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Sébastien Boucksom, Mattias Jonsson

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A NON-ARCHIMEDEAN APPROACH TO K-STABILITY

SÉBASTIEN BOUCKSOM AND MATTIAS JONSSON

Abstract. We study K-stability properties of a smooth Fano variety X using non-Archimedean geometry, specifically the Berkovich analytification of X with respect to the trivial absolute value on the ground field. More precisely, we view K-semistability and uniform K-stability as conditions on the space of plurisubharmonic (psh) metrics on the anticanonical bundle of X. Using the non-Archimedean Calabi–Yau theorem and the Legendre transform, this allows us to give a new proof that K-stability is equivalent to Ding stability. By choosing suitable psh metrics, we also recover the valuative criterion of K-stability by Fujita and Li. Finally, we study the asymptotic Fubini–Study operator, which associates a psh metric to any graded filtration (or norm) on the anticanonical ring. Our results hold for arbitrary smooth polarized varieties, and suitable adjoint/twisted notions of K-stability and Ding stability. They do not rely on the Minimal Model Program.

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Introduction

Consider a polarized, smooth projective variety (X, L) defined over a field k of characteristic zero. For the purposes of this introduction, we identify (X, L) with its analytification—in the sense of Berkovich—with respect to the *trivial* absolute value on k. As a set, X is thus the disjoint union of the spaces of real-valued valuations on the function field of each subvariety of X. Our goal in this paper is to use plurisubharmonic (psh) metrics on L to study problems involving K-stability.

We start by recalling some notions and results from [BoJ18]. There is a canonical map $L \to X$, with fibers being Berkovich affine lines. A metric on L is a function on L satisfying a certain homogeneity property on each fiber. Since k is trivially valued, L admits a canonical *trivial metric*, which we use to identify metrics on L with *functions* on X. For such metrics/functions, there are several natural positivity notions.

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First we have the class $\mathcal{H}(L)$ of *positive metrics* [BHJ17]. They are defined via ample *test configurations* [Don02] for (X, L). In [BoJ18] they are called *Fubini–Study metrics*, as each positive metric is built from a finite set of global sections of some multiple of L.

By taking uniform limits of positive metrics we obtain the class of *continuous psh metrics*, a notion going back to Zhang [Zha95] and Gubler [Gub98], and explored much more generally by Chambert-Loir and Ducros [CD12], see also [GK15, GM16]. By instead taking *decreasing* limits of positive metrics, we obtain the larger class $\mathrm{PSH}(L)$ of psh metrics on L. This class has many good properties, as studied in [BoJ18], see also [BFJ16a, BoJ18] for the discretely valued case. As in the usual complex case, psh metrics are not necessarily continuous, and may even be *singular* in the sense of taking the value $-\infty$ at certain points in X.

The class $\mathrm{PSH}(L)$ contains several natural subclasses. First, we have the continuous psh metrics. A larger class is formed by the *bounded* psh metrics, i.e. psh metrics that are bounded as functions on X. This class is, in turn, contained in the class $\mathcal{E}^1(L)$ of psh metrics φ of *finite energy*, $E(\varphi) > -\infty$. Here E is the *Monge–Ampère energy* functional. For positive metrics, it can be defined in terms of intersection numbers, is monotonous, and hence has an extension to all of $\mathrm{PSH}(L)$, with values in $\mathbf{R} \cup \{-\infty\}$. Our thesis is that the class $\mathcal{E}^1(L)$ is quite natural for problems in K-stability, to be discussed in detail below. The functional defined by $J(\varphi) = \sup_X \varphi - E(\varphi)$ acts as an exhaustion function on $\mathcal{E}^1(L)/\mathbf{R}$. In particular, $J(\varphi) \geq 0$, with equality iff φ is constant.

The *Monge–Ampère operator* assigns a Radon probability measure $\mathrm{MA}(\varphi)$ on X to any metric $\varphi \in \mathcal{E}^1(L)$. It is continuous under monotone limits and extends the operator introduced by Chambert-Loir [Cha06] and Gubler [Gub07]. The *Calabi–Yau* theorem [BFJ15, BoJ18] asserts that we have a bijection $\mathrm{MA}\colon \mathcal{E}^1(L)/\mathbf{R} \to \mathcal{M}^1(X)$, where $\mathcal{M}^1(X)$ is the space of Radon probability measures μ of *finite energy*, i.e. such that

$$E^*(\mu) := \sup_{\varphi \in \mathcal{E}^1(L)} \left(E(\varphi) - \int \varphi \, d\mu \right) < \infty.$$

K-stability and Ding stability. The notion of K-stability was introduced by Yau, Tian and Donaldson, as a conjectural criterion for the existence of special metrics in Kähler geometry; this conjecture was proved in the Fano case, see [CDS15, Tia15] and also [DS16, BBJ15, CSW15]. K-stability involves studying the sign of the *Donaldson–Futaki functional* on the space of ample test configurations for (X, L), i.e. on the space $\mathcal{H}(L)$. In [BHJ17] we showed that K-stability can be expressed in terms of a modified functional, the (non-Archimedean) *Mabuchi functional*, that has better properties with respect to base change, and which naturally extends to a functional $M : \mathcal{E}^1(L) \to \mathbf{R} \cup \{+\infty\}$.

In the Fano case, when $L = -K_X$, the Mabuchi functional factors through the Monge–Ampère operator as

$$M(\varphi) = \mathrm{Ent}\left(\mathrm{MA}(\varphi)\right) - E^*\left(\mathrm{MA}(\varphi)\right),$$

where $E^*(\mu)$ is the energy of $\mu \in \mathcal{M}^1(X)$, whereas $\mathrm{Ent}(\mu) := \int_X A \, d\mu$ is the *entropy* of μ, defined as the integral of the *log discrepancy* function $A\colon X \to \mathbf{R}_+ \cup \{+\infty\}$.

Another functional, the (non-Archimedean) *Ding functional* is also useful. First defined in [Berm16], it can be written as $D = L - E$, where E is the Monge–Ampère energy, and L is the Legendre transform of entropy: $L(\varphi) = \inf_X(\varphi + A)$, with A the log discrepancy.

In this language, a Fano manifold X is K-semistable (resp. uniformly K-stable) if $M \geq 0$ (resp. there exists $\varepsilon > 0$ such that $M \geq \varepsilon J$) on $\mathcal{H}(L)$. Similarly, X is Ding-semistable (resp. uniformly Ding-stable) if the corresponding inequalities hold with the Mabuchi functional M replaced by the Ding functional D. It was proved in [BBJ15] (see also [Fuj16]), using

techniques from the Minimal Model Program along the same lines as [LX14], that Ding semistability is equivalent to K-semistability, and similarly for the uniform versions.

Here we give a new proof of (a version of) the equivalence of Ding-stability and K-stability. Let L be any ample line bundle on X. The definitions of the functionals M and D still make sense; we call them the *adjoint Mabuchi functional* and *adjoint Ding functional*, respectively. We say that (X, L) is *K-semistable in the adjoint sense* if the adjoint Mabuchi functional is nonnegative on $\mathcal{E}^1(L)$. Similarly, we define adjoint versions of Ding semistability, and uniform Ding and K-stability.

Outside the Fano case, the adjoint Mabuchi functional differs from the Mabuchi functional discussed in the beginning of this section. Up to a conjectural approximation result (Conjecture 2.5), adjoint K-stability is equivalent to Dervan's notion of *twisted K-stability* [Der16] in the 'twisted Fano case', i.e. when the (not necessarily semipositive) twisting class T is defined so that $L = -(K_X + T)$, see §2.9.

Theorem A. *For any ample line bundle L on a smooth projective variety X, we have:*

(i) *L is K-semistable in the adjoint sense iff it is Ding-semistable in the adjoint sense;*
(ii) *L is uniformly K-stable in the adjoint sense iff it is uniformly Ding-stable in the adjoint sense.*

The proof is an adaptation to the non-Archimedean case of ideas of Berman [Berm13], who was inspired by thermodynamics. It is based on the non-Archimedean Calabi–Yau theorem together with the fact that the functionals L and E on $\mathcal{E}^1(L)$ are the Legendre transforms of the functionals Ent and E^*, respectively, on $\mathcal{M}^1(X)$.

The adjoint stability notions above were defined in terms of metrics of finite energy. This is because the Calabi–Yau theorem is an assertion about metrics and measures of finite energy. It is natural to ask whether the adjoint stability notions remain unchanged if we replace $\mathcal{E}^1(L)$ by the space $\mathcal{H}(L)$ (i.e. ample test configurations). The answer is 'yes' for adjoint Ding stability, since the Ding functional is continuous under decreasing limits. In the Fano case, the answer is also 'yes' by [BBJ15, Fuj16]. We expect the answer to be 'yes' in general, even though the Mabuchi functional is not continuous under decreasing limits.

Filtrations, norms, and metrics. Donaldson, Székelyhidi and others have suggested to strengthen the notion of K-stability by allowing more general objects than test configurations. One generalization is the class $\mathcal{E}^1(L)$ above. Another one, explored by Székelyhidi [Szé15], is given by (graded) *filtrations* of the section ring $R = R(X, L)$. Such filtrations are (see §3) in 1–1 correspondence with *graded norms* $\|\cdot\|_\bullet$ on R, i.e. the data of a non-Archimedean k-vector space norm $\|\cdot\|_m$ on $R_m := H^0(X, mL)$ for each $m \geq 1$ such that $\|s \otimes s'\|_{m+m'} \leq \|s\|_m \|s'\|_{m'}$ for $s \in R_m$, $s' \in R_{m'}$. A graded norm is *bounded* if there exists $C \geq 1$ such that $C^{-m} \leq \|\cdot\|_m \leq C^m$ on $R_m \setminus \{0\}$ for all m. To such a graded norm we associate a bounded psh metric on L, the *asymptotic Fubini–Study metric* $\mathrm{FS}(\|\cdot\|_\bullet)$.

Our next result characterizes the range of the asymptotic Fubini–Study operator. A bounded metric $\varphi \in \mathrm{PSH}(L)$ is *regularizable from below* if it is the limit of an increasing net of positive metrics. For example, any continuous psh metric is regularizable from below. The analogous notion on domains in $\mathbf{C}^n$ was studied by Bedford [Bed80].

Theorem B. *A psh metric on L lies in the image of the asymptotic Fubini–Study operator iff it is regularizable from below.*

It follows from the construction of the asymptotic Fubini–Study operator that any metric in its image is regularizable from below. To prove Theorem B we construct a one-sided

inverse, the *supremum graded norm* $\|\cdot\|_{\varphi,\bullet}$ of any bounded $\varphi \in \mathrm{PSH}(L)$. The formula $\varphi = \mathrm{FS}(\|\cdot\|_{\varphi,\bullet})$ can be viewed as a non-Archimedean L^∞-version of Bergman kernel asymptotics.

The asymptotic Fubini–Study operator is not injective in general, and we study the lack of injectivity. To each bounded graded norm is associated a *limit measure* [BC11], a probability measure on $\mathbf{R}$ that describes the asymptotic distribution of norms of vectors in R_m as $m \to \infty$. More generally, Chen and Maclean [CM15] showed that any two bounded graded norms on R induce a *relative limit measure*. The second moment of this measure (i.e. of the corresponding random variable) defines a semidistance d_2 on the space of bounded graded norms. More generally, there is a semidistance d_p for any $p \in [1, \infty)$. One can show that two bounded graded norms are at d_p-distance zero for some p iff they are so for all p; in this case the graded norms are called *equivalent*. For $p = 2$, the semidistance d_2 on bounded graded norms coincides with the limit pseudo-metric introduced in [Cod18].

Theorem C. *Two bounded graded norms induce the same associated Fubini–Study metric iff they are equivalent.*

This theorem is proved by exhibiting a formula for the d_1-semidistance:

$$d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = E(\varphi, Q(\varphi \wedge \varphi')) + E(\varphi', Q(\varphi \wedge \varphi')), \tag{0.1}$$

where $\varphi := \mathrm{FS}(\|\cdot\|_\bullet)$, $\varphi' := \mathrm{FS}(\|\cdot\|'_\bullet)$, and $Q(\varphi \wedge \varphi')$ is the largest psh metric that is regularizable from below and dominated by $\varphi \wedge \varphi' := \min\{\varphi, \varphi'\}$. In fact, the right-hand side of (0.1) defines a distance on the space $\mathrm{PSH}^\uparrow(L)$ of psh metrics regularizable from below. This is analogous to the *Darvas distance* in the complex analytic case [Dar15]. The asymptotic Fubini–Study operator now becomes an isometric bijection between the space of equivalence classes of bounded graded norms and the space $\mathrm{PSH}^\uparrow(L)$.

To prove (0.1), the main step is to establish the equality

$$E(\varphi, \varphi') = \mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet), \tag{0.2}$$

where the *relative volume* $\mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ is the barycenter of the relative limit measure. By taking the two graded norms as supremum norms of continuous metrics on L, we recover a version of the main results of [BE18, BGJKM16] in the trivially valued case.

The L^2-norm of a (graded) filtration introduced by Székelyhidi is equal to the variance of the relative limit measure with respect to the trivial graded filtration. It follows that *a filtration has L^2-norm zero iff its associated Fubini–Study metric is constant.* Székelyhidi also defined a notion of (Donaldson–)Futaki invariant of a (graded) filtration $\mathcal{F}$. We show that if X is uniformly K-stable, then the Donaldson–Futaki invariant is strictly positive for every filtration of positive norm.

A valuative criterion for adjoint K-stability. Next we study the valuative criterion for K-stability of Fujita [Fuj16] and Li [Li17] using psh metrics. There is a subset $X^{\mathrm{val}} \subset X$ consisting of *valuations* of the function field of X (the latter being in fact the disjoint union of Y^{val}, as Y ranges over irreducible subvarieties of X). To each point $x \in X^{\mathrm{val}}$, we can associate several invariants. First we have the log discrepancy $A(x) \in [0, +\infty]$. Second, given an ample line bundle L on X, the valuation x defines a graded norm on the section ring $R = R(X, L)$, see [BKMS16]. When bounded, this norm induces a limit measure on $\mathbf{R}$, whose barycenter is denoted by $S(x) \in [0, +\infty)$, and can be viewed as the *expected vanishing order* of elements of R along x. When the filtration is unbounded, we set $S(x) = +\infty$.

Theorem D. *For any point $x \in X^{\mathrm{val}}$, we have $S(x) = E^*(\delta_x)$, the energy of the Dirac mass at x. We have $S(x) = \infty$ iff the point x is pluripolar,* i.e. *there exists $\varphi \in \mathrm{PSH}(L)$ with $\varphi(x) = -\infty$. If $S(x) < \infty$, then the unique solution to the Monge–Ampère equation*

$$\mathrm{MA}(\varphi_x) = \delta_x$$

normalized by $\varphi_x(x) = 0$ is continuous.

The adjoint Mabuchi functional can be written as $M^{\mathrm{ad}}(\varphi) = \mathrm{Ent}(\mathrm{MA}(\varphi)) - E^*(\mathrm{MA}(\varphi))$. By the Calabi–Yau theorem, L is therefore K-semistable in the adjoint sense iff $\mathrm{Ent}(\mu) \geq E^*(\mu)$ for all $\mu \in \mathcal{M}^1(X)$. Note also that $M^{\mathrm{ad}}(\varphi_x) = A(x) - S(x)$. By studying the entropy and energy functionals, we prove

Theorem E. *The following conditions are equivalent:*

(i) *L is K-semistable in the adjoint sense;*
(ii) *$M^{\mathrm{ad}}(\varphi_x) \geq 0$ for all nonpluripolar $x \in X^{\mathrm{val}}$;*
(iii) *$A(x) \geq S(x)$ for all nonpluripolar $x \in X^{\mathrm{val}}$;*
(iv) *$A(x) \geq S(x)$ for all divisorial valuations $x \in X^{\mathrm{val}}$.*

Theorem E'. *The following conditions are equivalent:*

(i) *L is uniformly K-stable in the adjoint sense;*
(ii) *there exists $\varepsilon > 0$ such that $M^{\mathrm{ad}}(\varphi_x) \geq \varepsilon J(\varphi_x)$ for all nonpluripolar $x \in X^{\mathrm{val}}$;*
(iii) *there exists $\varepsilon > 0$ such that $A(x) \geq (1+\varepsilon)S(x)$ for all for all nonpluripolar $x \in X^{\mathrm{val}}$;*
(iv) *there exists $\varepsilon > 0$ such that $A(x) \geq (1+\varepsilon)S(x)$ for all divisorial valuations $x \in X^{\mathrm{val}}$.*

Theorem E and Theorem E' generalize Li and Fujita's valuative criterion [Li17, Fuj16] for K-stability to the adjoint setting. Our proof is completely different from theirs. However, we should mention that Fujita also proves that in (iv), it suffices to consider "dreamy" valuations x. These are points $x \in X^{\mathrm{div}}$ such that $\varphi_x \in \mathcal{H}(L)$.

Adjoint K-stability and uniform K-stability. Above we have used adjoint notions of K-semistability and uniform K-stability. As an intermediate notion, we say that L is *K-stable* in the adjoint sense if $M^{\mathrm{ad}}(\varphi) \geq 0$ for all $\varphi \in \mathcal{E}^1(L)$, with equality iff φ is constant.

Theorem F. *If $k = \mathbf{C}$, then L is K-stable in the adjoint sense iff L is uniformly K-stable in the adjoint sense.*

This relies on [BlJ17, Theorem E], which guarantees that the ratio $A(x)/S(x)$ attains its minimum at some nonpluripolar point $x \in X^{\mathrm{val}}$.

In the Fano case, it follows from [CDS15] and [BHJ16] that K-stability is equivalent to uniform K-stability. However, there seems to be no algebraic proof of this fact. We would get such a proof if we knew that the point x above was a "dreamy" divisorial valuation in the sense of Fujita, i.e. $\varphi_x \in \mathcal{H}(L)$.

Approaches to K-stability. We have tried to demonstrate that psh metrics form a useful tool for studying K-stability. They encompass not only test configurations, but also graded filtrations/norms on the section ring, and nonpluripolar valuations. Another way of studying K-stability of Fano varieties was introduced by Chi Li [Li15, Li17] and involves considering the cone over the variety. The cone point is a klt singularity, and there has been much recent activity on the study of general klt singularities [Blu16, LL16, LX17, BL18], but we shall not discuss this further here.

Organization of paper. After reviewing some background material from [BoJ18] in §1 we start in §2 by proving Theorem A on the equivalence of adjoint K-stability and Ding-stability. Then, in §§3–4, we study the relationship between graded norms/filtrations and psh metrics, proving Theorems B and C. Finally, the valuative criterion for K-stability (Theorems D, E and E') is proved in §5, as is Theorem F.

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1. Background

For details on the material in this section, see [BoJ18].

1.1. **Setup.** Throughout the paper, k is field of characteristic zero, equipped with the *trivial* absolute value. By a k-variety we mean an integral separated scheme of finite type over k. We fix a logarithm $\log\colon \mathbf{R}_+^\times \to \mathbf{R}$, with inverse $\exp\colon \mathbf{R} \to \mathbf{R}_+^\times$.

1.2. **Analytification.** The analytification functor in [Berk90, §3.5] associates to any k-variety a k-analytic space in the sense of Berkovich. Typically we write X for the analytification and X^{sch} for the underlying variety, viewed as a scheme. Let $\mathbf{A}^n = \mathbf{A}^n_k$, $\mathbf{P}^n = \mathbf{P}^n_k$ and $\mathbf{G}_m = \mathbf{G}_{m,k}$ be the analytifications of $\mathbb{A}^n = \mathbb{A}^n_k$, $\mathbb{P}^n = \mathbb{P}^n_k$, $\mathbb{G}_m = \mathbb{G}_{m,k}$, respectively.

The analytification X of X^{sch} consists of all pairs $x = (\xi, |\cdot|)$, where $\xi \in X^{\mathrm{sch}}$ is a point and $|\cdot| = |\cdot|_x$ is a multiplicative norm on the residue field $\kappa(\xi)$ extending the trivial norm on k. We denote by $\mathcal{H}(x)$ the completion of $\kappa(\xi)$ with respect to this norm. The surjective map $\ker\colon X \to X^{\mathrm{sch}}$ sending $(\xi, |\cdot|)$ to ξ is called the *kernel* map. The points in X whose kernel is the generic point of X^{sch} are the valuations of the function field of X^{sch} that are trivial on k. They form a subset $X^{\mathrm{val}} \subset X$.

There is a section $X^{\mathrm{sch}} \hookrightarrow X$ of the kernel map, defined by associating to $\xi \in X^{\mathrm{sch}}$ the point in X defined by the trivial norm on $\kappa(\xi)$. The image of the generic point of X^{sch} is called the generic point of X: it corresponds to the trivial valuation on $k(X)$.

If $X^{\mathrm{sch}} = \operatorname{Spec} A$ is affine, with A a finitely generated k-algebra, X consists of all multiplicative seminorms on A extending the trivial norm on k.

The *Zariski topology* on X is the weakest topology in which $\ker\colon X \to X^{\mathrm{sch}}$ is continuous. We shall work in *Berkovich* topology, the coarsest refinement of the Zariski topology for which the following holds: for any open affine $\mathcal{U} = \operatorname{Spec} A \subset X^{\mathrm{sch}}$ and any $f \in A$, the function $\ker^{-1}(\mathcal{U}) \ni x \to |f(x)| \in \mathbf{R}_+$ is continuous, where $f(x)$ denotes the image of f in $k(\xi) \subset \mathcal{H}(x)$, so that $|f(x)| = |f|_x$. The subset $X^{\mathrm{val}} \subset X$ is dense. In general, X is Hausdorff, locally compact, and locally path connected; it is compact iff X^{sch} is proper. We say X is projective (resp. smooth) if X^{sch} has the corresponding properties.

When X is compact, there is a *reduction map* $\mathrm{red}\colon X \to X^{\mathrm{sch}}$, defined as follows. Let $x \in X$ and set $\xi := \ker x \in X^{\mathrm{sch}}$, so that x defines valuation on $\kappa(\xi)$ that is trivial on k. If $\mathcal{Y}$ is the closure of ξ in X^{sch}, then $\eta = \mathrm{red}(x) \in \mathcal{Y} \subset X^{\mathrm{sch}}$ is the unique point such that $|f(x)| \le 1$ for $f \in \mathcal{O}_{\mathcal{Y},\eta}$ and $|f(x)| < 1$ when further $f(\eta) = 0$.

1.3. **Divisorial and quasimonomial points.** If $x \in X^{\mathrm{val}}$, let $s(x) := \mathrm{tr.deg}(\widetilde{\mathcal{H}(x)}/k)$ be the transcendence degree of x and $t(x) := \dim_{\mathbf{Q}} \sqrt{|\mathcal{H}(x)^\times|}$ the rational rank. The *Abhyankar inequality* says that $s(x) + t(x) \leq \dim X$. A point x is *quasimonomial* if equality holds. It is *divisorial* if $s(x) = \dim X - 1$ and $t(x) = 1$. We write X^{qm} and X^{div} for the set of quasimonomial and divisorial points, respectively. Thus $X^{\mathrm{div}} \subset X^{\mathrm{qm}} \subset X^{\mathrm{val}}$.

From now on, assume X is smooth and projective. Then any divisorial point is of the form $\exp(-c\,\mathrm{ord}_E)$, where $c > 0$ and E is a prime divisor on a smooth projective variety $\mathcal{Y}$ that admits a proper birational map onto X^{sch}.

Similarly, quasimonomial points can be geometrically described as follows, see [JM12]. A *log smooth pair* $(\mathcal{Y}, \mathcal{D})$ over X^{sch} is the data of a smooth k-variety $\mathcal{Y}$ together with a proper birational morphism $\pi\colon \mathcal{Y} \to X^{\mathrm{sch}}$, and $\mathcal{D}$ a reduced simple normal crossings divisor on $\mathcal{Y}$ such that π is an isomorphism outside the support of $\mathcal{D}$. To such a pair we associate a *dual cone complex* $\Delta(\mathcal{Y}, \mathcal{D})$, the cones of which are in bijection with strata (i.e. connected components of intersections of irreducible components) of $\mathcal{D}$. We embed $\Delta(\mathcal{Y}, \mathcal{D})$ into X^{val} as the set of monomial points with respect to local equations of the irreducible components of $\mathcal{D}$ at the generic point of the given stratum. The apex of $\Delta(\mathcal{Y}, \mathcal{D})$ is the generic point of X. We have $x \in X^{\mathrm{qm}}$ iff $x \in \Delta(\mathcal{Y}, \mathcal{D})$ for some log smooth pair $(\mathcal{Y}, \mathcal{D})$ over X^{sch}.

For many purposes, it is better to describe divisorial and quasimonomial points using snc test configurations, see below.

1.4. **Test configurations.** A *test configuration* $\mathcal{X}$ for X consists of the following data:

(i) a flat and projective morphism of k-schemes $\pi : \mathcal{X} \to \mathbb{A}^1_k$;
(ii) a $\mathbb{G}_m$-action on $\mathcal{X}$ lifting the canonical action on $\mathbb{A}^1_k$;
(iii) an isomorphism $\mathcal{X}_1 \xrightarrow{\sim} X$.

The *trivial* test configuration for X is given by the product $X^{\mathrm{sch}} \times \mathbb{A}^1_k$, with the trivial $\mathbb{G}_m$-action on X^{sch}. Given test configurations $\mathcal{X}$, $\mathcal{X}'$ for X there exists a unique $\mathbb{G}_m$-equivariant birational map $\mathcal{X}' \dashrightarrow \mathcal{X}$ extending the isomorphism $\mathcal{X}'_1 \simeq X \simeq \mathcal{X}_1$. We say that $\mathcal{X}'$ *dominates* $\mathcal{X}$ if this map is a morphism. Any two test configurations can be dominated by a third. Two test configurations that dominate each other will be identified.

For any test configuration $\mathcal{X}$ for X, we have a *Gauss embedding* $\sigma_{\mathcal{X}}\colon X \to \mathcal{X}^{\mathrm{an}}$, whose image consists of all $k^\times$-invariant points $y \in \mathcal{X}^{\mathrm{an}}$ satisfying $|\varpi(y)| = \exp(-1)$, where ϖ is the coordinate on $\mathbb{A}^1$. Each such point has a well-defined reduction $\mathrm{red}_{\mathcal{X}}(y) \in \mathcal{X}_0$. Somewhat abusively, we denote the composition $\mathrm{red}_{\mathcal{X}} \circ \sigma_{\mathcal{X}}\colon X \to \mathcal{X}_0$ by $\mathrm{red}_{\mathcal{X}}$. When $\mathcal{X}$ is the trivial test configuration, $\mathcal{X}_0 \simeq X$, and $\mathrm{red}_{\mathcal{X}}$ coincides with the reduction map considered earlier.

1.5. **Snc test configurations.** An *snc test configuration* is a test configuration $\mathcal{X}$ that dominates the trivial test configuration and whose central fiber $\mathcal{X}_0$ has strict normal crossing support. By Hironaka's theorem, the set $\mathrm{SNC}(X)$ of snc test configurations is directed and cofinal in the set of all test configurations.

To any snc test configuration $\mathcal{X}$ of X is associated a *dual complex* $\Delta_{\mathcal{X}}$, a simplicial complex whose simplices are in 1-1 correspondence with strata of $\mathcal{X}_0$, i.e. connected components of nonempty intersections of irreducible components of $\mathcal{X}_0$. For example, the dual complex of the trivial test configuration has a single vertex, corresponding to single stratum $X^{\mathrm{sch}} \times \{0\}$.

We can view $(\mathcal{X}, \mathcal{X}_0)$ as a log smooth pair over $\mathcal{X}$, whose dual cone complex $\Delta(\mathcal{X}, \mathcal{X}_0)$ is the cone over the dual complex $\Delta_{\mathcal{X}}$. The $\mathbb{G}_m$-equivariance of $\mathcal{X}$ implies that the image of $\Delta(\mathcal{X}, \mathcal{X}_0)$ in $\mathcal{X}^{\mathrm{an}}$ consists of $k^\times$-invariant points. We identify $\Delta_{\mathcal{X}}$ with the subset of $\Delta(\mathcal{X}, \mathcal{X}_0) \subset \mathcal{X}^{\mathrm{an}}$ cut out by the equation $|\varpi| = \exp(-1)$. Via the Gauss embedding

$\sigma_{\mathcal{X}}\colon X \to X^{\mathrm{an}}$, this allows us to view the dual complex $\Delta_{\mathcal{X}}$ as a subset of X. The points in $\Delta_{\mathcal{X}}$ are all quasimonomial, and every quasimonomial point belongs to some dual complex. Similarly we define a *retraction* $p_{\mathcal{X}}\colon X \to \Delta_{\mathcal{X}}$. The directed system $(p_{\mathcal{X}})_{\mathcal{X}\in\mathrm{SNC}(X)}$ induces a homeomorphism $X \xrightarrow{\sim} \varprojlim \Delta_{\mathcal{X}}$.

1.6. **Metrics on line bundles.** We view a line bundle L on X as a k-analytic space with a projection $p\colon L \to X$. Its fiber over $x \in X$ is isomorphic to the Berkovich affine line over the complete residue field $\mathcal{H}(x)$. Write $L^\times$ for L with the zero section removed. We typically work additively, so that a metric on L is function $\phi\colon L^\times \to \mathbf{R}$ such that $|\cdot|_\phi := \exp(-\phi)\colon L^\times \to \mathbf{R}_+^\times$ behaves like a norm on each fiber $p^{-1}(x)$.

If ϕ is a metric on L, any other metric is of the form $\phi + \varphi$, where φ is a function on X, that is, a metric on $\mathcal{O}_X$. If ϕ_i is a metric on L_i, $i = 1, 2$, then $\phi_1 + \phi_2$ is a metric on $L_1 + L_2$. If ϕ is a metric on L, then $-\phi$ is a metric on $-L$. If ϕ is a metric on mL, where $m \geq 1$, then $m^{-1}\phi$ is a metric on L. If ϕ_1, ϕ_2 are metrics on L, so is $\max\{\phi_1, \phi_2\}$.

If ϕ_1 and ϕ_2 are metrics on L inducing the same metric $m\phi_1 = m\phi_2$ on mL for some $m \geq 1$, then $\phi_1 = \phi_2$. We can therefore define a metric on a $\mathbf{Q}$-line bundle L as the choice of a metric ϕ_m on mL for m sufficiently divisible, with the compatibility condition $l\phi_m = \phi_{ml}$.

1.7. **Metrics as functions.** Any line bundle L on X admits a *trivial metric* ϕ_{triv} defined as follows. Given a point $x \in X$, set $\xi := \mathrm{red}(x) \in X^{\mathrm{sch}}$, and let t be a nonvanishing section of L on an open neighborhood $\mathcal{U} \subset X^{\mathrm{sch}}$ of ξ. Then t defines a nonvanishing analytic section of L on the Zariski open neighborhood $\mathcal{U}^{\mathrm{an}}$ of x in X, and $\phi_{\mathrm{triv}}(t(x)) = 0$.

As multiplicative notation for the trivial metric, we use $|\cdot|_0 = \exp(-\phi_{\mathrm{triv}})$. Given a global section s of L, the function $|s|_0$ on X has the following properties. Given $x \in X$, pick $\mathcal{U}$ and t as above, and write $s = ft$, where $f \in \Gamma(\mathcal{U}, \mathcal{O}_{X^{\mathrm{sch}}})$. Then $|s|_0(x) = |f(x)|$.

The trivial metric allows us to think of metrics on L as *functions* on X, and we shall frequently do so in what follows. Indeed, if ϕ is a metric on L, then $\varphi := \phi - \phi_{\mathrm{triv}}$ is a function on X. In this way, the trivial metric becomes the zero function.

1.8. **Metrics from test configurations.** A *test configuration* for a $\mathbf{Q}$-line bundle L on X consists of a test configuration $\mathcal{X}$ for X together with the following data:

(iv) a $\mathbb{G}_m$-linearized $\mathbf{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$;
(v) an isomorphism $\mathcal{L}|_{\mathcal{X}_1} \simeq L$.

If more precision is needed, we say that $(\mathcal{X}, \mathcal{L})$ is a test configuration for (X, L).

Given a $\mathbb{G}_m$-action on $(X^{\mathrm{sch}}, L^{\mathrm{sch}})$ we have a *product test configuration* $(X^{\mathrm{sch}} \times \mathbb{A}^1, L^{\mathrm{sch}} \times \mathbb{A}^1)$, with diagonal action. If the action on $(X^{\mathrm{sch}}, L^{\mathrm{sch}})$ is trivial, we obtain the *trivial* test configuration for (X, L). A test configuration $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{L}})$ *dominates* another test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L), if $\widetilde{\mathcal{X}}$ dominates $\mathcal{X}$ and $\widetilde{\mathcal{L}}$ is the pullback of $\mathcal{L}$ under $\widetilde{\mathcal{X}} \to \mathcal{X}$.

Any test configuration $\mathcal{L}$ for L induces a metric $\varphi_{\mathcal{L}}$ (viewed as function on $L^\times$) on L as follows. First assume $\mathcal{L}$ is a line bundle. Using the Gauss embedding $\sigma_{\mathcal{L}}\colon L \to \mathcal{L}^{\mathrm{an}}$ it suffices to define a metric $\varphi_{\mathcal{L}}$ on $\mathcal{L}^{\mathrm{an}}$. Pick any point $y \in \mathcal{X}^{\mathrm{an}}$ with $|\varpi(y)| = \exp(-1)$, write $\xi := \mathrm{red}_{\mathcal{X}}(y) \in \mathcal{X}_0$, and let s be a section of $\mathcal{L}$ on a Zariski open neighborhood of ξ in $\mathcal{X}$ such that $s(\xi) \neq 0$. Then s defines a section of $\mathcal{L}^{\mathrm{an}}$ on a Zariski open neighborhood of y, and we declare $\varphi_{\mathcal{L}}(s(y)) = 0$. One checks that this definition does not depend on the choice of s. Further, $\varphi_{m\mathcal{L}} = m\varphi_{\mathcal{L}}$, which allows us to define $\varphi_{\mathcal{L}}$ when $\mathcal{L}$ is a $\mathbf{Q}$-line bundle. The metric φ_L does not change when replacing $\mathcal{L}$ by a pullback. When $\mathcal{L}$ is the trivial test configuration for L, $\phi_{\mathcal{L}}$ is the trivial metric on L.

As in [BHJ17], the restriction of the function $\varphi = \varphi_{\mathcal{L}}$ to X^{div} can be described as follows. We may assume $\mathcal{L}$ dominates the trivial test configuration via $\rho\colon \mathcal{X} \to X^{\mathrm{sch}} \times \mathbb{A}^1$. Then $\mathcal{L} - \rho^*(L^{\mathrm{sch}} \times \mathbb{A}^1) = \mathcal{O}_{\mathcal{X}}(D)$ for a divisor D on $\mathcal{X}$ cosupported on $\mathcal{X}_0$, and

$$\varphi_{\mathcal{L}}(v_E) = \frac{\mathrm{ord}_E(D)}{\mathrm{ord}_E(\mathcal{X}_0)}$$

for every irreducible component E of $\mathcal{X}_0$, where $v_E \in X^{\mathrm{div}}$ is the restriction of $\mathrm{ord}_E / \mathrm{ord}_E(\mathcal{X}_0)$ from $k(\mathcal{X})$ to $k(X)$. Varying $\mathcal{X}$ determines $\varphi|_{X^{\mathrm{div}}}$ completely.

1.9. **Positive metrics.** Suppose L is ample. A test configuration $\mathcal{L}$ for L is *ample* (resp. semiample) if $\mathcal{L}$ is relatively ample (resp. semiample) for the canonical morphism $\mathcal{L} \to \mathbb{A}^1$. A metric on L is *positive* if it is defined by a semiample test configuration for L. Every positive metric is in fact associated to a unique normal ample test configuration for L (which may not dominate the trivial test configuration). A positive metric is the same thing as a *Fubini–Study metric* (or FS metric), i.e. a metric φ (viewed as function on X) of the form

$$\varphi := m^{-1} \max_{\alpha}(\log |s_\alpha|_0 + \lambda_\alpha), \tag{1.1}$$

where $m \geq 1$, (s_α) is a finite set of global sections of mL without common zero, and $\lambda_\alpha \in \mathbf{Z}$.

Denote by $\mathcal{H}(L)_{\mathbf{R}}$ the set of metrics of the form (1.1) with $\lambda_\alpha \in \mathbf{R}$ for each α. Such metrics are continuous and can be uniformly approximated by positive metrics.

1.10. **DFS metrics.** A *DFS metric* on a $\mathbf{Q}$-line bundle L on X is a metric of the form $\phi_1 - \phi_2$, with ϕ_i an FS metric on L_i, $i = 1, 2$, where $L = L_1 - L_2$. Equivalently, a DFS metric is a metric defined by a test configuration for L. The set of DFS metrics on L is dense in the space of continuous metrics in the topology of uniform convergence. It plays the role of smooth metrics in complex geometry.

1.11. **Monge–Ampère operator and energy functionals.** As a special case of the theory of Chambert-Loir and Ducros [CD12], there is a *mixed Monge–Ampère operator* that associates to any DFS metrics $\phi_1, \ldots, \phi_n$ on line bundles $L_1, \ldots, L_n$ on X, a signed finite atomic measure $dd^c\phi_1 \wedge \cdots \wedge dd^c\phi_n$ on X, supported on divisorial points, and of mass $(L_1 \cdot \ldots \cdot L_n)$. This measure is positive if the L_i are positive metrics.

As above, we think of the metrics as functions $\varphi_i = \phi_i - \phi_{\mathrm{triv}}$ on X, and write

$$dd^c\phi_i = \omega_{\varphi_i} = \omega + dd^c\varphi_i,$$

so that the mixed Monge–Ampère measure becomes $\omega_{\varphi_1} \wedge \cdots \wedge \omega_{\varphi_n}$. Here ω and ω_{φ_i} can be viewed as currents [CD12] or δ-forms [GK17] on X, but we do not need this terminology.

Now suppose $L_1 = \cdots = L_n =: L$ is an ample $\mathbf{Q}$-line bundle. We then write

$$\mathrm{MA}(\varphi) := V^{-1}\omega_\varphi^n,$$

where $V = (L^n)$. If $\varphi \in \mathcal{H}(L)$, this is a probability measure on X. Since $\varphi = 0$ corresponds to the trivial metric on L, $\mathrm{MA}(0) = \omega^n$ is a Dirac mass at the generic point of X.

If φ and ψ are DFS metrics on L, the *Monge–Ampère energy* of φ with respect to ψ is defined by

$$E(\varphi, \psi) = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int (\varphi - \psi)\omega_\varphi^j \wedge \omega_\psi^{n-j}.$$

In addition to the functional E, we set

$$I(\varphi,\psi)=\int(\varphi-\psi)(\mathrm{MA}(\psi)-\mathrm{MA}(\varphi))\quad\text{and}\quad J_\psi(\varphi)=\int(\varphi-\psi)\,\mathrm{MA}(\psi)-E(\varphi,\psi).$$

On positive metrics, the functionals I, J, and $I-J$ are nonnegative and comparable:

$$n^{-1}J\le I-J\le nJ. \tag{1.2}$$

The functionals E, I and J naturally have two arguments. Often it is convenient to fix the second argument ψ as the trivial metric, and write $E(\varphi)$, $I(\varphi)$ and $J(\varphi)$.

1.12. **Psh metrics.** For the rest of this section, L is an ample $\mathbf{Q}$-line bundle. A *psh* (plurisubharmonic) metric on L is the pointwise limit of any decreasing net of positive metrics, provided the limit is $\not\equiv-\infty$. We denote by $\mathrm{PSH}(L)$ the set of all psh metrics.

By subtracting the trivial metric, we can view the elements of $\mathrm{PSH}(L)$ as functions on X with values in $\mathbf{R}\cup\{-\infty\}$. This is analogous to the notion of ω-psh functions in complex geometry; we can think of ω as the curvature form of the trivial metric,

We equip $\mathrm{PSH}(L)$ with the topology of pointwise convergence on X^{qm}. If $\varphi\in\mathrm{PSH}(L)$ and $x\in X$, then the net $(\varphi\circ p_{\mathcal{X}}(x))_{X\in\mathrm{SNC}(X)}$ is decreasing, with limit $\varphi(x)$. Any $\varphi\in\mathrm{PSH}(L)$ takes its maximum value at the generic point of X. If $\varphi\in\mathrm{PSH}(L)$, then $\varphi+c\in\mathrm{PSH}(L)$ for all $c\in\mathbf{R}$. If $\varphi,\psi\in\mathrm{PSH}(L)$, then $\varphi\vee\psi:=\max\{\varphi,\psi\}\in\mathrm{PSH}(L)$.

If $(\varphi_j)_j$ is a decreasing net in $\mathrm{PSH}(L)$, and φ is the pointwise limit of (φ_j), then $\varphi\in\mathrm{PSH}(L)$, or $\varphi_j\equiv-\infty$. If instead $(\varphi_j)_j$ is an increasing net that is bounded from above, then φ_j converges in $\mathrm{PSH}(L)$ to some $\varphi\in\mathrm{PSH}(L)$. Thus $\varphi=\lim_j\varphi_j$ pointwise on X^{qm}, and in fact φ is the usc regularization of the pointwise limit of the φ_j.

The set of continuous psh metrics can be viewed as the set of *uniform* limits of positive metrics, in agreement with [Zha95, Gub98]. In particular, any metric in $\mathcal{H}(L)_{\mathbf{R}}$ is psh.

1.13. **Metrics of finite energy.** We extend the Monge–Ampère energy functional to all metrics $\varphi\in\mathrm{PSH}(L)$ by setting $E(\varphi):=\inf\{E(\psi)\mid\varphi\le\psi\in\mathcal{H}(L)\}\in\mathbf{R}\cup\{-\infty\}$ and define the class $\mathcal{E}^1(L)$ as the set of metrics of finite energy, $E(\varphi)>-\infty$.

The mixed Monge–Ampère operator and the functionals I, J extend to $\mathcal{E}^1(L)$ and are continuous under decreasing and increasing (but not arbitrary) limits, as is E. The inequalities (1.2) hold on $\mathcal{E}^1(L)$. If $\varphi,\psi\in\mathcal{E}^1(L)$, then $I(\varphi,\psi)=0$ iff $\varphi-\psi$ is a constant, and similarly for J and $I-J$.

1.14. **Measures of finite energy and the Calabi–Yau theorem.** The *energy* of a probability measure μ on X is defined by

$$E^*(\mu)=\sup\{E(\varphi)-\int\varphi\,d\mu\mid\varphi\in\mathcal{E}(L)\}\in\mathbf{R}\cup\{+\infty\}.$$

This quantity depends on the ample line bundle L, but the set $\mathcal{M}^1(X)$ of measures of finite energy, $E^*(\mu)<+\infty$, does not. The following result will be referred to as the *Calabi–Yau theorem*. It is proved in [BoJ18], see also [BFJ15, BFJ16b, YZ17], and is a trivially valued analogue of the fundamental results in [Yau78, GZ07, BBGZ13].

Theorem 1.1. [BoJ18] *The Monge–Ampère operator defines a bijection*

$$\mathrm{MA}\colon\mathcal{E}^1(L)/\mathbf{R}\to\mathcal{M}^1(X)$$

between plurisubharmonic metrics of finite energy modulo constants, and Radon probability measures of finite energy. For any $\varphi\in\mathcal{E}^1(L)$, we have $E^*(\mathrm{MA}(\varphi))=(I-J)(\varphi)$.

1.15. **Scaling action and homogeneity.** As k is trivially valued, there is a *scaling action* of the multiplicative group $\mathbf{R}_+^\times$ on X defined by powers of norms. We denote by x^t the image of x by $t \in \mathbf{R}_+^\times$. There is an induced action on functions. If φ is a metric on a $\mathbf{Q}$-line bundle (viewed as function on X) and $t \in \mathbf{R}_+^\times$, we denote by φ_t the metric on L defined by $\varphi_t(x^t) = t\varphi(x)$. This action preserves the classes $\mathcal{H}(L)_{\mathbf{R}}$, $\mathrm{PSH}(L)$, and $\mathcal{E}^1(L)$. The energy functional E is homogeneous in the sense $E(\varphi_t) = tE(\varphi)$ for $\varphi \in \mathcal{E}^1(L)$ and $t \in \mathbf{R}_+^\times$. The same is true for I and J.

There is also an induced action on Radon probability measures. We have $E^*(t_*\mu) = tE^*(\mu)$ for any $t \in \mathbf{R}_+^\times$, so the space $\mathcal{M}^1(X)$ is invariant under scaling. Further, $\mathrm{MA}(\varphi_t) = t_* \mathrm{MA}(\varphi)$ for $\varphi \in \mathcal{E}^1(L)$ and $t \in \mathbf{R}_+^\times$.

2. Adjoint K-stability and Ding-stability

In [Berm13], Berman introduced ideas from thermodynamics to the existence of Kähler–Einstein metrics on Fano manifolds. In particular, he used the Legendre transform, together with the (Archimedean) Calabi–Yau theorem to prove that the Mabuchi functional is proper on the space of smooth positive metrics on L if and only if the Ding functional is proper. When X admits no nontrivial vector fields, these two conditions are, further, equivalent to the existence of a Kähler–Einstein metric on X [BBEGZ16, DR17]

Here we adapt Berman's ideas to the non-Archimedean setting. Namely, we use the Legendre transform and the non-Archimedean Calabi–Yau theorem to prove that uniform Ding stability is equivalent to uniform K-stability. In fact, we work on an arbitrary polarized smooth variety, using adjoint versions of the Ding and Mabuchi functionals, the latter inducing Dervan's notion of twisted K-stability in the twisted Fano case [Der16]. The results here are used in [BBJ18], to give criteria for the existence of twisted Kähler–Einstein metrics.

In the rest of the paper, X is smooth and projective, and L is an ample $\mathbf{Q}$-line bundle.

2.1. **Log discrepancy.** There is a natural *log discrepancy* function

$$A = A_X \colon X \to \mathbf{R}_+ \cup \{+\infty\}.$$

This is perhaps most naturally viewed as a metric on the canonical bundle [Tem16], see [MMS], but here we present it differently. First consider a divisorial point $x \in X^{\mathrm{div}}$. There exists $c > 0$, a proper birational morphism $\mathcal{Y} \to X^{\mathrm{sch}}$, with $\mathcal{Y}$ smooth, and a prime divisor $\mathcal{D} \subset \mathcal{Y}$, such that $x = \exp(-c \operatorname{ord}_{\mathcal{D}})$. We then set

$$A(x) = c\left(1 + \operatorname{ord}_{\mathcal{D}}(K_{\mathcal{Y}/X^{\mathrm{sch}}})\right), \tag{2.1}$$

where $K_{\mathcal{Y}/X^{\mathrm{sch}}}$ is the relative canonical divisor.

The log discrepancy functional was extended to all of X^{val} in [JM12] (following earlier work in [FJ04, BFJ08]). Instead of explaining this construction here, we state the following characterization, which is proved in the appendix.

Theorem 2.1. *There exists a unique maximal lsc extension $A \colon X \to [0, +\infty]$ of the function $A \colon X^{\mathrm{div}} \to \mathbf{R}_+^\times$ defined above. Further, this function satisfies:*

(i) *$A = +\infty$ on $X \setminus X^{\mathrm{val}}$ and $A < +\infty$ on X^{qm};*
(ii) *$A(x^t) = tA(x)$ for all $x \in X$ and $t \in \mathbf{R}_+^\times$;*
(iii) *for any snc test configuration $\mathcal{X}$ for X, we have*
  (a) *A is continuous on the dual complex $\Delta_{\mathcal{X}}$ and affine on each simplex;*

(b) *for any point $x \in X$, we have $A(x) \geq A(p_{\mathcal{X}}(x))$, with equality iff $x \in \Delta_{\mathcal{X}}$;*

(iv) *$A = \sup_{\mathcal{X}} A \circ p_{\mathcal{X}}$, where $\mathcal{X}$ ranges over snc test configurations for X.*

Here $p_{\mathcal{X}} \colon X \to \Delta_{\mathcal{X}}$ denotes the retraction onto the dual complex.

2.2. **Entropy.** The *entropy* of a Radon probability measure μ on X is defined by

$$\mathrm{Ent}(\mu) = \int_X A(x)\, d\mu(x) \in [0, +\infty].$$

where A is the log discrepancy function on X. The integral is well-defined since A is lsc.

Remark 2.2. *If μ, ν are probability measures on a space X, then the classical entropy of μ with respect to ν is defined as $\int \log\left(\frac{d\mu}{d\nu}\right) d\mu$, when $\mu \ll \nu$, and $+\infty$ otherwise. Our notion of entropy can be seen as a non-Archimedean degeneration of the usual notion, see* [BHJ16].

The entropy behaves well with respect to regularizations of measures:

Lemma 2.3. *For any snc model $\mathcal{X}$ of X, set $\mu_{\mathcal{X}} := (p_{\mathcal{X}})_* \mu$. Then $(\mathrm{Ent}(\mu_{\mathcal{X}}))_{\mathcal{X}}$ forms an increasing net converging to $\mathrm{Ent}(\mu)$.*

Proof. Note that $\mathrm{Ent}(\mu_{\mathcal{X}}) = \int (A \circ p_{\mathcal{X}})\mu$. The result follows since μ is a Radon measure and $(A \circ p_{\mathcal{X}})_{\mathcal{X}}$ is an increasing net of lsc functions converging to A, see [Fol99, 7.12]. □

The entropy functional is a *linear* functional. In particular, it is convex. It is homogeneous and lsc on the space of Radon probability measures (with weak convergence) since A is lsc. However, it is not continuous, since A is not continuous.

2.3. **The twisted Mabuchi functional.** Given a $\mathbf{Q}$-line bundle T on X, define a relative Monge–Ampère energy functional E_T on $\mathcal{E}^1(L)$ by

$$E_T(\varphi) = (nV)^{-1} \sum_{j=0}^{n-1} \int \varphi\, \omega_\varphi^j \wedge \omega^{n-1-j} \wedge \eta.$$

Here ω and η are the curvature forms for the trivial metrics on L and T, respectively. When $\varphi \in \mathcal{H}(L)$, $E_T(\varphi)$ can be computed using intersection numbers, as in [BHJ17, §7.4]. That E_T is well-defined and finite on $\mathcal{E}^1(L)$ is seen by writing T as a difference between ample $\mathbf{Q}$-line bundles. We have $E_T(\varphi + c) = E_T(\varphi) + V^{-1}(T \cdot L^{n-1})c$ for $c \in \mathbf{R}$.

As in [BHJ17], we define the *Mabuchi* functional M on $\mathcal{E}^1(L)$ via the Chen–Tian formula:

$$M = H + E_{K_X} + \bar{S}E, \tag{2.2}$$

where $\bar{S} = -nV^{-1}(K_X \cdot L^n)$. Now, given a $\mathbf{Q}$-line bundle T on X, we define the *twisted Mabuchi functional* M_T on $\mathcal{E}^1(L)$ by replacing K_X by $K_X + T$ everywhere. This amounts to

$$M_T = M + nE_T - nV^{-1}(T \cdot L^n)E,$$

and is translation invariant: $M_T(\varphi + c) = M_T(\varphi)$ for $c \in \mathbf{R}$.

2.4. **The adjoint Mabuchi functional and free energy.** We are interested in the adjoint case $L = -(K_X + T)$, i.e. $T = -(K_X + L)$, which corresponds to the *twisted Fano case* in the terminology of [Der16]. In this situation, the *adjoint Mabuchi functional* $M^{\mathrm{ad}} := M_{-(K_X+L)}$ takes a particularly nice form:

$$M^{\mathrm{ad}} = H - (I - J), \tag{2.3}$$

where the functional $H\colon \mathcal{E}^1(L) \to \mathbf{R}_+ \cup \{+\infty\}$ is given by

$$H(\varphi) = \mathrm{Ent}(\mathrm{MA}(\varphi)) = \int_X A\,\mathrm{MA}(\varphi).$$

While I and J are continuous under decreasing nets, this is not true for H and M^{ad}.

Example 2.4. *Let $X = \mathbf{P}^1$ and $L = \mathcal{O}(1)$. Pick each $n \geq 1$, pick 2^n divisorial points $x_{n,j} \in X$, $1 \leq j \leq 2^n$ such that $\mathrm{red}(x_{n,i}) \neq \mathrm{red}(x_{n,j})$ for $i \neq j$, and $A(x_{n,j}) = 1$. Define $\varphi_n \in \mathrm{PSH}(L)$ by $\max(\varphi_n) = 2^{1-n}$ and $\mathrm{MA}(\varphi_n) = 2^{-n}\sum_{j=1}^{2^n} \delta_{x_{n,j}}$. Then $\varphi_n(x_{n,j}) = 2^{-n}$ for all n, j, from which it follows that $2^{-n} \leq \varphi_n \leq 2^{1-n}$ on X. In particular, the sequence $(\varphi_n)_1^\infty$ is decreasing, and converges to 0. Now $H(\varphi_n) = 1$ for all n, while $H(0) = 0$.*

For this reason—and in contrast to Lemma 2.9 below—we do not know whether $M^{\mathrm{ad}} \geq 0$ on $\mathcal{H}(L)$ implies $M^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$. However, this implication would follow from

Conjecture 2.5. *Given any metric $\varphi \in \mathcal{E}^1(L)$, there exists a decreasing net $(\varphi_j)_j$ of positive metrics converging to φ, such that $\lim_j \mathrm{Ent}(\mathrm{MA}(\varphi_j)) = \mathrm{Ent}(\mathrm{MA}(\varphi))$.*

In the Archimedean case, the corresponding conjecture is true: any metric φ of finite energy can be approximated by smooth positive metrics, see [BDL17, Lemma 3.1]. The proof in *loc. cit.* proceeds by approximating the Monge–Ampère measure of φ, and then using the Calabi–Yau theorem. The problem in the non-Archimedean case is that we don't know how to characterize the image of $\mathcal{H}(L)$ under the Monge–Ampère operator.

2.5. **Free energy.** Being a translation invariant functional on $\mathcal{E}^1(L)$, the adjoint Mabuchi functional factors through the Monge–Ampère operator: we have

$$M^{\mathrm{ad}}(\varphi) := F(\mathrm{MA}(\varphi)),$$

where the *free energy* functional $F\colon \mathcal{M}^1(X) \to \mathbf{R} \cup \{+\infty\}$ is given by

$$F = \mathrm{Ent} - E^*;$$

see [Berm13]. In other words,

$$F(\mu) = \int_X A\mu - E^*(\mu).$$

Note that while the space $\mathcal{M}(X)$ does not depend on L, the energy functional E^* does; hence the same is true for F.

2.6. **The Legendre transform of entropy.** Define $L\colon \mathcal{E}^1(L) \to \mathbf{R} \cup \{-\infty\}$ by

$$L(\varphi) = \inf_x (A(x) + \varphi(x)), \tag{2.4}$$

where the infimum is taken over divisorial valuations $x \in X^{\mathrm{div}}$. When X is Fano and $L = -K_X$, this extends the functional in [BHJ17, Definition 7.26].

Proposition 2.6. *The infimum in* (2.4) *is unchanged when taking it over quasimonomial points* $x \in X^{\mathrm{qm}}$*, or over points* $x \in X^{\mathrm{val}}$ *with* $A(x) < \infty$*. Further, we have*

$$L(\varphi) = \inf_{\mu \in \mathcal{M}^1(X)} \{\mathrm{Ent}(\mu) + \int \varphi \mu\}, \tag{2.5}$$

Proof. Let $L^{\mathrm{qm}}(\varphi)$ denote the infimum in (2.4) taken over quasimonomial points, and let $L'(\varphi)$ denote the right-hand side of (2.5). We will prove that $L(\varphi) = L^{\mathrm{qm}}(\varphi) = L'(\varphi)$, which implies the result since taking $\mu = \delta_x$ shows that the infimum in (2.4) over $x \in X^{\mathrm{val}}$ with $A(x) < \infty$ is bounded from below and above by $L(\varphi)$ and $L'(\varphi)$, respectively.

We have $L^{\mathrm{qm}}(\varphi) = L(\varphi)$ since for any snc test configuration $\mathcal{X}$ for X, the functions A and φ are continuous on the dual complex $\Delta_{\mathcal{X}}$, inside which divisorial points are dense.

Taking $\mu = \delta_x$ for $x \in X^{\mathrm{qm}}$ shows that $L'(\varphi) \leq L^{\mathrm{qm}}(\varphi)$. For the reverse inequality, pick $\varepsilon > 0$ and $\mu \in \mathcal{M}^1(X)$ such that $\mathrm{Ent}(\mu) + \int \varphi\, \mu \leq L'(\varphi) + \varepsilon$. Replacing μ by $p_{\mathcal{X}*}\mu$ for a large enough $\mathcal{X} \in \mathrm{SNC}(X)$, we may assume μ is supported on a dual complex $\Delta_{\mathcal{X}}$, see Lemma 2.3. But then it is clear that

$$\mathrm{Ent}(\mu) + \int \varphi\, \mu = \int_{\Delta_{\mathcal{X}}} (A + \varphi)\, d\mu \geq \inf_{\Delta_{\mathcal{X}}} (A + \varphi),$$

so $L^{\mathrm{qm}}(\varphi) \leq L'(\varphi) + \varepsilon$. □

Lemma 2.7. *The functional* L *is usc and non-increasing on* $\mathcal{E}^1(L)$*. As a consequence, it is continuous along decreasing nets in* $\mathcal{E}^1(L)$*.*

Proof. The only nontrivial statement is the upper semicontinuity, and this follows from the continuity of $\varphi \mapsto \varphi(x)$ for $x \in X^{\mathrm{qm}}$. □

We may think of (2.5) as saying that L is the *Legendre transform* of Ent. Now, the natural setting of the Legendre duality is between the space $\mathcal{M}'(X)$ of all signed Radon measures on X and the space $C^0(X)$ of continuous functions on X. Extend Ent to all of $\mathcal{M}'(X)$ by $\mathrm{Ent}(\mu) = \int A\mu$ when μ is a probability measure, and $\mathrm{Ent}(\mu) = +\infty$ otherwise. Then define $L(f)$ for $f \in C^0(X)$ by $L(f) = \inf_{x \in X}\{A(x) + f(x)\}$.

Proposition 2.8. *For any* $f \in C^0(X)$ *we have*

$$L(f) = \inf_{\mu \in \mathcal{M}'(X)} \{\mathrm{Ent}(\mu) + \int f\mu\}, \tag{2.6}$$

and for every $\mu \in \mathcal{M}'(X)$*, we have*

$$\mathrm{Ent}(\mu) = \sup_{f \in C^0(X)} \{L(f) - \int f\mu\}. \tag{2.7}$$

Proof. In (2.6) it clearly suffices to take the infimum over Radon probability measures μ. The equality then follows as in the proof of Proposition 2.6.

That (2.7) holds in now a formal consequence of A being lsc. Indeed, fix $\mu \in \mathcal{M}'(X)$ and let $\mathrm{Ent}'(\mu)$ be the right-hand side of (2.7). It follows from (2.6) that $\mathrm{Ent}'(\mu) \leq \mathrm{Ent}(\mu)$. To prove the reverse inequality, first suppose μ is a probability measure and pick $\varepsilon > 0$. Since $-A$ is usc, we can find a continuous function f on X with $f \geq -A$ and $\int f\, \mu \leq -\int A\, \mu + \varepsilon = -\mathrm{Ent}(\mu) + \varepsilon$. Thus $L(f) \geq 0$, and hence $\mathrm{Ent}'(\mu) \geq L(f) - \int f\, \mu \geq \mathrm{Ent}(\mu) - \varepsilon$.

Now suppose $\mu \in \mathcal{M}'(X)$ is not a probability measure. If $\mu(X) \neq 1$, picking $f \equiv \pm C$, where $C \gg 1$ gives $\mathrm{Ent}'(\mu) = +\infty$. If μ is not a positive measure, then there exists $g \in C^0(X)$ with $g \geq 0$ but $\int g\, \mu < 0$. Picking $f = Cg$ for $C \gg 0$ again gives $\mathrm{Ent}'(\mu) = +\infty$, □

2.7. **The adjoint Ding functional.** Define the *adjoint Ding functional* on $\mathcal{E}^1(L)$ by

$$D^{\mathrm{ad}} = L - E, \tag{2.8}$$

where L is the functional in (2.4), and E is the Monge–Ampère energy. When X is Fano and $L = -K_X$, the restriction of D^{ad} to $\mathcal{H}(L)$ coincides with the Ding functional of [Berm16, BHJ17]. The name derives from [Din88].

Lemma 2.9. *The adjoint Ding functional D^{ad} is continuous along decreasing nets in $\mathcal{E}^1(L)$. As a consequence, $D^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$ iff $D^{\mathrm{ad}} \geq 0$ on $\mathcal{H}(L)$.*

Proof. It suffices to prove the continuity assertion, since every element in $\mathcal{E}^1(L)$ is the limit of a decreasing net in $\mathcal{H}(L)$. Now, we know that E is continuous along decreasing nets, so the result follows from Lemma 2.7. □

2.8. **Non-Archimedean thermodynamics.** Using the Calabi–Yau theorem, we now relate the adjoint Ding and Mabuchi functionals. The following result can be viewed as a non-Archimedean version of [Berm13, Theorem 1.1].

Theorem 2.10. *Let L be an ample line bundle on a smooth projective variety X. Then we have $M^{\mathrm{ad}} \geq D^{\mathrm{ad}}$ on $\mathcal{E}^1(L)$. Further, the following conditions are equivalent:*

(i) *$F \geq 0$ on $\mathcal{M}^1(X)$;*
(ii) *$M^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$;*
(iii) *$D^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$.*

In the Archimedean situation, the conditions analogous to (ii) and (iii) are that the Mabuchi and Ding functionals are bounded from below. In the non-Archimedean setting, the presence of an $\mathbf{R}_+^\times$-action on $\mathcal{E}^1(L)$ under which M^{ad} and D^{ad} are homogeneous, shows that this is equivalent to (ii) and (iii).

Proof. Pick any $\varphi \in \mathcal{E}^1(L)$. If $D^{\mathrm{ad}}(\varphi) > -\infty$, then

$$M^{\mathrm{ad}}(\varphi) - D^{\mathrm{ad}}(\varphi) = \int (A + \varphi)\,\mathrm{MA}(\varphi) - \inf(A + \varphi) \geq 0,$$

so we get $M^{\mathrm{ad}} \geq D^{\mathrm{ad}}$ on $\mathcal{E}^1(L)$.

Thus (iii) implies (ii). By the Calabi-Yau theorem, it follows that (ii) implies (i). Hence it suffices to prove that (i) implies (iii). By (i) we have $F(\delta_x) \geq 0$ for every divisorial point $x \in X^{\mathrm{div}}$, which translates into $A(x) \geq E^*(\delta_x)$ for every $x \in X^{\mathrm{div}}$. By the definition of E^*, this implies $E(\varphi) - \varphi(x) \leq A(x)$ for every $\varphi \in \mathcal{E}^1(L)$. Taking the infimum over $x \in X^{\mathrm{div}}$ we get $L(\varphi) \geq E(\varphi)$, that is, $D^{\mathrm{ad}}(\varphi) \geq 0$ for all $\varphi \in \mathcal{E}^1(L)$. □

2.9. **Adjoint semistability.** We can reformulate Theorem 2.10 as follows. First, we say that L is *Ding semistable in the adjoint sense* if $D^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$, and *K-semistable in the adjoint sense* if $M^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$.

As in [BHJ17, Proposition 8.2], one shows that $M^{\mathrm{ad}} \geq 0$ on $\mathcal{H}(L)$ is equivalent to *twisted K-semistability* in the twisted Fano case, in the sense of [Der16]. According to Conjecture 2.5, $M^{\mathrm{ad}} \geq 0$ on $\mathcal{H}(L)$ should imply $M^{\mathrm{ad}} \geq 0$ on $\mathcal{E}^1(L)$, i.e. adjoint K-semistability.

Second, we define the *stability threshold* of L as

$$\delta(L) = \inf_{\mu \in \mathcal{M}(X)} \frac{\mathrm{Ent}(\mu)}{E^*(\mu)}. \tag{2.9}$$

We shall see in Theorem 5.16 that this invariant, which was suggested by Berman [Berm], coincides with the one defined in [FO16, BlJ17]. In particular, this will show that $\delta(L) > 0$.

Corollary 2.11. *For any ample line bundle L on X, the following are equivalent:*

(i) $\delta(L) \geq 1$;
(ii) *L is Ding semistable in the adjoint sense;*
(ii) *L is K-semistable in the adjoint sense.*

2.10. **Uniform adjoint stability.** In analogy with [BHJ17, Der16]) we say that L is *uniformly Ding-stable in the adjoint sense* if there exists $\varepsilon > 0$ such that $D^{\mathrm{ad}} \geq \varepsilon J$ on $\mathcal{E}^1(L)$. Similarly, L is *uniformly K-stable in the adjoint sense* if there exists $\varepsilon > 0$ such that $M^{\mathrm{ad}} \geq \varepsilon J$ on $\mathcal{E}^1(L)$. Here again, this is equivalent (at least up to Conjecture 2.5) to uniform twisted K-stability in the twisted Fano case, in the sense of [Der16].

Theorem 2.12. *For any ample line bundle L, the following are equivalent:*

(i) $\delta(L) > 1$;
(ii) *L is uniformly K-stable in the adjoint sense;*
(iii) *L is uniformly Ding-stable in the adjoint sense.*

Together with Corollary 2.11, this proves Theorem A in the introduction.

Proof. The Calabi–Yau theorem shows that if $\delta \geq 1$, then $\mathrm{Ent} \geq \delta E^*$ on $\mathcal{M}^1(X)$ iff $M^{\mathrm{ad}} \geq (\delta - 1)(I - J)$ on $\mathcal{E}^1(X)$. Since the functionals $I - J$ and J are comparable, see (1.2), this implies that (i)$\Leftrightarrow$(ii). The inequality $M^{\mathrm{ad}} \geq D^{\mathrm{ad}}$ shows that (iii) implies (ii).

It remains to prove that (i) implies (iii). We follow the proof of Theorem 2.10. Pick any $\delta \in (1, \delta(L))$. Then $\mathrm{Ent}(\mu) \geq \delta E^*(\mu)$ for any $\mu \in \mathcal{M}^1(X)$. When $\mu = \delta_x$, this gives $A(x) \geq \delta E^*(\delta_x)$ for any $x \in X^{\mathrm{div}}$. Now consider any $\varphi \in \mathcal{E}^1(L)$. Since $\delta > 1$ and $\mathcal{E}^1(L)$ is convex, we have $\delta^{-1}\varphi \in \mathcal{E}^1(L)$. This gives $E^*(\delta_x) \geq E(\delta^{-1}\varphi) - \delta^{-1}\varphi(x)$. Thus $A(x) + \varphi(x) \geq \delta E^*(\delta_x) + \varphi(x) \geq \delta E(\delta^{-1}\varphi)$. Taking the infimum over $x \in X^{\mathrm{div}}$ and subtracting $E(\varphi)$ gives $D^{\mathrm{ad}}(\varphi) \geq \delta E(\delta^{-1}\varphi) - E(\varphi)$. By translation invariance, we may assume $\sup_X \varphi = 0$. Then $E(\varphi) = -J(\varphi)$ and $E(\delta^{-1}\varphi) = -J(\delta^{-1}\varphi)$. By [BoJ18, Lemma 6.17], we have

$$J(\delta^{-1}\varphi) \leq \delta^{-(1+n^{-1})} J(\varphi).$$

This implies $D^{\mathrm{ad}}(\varphi) \geq \varepsilon J(\varphi)$, with $\varepsilon = (1 - \delta^{-n^{-1}})$. □

2.11. **The Fano case.** The adjoint stability notions above are defined in terms of the space $\mathcal{E}^1(L)$ of metrics of finite energy. This is a natural framework for applying the Calabi–Yau theorem. On the other hand, K-stability and Ding stability are usually expressed in terms of test configurations, that is, metrics in $\mathcal{H}(L)$. For Ding-stability, this makes no difference, in view of Lemma 2.9. If Conjecture 2.5 holds, then the same is true for K-stability.

When X is Fano and $L = -K_X$, the adjoint notions do coincide with the usual ones:

Theorem 2.13. *If X is a Fano manifold, then the following are equivalent:*

(i) *X is K-semistable (resp. uniformly K-stable);*
(ii) *X is K-semistable (resp. uniformly K-stable) in the adjoint sense;*
(iii) *X is Ding semistable (resp. uniformly Ding-stable);*
(iv) *X is Ding semistable (resp. uniformly Ding-stable) in the adjoint sense;*
(v) $\delta(-K_X) \geq 1$ *(resp. $\delta(-K_X) > 1$.*

Proof. The equivalence of (ii), (iv) and (v) follow from Corollary 2.11 and Theorem 2.12. Now Theorem 5.16 below shows that $\delta(-K_X)$ agrees with the invariant considered in [FO16, BlJ17]. The equivalence of (i), (iii) and (v) therefore follow from [BlJ17, Theorem B]. □

3. Graded norms and filtrations

In this section, we study the space of bounded graded norms (or, equivalently, filtrations) on the section ring $R = R(X, L)$. As before, k is a trivially valued field, whereas L is an ample line bundle (as opposed to a $\mathbf{Q}$-line bundle) on X.

Much of the material here is studied for more general valued fields k in [CM15, BE18], but we present the details for the convenience of the reader. There is also some overlap with the recent work of Codogni [Cod18].

3.1. **Norms and filtrations.** Let V be a k-vector space. By a *filtration* $\mathcal{F}$ of V we mean a family $(\mathcal{F}^\lambda V)_{\lambda\in\mathbf{R}}$ of k-vector subspaces of V, satisfying $\mathcal{F}^\lambda V = \bigcap_{\lambda'<\lambda} \mathcal{F}^{\lambda'} V$, $\bigcup_\lambda \mathcal{F}^\lambda V = V$, and $\bigcap_\lambda \mathcal{F}^\lambda V = 0$. A filtration is *bounded* if $\mathcal{F}^\lambda V = V$ for $\lambda \ll 0$ and $\mathcal{F}^\lambda V = 0$ for $\lambda \gg 0$.

Filtrations of V are in bijection with (non-Archimedean) *norms* on V, i.e. functions $\|\cdot\|\colon V \to \mathbf{R}_+$ satisfying $\|v\| = 0$ iff $v = 0$, $\|av\| = |a|\|v\|$ for $a \in k$, $v \in V$, and $\|v + w\| \leq \max\{\|v\|, \|w\|\}$ for $v, w \in V$. To a filtration $\mathcal{F}$ is associated the norm $\|v\| = \exp(-\sup\{\lambda \in \mathbf{R} \mid v \in \mathcal{F}^\lambda V\})$; conversely, a norm $\|\cdot\|$ on V induces the filtration $\mathcal{F}^\lambda V = \{v \in V \mid \|v\| \leq \exp(-\lambda)\}$. In what follows, we will usually work with norms rather than filtrations.

Bounded filtrations correspond to bounded norms, i.e. norms for which there exists $A > 0$ such that $A^{-1} \leq \|v\| \leq A$ for all $v \neq 0$. If V is finite dimensional, then any filtration/norm on V is bounded. The *trivial* norm on V is defined by $\|v\| = 1$ for $v \neq 0$. A norm is *almost trivial* if it is a multiple of the trivial norm.

If V is a normed vector space, any subspace $W \subset V$ is naturally equipped with the subspace norm, and the quotient V/W with the quotient norm defined by $\|v + W\| := \inf\{\|v + w\| \mid w \in W\}$. In general, this is only a seminorm on V/W (i.e. there may be nonzero elements of norm zero) but it is a norm when V is finite dimensional.

The space $\mathcal{N}_V$ of norms on V admits two natural operations. First, if $\|\cdot\|$ and $\|\cdot\|'$ are norms on V, so is their maximum $\|\cdot\| \vee \|\cdot\|' := \max\{\|\cdot\|, \|\cdot\|'\}$. Second, we have an action of $\mathbf{R}$ on $\mathcal{N}_V$ given by $(t, \|\cdot\|) \mapsto \exp(t)\|\cdot\|$.

3.2. **Relative successive minima and volume.** As k is trivially valued, any finite-dimensional normed k-vector space V admits a basis $\{e_j\}_j$ that is *orthogonal* for the norm in the sense that $\|\sum_j a_j e_j\| = \max_j |a_j|\|e_j\|$ for all $a_j \in k$. More generally, given any two norms $\|\cdot\|$, $\|\cdot\|'$ on a finite-dimensional k-vector space V, there exists a basis $\{e_j\}_{j=1}^N$ for V that is orthogonal for both norms. The numbers

$$\lambda_j := \log \frac{\|e_j\|'}{\|e_j\|}, \quad 1 \leq j \leq N,$$

are called the *relative successive minima* of $\|\cdot\|$ with respect to $\|\cdot\|'$. They do not depend on the choice of orthogonal basis[1].

Following [BE18], the average of the relative successive minima,

$$\mathrm{vol}(\|\cdot\|, \|\cdot\|') := N^{-1} \sum_{j=1}^N \lambda_j, \tag{3.1}$$

[1]This is not completely obvious: see [BE18, §3.1]

is called the (logarithmic) *relative volume* of $\|\cdot\|$ with respect to $\|\cdot\|'$. It can be described as follows. The norms $\|\cdot\|$, $\|\cdot\|'$ on V canonically induce norms $\det\|\cdot\|$, $\det\|\cdot\|'$ on the determinant line $\det V$, and

$$\mathrm{vol}(\|\cdot\|,\|\cdot\|') = N^{-1}(\log\det\|\eta\|' - \log\det\|\eta\|) \tag{3.2}$$

for any nonzero element $\eta \in \det V$. As a consequence we have the *cocycle condition*

$$\mathrm{vol}(\|\cdot\|,\|\cdot\|') + \mathrm{vol}(\|\cdot\|',\|\cdot\|'') = \mathrm{vol}(\|\cdot\|,\|\cdot\|'') \tag{3.3}$$

for any three norms on V.

When $\|\cdot\|'$ is the trivial norm, we drop the term "relative" and simply say *successive minima* and *volume*, and write $\mathrm{vol}(\|\cdot\|)$ for the latter. The successive minima of a norm are exactly the *jumping numbers* of the associated filtration, i.e. the $\lambda \in \mathbf{R}$ such that $\mathcal{F}^\lambda \supsetneq \mathcal{F}^{\lambda'}$ for any $\lambda' > \lambda$ (counted with multiplicity).

3.3. **Distances.** We use the relative successive minima to define a distances d_p, $1 \le p \le \infty$ on the space $\mathcal{N}_V$ of norms on V. Namely, we set

$$d_p(\|\cdot\|,\|\cdot\|') := (N^{-1}\sum_{j=1}^N |\lambda_j|^p)^{1/p} \tag{3.4}$$

for $p \in [1,\infty)$, and $d_\infty(\|\cdot\|,\|\cdot\|') := \max_{1\le j\le N}|\lambda_j|$. One can prove that d_p satisfies the triangle inequality, see [BE18, §3.1]. Note that

$$d_p(\|\cdot\|,\|\cdot\|')^p = d_p(\|\cdot\|,\|\cdot\|\vee\|\cdot\|')^p + d_p(\|\cdot\|',\|\cdot\|\vee\|\cdot\|')^p \tag{3.5}$$

for $p \in [1,\infty)$. There is a similar formula when $p=\infty$.

The distance d_1 is easier to control than the others, because of its close relationship to the relative volume. Indeed, if $\|\cdot\| \le \|\cdot\|'$ pointwise on V, then

$$d_1(\|\cdot\|,\|\cdot\|') = \mathrm{vol}(\|\cdot\|,\|\cdot\|'). \tag{3.6}$$

We will later need the following estimate.

Lemma 3.1. *Let $\|\cdot\|_i$, $\|\cdot\|_i'$, $i=1,2$, be norms on V. Then*

$$d_1(\|\cdot\|_1\vee\|\cdot\|_2,\|\cdot\|_1'\vee\|\cdot\|_2') \le d_1(\|\cdot\|_1,\|\cdot\|_1') + d_1(\|\cdot\|_2,\|\cdot\|_2'). \tag{3.7}$$

Proof. To begin, consider the case when $\|\cdot\|_i \le \|\cdot\|_i'$, $i=1,2$.

Further, we first assume there exists a basis $e=(e_1,\dots,e_N)$ for V that is orthogonal for all four norms. Write $\|e_j\|_i = \exp(a_{i,j})$ and $\|e_j\|_i' = \exp(a_{i,j}')$ for $1\le j\le N$ and $i=1,2$. Then $a_{i,j} \le a_{i,j}'$ for all i,j, and we must show that

$$\sum_{j=1}^N a_{1,j}'\vee a_{2,j}' - \sum_{j=1}^N a_{1,j}\vee a_{2,j} \le \sum_{j=1}^N (a_{1,j}'-a_{1,j}) + \sum_{j=1}^N (a_{2,j}'-a_{2,j});$$

this is straightforward.

When no such basis exists, we use the following construction: see [BE18, §3.1] for details. For any basis $e=(e_1,\dots,e_N)$ of V there is a "projection" $\rho_e\colon \mathcal{N}_V \to \mathcal{N}_V$ with the following properties: (1) $\rho_e(\|\cdot\|) = \|\cdot\|$ iff $\|\cdot\|$ is orthogonal for e; (2) $\rho_e\circ\rho_e = \rho_e$; (3) $\det\rho_e(\|\cdot\|) = \det\|\cdot\|$; and (4) if $\|\cdot\|\le\|\cdot\|'$, then $\rho_e(\|\cdot\|)\le\rho_e(\|\cdot\|')$.

Now assume e is orthogonal for $\|\cdot\|_i'$, $i=1,2$. Replacing $\|\cdot\|_i$ by $\rho_e(\|\cdot\|_i)$, $i=1,2$ does not change the right-hand side of (3.7). As for the left-hand side, (2) and (4) above imply

$$\rho_e(\|\cdot\|_1)\vee\rho_e(\|\cdot\|_2) \le \rho_e(\|\cdot\|_1\vee\|\cdot\|_2) \le \|\cdot\|_1'\vee\|\cdot\|_2',$$

which in view of (3) and (3.2) implies that the left-hand side of (3.7) can only increase upon replacing $\|\cdot\|_i$ by $\rho_e(\|\cdot\|_i)$, $i = 1, 2$.

Finally consider arbitrary norms. Set $\|\cdot\|_i'' = \|\cdot\|_i \vee \|\cdot\|_i'$ for $i = 1, 2$. By (3.5) we have $d_1(\|\cdot\|_1 \vee \|\cdot\|_2, \|\cdot\|_1' \vee \|\cdot\|_2') = d_1(\|\cdot\|_1 \vee \|\cdot\|_2, \|\cdot\|_1'' \vee \|\cdot\|_2'') + d_1(\|\cdot\|_1' \vee \|\cdot\|_2', \|\cdot\|_1'' \vee \|\cdot\|_2'')$ and $\|\cdot\|_i, \|\cdot\|_i' \leq \|\cdot\|_i''$, for $i = 1, 2$, so (3.7) follows from (3.5) and the case just considered. □

3.4. **Graded norms and filtrations.** Let L be an ample line bundle. For $m \in \mathbf{N}$, write $R_m := H^0(X, mL)$. Thus $R_m \neq 0$ for $m \gg 0$. Consider the section ring $R = \bigoplus_m R_m$.

A *graded norm* $\|\cdot\|_\bullet$ on R is the data of a norm $\|\cdot\|_m$ on the k-vector space R_m for each m, satisfying $\|s \otimes s'\|_{m+m'} \leq \|s\|_m \cdot \|s'\|_{m'}$ for $s \in R_m$, $s' \in R_{m'}$. Since R is finitely generated, there exists $C \geq 0$ such that $\|\cdot\|_m \leq \exp(Cm)$ on R_m for all m. We say $\|\cdot\|$ is (exponentially) *bounded* if $\|\cdot\|_m \geq \exp(-Cm)$ on $R_m \setminus \{0\}$ for some $C \geq 0$ and all m.

If $\|\cdot\|_\bullet$ and $\|\cdot\|_\bullet'$ are (bounded) graded norms on V, so is their maximum $\|\cdot\|_\bullet \vee \|\cdot\|_\bullet'$. If $\|\cdot\|_\bullet$ is a (bounded) graded norm and $c \in \mathbf{R}$, then $\exp(c\bullet)\|\cdot\|_\bullet$, defined by $\exp(cm)\|\cdot\|_m$ on R_m, is a (bounded) graded norm on V. The *trivial graded norm* on R is the graded norm for which $\|\cdot\|_m$ is the trivial norm on R_m for every m.

A graded norm is *generated in degree one* if R is generated in degree one, that is, the canonical morphism $S^m R_1 \to R_m$ is surjective for all $m \geq 1$, and the associated norm on R_m is equal to the quotient norm from this morphism. If R is generated in degree 1, and $\|\cdot\|_1$ is any norm on R_1, then R admits a unique graded norm that is generated in degree one and extends $\|\cdot\|_1$. A graded norm is *finitely generated* if the induced graded norm on $R(X, rL)$ is generated in degree 1 for some $r \geq 1$.

A *graded filtration* on R is the collection of a filtration $(\mathcal{F}^\lambda R_m)_\lambda$ of R_m for all m, satisfying $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^{\lambda'} R_{m'} \subset \mathcal{F}^{\lambda+\lambda'} R_{m+m'}$. As above, graded norms on R are in bijection with graded filtrations of R, and bounded graded norms correspond to (linearly) bounded graded filtrations, i.e. graded filtrations for which there exists $C \in \mathbf{R}$ such that $\mathcal{F}^\lambda R_m = 0$ for $\lambda \geq Cm$ and $\mathcal{F}^\lambda R_m = R_m$ for $\lambda \leq -Cm$. We say that a graded filtration is generated in degree one if the associated graded norm is generated in degree one. The trivial graded norm on R corresponds to the trivial graded filtration of R, defined by $\mathcal{F}^\lambda R_m = R_m$ for $\lambda \leq 0$ and $\mathcal{F}^\lambda R_m = 0$ for $\lambda > 0$.

A graded filtration $\mathcal{F}$ of $R(X, L)$ is a graded $\mathbf{Z}$-*filtration* if all jumping numbers are integers, i.e. $\mathcal{F}^\lambda R_m = \mathcal{F}^{\lceil\lambda\rceil} R_m$ for all λ and m. They correspond to graded norms taking values in $\{0\} \cup \exp(\mathbf{Z})$. Any graded filtration $\mathcal{F}$ induces a graded $\mathbf{Z}$-filtration $\mathcal{F}_{\mathbf{Z}}$ by setting $\mathcal{F}_{\mathbf{Z}}^\lambda R_m := \mathcal{F}^{\lceil\lambda\rceil} R_m$. There is a similar operation on graded norms.

Finitely generated bounded $\mathbf{Z}$-filtrations of $R(X, L)$ are in 1-1 correspondence with ample test configurations for (X, L), see [BHJ17, §2.5].

3.5. **Relative limit measures.** Let $\|\cdot\|_\bullet$ and $\|\cdot\|_\bullet'$ be bounded graded norms on $R(X, L)$. For $m \geq 1$, let $\lambda_{m,j}$, $1 \leq j \leq N_m$ be the relative successive minima of $\|\cdot\|_m$ with respect to $\|\cdot\|_m'$. Since the graded norms are bounded, there exists $C > 0$ such that $|\lambda_{m,j}| \leq Cm$ for all m, j. The following result was proved by Chen and Maclean [CM15], building upon [BC11].

Theorem 3.2. *There exists a compactly supported Borel probability measure ν on $\mathbf{R}$ such that the probability measures*

$$\nu_m := \frac{1}{N_m} \sum_{j=1}^{N_m} \delta_{\lambda_{m,j}/m}$$

converge weakly to ν as $m \to \infty$.

We call ν is the *relative limit measure* of $\|\cdot\|_\bullet$ with respect to $\|\cdot\|'_\bullet$. To indicate the dependence on the graded norms, we write $\nu = \mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$. When $\|\cdot\|'_\bullet$ is the trivial graded norm, we write $\nu = \mathrm{LM}(\|\cdot\|_\bullet)$ and call it the *limit measure* of $\|\cdot\|_\bullet$ following [BC11].

Proof. Since our setting and notation differs slightly from [CM15], we sketch the proof. The idea is to reduce to the case when $\|\cdot\|'_\bullet$ is the trivial graded norm; this case was treated in [BC11] (see also [Bou14]) using the technique of Okounkov bodies.

Since L is ample, $R_m \neq 0$ for $m \gg 0$. As already noted, there exists $C > 0$ such that ν_m is supported in $[-C, C]$ for $m \gg 0$. It suffices to prove that

$$\int_{\mathbf{R}} \max\{\lambda, c\}\, d\nu_m(\lambda) = \frac{1}{mN_m} \sum_{j=1}^{N_m} \max\{\lambda_{j,m}, mc\}$$

converges as $m \to \infty$, for all $c \in \mathbf{R}$, see [CM15, Proposition 5.1]. But the numbers $\max\{\lambda_{j,m}, cm\}$, $1 \leq j \leq N_m$, are the relative successive minima of $\|\cdot\|_m$ with respect to the norm $\|\cdot\|'_m \vee \exp(cm)\|\cdot\|_m$ on R_m. Replacing $\|\cdot\|'_\bullet$ by $\|\cdot\|'_\bullet \vee \exp(c\,\bullet)\|\cdot\|_\bullet$, we are reduced to proving that the *barycenters* $(mN_m)^{-1}\sum_j \lambda_{m,j}$ of the measures ν_m converge as $m \to \infty$. But if $\{e_{m,j}\}_j$ is a basis for R_m that is orthogonal for both norms, then

$$\frac{1}{mN_m}\sum_{j=1}^{N_m} \lambda_{m,j} = \frac{1}{mN_m}\sum_{j=1}^{N_m} \log \frac{\|e_{m,j}\|'}{\|e_{m,j}\|} = \frac{1}{mN_m}\sum_{j=1}^{N_m} \log \frac{1}{\|e_{m,j}\|} - \frac{1}{mN_m}\sum_{j=1}^{N_m} \log \frac{1}{\|e_{m,j}\|'}$$

is the difference of the barycenters of the probability measures defined by the successive minima of $\|\cdot\|_m$ and $\|\cdot\|'_m$ respectively, and hence converges to the difference of the barycenters of the limit measures of $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$. □

Corollary 3.3. *For any two bounded graded norms $\|\cdot\|$ and $\|\cdot\|'$ on R, the limit*

$$\mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) := \lim_{m\to\infty} m^{-1}\,\mathrm{vol}(\|\cdot\|_m, \|\cdot\|'_m)$$

exists, and equals the barycenter of the relative limit measure $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$.

We call $\mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ the *relative volume* of $\|\cdot\|_\bullet$ with respect to $\|\cdot\|'_\bullet$. It satisfies a cocycle condition as in (3.3). The proof of Theorem 3.2 shows that if $\nu = \mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$, then

$$\int \max\{\lambda, c\}\, d\nu(\lambda) = \mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet \vee \exp(c\,\bullet)\|\cdot\|_\bullet). \tag{3.8}$$

Proof of Corollary 3.3. This follows from Theorem 3.2 since $m^{-1}\,\mathrm{vol}(\|\cdot\|_m, \|\cdot\|'_m)$ is the barycenter of the measure ν_m, which converges weakly to $\nu := \mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$. Indeed, the supports of all the ν_m are all contained in a fixed interval $[-C, C]$. □

The next results show how relative limit measures behave under operations on graded norms. They follow from elementary computations of relative successive minima in bases for R_m that are orthogonal for both $\|\cdot\|_m$ and $\|\cdot\|'_m$. The details are left to the reader.

Proposition 3.4. *Let $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ be bounded graded norms on R, and $c \in \mathbf{R}$. Then:*

(i) $\mathrm{RLM}(\|\cdot\|'_\bullet, \|\cdot\|_\bullet)$ *is the pushforward of* $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ *under* $\lambda \mapsto -\lambda$.
(ii) $\mathrm{RLM}(\exp(c\,\bullet)\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ *is the pushforward of* $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ *under* $\lambda \mapsto \lambda - c$.
(iii) $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|_\bullet \vee \|\cdot\|'_\bullet)$ *is the pushforward of* $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ *under* $\lambda \mapsto \max\{\lambda, 0\}$.

Proposition 3.5. *Let $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ be bounded graded norms on R. Given $r \geq 1$, let $\|\cdot\|_\bullet^{(r)}$ and $\|\cdot\|_\bullet'^{(r)}$ be their restrictions to $R(X, rL)$. Then $\mathrm{RLM}(\|\cdot\|_\bullet^{(r)}, \|\cdot\|_\bullet'^{(r)})$ is the pushforward of $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ under $\lambda \mapsto r\lambda$.*

3.6. **Equivalence of bounded graded norms.** Theorem 3.2 yields

Corollary 3.6. *Let $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ be bounded graded norms on R. For $p \in [1, \infty)$, the limit*

$$d_p(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = \lim m^{-1} d_{m,p}(\|\cdot\|_m, \|\cdot\|'_m)$$

exists, and we have

$$d_p(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = \int_{\mathbf{R}} |\lambda|^p \, d\nu(\lambda), \tag{3.9}$$

where $\nu = \mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ is the relative limit measure of $\|\cdot\|_\bullet$ with respect to $\|\cdot\|'_\bullet$.

The function d_p on pairs of bounded graded norms is symmetric and satisfies the triangle inequality. Further (3.5) implies

$$d_p(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)^p = d_p(\|\cdot\|_\bullet, \|\cdot\|_\bullet \vee \|\cdot\|'_\bullet)^p + d_p(\|\cdot\|'_\bullet, \|\cdot\|_\bullet \vee \|\cdot\|'_\bullet)^p. \tag{3.10}$$

In particular, the distance d_1 can be computed in terms of relative volumes:

$$d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = \mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|_\bullet \vee \|\cdot\|'_\bullet) + \mathrm{vol}(\|\cdot\|'_\bullet, \|\cdot\|_\bullet \vee \|\cdot\|'_\bullet). \tag{3.11}$$

We'd like to analyze how far d_p is from being a distance. If $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ are bounded graded norms on R, then $p \mapsto d_p(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ is increasing. Further, if $C \geq 0$ is such that

$$\exp(-Cm) \leq \|\cdot\|_m, \|\cdot\|'_m \leq \exp(Cm)$$

on R_m for all $m \geq 1$, then $d_p(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) \leq C^{p-1} d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ for all $p \in [1, \infty)$. In particular, for any two graded norms and any $p \in [1, \infty)$, we have $d_p(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = 0$ iff $d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = 0$. In this case we say that the two graded norms are *equivalent*. As a consequence of Corollary 3.6 we have:

Corollary 3.7. *If $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ are bounded graded norms on R, then $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ are equivalent iff the relative limit measure $\mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ is a Dirac mass at the origin.*

Remark 3.8. *The equivalence notion above is the natural one in our study, but there other possibilities. For example, let us say two bounded graded norms $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ are ∞-*equivalent *if $\lim m^{-1} d_\infty(\|\cdot\|_m, \|\cdot\|'_m) = 0$. This implies equivalence in the sense above, but the converse is not true. Indeed, let $X = \mathbf{P}^1$ and $L = \mathcal{O}(1)$, so that $R = k[S, T]$. Let $\|\cdot\|'_\bullet$ be the trivial graded norm, and define a graded norm $\|\cdot\|_\bullet$ on R by $\|Q(S, T)\|_m = \exp(m)$ if $S \nmid Q$ and $\|Q(S, T)\|_m = 1$ if $S \mid Q$. The measure in Theorem 3.2 is given by $\nu_m = \frac{1}{m+1}\delta_{-1} + \frac{m}{m+1}\delta_0$. Thus the relative limit measure is equal to δ_0, but $d_\infty(\|\cdot\|_m, |\cdot\|'_m) = m$ for all m. See also* [Szé15, p.458].

All the constructions so far involving graded norms only depend on equivalence classes:

Theorem 3.9. *Let $\|\cdot\|_{i,\bullet}$ and $\|\cdot\|'_{i,\bullet}$, $i = 1, 2$, be bounded graded norms on R such that $\|\cdot\|_{i,\bullet}$ is equivalent to $\|\cdot\|'_{i,\bullet}$. Then $\mathrm{RLM}(\|\cdot\|_{1,\bullet}, \|\cdot\|_{2,\bullet}) = \mathrm{RLM}(\|\cdot\|'_{1,\bullet}, \|\cdot\|'_{2,\bullet})$. As a consequence, $\mathrm{vol}(\|\cdot\|_{1,\bullet}, \|\cdot\|_{2,\bullet}) = \mathrm{vol}(\|\cdot\|'_{1,\bullet}, \|\cdot\|'_{2,\bullet})$.*

Lemma 3.10. *With notation and assumptions as in Theorem 3.9, we have:*

(i) *$\|\cdot\|_{1,\bullet} \vee \|\cdot\|_{2,\bullet}$ is equivalent to $\|\cdot\|'_{1,\bullet} \vee \|\cdot\|'_{2,\bullet}$;*
(ii) *For $i = 1, 2$ and any $c \in \mathbf{R}$, $\exp(c\,\bullet)\|\cdot\|_{i,\bullet}$ and $\exp(c\,\bullet)\|\cdot\|'_{i,\bullet}$ are equivalent.*

Proof. The estimate in Lemma 3.1 implies a corresponding estimate for bounded graded norms, and yields (i). The statement in (ii) follows from the equality

$$d_1(\exp(c\,\bullet)\|\cdot\|_\bullet, \exp(c\,\bullet)\|\cdot\|'_\bullet) = d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$$

for any two bounded graded norms. □

Proof of Theorem 3.9. Set $\nu := \mathrm{RLM}(\|\cdot\|_{1,\bullet}, \|\cdot\|_{2,\bullet})$ and $\nu' := \mathrm{RLM}(\|\cdot\|'_{1,\bullet}, \|\cdot\|'_{2,\bullet})$. We must show that $\nu = \nu'$. It suffices to prove that $\int \max\{\lambda, c\}\, d\nu(\lambda) = \int \max\{\lambda, c\}\, d\nu'(\lambda)$ for all $c \in \mathbf{R}$. This follows from (3.8) and Lemma 3.10. □

3.7. **Almost trivial graded norms and filtrations.** Given $p \in [1, \infty)$ we define the *pth central moment* $\|\mathcal{F}\|_p$ of a graded filtration $\mathcal{F}$ of R as the pth central moment of its limit measure. Thus $\|\mathcal{F}\|_p = (\int |\lambda - \overline{\lambda}|^p\, d\nu(\lambda))^{1/p}$, where ν is the limit measure and $\overline{\lambda} = \int \lambda\, d\nu(\lambda)$ is its barycenter. We define pth central moments of bounded graded norms in the same way, although the notation becomes rather cumbersome!

When $\mathcal{F}$ is associated to a test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L), $\|\mathcal{F}\|_p$ coincides with the L^p-norm of the test configuration as defined in [Don05, WN12, His16]. For a general bounded graded filtration, $\|\mathcal{F}\|_2$ equals the L^2-norm considered by Székelyhidi [Szé15].

We say that a bounded graded norm (or filtration) is *almost trivial* if its pth central moment is zero, i.e. its limit measure is a Dirac mass. This means that it is equivalent to a graded norm in the orbit of the trivial graded norm under the $\mathbf{R}$-action. By what precedes, a bounded graded filtration is almost trivial iff its L^2-norm in the sense of [Szé15] is zero.

4. The asymptotic Fubini–Study operator

In this section we define and study the asymptotic Fubini–Study operator. This associates a bounded psh metric on L to any graded norm on the section ring $R = R(X, L)$. In particular, we prove Theorems B and C in the introduction. As in §3, L is an ample line bundle (as opposed to $\mathbf{Q}$-line bundle) on X.

4.1. **Psh metrics regularizable from below.** A bounded psh metric $\varphi \in \mathrm{PSH}(L)$ is *regularizable from below* if there exists an increasing net $(\varphi_j)_{j\in J}$ of metrics in $\mathcal{H}(L)$ converging to φ in $\mathrm{PSH}(L)$. Thus $\varphi = \lim_j \varphi_j$ pointwise on X^{qm}, and φ is the usc regularization of the pointwise limit of the φ_j. It is equivalent to demand the same property with the φ_j being continuous psh metrics. We write $\mathrm{PSH}^{\uparrow}(L)$ for the set of metrics regularizable from below.

Remark 4.1. *Inspired by the main result in* [Bed80]*, we conjecture that a psh metric is regularizable from below iff it is continuous outside a pluripolar set, see Definition 5.1.*

Not every bounded psh metric is regularizable from below.

Example 4.2. *Let $X = \mathbf{P}^1_k$ and $L = \mathcal{O}(1)$, where k is an infinite field, and let $(x_n)_1^\infty$ be a Zariski dense set in $X(k)$. There exists a metric $\psi \in \mathrm{PSH}(L)$ such that $\max \psi = 0$ and $\psi(x_n) = -\infty$ for all j. (We can take ψ with $\mathrm{MA}(\psi) = \sum_j c_n \delta_{x_n}$ where $c_n > 0$ and $\sum_j c_n = 1$). Then $\varphi := \exp(\psi) \in \mathrm{PSH}(L)$ is bounded, $\sup_X \varphi = 1$, and $\varphi(x_n) = 0$ for all j. Suppose $\varphi = \lim_j \varphi_j$ for an increasing net $(\varphi_j)_j$ of positive metrics. Then $\varphi_j(x_n) \le \varphi(x_n) = 0$ for all j and n, so since φ_j is continuous and x_n converges to the generic point of X, we must have $\varphi_j \le 0$ for all j, and hence $\varphi \le 0$, a contradiction.*

4.2. **Psh envelopes of bounded metrics.** Next we introduce

Definition 4.3. *For any bounded metric φ on L we set*

$$Q(\varphi) = \operatorname{usc}\sup\{\psi \in \mathcal{H}(L) \mid \psi \leq \varphi\}, \tag{4.1}$$

In (4.1) it would be equivalent to replace $\mathcal{H}(L)$ by $\mathcal{H}(L)_{\mathbf{R}}$ or $\mathrm{PSH}(L) \cap C^0(X)$, since any continuous psh metric can be uniformly approximated by positive metrics. Also note that $Q(\varphi) = Q(\operatorname{lsc}\varphi)$. Indeed, if $\psi \in \mathcal{H}(L)$, then $\psi \leq \varphi$ iff $\psi \leq \operatorname{lsc}\varphi$.

Lemma 4.4. *For any bounded metric φ, we have $Q(\varphi) \in \mathrm{PSH}^{\uparrow}(L)$. Further, if φ is usc, then $Q(\varphi) \leq \varphi$, with equality iff $\varphi \in \mathrm{PSH}^{\uparrow}(L)$.*

Proof. The set J of metrics $\psi \in \mathcal{H}(L)$ such that $\psi \leq \varphi$ is directed in the obvious way: $\psi \leq \psi'$ iff $\psi(x) \leq \psi'(x)$ for all $x \in X$. We can then view J as an increasing net in $\mathcal{H}(L)$, indexed by itself. This net converges to $Q(\varphi)$ in $\mathrm{PSH}(L)$, so $Q(\varphi) \in \mathrm{PSH}^{\uparrow}(L)$. The remaining statements are clear. □

In [BoJ18] we considered a different envelope: for any bounded metric φ on L, set

$$P(\varphi) = \sup\{\psi \in \mathrm{PSH}(L) \mid \psi \leq \varphi\}. \tag{4.2}$$

In view of Example 4.2, we can have $P(\varphi) \neq Q(\varphi)$, even when $\varphi \in \mathrm{PSH}(L)$.

Proposition 4.5. *Let φ be a bounded metric on L.*

(i) *If φ is continuous, so if $P(\varphi)$, and $Q(\varphi) = P(\varphi)$;*
(ii) *In general, $Q(\varphi) = \operatorname{usc} P(\operatorname{lsc}\varphi)$.*

Proof. First assume φ is continuous. By Corollary 5.28 in [BoJ18], $P(\varphi)$ is a continuous psh metric. Since the supremum in (4.2) is taken over a larger set of metrics than in (4.1), we have $Q(\varphi) \leq \operatorname{usc}(P(\varphi)) = P(\varphi)$. On the other hand, since $P(\varphi)$ can be uniformly approximated by positive metrics, it follows from (4.1) that $Q(\varphi) \geq P(\varphi)$. This proves (i).

To prove (ii), we may assume φ is lsc, since $Q(\varphi) = Q(\operatorname{lsc}\varphi)$. We must then prove that $Q(\varphi) = \operatorname{usc} P(\varphi)$. The inequality $Q(\varphi) \leq \operatorname{usc} P(\varphi)$ is clear from the definitions. Being lsc, φ is the limit of an increasing net φ_j of continuous metrics on L. Pick $\varepsilon > 0$ and suppose $\psi \in \mathrm{PSH}(L)$ satisfies $\psi \leq \varphi$. Then $\psi < \varphi_j + \varepsilon$ for $j \gg 0$. Thus (i) gives

$$\psi = P(\psi) \leq P(\varphi_j + \varepsilon) = P(\varphi_j) + \varepsilon = Q(\varphi_j) + \varepsilon \leq Q(\varphi) + \varepsilon,$$

Taking the supremum over ψ yields $P(\varphi) \leq Q(\varphi) + \varepsilon$, and hence $\operatorname{usc} P(\varphi) \leq Q(\varphi) + \varepsilon$. □

4.3. **The asymptotic Fubini–Study operator.** First suppose L is globally generated, and consider a norm $\|\cdot\|$ on $H^0(X, L)$. Define a metric $\mathrm{FS}(\|\cdot\|)$ on L by setting

$$\mathrm{FS}(\|\cdot\|) := \max\{\log(|s|_0/\|s\|) \mid s \in H^0(X, L) \setminus \{0\}\}.$$

This can more concretely be described as follows. Pick a basis $\{s_j\}_{j=1}^N$ for $H^0(X, L)$ that is orthogonal for $\|\cdot\|$ and write $\lambda_j := -\log\|s_j\|$. Then

$$\mathrm{FS}(\|\cdot\|) = \max_j(\log|s_j|_0 + \lambda_j) \tag{4.3}$$

Clearly $\mathrm{FS}(\|\cdot\|)$ belongs to $\mathcal{H}(L)_{\mathbf{R}}$, and is hence a continuous psh metric.

Now assume L is an ample bundle. Then mL is globally generated for $m \gg 0$.

Lemma 4.6. *The set $\mathcal{H}(L)_{\mathbf{R}}$ coincides with the set of metrics of the form $m^{-1}\,\mathrm{FS}(\|\cdot\|)$, where $m \gg 0$ and $\|\cdot\|$ is a norm on R_m.*

Proof. It is clear from (4.3) that any metric of the form $m^{-1}\,\mathrm{FS}(\|\cdot\|)$ belongs to $\mathcal{H}(L)_{\mathbf{R}}$. Conversely, any metric $\varphi \in \mathcal{H}(L)_{\mathbf{R}}$ is of the form $\varphi = m^{-1}\max_{1\le j\le N'}(\log|s_j|_0 + \lambda_j)$, where $m \ge 1$, $s_1,\dots,s_{N'}$ are global sections of mL without common zero, and $\lambda_j \in \mathbf{R}$, $1 \le j \le N'$. We may assume $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_{N'}$. We may also assume that s_j does not lie in the linear span of $(s_i)_{i<j}$ for $1 \le j \le N'$, or else we could remove the entry $\log|s_j| + \lambda_j$ from the max defining φ. Now $(s_j)_{j\le N'}$ are linearly independent, and can be completed to a basis $(s_j)_{j\le N_m}$ of R_m. If we pick $\lambda_j \ll 0$ for $j > N'$, then $\varphi = m^{-1}\max_{j\le N_m}(\log|s_j| + \lambda_j)$. Define a norm $\|\cdot\|$ on R_m by $\|\sum a_j s_j\| = \max_j |a_j|\exp(-\lambda_j)$. Then $\varphi = m^{-1}\,\mathrm{FS}(\|\cdot\|)$. □

Now consider a bounded graded norm $\|\cdot\|_\bullet$ on R. Then $\varphi_m := m^{-1}\,\mathrm{FS}(\|\cdot\|_m) \in \mathcal{H}(L)_{\mathbf{R}}$ for $m \gg 0$. The fact that $\|\cdot\|_\bullet$ is bounded means that there exists $C > 0$ such that $|\varphi_m| \le C$ for all $m \gg 0$. The submultiplicative property $\|s\otimes s'\|_{m+m'} \le \|s\|_m \cdot \|s'\|_{m'}$ implies $(m+m')\varphi_{m+m'} \ge m\varphi_m + m'\varphi_{m'}$ pointwise on X. By Fekete's lemma, the limit $\lim_{m\to\infty}\varphi_m$ therefore exists pointwise on X, is finite, and equals $\sup_m \varphi_m$. Now set

$$\mathrm{FS}(\|\cdot\|_\bullet) := \mathrm{usc}(\lim_{m\to\infty}\varphi_m) = \mathrm{usc}(\sup_m \varphi_m),$$

where usc denotes upper semicontinuous regularization. Thus $\mathrm{FS}(\|\cdot\|_\bullet)$ is a bounded psh metric on L, called the *asymptotic Fubini–Study metric* associated to $\|\cdot\|_\bullet$.

Remark 4.7. *It will be useful to consider $\mathbf{Z}_{>0}$ as a set directed by $r_1 \le r_2$ iff $r_1 \mid r_2$; then $(\varphi_r)_r$ is an increasing net in $\mathrm{PSH}(L)$ that converges to φ.*

The next result shows how the asymptotic Fubini–Study operator interacts with natural operations on graded norms and metrics. The proof is left to the reader. See Proposition 4.20 for a deeper result.

Proposition 4.8. *Let $\|\cdot\|_\bullet$ be a bounded graded norm on R, $c \in \mathbf{R}$, and $r \in \mathbf{Z}_{>0}$. Then:*

(i) $\mathrm{FS}(\exp(c\bullet)\|\cdot\|_\bullet) = \mathrm{FS}(\|\cdot\|_\bullet) - c$;

(ii) *if $\|\cdot\|_\bullet^{(r)}$ is the restriction of $\|\cdot\|_\bullet$ to $R(X, rL)$, then $\mathrm{FS}(\|\cdot\|_\bullet) = r^{-1}\,\mathrm{FS}(\|\cdot\|_\bullet^{(r)})$*

4.4. **The range of the asymptotic Fubini–Study operator.** We now turn to the proof of Theorem B in the introduction. It is clear that any asymptotic Fubini–Study metric is regularizable from below. To prove the converse, we associate to any (not necessarily psh) bounded metric φ on L its *graded supremum norm* $\|\cdot\|_{\varphi,\bullet}$ defined by

$$\|\cdot\|_{\varphi,m} := \sup_X |\cdot|_0 \exp(-m\varphi)$$

for $m \ge 1$. This is a bounded graded norm on R, and it follows from the definition that

$$\|\cdot\|_{\varphi,\bullet} \vee \|\cdot\|_{\varphi,\bullet} = \|\cdot\|_{\varphi\wedge\varphi',\bullet} \tag{4.4}$$

for bounded metrics φ and φ' on L.

Proposition 4.9. *For any bounded metric φ on L we have $Q(\varphi) = \mathrm{FS}(\|\cdot\|_{\varphi,\bullet})$.*

Remark 4.10. *As a consequence, it would have sufficed to use sequences rather than nets in Definition 4.3. This can be viewed as an instance of Choquet's lemma.*

Proof. We will use the fact that (4.1) remains true with $\mathcal{H}(L)$ replaced by $\mathcal{H}(L)_{\mathbf{R}}$. Write $\varphi_m := m^{-1}\,\mathrm{FS}(\|\cdot\|_{\varphi,m})$ for $m \ge 1$ so that $\mathrm{FS}(\|\cdot\|_{\varphi,\bullet}) = \mathrm{usc}\sup_m \varphi_m$. On the one hand, $\varphi_m \in \mathcal{H}(L)_{\mathbf{R}}$ and $\varphi_m \le \varphi$ for all $m \gg 0$, so $\mathrm{FS}(\|\cdot\|_{\varphi,\bullet}) \le Q(\varphi)$. On the other hand, suppose $\psi \in \mathcal{H}(L)_{\mathbf{R}}$ and $\psi \le \varphi$. Write $\psi = m^{-1}\max_{1\le j\le N}(\log|s_j|_0 + \lambda_j)$, where $s_j \in R_m$, $\lambda_j \in \mathbf{R}$ and the s_j have no common zero. Since $\psi \le \varphi$, we have $\|s_j\|_{m\varphi} = \sup|s_j|_0\exp(-m\varphi) \le$

$\exp(-m\lambda_j)$, and hence $\log(|s_j|_0/\|s_j\|_{m\varphi}) \geq \log|s_j|_0 + \lambda_j$ for all j. Taking the max over j gives

$$\psi \leq m^{-1}\max_j \log(|s_j|_0/\|s_j\|_{m\varphi}) \leq m^{-1}\,\mathrm{FS}(\|\cdot\|_{\varphi,m}) = \varphi_m \leq \mathrm{FS}(\|\cdot\|_{\varphi,\bullet}).$$

This gives $Q(\varphi) \leq \mathrm{FS}(\|\cdot\|_{\varphi,\bullet})$ and concludes the proof. □

As a consequence, we obtain the following result, which implies Theorem B.

Corollary 4.11. *If $\varphi \in \mathrm{PSH}(L)$ is regularizable from below, then $\varphi = Q(\varphi) = \mathrm{FS}(\|\cdot\|_{\varphi,\bullet})$.*

The following result is more restrictive but also more precise.

Lemma 4.12. *Suppose that R is generated in degree 1, and that $\varphi = \varphi_{\mathcal{L}}$ is defined by a globally generated test configuration $(\mathcal{X}, \mathcal{L})$ for (X, L). Then $\varphi = Q(\varphi) = \mathrm{FS}(\|\cdot\|_{\varphi,1})$.*

Proof. Since $\mathrm{FS}(\|\cdot\|_{\varphi,1}) \leq Q(\varphi) \leq \varphi$, it suffices to prove $\mathrm{FS}(\|\cdot\|_{\varphi,1}) = \varphi$.

By assumption, $\mathcal{L}$ defines a $\mathbb{G}_m$-equivariant morphism of $\mathcal{X}$ into a product test configuration $\mathbb{P} \times \mathbb{A}^1_k$, where $\mathbb{P} = \mathbb{P}(H^0(X, L))$, and $\mathcal{L}$ is the pullback of $\mathcal{O}_{\mathbb{P}\times\mathbb{A}^1_k}(1)$. Since φ and $\mathrm{FS}(\|\cdot\|_{\varphi,1})$ are the pullbacks of the corresponding metrics on $\mathcal{O}_{\mathbf{P}}(1)$, it suffices to consider the case when X is a projective space, $L = \mathcal{O}_{\mathbf{P}}(1)$, and the test configuration for (X, L) is a product (but not necessarily trivial) test configuration. Pick a basis $s_0, \ldots, s_N$ of $H^0(X, L)$ that diagonalizes the $\mathbb{G}_m$-action, with the weights being $\lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_N$.

On the one hand, $\mathrm{FS}(\|\cdot\|_{\varphi,1}) = \max_j \log|s_j|_0 + \lambda_j$. On the other hand, for each j, s_j extends as a $\mathbb{G}_m$-equivariant section $\bar{s}_j$ of $\mathcal{L}$ over $\mathbb{P} \times \mathbb{G}_m$. Then $\varpi^{-\lambda_j}\bar{s}_j$, $1 \leq j \leq N$ are global sections of $\mathcal{L}$ without common zero on $\mathbb{P} \times \mathbb{A}^1_k$. Thus the metric $\varphi_{\mathcal{L}}$ defined by the test configuration $(\mathcal{X}, \mathcal{L})$ is given by $\varphi_{\mathcal{L}} = \max_j(|s_j|_0 + \lambda_j)$, which completes the proof. □

4.5. **Monge–Ampère energy and relative limit measures.** Next we show that the relative Monge–Ampère energy of two asymptotic Fubini–Study metrics is equal to the relative volume of the graded norms, as well as the barycenter of their relative limit measure.

Theorem 4.13. *Let $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ be bounded graded norms on $R = R(X, L)$, and set $\varphi := \mathrm{FS}(\|\cdot\|_\bullet)$, $\varphi' := \mathrm{FS}(\|\cdot\|'_\bullet)$. Then we have*

$$E(\varphi, \varphi') = \int_{\mathbf{R}} \lambda\, d\nu(\lambda) = \mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet), \tag{4.5}$$

where $\nu = \mathrm{RLM}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet)$ is the relative limit measure of $\|\cdot\|_\bullet$ with respect to $\|\cdot\|'_\bullet$.

Corollary 4.14. *If φ and φ' are bounded metrics on L, then*

$$E(Q(\varphi), Q(\varphi')) = \mathrm{vol}(\|\cdot\|_{\varphi,\bullet}, \|\cdot\|_{\varphi',\bullet}). \tag{4.6}$$

Proof. This follows since $Q(\varphi) = \mathrm{FS}(\|\cdot\|_{\varphi,\bullet})$ and $Q(\varphi') = \mathrm{FS}(\|\cdot\|_{\varphi',\bullet})$. □

If φ and φ' are *continuous* metrics on L, then by Proposition 4.5, the envelopes $Q(\varphi)$, $Q(\varphi')$ coincide with the envelopes considered in [BE18, BGJKM16], namely $P(\varphi)$, $P(\varphi')$, so Corollary 4.14 recovers the main result of *loc. cit.* in the case of a trivially valued field.

Proof of Theorem 4.13. The last equality in (4.5) follows from Corollary 3.3. By the cocycle properties of E and vol, we may assume $\|\cdot\|'_\bullet$ is the trivial graded norm, and φ' the trivial metric on L.

After replacing L by a multiple, and using Propositions 3.5 and 4.8, we may assume that L is a line bundle, and that the canonical map $S^m R_r \to R_{mr}$ is surjective for all $r, m \geq 1$.

Let $\mathcal{F}$ be the graded filtration associated to $\|\cdot\|_\bullet$. It is easy to see that replacing $\mathcal{F}$ by the associated graded $\mathbf{Z}$-filtration $\mathcal{F}_{\mathbf{Z}}$ (and modifying the graded norm accordingly) does not change φ, ν, or $\mathrm{vol}(\|\cdot\|_\bullet)$. We may therefore assume that $\mathcal{F}$ is a graded $\mathbf{Z}$-filtration.

First assume $\mathcal{F}$ is generated in degree 1. In this case, $\mathcal{F}$ is associated to an ample test configuration $(\mathcal{X},\mathcal{L})$ for (X,L), and $\varphi=\varphi_{\mathcal{L}}$ is the metric defined by $(\mathcal{X},\mathcal{L})$, see Lemma 4.12. The first equality in (4.5) then follows from [BHJ17, Proposition 5.9].

Now consider a general bounded graded norm $\|\cdot\|_\bullet$ whose associated graded filtration $\mathcal{F}$ is a graded $\mathbf{Z}$-filtration. Since the canonical map $S^m R_r \to R_{rm}$ is surjective for all $r,m\geq 1$, we can equip R_{rm} with the quotient norm from the norm $\|\cdot\|_r$ on R_r. This defines a bounded graded norm $\|\cdot\|_\bullet^{(r)}$ on the section ring $R(X,rL)$. By definition, this graded norm is generated in degree 1. Let $\mathcal{F}^{(r)}$ be the associated graded filtration of $R(X,rL)$, $\nu_r := r_*^{-1}\,\mathrm{LM}(\|\cdot\|_\bullet^{(r)})$ its (scaled) limit measure, and $\varphi_r := r^{-1}\,\mathrm{FS}(\|\cdot\|_\bullet^{(r)})$ the corresponding metric on L. By what precedes, we have $E(\varphi_r)=\int\lambda\,d\nu_r(\lambda)$ for each $r\geq 1$.

As in Remark 4.7, consider $\mathbf{Z}_{>0}$ as a set directed by $r_1\leq r_2$ iff $r_1\mid r_2$. Then $(\varphi_r)_r$ is an increasing net in $\mathrm{PSH}(L)$ that converges to φ. Since the Monge–Ampère energy is continuous along increasing nets, we get $\lim_r E(\varphi_r)=E(\varphi)$.

It remains to prove that $\lim_{r\to\infty}\int\lambda\,d\nu_r(\lambda)=\int\lambda\,d\nu(\lambda)$. Since the measures ν_r have support contained in a fixed compact interval, it suffices to prove that $\lim_r\nu_r=\nu$ weakly. To do so, we recall the construction of the limit measure from [BC11]. For each $\lambda\in\mathbf{R}$, consider the graded subalgebra $V_\bullet^{(\lambda)}$ of $R(X,L)$ defined by $V_m^{(\lambda)}=\{s\in R_m\mid \|s\|_m\leq\exp(-m\lambda)\}$. Using Okounkov bodies [LM09], one shows that the limit $\mathrm{vol}(V_\bullet^{(\lambda)}):=\lim_{m\to\infty}n!\,\mathrm{vol}(V_m^{(\lambda)})/m^n$ exists, and is a decreasing function of λ. The limit measure then satisfies

$$\nu=-\,\mathrm{vol}(L)^{-1}\frac{d}{d\lambda}\,\mathrm{vol}(V_\bullet^{(\lambda)})$$

in the sense of distributions. Similarly, for $r\geq 1$ and $\lambda\in\mathbf{R}$, we have a graded subalgebra $W_\bullet^{(r,\lambda)}$ of $R(X,rL)$, given by $W_m^{(r,\lambda)}=\{s\in R_{rm}\mid \|s\|_m^{(r)}\leq\exp(-rm\lambda)\}$. Again, the limit $\mathrm{vol}(W_\bullet^{(r,\lambda)}):=\lim_{m\to\infty}n!\,\mathrm{vol}(W_m^{(r,\lambda)})/m^n$ exists, and is a decreasing function of λ, and

$$\nu_r=-\,\mathrm{vol}(rL)^{-1}\frac{d}{d\lambda}\,\mathrm{vol}(W_\bullet^{(r,\lambda)})=-\,\mathrm{vol}(L)^{-1}\frac{d}{d\lambda}r^{-n}\,\mathrm{vol}(W_\bullet^{(r,\lambda)}).$$

Set $g(\lambda):=\mathrm{vol}(V_\bullet^{(\lambda)})$ and $g_r(\lambda):=r^{-n}\,\mathrm{vol}(W_\bullet^{(r,\lambda)})$. By Lemma 4.15 below, $g_r\to g$ pointwise on $\mathbf{R}$. Since $0\leq g_r,g\leq\mathrm{vol}(L)$, it follows from dominated convergence that $g_r\to g$ in $L^1_{\mathrm{loc}}(\mathbf{R})$. Hence $\nu_r\to\nu$ in the sense of distributions, which completes the proof. □

Lemma 4.15. *Let $V_\bullet\subset R(X,L)$ be a graded subalgebra. Suppose that for $r\geq 1$ we have a graded subalgebra $W_\bullet^{(r)}\subset V_\bullet^{(r)}$, where $V_m^{(r)}:=V_{rm}$ for $m\geq 1$, satisfying $W_1^{(r)}=V_1^{(r)}$. Then $\lim_{r\to\infty}r^{-n}\,\mathrm{vol}(W_\bullet^{(r)})=\mathrm{vol}(V_\bullet)$.*

Proof. We use Okounkov bodies, following [Bou14]. Pick a valuation $\mu\colon k(X)^\times\to\mathbf{Z}^n$ of rational rank n. Set $\Gamma_m:=\mu(V_m\setminus\{0\})$ and $\Gamma_m^{(r)}:=\mu(W_m^{(r)}\setminus\{0\})$ for $m\geq 1$. Let $\Delta(V_\bullet)$ and $\Delta(W_\bullet^{(r)})$ be the closed convex hull inside $\mathbf{R}^n$ of $\bigcup_m m^{-1}\Gamma_m$ and of $\bigcup_m m^{-1}\Gamma_m^{(r)}$, respectively. Then $\mathrm{vol}(V_\bullet)=n!\,\mathrm{vol}(\Delta(V_\bullet))$ and $\mathrm{vol}(W_\bullet^{(r)})=n!\,\mathrm{vol}(\Delta(W_\bullet^{(r)}))$, so it suffices to prove that $\lim_{r\to\infty}\mathrm{vol}(r^{-1}\Delta(W_\bullet^{(r)}))=\mathrm{vol}(\Delta(V_\bullet))$.

Since $W_m^{(r)}\subset V_{rm}$, we get $\Gamma_m^{(r)}\subset\Gamma_{rm}$ for all r,m, and hence $r^{-1}\Delta(W_\bullet^{(r)})\subset\Delta(V_\bullet)$ for all r. If $\mathrm{vol}(\Delta(V_\bullet))=0$, we are done, so we may assume $\Delta(V_\bullet)$ has nonempty interior.

Pick compact subsets K and L of $\mathbf{R}^n$ with $K \subset\subset L \subset\subset \Delta(V_\bullet)$. It suffices to prove that $r^{-1}\Delta(W_\bullet^{(r)}) \supset K$ for $r \gg 1$. Now $r^{-1}\mathbf{Z}^n \cap L = r^{-1}\Gamma_r \cap L$ for $r \gg 1$, see [Bou14, Lemme 1.13]. If Δ_r is the convex hull of $r^{-1}\Gamma_r$, it follows that $\Delta_r \supset K$ for $r \gg 1$. But $W_1^{(r)} = V_r$, so $\Gamma_1^{(r)} = \Gamma_r$, and hence $r^{-1}\Delta(W_\bullet^{(r)}) \supset \Delta_r \supset K$, which completes the proof. □

4.6. **Injectivity of the asymptotic Fubini–Study operator.** The next result is a reformulation of Theorem C in the introduction.

Theorem 4.16. *Let $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ be bounded graded norms on $R(X,L)$ and write $\varphi := \mathrm{FS}(\|\cdot\|_\bullet)$, $\varphi' := \mathrm{FS}(\|\cdot\|'_\bullet)$. Then $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ are equivalent iff $\varphi = \varphi'$.*

We will prove this using the Monge–Ampère energy. For this we need some preparation.

Lemma 4.17. *If $\varphi, \varphi' \in \mathcal{E}^1(L)$ and $\varphi \geq \varphi'$, then $\varphi = \varphi'$ iff $E(\varphi, \varphi') = 0$.*

Proof. The direct implication is trivial. For the reverse direction, note that

$$E(\varphi, \varphi') = \frac{1}{(n+1)V} \sum_{j=0}^{n} \int (\varphi' - \varphi)\omega_\varphi^j \wedge \omega_\psi^{n-j} \geq \frac{1}{n+1} I(\varphi, \varphi').$$

This implies $I(\varphi, \varphi') = 0$, and hence $\varphi - \varphi'$ is constant, say $\varphi = \varphi' + c$, with $c \geq 0$. But then $c = E(\varphi, \varphi') = 0$, completing the proof. □

Corollary 4.18. *The assertion of Theorem 4.16 holds when $\|\cdot\|_\bullet \leq \|\cdot\|'_\bullet$.*

Proof. Indeed, in this case, $\varphi \geq \varphi'$, and (3.11), (4.5) imply that

$$d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = \mathrm{vol}(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = E(\varphi, \varphi'),$$

which concludes the proof in view of Lemma 4.17. □

Lemma 4.19. *If $\|\cdot\|_\bullet$ is a bounded graded norm, and $\varphi := \mathrm{FS}(\|\cdot\|_\bullet)$ its asymptotic Fubini–Study metric, then the supremum graded norm $\|\cdot\|_{\varphi,\bullet}$ is equivalent to $\|\cdot\|_\bullet$. It is the smallest bounded graded norm equivalent to $\|\cdot\|_\bullet$.*

Proof. We first prove that $\|\cdot\|_{\varphi,\bullet} \leq \|\cdot\|_\bullet$. Recall that $\varphi = \sup_m \varphi_m$, where $\varphi_m := m^{-1} \sup_{s \in R_m \setminus \{0\}} \log \frac{|s|_0}{\|s\|_m}$. Thus, for every $s \in R_m \setminus \{0\}$,

$$\|s\|_{\varphi,m} = \sup_X |s|_0 \exp(-m\varphi) \leq \sup_X |s|_0 \exp(-mm^{-1} \log \frac{|s|_0}{\|s\|_m}) = \|s\|_m.$$

Since $\varphi \in \mathrm{PSH}^\uparrow(L)$, Corollary 4.11 shows that $\mathrm{FS}(\|\cdot\|_{\varphi,\bullet}) = \varphi = \mathrm{FS}(\|\cdot\|_\bullet)$. Thus $\|\cdot\|_\bullet$ and $\|\cdot\|_{\varphi,\bullet}$ are equivalent by Corollary 4.18. □

Proposition 4.20. *Let $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ be bounded graded norms on R. Then*

$$\mathrm{FS}(\|\cdot\|_\bullet \vee \|\cdot\|'_\bullet) = Q(\varphi \wedge \varphi'), \tag{4.7}$$

where $\varphi := \mathrm{FS}(\|\cdot\|_\bullet)$ and $\varphi' := \mathrm{FS}(\|\cdot\|'_\bullet)$.

Proof. By Lemma 4.19, the graded norms $\|\cdot\|_\bullet$ and $\|\cdot\|_{\varphi,\bullet}$ are equivalent, as are $\|\cdot\|'_\bullet$ and $\|\cdot\|_{\varphi',\bullet}$. Lemma 3.10 then shows that $\|\cdot\|_\bullet \vee \|\cdot\|'_\bullet$ is equivalent to $\|\cdot\|_{\varphi,\bullet} \vee \|\cdot\|_{\varphi',\bullet}$. Thus we may assume $\|\cdot\|_\bullet = \|\cdot\|_{\varphi,\bullet}$ and $\|\cdot\|'_\bullet = \|\cdot\|_{\varphi',\bullet}$. Now $\|\cdot\|_\bullet \vee \|\cdot\|'_\bullet = \|\cdot\|_{\varphi\wedge\varphi',\bullet}$, see (4.4), so the result follows from Proposition 4.9. □

Corollary 4.21. *If $\|\cdot\|_\bullet$ and $\|\cdot\|'_\bullet$ are bounded graded norms on $R(X,L)$, then*

$$d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = E(\varphi, Q(\varphi\wedge\varphi')) + E(\varphi', Q(\varphi\wedge\varphi')), \tag{4.8}$$

where $\varphi := \mathrm{FS}(\|\cdot\|_\bullet)$ and $\varphi' := \mathrm{FS}(\|\cdot\|'_\bullet)$.

Proof. This follows from Theorem 4.13, (3.11) and (4.7). □

Proof of Theorem 4.16. By definition, the graded norms are equivalent iff $d_1(\|\cdot\|_\bullet, \|\cdot\|'_\bullet) = 0$. This happens iff both terms in the right hand side of (4.8) vanish, since the terms are nonnegative. By Lemma 4.17, this occurs iff $\varphi = Q(\varphi\wedge\varphi') = \varphi'$. □

We also mention a related result. Let $\mathcal{F}$ be a bounded graded filtration, with associated graded norm $\|\cdot\|_\bullet$ and metric $\varphi = \mathrm{FS}(\|\cdot\|_\bullet) \in \mathrm{PSH}^\uparrow(L)$. Then

$$c_n J(\varphi) \le \|\mathcal{F}\|_1 \le 2J(\varphi), \tag{4.9}$$

where $c_n = 2n^n/(n+1)^{n+1}$ and $\|\cdot\|_1$ is the first central moment defined in §3.7. Indeed, by [BHJ17, Theorem 7.9], this holds when $\mathcal{F}$ is induced by an ample test configuration, and the general case is reduced to this case by approximation, as in the proof of Theorem 4.13.

4.7. **The Darvas distance.** If $\varphi, \varphi' \in \mathrm{PSH}(L)$ are regularizable from below, then we set

$$d_1(\varphi,\varphi') := E(\varphi, Q(\varphi\wedge\varphi')) + E(\varphi', Q(\varphi\wedge\varphi')).$$

It follows that d_1 defines a distance on the space $\mathrm{PSH}^\uparrow(L)$. Further, the asymptotic Fubini–Study operator gives an isometric bijection between the space of equivalence classes of bounded graded norms on R, and the space $\mathrm{PSH}^\uparrow(L)$.

The metric on $\mathrm{PSH}^\uparrow(L)$ is analogous to the metric on $\mathcal{E}^1(L)$ considered by Darvas in the Archimedean case. Indeed, in our setting, if φ and φ' are continuous, then $d_1(\varphi,\varphi') := E(\varphi, P(\varphi\wedge\varphi')) + E(\varphi', P(\varphi\wedge\varphi'))$, in accordance with [Dar15, Corollary 4.14]. A positive answer to the following conjecture would make the analogy even stronger.

Conjecture 4.22. *We have $P(\varphi\wedge\varphi') = Q(\varphi\wedge\varphi')$ for $\varphi,\varphi' \in \mathrm{PSH}^\uparrow(L)$.*

4.8. **K-stability and filtrations.** Székelyhidi proved in [Szé15] that if the (reduced) automorphism group of (X,L) is finite and X admits a cscK metric in $c_1(L)$, then (X,L) satisfies a condition—involving filtrations—that is stronger than the usual notion of K-stability.

For simplicity assume $R(X,L)$ is generated in degree 1. Consider a bounded graded $\mathbf{Z}$-filtration $\mathcal{F}$ of $R(X,L)$. For $r \ge 1$, the filtration of R_r induces a filtration $\mathcal{F}^{(r)}$ of $R(X,rL)$ that is generated in degree one. This filtration defines an ample test configuration $\mathcal{L}_r$ for L. Let $\mathrm{DF}(\mathcal{L}_r)$ be its Donaldson–Futaki invariant, and set $\mathrm{Fut}(\mathcal{F}) := \liminf_r \mathrm{DF}(\mathcal{L}_r)$.

Now suppose (X,L) is uniformly K-stable in the strong sense that there exists $\varepsilon > 0$ such that $M(\varphi) \ge \varepsilon J(\varphi)$ for all $\varphi \in \mathcal{E}^1(L)$. Pick $\varphi = \mathrm{FS}(\|\cdot\|_\bullet)$, where $\|\cdot\|_\bullet$ is the graded norm associated to $\mathcal{F}$. By (4.9), we have $M(\varphi) \ge \frac{\varepsilon}{2}\|\mathcal{F}\|_1$. For $r \ge 1$, $\varphi_r := r^{-1}\,\mathrm{FS}(\|\cdot\|_r)$ is the positive metric associated to $\mathcal{L}_r$. As in Remark 4.7, partially order $\mathbf{Z}_{>0}$ by $r \le r'$ iff $r \mid r'$. Then the net $(\varphi_r)_r$ increases to φ. Thus $\lim_r E(\varphi_r) = E(\varphi)$, and similarly $E_{K_X}(\varphi_r) \to E_{K_X}(\varphi)$. Further $\mathrm{MA}(\varphi_r) \to \mathrm{MA}(\varphi)$, which implies $\liminf_r H(\varphi_r) \ge H(\varphi)$, since $H(\varphi) = \mathrm{Ent}(\mathrm{MA}(\varphi))$, and Ent is lsc. In view of (2.2), this gives $\liminf_r M(\varphi_r) \ge M(\varphi) \ge \frac{\varepsilon}{2}\|\mathcal{F}\|_1$. Now $\mathrm{DF}(\mathcal{L}_r) \ge M(\varphi_r)$ by [BHJ17, (7.7)], so we get

$$\mathrm{Fut}(\mathcal{F}) \ge \tfrac{\varepsilon}{2}\|\mathcal{F}\|_1.$$

Since $\|\mathcal{F}\|_2 = 0$ iff $\|\mathcal{F}\|_1 = 0$, we conclude that $\mathrm{Fut}(\mathcal{F}) > 0$ whenever $\|\mathcal{F}\|_2 > 0$; this is the condition considered in [Szé15].

If the (reduced) automorphism group of (X, L) is finite and X admits a cscK metric in $c_1(L)$, then (X, L) is uniformly K-stable in the usual sense, see [BDL16, Theorem 1.7].

5. A valuative criterion of K-stability

In [Fuj16], Fujita gave a valuative criterion of K-semistability and uniform K-stability of Fano varieties. The case of K-semistability was treated independently by C. Li [Li17]. Here we use psh metrics to prove a version of the valuative criterion in the general adjoint setting of an ample $\mathbf{Q}$-line bundle L on a smooth projective variety X. In particular, we prove Theorems D, E and E' of the introduction.

5.1. **Alexander-Taylor capacity.** Inspired by [AT84] we define the *Alexander-Taylor capacity*[2] of a point $x \in X$ (relative to L) as

$$T(x) = \sup\{\sup_X \varphi - \varphi(x) \mid \varphi \in \mathrm{PSH}(L)\}.$$

Here it is equivalent to take the supremum over metrics $\varphi \in \mathcal{H}(L)$ and/or normalized by $\varphi(x) \leq 0$. The following definition will be used in what follows.

Definition 5.1. *A subset $E \subset X$ is* pluripolar *if there exists $\varphi \in \mathrm{PSH}(L)$ with $\varphi|_E \equiv -\infty$.*

This notion does not depend on the choice of ample $\mathbf{Q}$-line bundle L. Indeed, if L' is another ample $\mathbf{Q}$-line bundle, then there exists $\varepsilon > 0$ rational such that $L'' := L' - \varepsilon L$ is ample. If $\varphi'' \in \mathcal{H}(L'')$, then $\varphi' := \varepsilon\varphi + \varphi'' \in \mathrm{PSH}(L'')$ and $\varphi' \equiv -\infty$ on E.

We say that a point $x \in X$ is pluripolar if $\{x\}$ is pluripolar. Any point $x \in X \setminus X^{\mathrm{val}}$ is pluripolar. Indeed, there exists a subvariety $Y \subsetneq X$ containing x. Since L is ample, there exists $m \geq 1$ and a nonzero section $s \in H^0(X, mL)$ vanishing along Y. Then $\varphi := m^{-1}\log|s|_0 \in \mathrm{PSH}(L)$ and $\varphi(x) = -\infty$. There also exist pluripolar points $x \in X^{\mathrm{val}}$. A simple example from [ELS03] is explained in [BKMS16, Remark 2.19].

Proposition 5.2. *Let $x \in X$ be any point. Then:*

(i) *$T(x) \geq 0$, with equality iff x is the generic point of X;*
(ii) *$T(x) = \infty$ iff x is pluripolar.*

Proof of Proposition 5.2. Clearly, $T(x) \geq 0$. If x is the generic point, then $\varphi(x) = \sup_X \varphi$ for every $\varphi \in \mathrm{PSH}(L)$, so $T(x) = 0$. Now suppose x is not the generic point of X and set $\xi = \mathrm{red}(x)$. There exists $m \geq 1$ and a nonzero section $s \in H^0(X, mL)$ vanishing at ξ. Then $\varphi := m^{-1}\log|s|_0 \in \mathrm{PSH}(L)$ satisfies $\sup_X \varphi = 0$ and $\varphi(x) < 0$, so $T(x) > 0$. This proves (i).

As for (ii), it is clear that $T(x) = \infty$ when x is pluripolar. Conversely, if $T(x) = \infty$, there exists a sequence $(\varphi_j)_1^\infty$ of metrics in $\mathrm{PSH}(L)$ such that $\sup_X \varphi_j = 0$ and $\varphi_j(x) \leq -2^j$. Then $\varphi := \sum_j 2^{-j}\varphi_j \in \mathrm{PSH}(L)$ satisfies $\varphi(x) = -\infty$, so x is pluripolar. □

We shall later use the following description of the Alexander-Taylor capacity.

Lemma 5.3. *Given a subset $P \subset \mathrm{PSH}(L)$, set $T_P(x) := \sup_{\varphi \in P}\{\sup_X \varphi - \varphi(x)\}$. Then we have $T_{\mathcal{H}(L)} = T_{P_1}(x) = T_{P_2}(x) = T(x)$, where*

$$P_1 = \{\varphi = m^{-1}\log|s|_0 \mid m \geq 1, s \in H^0(X, mL) \setminus \{0\}\},$$

and $P_2 = \{f = \psi + r \mid \psi \in P_1, r \in \mathbf{Q}\}$.

[2] It is the "multiplicative" version $e^{-T(x)}$ of $T(x)$ that behaves like the capacity T in [AT84].

Proof. That $T_{\mathcal{H}(L)}(x) = T_{\mathrm{PSH}(L)}(x)$ is clear since every function in $\mathrm{PSH}(L)$ is a decreasing limit of functions in $\mathcal{H}(L)$. That $T_{P_1}(x) = T_{P_2}(x)$ is also clear since $\varphi \mapsto \sup_X \varphi - \varphi(x)$ is translation invariant. Finally, since every function in $\mathcal{H}(L)$ is the maximum of finitely many functions in P_2, it follows that $T_{\mathcal{H}(L)} = T_{P_2}(x)$. This completes the proof. □

5.2. **Monge–Ampère energy.** The *Monge–Ampère energy* of a point $x \in X$ is

$$S(x) := E^*(\delta_x) \in \mathbf{R}_+ \cup \{+\infty\}.$$

We can write this as

$$S(x) = \sup\{E(\varphi) - \varphi(x) \mid \varphi \in \mathcal{E}^1(L)\},$$

where we may equivalently take the supremum over $\varphi \in \mathcal{H}(L)$. For such φ we have

$$\frac{n}{n+1} \min \varphi + \frac{1}{n+1} \sup_X \varphi \leq E(\varphi) \leq \sup_X \varphi, \tag{5.1}$$

so that $(n+1)^{-1}T(x) \leq S(x) \leq T(x)$. (See (5.3) for a better estimate.) In particular,

Lemma 5.4. *For any $x \in X$, we have $S(x) = \infty$ iff $T(x) = \infty$ iff x is pluripolar.*

The following result will be used later on.

Proposition 5.5. *We have $E^*(\mu) \leq \int S(x)\, d\mu(x)$ for any Radon probability measure μ on X.*

Proof. For any $x \in X$ and any $\varphi \in \mathcal{H}(L)$, we have $E(\varphi) \leq S(x) + \varphi(x)$. Integrating with respect to x, we get $E(\varphi) \leq \int S(x)\, d\mu(x) + \int \varphi\, d\mu$, and taking the supremum over $\varphi \in \mathcal{H}(L)$ gives the desired inequality. □

5.3. **Continuity properties.** We now study S and T as functions on X.

Proposition 5.6. *Any point in X^{qm} is nonpluripolar. Further, S and T are continuous on the dual complex $\Delta_{\mathcal{X}}$ of any snc test configuration $\mathcal{X}$ for X.*

Proof. For any $\mathcal{X} \in \mathrm{SNC}(X)$, it follows from [BoJ18, Theorem 5.29] that the restriction to $\Delta_{\mathcal{X}}$ of the family $\{\varphi - \sup_X \varphi \mid \varphi \in \mathcal{H}(L)\}$ is equicontinuous. In view of (5.1), the same holds for the family $\{\varphi - E(\varphi) \mid \varphi \in \mathcal{H}(L)\}$. The result follows. □

Remark 5.7. *There exist nonpluripolar points in $X^{\mathrm{val}} \backslash X^{\mathrm{qm}}$. See Example 2.14 in* [BKMS16], *and also Example 5.24 below.*

Lemma 5.8. *For any $x \in X$, the nets $(S(p_{\mathcal{X}}(x)))_{\mathcal{X} \in \mathrm{SNC}(X)}$ and $(T(p_{\mathcal{X}}(x)))_{\mathcal{X} \in \mathrm{SNC}(X)}$ are increasing, with limits $S(x)$ and $T(x)$, respectively.*

Proof. For every $\varphi \in \mathrm{PSH}(L)$, the net $(\varphi(p_{\mathcal{X}}(x)))_{\mathcal{X}}$ is decreasing, with limit $\varphi(x)$. This implies both results. □

Corollary 5.9. *The functions S and T are lsc on X.*

We also note

Proposition 5.10. *The functions S and T are homogeneous on X in the sense that $S(x^t) = tS(x)$ and $T(x^t) = tT(x)$ for $x \in X$ and $t \in \mathbf{R}_+^\times$.*

Proof. By [BoJ18, Proposition 7.12] we have $S(x^t) = E^*(t \cdot \delta_x) = tE^*(\delta_x) = tS(x)$. Second, if $\varphi \in \mathrm{PSH}(L)$, the function φ_t on X defined by $\varphi_t(y) = t\varphi(y^{1/t})$ also lies in $\mathrm{PSH}(L)$, see [BoJ18, Proposition 5.13]. Further, $\varphi_t \leq 0$ when $\varphi \leq 0$. This easily implies $T(x^t) \geq tT(x)$. Replacing (x, t) by (x^t, t^{-1}) shows that equality must hold. □

5.4. **Associated graded norm.** If L is an ample line bundle (as opposed to $\mathbf{Q}$-line bundle), any point $x \in X^{\mathrm{val}}$ defines a graded norm $\|\cdot\|_{x,\bullet}$ on $R = R(X, L)$ given by

$$\|s\|_m = |s|_0(x)$$

for $s \in R_m$. Its associated graded filtration was studied in detail in [BKMS16].

Proposition 5.11. *The graded norm $\|\cdot\|_{x,\bullet}$ is bounded iff x is nonpluripolar. Further, the Alexander-Taylor capacity $T(x)$ coincides with the invariant T in* [BlJ17].

The latter invariant was already introduced (but denoted differently) in [BKMS16]. It is called the *maximum order of vanishing* in [BlJ17]. We will see below that the Monge–Ampère energy $S(x)$ coincides with the *expected order of vanishing* as studied in [BKMS16, BlJ17].

Proof. For $m \geq 1$, consider the successive minima $0 = \lambda_{m,1} \leq \lambda_{m,2} \leq \cdots \leq \lambda_{m,N_m}$ of the norm $\|\cdot\|_{x,m}$ on R_m. The maximum order of vanishing in [BKMS16, BlJ17] is then given by $\lim_{m\to\infty} m^{-1}\lambda_{m,N_m} = \sup_m m^{-1}\lambda_{m,N_m}$, and hence equals $T(x)$ in view of Lemma 5.3. □

5.5. **Metrics associated to points.** Given any point $x \in X$, define

$$\varphi_x := \sup\{\varphi \in \mathrm{PSH}(L) \mid \varphi(x) \leq 0\}.$$

This is a function on X with values in $[0, +\infty]$. We shall see that the behavior of φ_x is vastly different, depending upon whether x is pluripolar or not.

Proposition 5.12. *If x is pluripolar, then the usc envelope of φ_x is equal to $+\infty$ on X.*

Proof. Since X^{qm} is dense in X, it suffices to prove that $\varphi_x(y) = +\infty$ for every point $y \in X^{\mathrm{qm}}$. But by Izumi's inequality [BoJ18, Theorem 2.21], there exists a constant $C = C(y) > 0$ such that $\varphi(y) \geq \sup_X \varphi - C$ for all $\varphi \in \mathrm{PSH}(L)$. This implies that $\varphi_x(y) = +\infty$. □

The following result is a more precise version of Theorem D.

Theorem 5.13. *Let L be an ample $\mathbf{Q}$-line bundle. For any nonpluripolar point $x \in X$, φ_x is a continuous psh metric on L, and satisfies $\mathrm{MA}(\varphi_x) = \delta_x$, $\varphi_x(x) = 0$. Further,*

$$E(\varphi_x) = S(x), \quad I(\varphi_x) = T(x), \quad \text{and} \quad J(\varphi_x) = T(x) - S(x). \tag{5.2}$$

When L is a line bundle, φ_x is the asymptotic Fubini–Study metric of the graded norm $\|\cdot\|_{x,\bullet}$ defined by x, and we have $S(x) = \mathrm{vol}(\|\cdot\|_{x,\bullet})$.

Before starting the proof, we make a few remarks. First, the continuity of φ_x can be interpreted as the singleton $\{x\}$ being a *regular* compact set in the sense of pluripotential theory. Second, the last equality shows that $S(x)$ agrees with the expected order of vanishing in [BKMS16, BlJ17], see also [MR15]. Third, combining (5.2) and (1.2) gives

$$\frac{1}{n+1}T(x) \leq S(x) \leq \frac{n}{n+1}T(x). \tag{5.3}$$

for any nonpluripolar point $x \in X$. In fact, this also holds when x is pluripolar, since then $S(x) = T(x) = \infty$. The inequality (5.3) is reminiscent of the Alexander-Taylor inequality [AT84, Theorem 2.1]. It was proved in [Fuj17] for a divisorial point. The proof here is quite different; however, both proofs ultimately reduce to the Hodge Index Theorem.

Proof of Theorem 5.13. During the proof we denote the generic point of X by x_g.

First assume x is quasimonomial. To prove that φ_x is continuous and that $\mathrm{MA}(\varphi_x) = \delta_x$, we can use ground field extension to reduce to [BFJ15, §8.4], which treats the case of a discretely valued ground field, but let us give a direct proof, based on the argument in *loc. cit.*

Given any open neighborhood U of x in X, we may find an snc test configuration $\mathcal{X}$ for X, with dual complex $\Delta = \Delta_{\mathcal{X}}$, such that $p_{\mathcal{X}}^{-1}(\{x\}) \subset U$. This follows from the homeomorphism $X \xrightarrow{\sim} \varprojlim_{\mathcal{X}} \Delta_{\mathcal{X}}$. Let Δ' be a (possibly irrational) simplicial subdivision of Δ such that x is a vertex of Δ'. (Thus Δ' and and Δ have the same underlying space.) Fix a constant $C \gg 0$ and let f_x be the unique function on Δ' that is affine on each face, $f_x(x) = 0$, and $f_x(y) = C$ for every vertex y of Δ' with $y \neq x$. Extend f_x to all of X by demanding $f_x = f_x \circ p_{\mathcal{X}}$. Then f_x is continuous.

We claim that $\varphi_x = P(f_x)$, where $P(f_x) = \sup\{\varphi \in \mathrm{PSH}(L) \mid \varphi \leq f_x\}$ is the psh envelope of f_x, see §4.2. To see this, note that $P(f_x) \leq \varphi_x$, so it suffices to prove that if $\varphi \in \mathrm{PSH}(L)$ and $\varphi(x) \leq 0$, then $\varphi \leq f_x$. Now, if $C \gg 0$, it follows from the Izumi estimate in [BoJ18, Theorem 2.21] that $\varphi(y) < C = f_x(y)$ for all vertices $y \neq x$ of Δ'. Further, φ is convex on each face of Δ (see [BoJ18, Theorem 5.29]), and hence also on each face of Δ', so we must have $\varphi \leq f_x$ on Δ, with equality only at x. Since $\varphi \leq \varphi \circ p_{\mathcal{X}}$ by *loc. cit.*, this gives $\varphi \leq \varphi_x$ on X, with strict inequality outside $p_{\mathcal{X}}^{-1}\{x\} \subset U$.

Now f_x is continuous, so $\varphi_x = P(f_x)$ is also continuous by [BoJ18, Corollary 5.28]. By the orthogonality property (see [BoJ18, Theorem 6.30]), the support of $\mathrm{MA}(\varphi_x)$ is contained in the locus where $\varphi_x = f_x$. But this locus is contained in U, which was an arbitrary neighborhood of x, so we must have $\mathrm{MA}(\varphi_x) = \delta_x$, and also $\varphi_x(x) = 0$.

We further have $\varphi_x(x_g) = \sup_X \varphi_x = T(x)$. Indeed, $T(x) \geq \varphi_x(x_g) - \varphi_x(x) = \varphi_x(x_g)$, and if $\varphi \in \mathcal{H}(L)$, then $\varphi - \varphi(x) \leq \varphi_x$, so $\varphi(x_g) - \varphi(x) \leq \varphi_x(x_g)$, and hence $T(x) \leq \varphi_x(x_g)$, so that $T(x) = \varphi_x(x_g)$. Using $\mathrm{MA}(\varphi_x) = \delta_x$, this implies $I(\varphi_x) = \varphi_x(x_g) - \varphi_x(x) = T(x)$. We also have $(I - J)(\varphi_x) = E^*(\delta_x) = S(x)$. As a consequence, $J(\varphi_x) = T(x) - S(x)$, and hence $E(\varphi_x) = J(\varphi_x) - \varphi_x(x_g) = S(x)$.

Now consider a general nonpluripolar point $x \in X$. Thus $S(x), T(x) < \infty$. To simplify notation, denote by $J := \mathrm{SNC}(X)$ the directed set of snc test configurations for X. For $j \in J$, we have a retraction $r_j \colon X \to X$ onto the corresponding dual complex. Set $x_j := r_j(x)$. Then $\lim_j x_j = x$, and by Lemma 5.8, $S(x_j)$ and $T(x_j)$ increase to $S(x)$ and $T(x)$, respectively.

The net $(\varphi_{x_j})_j$ is increasing and bounded from above by $T(x)$; hence it converges in $\mathrm{PSH}(L)$ to the solution φ'_x of $\mathrm{MA}(\varphi'_x) = \delta_x$, normalized by $\sup_X \varphi'_x = T(x)$. Thus $E(\varphi_{x_j}) \to E(\varphi'_x)$. We claim that $\varphi'_x = \varphi_x$. To see this, note that since $\mathrm{MA}(\varphi'_x) = \delta_x$, we have

$$S(x) = E^*(\delta_x) = E(\varphi'_x) - \int \varphi'_x \delta_x = E(\varphi'_x) - \varphi'_x(x)$$

and similarly $S(x_j) = E(\varphi_{x_j}) - \varphi_{x_j}(x_j) = E(\varphi_{x_j})$. Since $\lim_j S(x_j) = S(x)$ and $\lim_j E(\varphi_j) = E(\varphi'_x)$, we conclude that $\varphi'_x(x) = 0$. Now consider any metric $\varphi \in \mathrm{PSH}(L)$ with $\varphi(x) \leq 0$. Set $\psi := \max\{\varphi, \varphi'_x\}$. Then $\psi \geq \varphi'_x$ so $E(\psi) \geq E(\varphi'_x)$. Further, $\psi(x) = 0$, so

$$E(\psi) - \int \psi \delta_x \geq E(\varphi'_x) = E^*(\delta_x).$$

Thus ψ maximizes the functional $\varphi \mapsto E(\varphi) - \int \varphi \delta_x$, whose supremum equals $E^*(\delta_x)$ and is also maximized by φ'_x. Since the maximizer is unique up to an additive constant, and since $\psi(x) = \varphi'_x(x) = 0$, we must have $\psi = \varphi'_x$. This amounts to $\varphi \leq \varphi'_x$, and taking the

supremum over all φ gives $\varphi_x \leq \varphi'_x$. On the other hand, φ'_x is a competitor in the definition of φ_x, so $\varphi'_x \leq \varphi_x$, and we conclude that $\varphi'_x = \varphi_x$, as claimed.

It remains to prove that φ_x is continuous. Since φ_x is psh, $\varepsilon_j := \varphi_x(x_j)$ decreases to $\varphi_x(x) = 0$. Now $\varphi_{x_j} \leq \varphi_x \leq \varphi_{x_j} + \varepsilon_j$ on X, so we see that φ_{x_j} converges uniformly to φ_x. Since φ_{x_j} is continuous for all j, φ_x is continuous.

Finally suppose L is a line bundle. We must prove $\varphi_x = \mathrm{FS}(\|\cdot\|_{x,\bullet}) =: \psi_x$. In the definition of φ_x, it suffices to take the supremum over positive metrics. Recall that $\psi_x = \operatorname{usc}\sup_m \varphi_m$, where $\varphi_m = m^{-1}\max\{\log\frac{|s|_0}{|s(x)|}\ s \in R_m \setminus \{0\}\}$. Since $\varphi_m(x) = 0$, this gives $\varphi_m \leq \varphi_x$ and hence $\psi_x \leq \varphi_x$ since φ_x is continuous. On the other hand, if $\varphi \in \mathcal{H}(L)$ and $\varphi(x) \leq 0$, then $\varphi \leq \varphi_m$ for some m, and hence $\varphi \leq \psi_x$. Thus $\varphi_x = \psi_x$. The equality $E(\varphi_x) = \mathrm{vol}(\|\cdot\|_{x,\bullet})$ now follows from Theorem 4.13 (with $\|\cdot\|'_\bullet$ the trivial graded norm). □

Remark 5.14. *Suppose $(\mathcal{X}, \mathcal{L})$ is a normal test configuration for (X, L) with irreducible central fiber $\mathcal{X}_0$. This includes the case of "special" test configurations of* [LX14]*. Then the associated metric $\varphi = \varphi_{\mathcal{X},\mathcal{L}}$ on L satisfies $\mathrm{MA}(\varphi) = \delta_x$, where $x \in X^{\mathrm{div}}$ is the divisorial point corresponding to $\mathcal{X}_0$. Therefore $\varphi_x = \varphi + c$ is a positive metric in this case. More generally, if x is a "dreamy" divisorial valuation in the sense of* [Fuj16]*, then $\varphi_x \in \mathcal{H}(L)$. However, given* [Kür03]*, we do not expect $\varphi_x \in \mathcal{H}(L)$ for a general divisorial point $x \in X^{\mathrm{div}}$.*

Note that $x \mapsto \varphi_x$ is equivariant for the $\mathbf{R}_+^\times$-actions on X and $\mathrm{PSH}(L)$: we have $\varphi_{x^t}(y^t) = t\varphi_x(y)$ for $x, y \in X$, $t \in \mathbf{R}_+^\times$.

5.6. **Thresholds.** Recall (see e.g. [CS08]) that the *log canonical threshold* $\alpha(L)$ of L is the infimum of the log canonical threshold $\mathrm{lct}(\mathcal{D})$, where $\mathcal{D}$ ranges over effective $\mathbf{Q}$-divisors $\mathbf{Q}$-linearly equivalent to L. We have $\alpha(L) > 0$, see for instance [BHJ17, Theorem 9.14]. In [BlJ17, Theorem E] the following valuative formula was given.

Theorem 5.15. *We have $\alpha(L) = \inf_x \frac{A(x)}{T(x)}$, where x may range over either divisorial points or nonpluripolar points. Here A is the log discrepancy and T the Alexander-Taylor capacity.*

Indeed, this follows from [BlJ17, Theorem E] since $T(x)$ is equal to the maximum order of vanishing of x.

In §2 we defined the adjoint stability threshold $\delta(L)$ using measures of finite energy, see (2.9). We now show that this definition coincides with the one in [BlJ17].

Theorem 5.16. *We have $\delta(L) = \inf_x \frac{A(x)}{S(x)}$, where x may range over either divisorial points or nonpluripolar points. Here A is the log discrepancy and S the Monge–Ampère energy.*

Proof. We have seen that a point $x \in X^{\mathrm{val}}$ is nonpluripolar iff $S(x) < \infty$ iff $T(x) < \infty$ iff the valuation x has linear growth in the sense of [BKMS16]. Further, for such x, the Monge–Ampère energy $S(x) = E^*(\delta_x)$ coincides with the expected order of vanishing as in [BKMS16]. It therefore follows from [BlJ17, Theorem E] that the infima of $A(x)/S(x)$ over the set of divisorial points and the set of nonpluripolar points agree. Denote the common infimum by $\delta'(L)$. It remains to show that $\delta'(L) = \delta(L)$.

By considering Dirac masses $\mu = \delta_x$, we get $\delta(L) \leq \delta'(L)$. By definition of $\delta'(L)$, we have $A(x) \geq \delta'(L)S(x)$ for all divisorial points x, and hence for all $x \in X$ by (iv) in Theorem 2.1 and Lemma 5.8. On the other hand, for any measure $\mu \in \mathcal{M}^1(X)$, we have $\mathrm{Ent}(\mu) = \int_X A(x)\,d\mu(x) = \int_X \mathrm{Ent}(\delta_x)\,d\mu(x)$ and $E^*(\mu) \leq \int_X E^*(\delta_x)\,d\mu(x)$, see Proposition 5.5. This implies $\delta'(L) \leq \delta(L)$ and completes the proof. □

The Alexander-Taylor inequality (5.3) together with Theorems 5.15 and (5.16) immediately shows that

$$\frac{n+1}{n}\alpha(L) \leq \delta(L) \leq (n+1)\alpha(L); \tag{5.4}$$

in particular, $\delta(L) > 0$.

5.7. **A valuative criterion of adjoint K-stability.** In this section we prove Theorems E and E'. Recall that these say that to test K-stability, it suffices to consider metrics in $\mathcal{E}^1(L)$ of the form $\varphi = \varphi_x$, where x is a nonpluripolar (or even divisorial) point.

First we note the following consequence of Theorem 5.13.

Corollary 5.17. *For any nonpluripolar point $x \in X$,*

$$H(\varphi_x) = A(x) \quad \textit{and} \quad M^{\mathrm{ad}}(\varphi_x) = A(x) - S(x).$$

Indeed, $\mathrm{MA}(\varphi_x) = \delta_x$ implies $H(\varphi_x) = \mathrm{Ent}(\delta_x) = A(x)$, $(I-J)(x) = E^*(\delta_x) = S(x)$, and hence $M^{\mathrm{ad}}(\varphi_x) = A(x) - S(x)$.

To prove Theorem E we use Corollary 2.11, which says that L is K-semistable in the adjoint sense iff $\delta(L) \geq 1$. By the valuative formula for $\delta(L)$ in Theorem 5.16, this is equivalent to $A(x) \leq S(x)$ for all divisorial points x, or, equivalently, for all nonpluripolar points x. Since $M^{\mathrm{ad}}(\varphi_x) = S(x) - A(x)$, this proves Theorem E.

The proof of Theorem E' is similar. By Theorem 2.12, L is uniformly K-stable in the adjoint sense iff $\delta(L) > 1$, and this is equivalent to the existence to $\varepsilon > 0$ such that $(1+\varepsilon)S(x) \leq A(x)$ for all divisorial (resp. nonpluripolar) points $x \in X$. This is in turn equivalent to $M^{\mathrm{ad}}(\varphi_x) \geq \varepsilon(I-J)(\varphi_x)$, and the proof is complete since the functionals $I-J$ and J are comparable, see (1.2).

Remark 5.18. *For the convenience of the reader, we compare our notation with the one in Fujita* [Fuj16, Fuj17]. *Assume X is Fano, and $L = -K_X$. Let F be a prime divisor over X, and write $x = \exp(-\operatorname{ord}_F) \in X^{\mathrm{div}}$. Then $\tau(F) = T(x)$, $j(F) = V(T(x) - S(x))$, $\beta(F) = V(A(x) - S(x))$, and $\hat{\beta}(F) = 1 - S(x)/A(x)$.*

5.8. **Adjoint K-stability and uniform K-stability.** Suppose that $k = \mathbf{C}$ (or k algebraically closed and uncountable). By [BlJ17, Theorem E], there exists a nonpluripolar point $x \in X^{\mathrm{val}}$ such that $A(x) = \delta(L)S(x) > 0$. This implies Theorem F in the introduction. Indeed, if L is K-stable in the adjoint sense, it is K-semistable in the adjoint sense, so $\delta(L) \geq 1$ by Corollary 2.11. In view of Theorem 2.12 we must show that $\delta(L) > 1$. But if $\delta(L) = 1$, pick $x \in X^{\mathrm{val}}$ as above. Then φ_x is a nonconstant metric in $\mathcal{E}^1(L)$, and $M^{\mathrm{ad}}(\varphi_x) = 0$, contradicting L being K-stable in the adjoint sense.

If L is not K-stable in the adjoint sense, i.e. $\delta(L) < 1$, and $x \in X^{\mathrm{val}}$ is a nonpluripolar point with $A(x) = \delta(L)S(x) > 0$, then $\varphi_x \in \mathcal{E}^1(L)$ can be viewed as a "maximally destabilizing metric". In general, we don't know whether $\varphi_x \in \mathcal{H}(L)$.

5.9. **The Tian–Odaka–Sano criterion.** In [Tia97], Tian gave an analytic sufficient condition for the existence of Kähler–Einstein metrics on Fano manifolds. An algebraic version of this was proved by Odaka and Sano [OS12]; see also [Der16, BHJ17]. Our next result gives a generalization to the general adjoint setting.

Corollary 5.19. *Let L be an ample $\mathbf{Q}$-line bundle. If $\alpha(L) \geq \frac{n}{n+1}$ (resp. $\alpha(L) > \frac{n}{n+1}$), then L is K-semistable (resp. uniformly K-stable) in the adjoint sense.*

Proof. Immediate consequence of (5.4), Corollary 2.11 and Theorem 2.12. □

The same proof also gives a necessary condition for adjoint K-stability.

Corollary 5.20. *Let L be an ample $\mathbf{Q}$-line bundle. If $\alpha(L) < \frac{1}{n+1}$ (resp. $\alpha(L) \leq \frac{1}{n+1}$), then L cannot be K-semistable (resp. uniformly K-stable) in the adjoint sense.*

In the case of K-semistability, this generalizes [FO16, Theorem 3.5].

5.10. **A criterion of finite energy.** Using the positivity of the stability threshold, we now give a criterion for a measure to have finite energy.

Corollary 5.21. *Any Radon probability measure of finite entropy has finite energy.*

Proof. Indeed, if $\mathrm{Ent}(\mu) < \infty$, then $E^*(\mu) \leq \delta(L)^{-1}\,\mathrm{Ent}(\mu) < \infty$. □

Corollary 5.22. *If μ is a Radon probability measure on X such that the log discrepancy A is bounded above on the support of μ, then $E^*(\mu) < \infty$.*

Applying this result to a Dirac mass $\mu = \delta_x$ and using Theorem 5.13, we get

Corollary 5.23. *If $x \in X^{\mathrm{val}}$ and $A(x) < \infty$, then x is nonpluripolar.*

Remark 2.19 in [BKMS16] contains an example of a point $x \in X^{\mathrm{val}}$ whose associated graded norm is not bounded; hence x is pluripolar. In this example, $A(x) = \infty$. On the other hand, it is also possible that x is nonpluripolar even though $A(x) = \infty$.

Example 5.24. *Suppose* $\dim X = 2$ *and that $\xi = \mathrm{red}(x)$ is a closed point of X^{sch}. Thus $v := -\log|\cdot|_x$ is a valuation (in the additive sense) on the local ring $\mathcal{O}_{X^{\mathrm{sch}},\xi}$ centered at ξ, and after scaling we may assume $\min_{f\in\mathfrak{m}} v(f) = 1$, where $\mathfrak{m}$ is the maximal ideal. This means v can be viewed as a point in the* valuative tree $\mathcal{V} = \mathcal{V}_\xi$ *at ξ, see* [FJ04] *and also* [Jon12]. *The log discrepancy $A(x)$ equals the thinness $A(v)$ considered in* [FJ04]. *By* [FJ04, Remark A.4] *we can find v of infinite thinness, $A(v) = \infty$ but finite skewness, $\alpha(v) < \infty$. The latter condition means that there exists $C > 0$ such that $v \leq C\,\mathrm{ord}_\xi$ on $\mathcal{O}_{X^{\mathrm{sch}},\xi}$, and implies that the associated graded norm is bounded, see* [BKMS16, Theorem 2.16], *and hence $T(x) < \infty$.*

Corollary 5.25. *If μ is a Radon probability measure on X whose support is contained in the dual cone complex $\Delta(\mathcal{Y},\mathcal{D})$ of some log smooth pair $(\mathcal{Y},\mathcal{D})$ over X, then $E^*(\mu) < \infty$.*

Proof. Indeed, the support of μ must then be a compact subset of $\Delta(\mathcal{Y},\mathcal{D})$, so since the log discrepancy is continuous on $\Delta(\mathcal{Y},\mathcal{D})$, it must be bounded on the support. □

Appendix A. Properties of the log discrepancy

In this section we prove Theorem 2.1. We first *define* the log discrepancy function A_X. On $X \setminus X^{\mathrm{val}}$, we declare $A_X \equiv \infty$, and on X^{val} we define A_X as in [JM12, §5]. Let us recall how this is done. Consider a log smooth pair $(\mathcal{Y},\mathcal{D})$ over X^{sch}. The function A_X is defined on the dual cone complex $\Delta(\mathcal{Y},\mathcal{D})$ by (2.1) on the one-dimensional cones, and by linearity on the higher-dimensional cones. It is shown in [JM12] that this gives a well-defined function on X^{qm}. To extend A_X to X^{val}, we use the fact that the set of log smooth pairs is directed, and that we have a homeomorphism

$$X^{\mathrm{val}} \xrightarrow{\sim} \varprojlim_{(\mathcal{Y},\mathcal{D})} \Delta(\mathcal{Y},\mathcal{D}),$$

where $p_{\mathcal{Y},\mathcal{D}}\colon X^{\mathrm{val}} \to \Delta(\mathcal{Y},\mathcal{D})$ is a natural retraction defined by evaluation. One can then show that the net $A_X \circ p_{\mathcal{Y},\mathcal{D}}$ is increasing and set

$$A_X(x) = \lim_{\mathcal{Y},\mathcal{D}} A_X(p_{\mathcal{Y},\mathcal{D}}(x)) = \sup_{\mathcal{Y},\mathcal{D}} A_X(p_{\mathcal{Y},\mathcal{D}}(x))$$

for $x \in X^{\mathrm{val}}$. Finally we set $A_X = \infty$ on $X \setminus X^{\mathrm{val}}$.

We claim that A_X is lsc on X. Since $A_X \circ p_{\mathcal{Y},\mathcal{D}}$ is continuous for every $(\mathcal{Y},\mathcal{D})$, the restriction of A_X to X^{val} is lsc. Now $A_X = \infty$ on $X \setminus X^{\mathrm{val}}$, so this implies that A_X is lsc at every point in X^{val}. It remains to prove that A_X is lsc at any point $x \in X \setminus X^{\mathrm{val}}$. There exists an ample line bundle L on X and a nonzero global section s of L such that $|s(x)| = 0$. Now there exists $C = C(L) > 0$ such that $\mathrm{ord}_\xi(s) \le C$ for all $\xi \in X^{\mathrm{sch}}$, cf [BHJ17, p.830]. By the Izumi inequality, see e.g. [JM12, Proposition 5.10], this gives

$$|s(x')| \ge \exp(-A_X(x')\,\mathrm{ord}_\xi(x')) \ge \exp(-CA_X(x')) \tag{A.1}$$

for every $x' \in X^{\mathrm{val}}$, where $\xi = \mathrm{red}\, x' \in X^{\mathrm{sch}}$. The same inequality trivially holds also when $x' \in X \setminus X^{\mathrm{val}}$, since $A_X(x') = \infty$ in this case

Now, given any $B > 0$, we have $|s| < \exp(-CB)$ on any sufficiently small neighborhood of x in X, and hence $A_X \ge B$ on the same neighborhood. This proves that $\lim_{x' \to x} A_X(x') = A_X(x) = \infty$. In particular, A_X is lsc at x, and hence lsc everywhere on X.

Next we prove that A_X is the largest lsc extension of $A_X\colon X^{\mathrm{div}} \to \mathbf{R}$. (This of course implies the uniqueness statement in Theorem 2.1.) Let $A' : X \to [0,\infty]$ be any other lsc extension. Since A_X is continuous on any dual cone complex $\Delta = \Delta(\mathcal{Y},\mathcal{D})$, and X^{div} is dense in Δ, we have $A_X \ge A'$ on Δ; hence $A_X \ge A'$ on X^{qm}. Now suppose $x \in X \setminus X^{\mathrm{qm}}$. By construction, there exists a a net $(x_j)_j$ in X^{qm} such that $\lim A_X(x_j) = A_X(x)$. Then $A'(x_j) \le A_X(x_j)$ for all j, so $A'(x) \le A_X(x)$ since A' is lsc.

It remains to prove that A satisfies (iii) and (iv) in Theorem 2.1. For this, we use the Gauss extension $\sigma\colon X \to X \times \mathbf{P}^1$. Its image consists of all $\mathbb{G}_m$-invariant points satisfying $\log|\varpi| = -1$, and $\sigma(X^{\mathrm{val}}) \subset (X \times \mathbf{P}^1)^{\mathrm{val}}$. If $x \in X^{\mathrm{val}}$, then $\sigma(x)$ is divisorial iff x is either divisorial or the generic point of x.

Consider $\mathcal{X} \in \mathrm{SNC}(X)$. We view $(\mathcal{X},\mathcal{X}_0)$ as a log smooth pair over $X^{\mathrm{sch}} \times \mathbb{P}^1$. The dual cone complex $\Delta(\mathcal{X},\mathcal{X}_0)$ embeds in $(X \times \mathbf{P}^1)^{\mathrm{val}} \subset X \times \mathbf{P}^1$, σ maps $\Delta_{\mathcal{X}}$ homeomorphically onto the subset of $\Delta(\mathcal{X},\mathcal{X}_0)$ cut out by the equation $\log|\varpi| = -1$, and $\sigma^{-1}(\Delta(\mathcal{X},\mathcal{X}_0)) = \Delta_{\mathcal{X}}$. The embedding $\sigma\colon \Delta_{\mathcal{X}} \hookrightarrow \Delta(\mathcal{X},\mathcal{X}_0)$ is affine in the sense that if $f \in C^0(\Delta(\mathcal{X},\mathcal{X}_0))$ is linear on each cone, then $f \circ \sigma$ is affine on each simplex of $\Delta_{\mathcal{X}}$. The retractions $p_{\mathcal{X}}\colon X \to \Delta_{\mathcal{X}}$ and $p_{\mathcal{X},\mathcal{X}_0}\colon (X \times \mathbf{P}^1)^{\mathrm{val}} \to \Delta(\mathcal{X},\mathcal{X}_0)$ satisfy

$$p_{\mathcal{X},\mathcal{X}_0} \circ \sigma = \sigma \circ p_{\mathcal{X}} \quad \text{on } X^{\mathrm{val}}. \tag{A.2}$$

Lemma A.1. *For any $x \in X$, we have $A_{X\times\mathbf{P}^1}(\sigma(x)) = A_X(x) + 1$.*

Granted this lemma, property (iii) follows from (A.2). Indeed, $A_{X\times\mathbf{P}^1}$ is affine on each cone of $\Delta(\mathcal{X},\mathcal{X}_0)$, so since σ is affine, $A_X = A_{X\times\mathbf{P}^1} \circ \sigma - 1$ is affine on each simplex of $\Delta_{\mathcal{X}}$, proving (a). Similarly, by [JM12] we have $A_{X\times\mathbf{P}^1}(y) \ge A_{X\times\mathbf{P}^1}(p_{\mathcal{X},\mathcal{X}_0}(y))$ for $y \in (X \times \mathbf{P}^1)^{\mathrm{val}}$, with equality iff $y \in \Delta(\mathcal{X},\mathcal{X}_0)$. Using Lemma A.1, this translates into (iii) (b). Property (iv) now follows formally. Indeed, if $x \in X$, then $x = \lim_{\mathcal{X}} p_{\mathcal{X}}(x)$, so $A_X(x) \le \underline{\lim}_{\mathcal{X}} A_X(p_{\mathcal{X}}(x))$ since A_X is lsc. But $A_X(x) \le A_X(p_{\mathcal{X}}(x))$ for all $\mathcal{X}$, so (iv) follows.

Proof of Lemma A.1. For x divisorial or the generic point of X, the equality is a special case of [BHJ17, Proposition 4.11]. Set $A' := A_{X\times\mathbf{P}^1}\circ\sigma - 1$. Then A' is lsc and $A' = A_X$ on X^{div}, so $A' \leq A_X$ by what precedes.

Now consider $\mathcal{X} \in \mathrm{SNC}(X)$, let S be a simplex in $\Delta_{\mathcal{X}}$, and $\hat{S}$ the corresponding cone in $\Delta(\mathcal{X}, \mathcal{X}_0)$. Then $A_{X\times\mathbf{P}^1}$ is continuous on $\hat{S}$, so A' is continuous on S. Further, $A' = A_X$ on a dense subset of S, so $A_X \leq A$ on S, since A_X is lsc. Thus $A' = A_X$ on X^{qm}.

Finally consider an arbitrary point $x \in X$. If x has nontrivial kernel, so does $\sigma(x)$ and $A'(x) = A_X(x) = \infty$. Now suppose $x \in X^{\mathrm{val}}$. The net $(p_{\mathcal{X}}(x))_{\mathcal{X}}$ converges to x, so $p_{\mathcal{X},\mathcal{X}_0}(x) = \sigma(p_{\mathcal{X}}(x))$ converges to $\sigma(x)$, by the continuity of σ. Now $A_{X\times\mathbf{P}^1}(p_{\mathcal{X},\mathcal{X}_0}(\sigma(x))) \leq A_{X\times\mathbf{P}^1}(\sigma(x))$ for every x, so $\lim_{\mathcal{X}} A_{X\times\mathbf{P}^1}(p_{\mathcal{X},\mathcal{X}_0}(\sigma(x)) = A_{X\times\mathbf{P}^1}(\sigma(x))$ by lower semicontinuity. Thus $A'(x) = \lim_{\mathcal{X}} A'(p_{\mathcal{X}}(x)) = \lim_{\mathcal{X}} A_X(p_{\mathcal{X}}(x)) \geq A_X(x)$, where the last inequality follows since A_X is lsc. Since $A' \leq A_X$, this completes the proof. □

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CNRS–CMLS, École Polytechnique, F-91128 Palaiseau Cedex, France
E-mail address: `boucksom@math.polytechnique.fr`

Dept of Mathematics, University of Michigan, Ann Arbor, MI 48109-1043, USA
E-mail address: `mattiasj@umich.edu`