \in \{1, \dots, m\}$; suppose that $bX^{m-k}Y^k \in \Lambda_2$ for some $b \in F$. Then we get that

$$\Lambda_2 \ni n_z^k \cdot bX^{m-k}Y^k = bk!z^k X^m$$

so that $bk!z^m \in \mathcal{O}_D$. Thus we get that $\Lambda_2 \subset (z^m m!)^{-1}\Lambda_0$. On the other hand the proof of Lemma 7.2 yields that $[\Lambda_0 : \Lambda_3] \leq z^m m!$, hence $\Lambda_2 \subset (z^m m!)^{-2}\Lambda_3$. \square

Proof of Proposition 7.4. Let \mathcal{L}_i be the local system on X induced by the lattice Λ_i . We have by (7.2) that

$$\frac{R^2(X, \mathcal{L}_1) \cdot |H^1(\Gamma, \Lambda_1)_{tors}|}{R^1(X, \mathcal{L}_1) \cdot |H^2(\Gamma, \Lambda_1)_{tors}|} = \tau_{Eis}(X, \rho_m) = \frac{R^2(X, \mathcal{L}_2) \cdot |H^1(\Gamma, \Lambda_2)_{tors}|}{R^1(X, \mathcal{L}_2) \cdot |H^2(\Gamma, \Lambda_2)_{tors}|}.$$

By Lemmas 4.2 and 7.2 we have that H^1, R^2 are $\ll m \log(m)$ for whichever lattice, and by Lemmas 7.5 and 7.6 we get that

$$\left| \log \left(\frac{R^1(X, \mathcal{L}_1)}{R^1(X, \mathcal{L}_2)} \right) \right| \ll m \log(m)$$

and we can thus conclude that the remaining terms $\log(H^2(\Gamma, \Lambda_i))$ in the Reidemeister torsion differ by at most $Cm \log(m)$ for some $C > 0$ depending on Γ . \square

7.4. Conclusion. Let $\bar{\Lambda}'(m)$ be the lattice $\Lambda(m) \oplus \Lambda(m)$ in $V(m) \oplus V(m)$. By Propositions 7.4 and 7.3 we get that

$$\liminf \frac{\log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}'(m))_{tors}|}{m^2} \geq \frac{\text{vol}(X_{\mathfrak{a}})}{\pi} \left(1 - \frac{N(\mathfrak{a}_0)}{N(\mathfrak{a})} \right).$$

Since $H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}'(m)) \cong H^2(\Gamma(\mathfrak{a}), \Lambda(m))^2$ we get (1.1). The estimate (1.3) is proven exactly as in Lemma 7.2.

8. BOUNDING THE TORSION FROM ABOVE

In this section we proof equation (1.2) from Theorem A : we give the proof for $\bar{\Lambda}(m)$ -coefficients, the case of $\Lambda(m)$ follows immediately by Proposition 7.4. The main ingredient is the following lemma, usually attributed to O. Gabber and C. Soulé (we note that it is also an important tool in V. Emery's a priori bound for the torsion in the homology of certain arithmetic lattices, see [Eme14]). We refer the reader to [Sau16, Lemma 3.2] for a proof.

Lemma 8.1. *Let $A := \mathbb{Z}^a$ with standard basis $(e_i)_{i=1,\dots,a}$ and let $B := \mathbb{Z}^b$. Equip $B \otimes_{\mathbb{Z}} \mathbb{R}$ with the Euclidan norm $\|\cdot\|$. Let $\phi : A \rightarrow B$ be \mathbb{Z} -linear and assume that there exists $\alpha \in \mathbb{R}$ such that $\|\phi(e_i)\| \leq \alpha$ for each $i = 1, \dots, a$. Then one has*

$$|\text{coker}(\phi)_{\text{tors}}| \leq \alpha^{\min(a,b)}$$

Next, we have the following elementary Lemma.

Lemma 8.2. *Let $\{v_j := e_1^{m-i} e_2^j : i = 0, \dots, m\}$, denote the standard integral basis of the lattice $\Lambda_m \subset V(m)$. Equip $V(m)$ with the inner product such that the v_i form an orthonormal basis and let $\|\cdot\|_{\text{End}(V_m)}$ denote the corresponding norm on $\text{End}(V_m)$. Then for each $\gamma \in \Gamma_D$ one has*

$$\|\rho_m(\gamma)\|_{\text{End}(V_m)} \leq \|\rho_1(\gamma)\|_{\text{End}(V_1)}^m$$

Proof. This follows immediately from the definitions. \square

Now we can estimate the torsion from above as follows.

Proposition 8.3. *Let Γ be a finite index, torsion-free subgroup of Γ_D . Then there exists a constant c_Γ such that one can estimate*

$$\log |H^2(\Gamma, \bar{\Lambda}(m))_{\text{tors}}| \leq c_\Gamma m^2$$

for each $m \in \mathbb{N}$.

Proof. Let $X := \Gamma \backslash \mathbb{H}^3$. Let \mathcal{K} be a smooth triangulation of X and let $\tilde{\mathcal{K}}$ denote its lift to a smooth triangulation of \mathbb{H}^3 . For each q let $C_q(\mathcal{K}) := \{\sigma_{1,q}, \dots, \sigma_{N(\Gamma,q),q}\}$ denote the simplicial q -chains of \mathcal{K} , where $N(\Gamma, q) \in \mathbb{N}$ depends on \mathcal{K} . Let $C_q(\tilde{\mathcal{K}})$ denote the simplicial q -chains of $\tilde{\mathcal{K}}$ and let $\tilde{\partial}_q : C_q(\tilde{\mathcal{K}}) \rightarrow C_{q-1}(\tilde{\mathcal{K}})$ be the corresponding boundary operator. For each $\sigma_{i,q}$ we fix a $\tilde{\sigma}_{i,q} \in \tilde{\mathcal{K}}$ such that $\pi_*(\tilde{\sigma}_{i,q}) = \sigma_{i,q}$, where $\pi : \tilde{X} \rightarrow X$ is the covering map. The group Γ acts on $C_q(\mathcal{K})$ and we denote the corresponding action simply by \cdot . For each $\sigma_{i,q}$ there exist elements $\gamma_{k,q-1} \in \Gamma$, $k = 1, \dots, N(q-1, \Gamma)$, such that

$$\tilde{\partial}_q(\tilde{\sigma}_{i,q}) = \sum_{k=1}^{N(q-1,\Gamma)} \gamma_{k,q-1} \cdot \tilde{\sigma}_{k,q-1}.$$

Let $C_q(\mathcal{K}; \Lambda_m) := C_q(\mathcal{K}) \otimes_{\mathbb{Z}[\Gamma]} \Lambda_m$. Then the homology groups $H_*(\Gamma; \Lambda_m)$ are isomorphic to the homology groups $H_*(C_*(\mathcal{K}; \Lambda_m))$ of the complex

$$(C_*(\mathcal{K}; \Lambda_m), \partial_{*,\rho_m}) := (C_*(\mathcal{K}) \otimes_{\mathbb{Z}[\Gamma]} \Lambda_m, \tilde{\partial}_* \otimes \text{Id}).$$

Let $\{v_0, \dots, v_m\}$ denote the standard integral basis of Λ_m as in Lemma 8.2. Then an integral basis of $C_q(\mathcal{K}; \Lambda_m)$ is given by

$$B_q(\mathcal{K}; \Lambda_m) := \{\tilde{\sigma}_{i,q} \otimes v_j : i = 1, \dots, N(\Gamma, q) : j = 0, \dots, m\}.$$

We equip $C_q(\mathcal{K}; \Lambda_m) \otimes_{\mathbb{Z}} \mathbb{R}$ with the inner product for which $B_q(\mathcal{K}; \Lambda_m)$ is an orthonormal basis and denote the corresponding norm by $\|\cdot\|_{C_q(\mathcal{K}; V(m))}$. Then we have

$$\partial_{q;\rho_m}(\tilde{\sigma}_{i,q} \otimes v_j) = \sum_{k=1}^{N(q-1, \Gamma)} \tilde{\sigma}_{k,q-1} \otimes (\rho_m(\gamma_{k,q-1}^{-1})v_j)$$

and thus by the definition of the norms we have

$$(8.1) \quad \|\partial_{q;\rho_m}(\tilde{\sigma}_{i,q} \otimes v_j)\|_{C_{q-1}(\mathcal{K}; V(m))} \leq N(q-1, \Gamma) \max_{k=1, \dots, N(q-1, \Gamma)} \|\rho_m(\gamma_{k,q-1}^{-1})\|_{\text{End}(V_m)} \\ = N(q-1, \Gamma) \left(\max_{k=1, \dots, N(q-1, \Gamma)} \|\rho_1(\gamma_{k,q-1}^{-1})\|_{\text{End}(V_1)} \right)^m,$$

where the last step follows from Lemma 8.2. We put

$$c_0(\Gamma) := \max_q \max_{k=1, \dots, N(q-1, \Gamma)} \|\rho_1(\gamma_{k,q-1}^{-1})\|_{\text{End}(V_1)}.$$

Then $c_0(\Gamma)$ depends on Γ and the triangulation \mathcal{K} , but not on the local system Λ_m . If we apply Lemma 8.1 with $A := C_q(\mathcal{K}; \Lambda_m)$, $B := C_{q-1}(\mathcal{K}; \Lambda_m)$, $\phi := \partial_{q;\rho_m}$ and $\alpha := N(\Gamma, q-1)c_0(\Gamma)^m$, then, using that $\text{rk}_{\mathbb{Z}} A = (m+1)N(q, \Gamma)$, $\text{rk}_{\mathbb{Z}} B = (m+1)N(q-1, \Gamma)$ we obtain from (8.1) that

$$\left| (\text{coker}(\partial_{q;\rho_m}))_{\text{tors}} \right| \leq (N(\Gamma, q-1)c_0(\Gamma)^m)^{(m+1) \min\{N(q, \Gamma), N(q-1, \Gamma)\}}$$

For Λ_m^{\vee} one argues in the same way. Thus the proposition follows by applying the universal coefficient theorem. \square

Remark 1. Using the *KAK*-decomposition, it should be possible to generalize Lemma 8.2 and thus the proof of Proposition 8.3 to arithmetic subgroups Γ of arbitrary connected semisimple Liegroups G defined over \mathbb{Q} which satisfy $\delta(G) = 1$. For suitable rays $\rho_{\lambda}(m)$ of \mathbb{Q} -rational representations of G of highest weight $m\lambda$ with Γ -invariant integral lattices $\Lambda(\rho_{\lambda}(m))$, this should give an upper bound of the corresponding sizes of all cohomological torsion subgroups $H_{\text{tors}}^*(\Gamma, \Lambda(\rho_{\lambda}(m)))$ by $C(\Gamma)m \dim \rho_{\lambda}(m)$. Such a bound can be regarded as complementary to the lower bound obtained in the compact case in [MP14b].

Let $\mathfrak{a}_0 \in \mathcal{A}$ be as in the previous section. If we apply Proposition 8.3 for the group $\Gamma(\mathfrak{a}_0)$ instead of the group $\Gamma(\mathfrak{a})$ and also use Proposition 3.2, we can improve the constant in the upper bound of the size of $m^{-2} \log |H^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))_{\text{tors}}|$ and thus prove (1.2). Namely,

arguing similar as in the proof of (1.1) given in the previous section, we obtain

$$\begin{aligned} & \limsup_{m \rightarrow \infty} m^{-2} \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*|N(\mathfrak{a}_0)} \log |H_{tors}^2(\Gamma(\mathfrak{a}), \bar{\Lambda}(m))| - \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*|N(\mathfrak{a})} c(\Gamma_0) \\ & \leq \limsup_{m \rightarrow \infty} m^{-2} \left(-\frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*|N(\mathfrak{a}_0)} \log \tau_{Eis}(X_{\mathfrak{a}}; E_{\bar{\rho}(m)}) + \frac{[\Gamma_D : \Gamma(\mathfrak{a})]}{|\mathcal{O}_D^*|N(\mathfrak{a})} \log \tau_{Eis}(X_{\mathfrak{a}_0}; E_{\bar{\rho}(m)}) \right) \\ & \quad + \frac{[\Gamma_D : \Gamma(\mathfrak{a}_0)]}{|\mathcal{O}_D^*|N(\mathfrak{a}_0)} c(\Gamma(\mathfrak{a}_0)). \end{aligned}$$

Invoking Proposition 3.2, equation (1.2) follows.

APPENDIX A. GETTING EFFECTIVE BOUNDS FROM DAMERELL'S PAPER

Here we give a proof of Lemma 6.4 following [Dam70]. Let us recall the statement first. We are given an imaginary quadratic field F , and want to prove that there is $\Omega \in \mathbb{C}^\times$ such that for any even integer n and unitary Hecke character χ of F with infinite part $\chi_\infty(z) = (\bar{z}/z)^{n/2}$, for all integers $s \in \{0, \dots, n/2\}$ we have that the normalised L -value

$$\pi^{n/2-s} L(\chi, s) / \Omega^n$$

is an algebraic number, whose degree over \mathbb{Q} is bounded by a constant depending only on F and the conductor \mathfrak{f} of χ and whose absolute norm is bounded by $C(n!)^A$ for positive integers C, A depending only on F .

Before detailing Damerell's arguments we give a brief explanation: the main step is to express the L -function as a rational fraction whose arguments are elliptic functions associated to the elliptic curves \mathbb{C}/Λ where Λ are ideals in \mathcal{O}_F (see (†) and (*) below). When evaluating this expression we will evaluate the elliptic functions at finite-order points, where they take algebraic values of controlled degree.

From now on all our numbered references will be to Damerell's paper [Dam70]. Damerell's statement includes only the algebraicity, but his arguments give the full statement above as we shall now explain. Formula (6.2) yields the following expression for the normalised L -value occurring above :

$$(†) \quad \pi^{n/2-s} L(\chi, s) / \Omega^n = M \cdot \sum_{i=1}^h \hat{\mathfrak{A}}_i \chi_f(\hat{\mathfrak{A}}_i)^{-1} \sum_{\beta \in \hat{\mathfrak{A}}_i / \mathfrak{B}_i} \chi_f(\beta) \psi^p F_n(\beta \Omega, s, \Omega \mathfrak{B}_i)$$

where:

- M is an algebraic number depending on \mathfrak{f} and F ;
- $\mathfrak{A}_1, \dots, \mathfrak{A}_h$ are integral ideals representing the elements in the class-group of F ;
- $\mathfrak{B}_i = \mathfrak{f} \mathfrak{A}_i$;
- ψ is a number depending on F and $p = n/2 - s$;
- F_n is a particular function which we will analyse below ;
- $\Omega \in \mathbb{C}^\times$ is a particular number depending only on F .

The next step is Lemma 5.2, which yields an expression of $F_n(\cdot, \cdot, \Lambda)$ in terms of arithmetic invariants of the elliptic curve \mathbb{C}/Λ . More precisely, let \wp be the Weierstrass function associated to this elliptic curve, and let $s > 1$. Then we have :

$$(*) \quad \psi^p F_n(z, s, \Lambda) = \sum_{t+u+v=p} \frac{p!}{t!u!v!} h(z)^t (-\wp)^u (-1)^v K_{q-u}^v(z).$$

where :

- $h(z)$ is a rational fraction (with coefficients in \mathbb{Q}) in $\wp(kz), \wp'(kz), \wp''(kz)$ where $k = 1, \dots, \ell - 2$, with ℓ the exponent of the finite abelian group $\mathfrak{A}_i/\mathfrak{B}_i$ (see Lemma 4.3) ;
- K_j^i is a polynomial (with coefficients in \mathbb{Q}) in $h, \wp(z), \wp'(z), g_2(\Lambda)$ and $g_3(\Lambda)$ and φ (Corollary 4.1) ;
- φ is a constant depending on the curve \mathbb{C}/Λ .

The key fact is then that all points at which the various \wp occurring in (\dagger) are estimated are of bounded finite order on the elliptic curves: this yields algebraic equations for the relevant values of \wp and its derivative whose degree is bounded (depending on F and \mathfrak{f}). The values of g_2 and g_3 are algebraic of degree depending on the elliptic curve (this is where the choice of x enters, see the remark after Lemma 2.1). It then follows that the values of \wp'' are algebraic of bounded degree, as we can see by differentiating in z the equation

$$(d\wp/dz)^2 = 4\wp^3 - g_2\wp - g_3$$

satisfied by \wp , which yields

$$(b) \quad d^2\wp/dz^2 = 6\wp^2 - g_2/2$$

(cf. (3.10),(3.11)). All of this proves that the factors $h(z)$ and K_{q-u}^v in $(*)$ are algebraic of bounded (depending on F, \mathfrak{f}) degree. It remains to deal with φ . Choosing a $\tau \in \mathcal{O}_F$, equation (6.1) yields the expression

$$\varphi = (\tau\bar{\tau} - \tau^2)^{-1} \sum_{\substack{\rho \in \tau b\mathfrak{B}_i/x\mathfrak{B}_i \\ \rho \neq 0}} \wp(\rho/\tau)$$

which is algebraic of bounded degree. This finishes the proof that all factors of the summands in $(*)$ are algebraic of bounded degree. To finish the proof that the normalised L -value itself is so we need only note that since the character χ_f is of bounded finite order (depending on \mathfrak{f}), its values in (\dagger) are roots of unity of bounded degree. Thus all terms in (\dagger) are algebraic integers of bounded degree.

Now we must bound the absolute norm of the right-hand side in (\dagger) . It is obvious from $(\dagger), (*)$ and the proof that the degree is bounded that it suffices to prove that the valuation of K_i^j , for $2 \leq i + 2 \leq j \leq n$ is bounded by $C(n!)^4$ for some constant C . To do this we must return to the arguments of Damerell; in the proof of Lemma 3.3 he shows that for $j \geq 2$ one has

$$K_j^0(z) = (-1)^j \frac{d^{j-2}\wp(z)}{dz^{j-2}}.$$

From this an easy recursive argument using the identity (b) allows to prove that for $j \geq 2$ $K_j^0(z)$ is a polynomial in $\wp(z)$, $\wp'(z)/2$ and $g_2/12$ of degree less than $2j$ in each variable, with coefficients in \mathbb{Z} that are $\ll (2j)! \cdot N^j$ for some integer $N \in \mathbb{Z}_{>0}$; we will denote this polynomial by $P_j^0 \in \mathbb{Z}[X_1, \dots, X_4]$ (the last variable represents $g_3/4$).

Damerell proves that for $0 \leq i < j$ there is a polynomial $P_j^i \in \mathbb{Z}[X_1, \dots, X_4]$ such that $K_j^i(z) = P_j^i(\wp(z), \wp'(z)/2, g_2/12, g_3/4)$. For this he uses the recurrence relation (3.12), which is :

$$(\sharp) \quad K_{j+1}^{i+1}(z) = i \frac{\wp'(z)}{2} \cdot \frac{1}{j} K_j^{i-1}(z) - i \frac{g_2}{12} \cdot \frac{1}{j} K_{j-1}^{i-1}(z) - \frac{1}{j} D K_j^i(z)$$

where D is a differential operator (in both second variables of \wp). It is given explicitly for \wp, \wp', g_2 and g_3 in the equalities (3.7), which we rewrite here :

$$\begin{aligned} Dg_2 &= -6g_3, & Dg_3 &= -\frac{1}{3}g_2^2, \\ D\wp &= -2\wp^2 - \frac{g_2}{3}, & D\wp' &= -3\wp \cdot \wp'. \end{aligned}$$

Together with (\sharp) these finally yield that the degree of P_j^i in each variable is less than $2(i+j)$ and the coefficients are majorised by $2(i+j)!N^{(i+j)}$ for some $N \in \mathbb{Z}_{>0}$. It follows that for the values of z occuring in (\dagger) we have $|K_{q-u}^v(z)| \ll n!N^n$ at each place, hence the absolute norm is bounded by $(n!N^n)^{[E:\mathbb{Q}]}$. This finishes the proof of our statement.

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