A NOTE ON LANG'S CONJECTURE FOR QUOTIENTS OF BOUNDED DOMAINS

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"... Voglio una vita che non è mai tardi, di quelle che non dormono mai..."

Abstract. It was conjectured by Lang that a complex projective manifold is Kobayashi hyperbolic if and only if it is of general type together with all of its subvarieties. We verify this conjecture for projective manifolds whose universal cover carries a bounded, strictly plurisubharmonic function. This includes in particular compact free quotients of bounded domains.

Introduction

For a compact complex space X, Kobayashi hyperbolicity is equivalent to the fact that every holomorphic map $\mathbb{C} \to X$ is constant, thanks to a classical result of Brody. When X is moreover projective (or, more generally, compact Kähler), hyperbolicity is further expected to be completely characterized by (algebraic) positivity properties of X and of its subvarieties. More precisely, we have the following conjecture, due to S. Lang.

Conjecture. [Lan86, Conjecture 5.6] *A projective variety X is hyperbolic if and only if every subvariety (including X itself) is of general type.*

Recall that a projective variety X is of general type if the canonical bundle of any smooth projective birational model of X is big, *i.e.* has maximal Kodaira dimension. This is for instance the case when X is smooth and *canonically polarized*, *i.e.* with an ample canonical bundle K_X.

Note that Lang's conjecture in fact implies that every smooth hyperbolic projective manifold X is canonically polarized, as conjectured in 1970 by S. Kobayashi. It is indeed a well-known consequence of the Minimal Model Program that any projective manifold of general type without rational curves is canonically polarized (see for instance [BBP13, Theorem A]).

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Besides the trivial case of curves and partial results for surfaces [MM83, Des79, GG80, McQ98], Lang's conjecture is still almost completely open in higher dimension as of this writing. General projective hypersurfaces of high degree in projective space form a remarkable exception: they are known to be hyperbolic [Bro17] (see also [McQ99, DEG00, DT10, Siu04, Siu15, RY18]), and they satisfy Lang's conjecture [Cle86, Ein88, Xu94, Voi96, Pac04].

It is natural to test Lang's conjecture for the following two basic classes of manifolds, known to be hyperbolic since the very beginning of the theory:

(N) compact Kähler manifolds X with negative holomorphic sectional curvature;
(B) compact, free quotients X of bounded domains $\Omega \Subset \mathbb{C}^n$.

In case (N), ampleness of K_X was established in [WY16a, WY16b, TY17] (see also [DT16]). By curvature monotonicity, this implies that every smooth subvariety of X also has ample canonical bundle. More generally, Guenancia recently showed [Gue18] that each (possibly singular) subvariety of X is of general type, thereby verifying Lang's conjecture in that case. One might even more generally consider the case where X carries an arbitrary Hermitian metric of negative holomorphic sectional curvature, which seems to be still open.

In this note, we confirm Lang's conjecture in case (B). While the case of quotients of bounded *symmetric* domains has been widely studied (see, just to cite a few, [Nad89, BKT13, Bru16, Cad16, Rou16, RT18]), the general case seems to have somehow passed unnoticed. Instead of bounded domains, we consider more generally the following class of manifolds, which comprises relatively compact domains in Stein manifolds, and has the virtue of being stable under passing to an étale cover or a submanifold.

Definition. We say that a complex manifold M is *of bounded type* if it carries a bounded, strictly plurisubharmonic function φ.

By a well-known result of Richberg, any *continuous* bounded strictly psh function on a complex manifold M can be written as a decreasing limit of smooth strictly psh functions, but this fails in general for discontinuous functions [FsSn87, p.66], and it is thus unclear to us whether every manifold of bounded type should carry also a *smooth* bounded strictly psh function.

Theorem A. *Let X be a compact Kähler manifold admitting an étale (Galois) cover $\tilde{X} \to X$ of bounded type. Then:*

(i) *X is Kobayashi hyperbolic;*
(ii) *X has large fundamental group;*
(iii) *X is projective and canonically polarized;*
(iv) *every subvariety of X is of general type.*

Note that $\tilde{X}$ can always be replaced with the universal cover of X, and hence can be assumed to be Galois.

By [Kob98, 3.2.8], (i) holds iff $\tilde{X}$ is hyperbolic, which follows from the fact that manifolds of bounded type are Kobayashi hyperbolic [Sib81, Theorem 3]. Alternatively, any entire curve $f : \mathbb{C} \to X$ lifts to $\tilde{X}$, and the pull-back to $\mathbb{C}$ of the bounded, strictly psh function carried by $\tilde{X}$ has to be constant, showing that f itself is constant.

By definition, (ii) means that the image in $\pi_1(X)$ of the fundamental group of any subvariety $Z \subseteq X$ is infinite [Kol95, §4.1], and is a direct consequence of the fact that manifolds of bounded type do not contain non-trivial compact subvarieties. According to the Shafarevich conjecture, $\tilde{X}$ should in fact be Stein; in case $\tilde{X}$ is a bounded domain of $\mathbb{C}^n$, this is indeed a classical result of Siegel [Sie50] (see also [Kob59, Theorem 6.2]).

By another classical result, this time due to Kodaira [Kod54], any compact complex manifold X admitting a Galois étale cover $\tilde{X} \to X$ biholomorphic to a bounded domain in $\mathbb{C}^n$ is projective, with K_X ample. Indeed, the Bergman metric of $\tilde{X}$ is non-degenerate, and it descends to a positively curved metric on K_X. Our proof of (iii) and (iv) is a simple variant of this idea, inspired by [CZ02]. For each subvariety $Y \subseteq X$ with desingularization $Z \to Y$ and induced Galois étale cover $\tilde{Z} \to Z$, we use basic Hörmander–Andreotti–Vesentini–Demailly L^2-estimates for $\overline{\partial}$ to show that the Bergman metric of $\tilde{Z}$ is generically non-degenerate. It then descends to a psh metric on K_Z, smooth and strictly psh on a nonempty Zariski open set, which is enough to conclude that K_Z is big, by [Bou02].

As a final comment, note that Kähler hyperbolic manifolds, *i.e.* compact Kähler manifolds X carrying a Kähler metric ω whose pull-back to the universal cover $\pi : \tilde{X} \to X$ satisfies $\pi^*\omega = d\alpha$ with α bounded, also satisfy (i)–(iii) in Theorem A [Gro91]. It would be interesting to check Lang's conjecture for such manifolss as well.

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1. The Bergman metric and manifolds of general type

1.1. Non-degeneration of the Bergman metric.

Recall that the *Bergman space* of a complex manifold M is the separable Hibert space $\mathcal{H} = \mathcal{H}(M)$ of holomorphic forms $\eta \in H^0(M, K_M)$ such that

$$\|\eta\|^2_{\mathcal{H}} := i^{n^2} \int_{\tilde{X}} \eta \wedge \bar{\eta} < \infty,$$

with $n = \dim M$. Assuming $\mathcal{H} \neq \{0\}$, we get an induced (possibly singular) psh metric h_M on K_M, invariant under $\mathrm{Aut}(M)$, characterized pointwise by

$$h/h_M = \sup_{\eta\in\mathcal{H}\setminus\{0\}} \frac{|\eta|_h^2}{\|\eta\|_{\mathcal{H}}^2} = \sum_j |\eta_j|_h^2,$$

for any choice of smooth metric h on K_M and orthonormal basis (η_j) for $\mathcal{H}$ (see for instance [Kob98, §4.10]).

The curvature current of h_M is classically called the "Bergman metric" of M; it is a *bona fide* Kähler form precisely on the Zariski open subset of M consisting of points at which $\mathcal{H}$ generates 1-jets [Kob98, Proposition 4.10.11].

Definition 1.1. We shall say that a complex manifold M has a *non-degenerate (resp. generically non-degenerate) Bergman metric* if its Bergman space $\mathcal{H}$ generates 1-jets at each (resp. some) point of M.

We next recall the following standard consequence of L^2-estimates for $\overline{\partial}$.

Lemma 1.2. *Let M be a complete Kähler manifold with a bounded psh function φ. If φ is strictly psh on M (resp. at some point of M), then the Bergman metric of M is non-degenerate (resp. generically non-degenerate).*

Proof. Pick a complete Kähler metric ω on M. Assume φ strictly psh at $p \in M$, and fix a coordinate ball (U, z) centered at p with φ strictly psh near $\overline{U}$. Pick also $\chi \in C_c^\infty(U)$ with $\chi \equiv 1$ near p. Since $\chi \log |z|$ is strictly psh in an open neighbourhood V of p, smooth on $U \setminus \overline{V}$, and compactly supported in U, we can then choose $A \gg 1$ such that

$$\psi := (n+1)\chi \log |z| + A\varphi$$

is psh on M, with $dd^c\psi \geq \omega$ on U. Note that ψ is also bounded above on M, φ being assumed to be bounded.

For an appropriate choice of holomorphic function f on U, the smooth $(n, 0)$-form $\eta := \chi f\, dz_1 \wedge \cdots \wedge dz_n$, which is compactly supported in U and holomorphic in a neighborhood of x, will have any prescribed jet at p. The $(n, 1)$-form $\bar\partial\eta$ is compactly supported in U, and identically zero in a neighborhood of p, so that $|\overline{\partial}\eta|_\omega e^{-\psi} \in L^2(U)$. Since $dd^c\psi \geq \omega$ on U, [Dem82, Théorème 5.1] yields an L^2_{loc} $(n, 0)$-form u on M such that $\overline{\partial}u = \overline{\partial}\eta$ and

$$i^{n^2} \int_M u \wedge \bar{u}\, e^{-2\psi} \leq \int_U |\overline{\partial}\eta|_\omega^2 e^{-2\psi} dV_\omega. \tag{1.1}$$

As a result, $v := \eta - u$ is a holomorphic n-form on X. Since $u = \eta - v$ is holomorphic at x and ψ has an isolated singularity of type $(n + 1) \log |z|$ at x, (1.1) forces u to vanish to order 2 at p, so that v and η have the same 1-jet at p. Finally, (1.1) and the fact that ψ is bounded above on M shows that u is L^2. Since η is clearly L^2 as well, v belongs to the Bergman space $\mathcal{H}$, with given 1-jet at p, and we are done. □

1.2. **Manifolds of general type.** Let X be a compact complex manifold, $\tilde{X} \to X$ a Galois étale cover, and assume that the Bergman metric of $\tilde{X}$ is non-degenerate, so that the canonical metric $h_{\tilde{X}}$ on $K_{\tilde{X}}$ defined by $\mathcal{H}(\tilde{X})$ is smooth, strictly psh. Being invariant under automorphisms, this metric descends to a smooth, strictly psh metric on K_X, and the latter is thus ample by [Kod54]. This argument, which goes back to the same paper by Kodaira, admits the following variant.

Lemma 1.3. *Let X be a compact Kähler manifold admitting a Galois étale cover $\tilde{X} \to X$ with generically non-degenerate Bergman metric. Then X is projective and of general type.*

Proof. The assumption now means that the psh metric $h_{\tilde{X}}$ on $K_{\tilde{X}}$ is smooth and strictly psh on a non-empty Zariski open subset. It descends again to a psh metric on K_X, smooth and strictly psh on a non-empty Zariski open subset, and we conclude that K_X is big by [Bou02, §2.3] (see also [BEGZ10, §1.5]). Being both Moishezon and Kähler, X is then projective. □

2. Proof of Theorem A

Let X be a compact Kähler manifold with an étale cover $\pi : \tilde{X} \to X$ of bounded type, which may be assumed to be Galois after replacing $\tilde{X}$ by the universal cover of X. Since $\tilde{X}$ is also complete Kähler, its Bergman metric is non-degenerate by Lemma 1.2, and X is thus projective and canonically polarized by [Kod54].

Now let $Y \subseteq X$ be an irreducible subvariety. On the one hand, pick any connected component $\tilde{Y}$ of the preimage $\pi^{-1}(Y) \subset \tilde{X}$, so that π induces a Galois étale cover $\pi|_{\tilde{Y}} \colon \tilde{Y} \to Y$. On the other hand, let $\mu \colon Z \to Y$ be a projective modification with Z smooth and μ isomorphic over Y_{reg}, whose existence is guaranteed by Hironaka. Since Y is Kähler and μ is projective, Z is then a compact Kähler manifold. The fiber product $\tilde{Z} = Z \times_Y \tilde{Y}$ sits in the following diagram

$$\begin{tikzcd}
\tilde{Z} \arrow[dd, "\nu"'] \arrow[dr, "\tilde{\mu}"] & & \\
& \tilde{Y} \arrow[r, hook] \arrow[dd, "\pi|_{\tilde{Y}}"] & \tilde{X} \arrow[dd, "\pi"] \\
Z \arrow[dr, "\mu"'] & & \\
& Y \arrow[r, hook] & X.
\end{tikzcd}$$

Being a base change of a Galois étale cover, ν is a Galois étale cover, and $\tilde{\mu}$ is a resolution of singularities of $\tilde{Y}$. Since π is étale, we have $\tilde{Y}_{\mathrm{reg}} = \pi^{-1}(Y_{\mathrm{reg}})$, and $\tilde{\mu}$ is an isomorphism over $\tilde{Y}_{\mathrm{reg}}$. The pull-back of φ to $\tilde{Z}$ is thus a bounded psh function, strictly psh at any point $p \in \tilde{\mu}^{-1}(\tilde{Y}_{\mathrm{reg}})$. Since Z is compact Kähler, $\tilde{Z}$ is complete Kähler. By Lemma 1.2, the Bergman metric of Z is generically non-degenerate, and Z is thus of general type, by Lemma 1.3.

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