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JORDAN BLOCKS OF CUSPIDAL REPRESENTATIONS OF SYMPLECTIC GROUPS

CORINNE BLONDEL, GUY HENNIART, AND SHAUN STEVENS

Abstract. Let G be a symplectic group over a nonarchimedean local field of characteristic zero and odd residual characteristic. Given an irreducible cuspidal representation of G, we determine its Langlands parameter (equivalently, its *Jordan blocks* in the language of Mœglin) in terms of the local data from which the representation is explicitly constructed, up to a possible unramified twist in each block of the parameter. We deduce a Ramification Theorem for G, giving a bijection between the set of *endo-parameters* for G and the set of restrictions to wild inertia of discrete Langlands parameters for G, compatible with the local Langlands correspondence. The main tool consists in analysing the Hecke algebra of a good cover, in the sense of Bushnell–Kutzko, for parabolic induction from a cuspidal representation of $G \times \mathrm{GL}_n$, seen as a maximal Levi subgroup of a bigger symplectic group, in order to determine reducibility points; a criterion of Mœglin then relates this to Langlands parameters.

Introduction

0.1. Let F be a locally compact nonarchimedean local field of odd residual characteristic and denote by W_F the Weil group of F. Let G be the symplectic group preserving a nondegenerate alternating form on a $2N$-dimensional F-vector space. The local Langlands conjectures for G (now a theorem of Arthur [2] when F has characteristic zero) stipulate that to an irreducible (smooth, complex) representation π of G is attached a Langlands parameter, and the representations with a given parameter form a finite set of isomorphism classes, called an L-packet for G.

Since the symplectic group is split, Langlands parameters for G are simply continuous homomorphisms ϕ from $W_F \times \mathrm{SL}_2(\mathbb{C})$ into the dual group $\hat{G} = \mathrm{SO}_{2N+1}(\mathbb{C})$, taken up to conjugation, such that the $(2N+1)$-dimensional representation $\iota \circ \phi$ of $W_F \times \mathrm{SL}_2(\mathbb{C})$, obtained from the inclusion ι of $\hat{G}$ into $\mathrm{GL}_{2N+1}(\mathbb{C})$, is semisimple. If π is a discrete series

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representation of G, then its parameter ϕ is discrete, that is, the image of ϕ is not contained in a proper parabolic subgroup of $\hat{G}$; equivalently, $\iota \circ \phi$ is the direct sum of inequivalent irreducible orthogonal representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$, and has determinant 1. In that case giving $\iota \circ \phi$ up to equivalence is the same as giving ϕ up to conjugation in $\hat{G}$.

On the other hand, we have an explicit description of the cuspidal representations of G via the theory of types [35], in the spirit of the classification of the irreducible representations of $\mathrm{GL}_n(F)$ of Bushnell–Kutzko [10]. It is our goal in this paper to describe as much as possible of the Langlands parameter of a cuspidal representation of G from its explicit construction. We will denote by $\mathrm{Cusp}(G)$ the set of equivalence classes of cuspidal representations of G, and by $\Phi^{\mathsf{cusp}}(G)$ the subset of discrete Langlands parameters consisting of those parameters with a cuspidal representation in the corresponding L-packet (see paragraph 0.5 below for a more detailed description).

0.2. At the technical and arithmetic heart of the construction of cuspidal representations of G and $\mathrm{GL}_n(F)$ is the theory of *endo-classes* of simple characters – families of very special characters of compact open subgroups. An irreducible cuspidal representation of $\mathrm{GL}_n(F)$ contains, up to conjugacy, a unique such simple character and thus determines an endo-class. By considering the endo-classes in its cuspidal support, an arbitrary irreducible representation of $\mathrm{GL}_n(F)$ then determines a formal sum of endo-classes (with multiplicities), which we call an *endo-parameter of degree* n (see paragraph 2.7). We write $\mathcal{EE}_n(F)$ for the set of endo-parameters of degree n.

Similarly, an irreducible cuspidal representation of G is constructed from a *semi*simple character, and thus also comes from an endo-parameter, the weighted formal sum of the endo-classes of its simple components; moreover, the semisimple character is self-dual so that every endo-class appearing must also be self-dual. Thus the construction of an irreducible cuspidal representation of G gives rise to a self-dual endo-parameter of degree $2N$. We write $\mathcal{EE}^{\mathsf{sd}}_{2N}(F)$ for the set of these self-dual endo-parameters.

0.3. The notions of endo-class and endo-parameter admit an instructive interpretation via the local Langlands correspondence. Denote by $\mathscr{P}_F$ the wild ramification subgroup of the Weil group W_F. Then the (First) Ramification Theorem [6, 8.2 Theorem] says that there is a unique bijection between the set of endo-classes over F and the set of W_F-orbits of irreducible complex representations of $\mathscr{P}_F$, which is compatible with the local Langlands correspondence for general linear groups. This then induces a bijection, again compatible with the Langlands correspondence, between the set of endo-parameters of degree n and the set of equivalence classes of n-dimensional complex representations of $\mathscr{P}_F$ which are invariant under conjugation by W_F (see 7.3 Theorem for a precise statement). We call these representations of $\mathscr{P}_F$ *wild parameters*.

Our first main result (or, rather, the last in the scheme of proof) is an analogous ramification theorem for the symplectic group G. First we see that the bijection above restricts to a bijection between self-dual endo-classes and self-dual W_F-orbits of irreducible complex

representations of $\mathscr{P}_F$. (Note that we really mean that the *orbit* is self-dual: the only self-dual irreducible complex representation of $\mathscr{P}_F$ is the trivial representation, since p is odd.) We say that a $(2N+1)$-dimensional wild parameter is *discrete self-dual* if it is a sum of self-dual W_F-orbits of irreducible complex representations of $\mathscr{P}_F$, and write $\Psi^{\mathsf{sd}}_{2N+1}(F)$ for the set of such wild parameters. These are precisely the restrictions to wild inertia of discrete Langlands parameters. We prove the following Ramification Theorem for G (see the end of the introduction for remarks on the characteristic).

Theorem (7.6 Theorem)**.** *Suppose F is of characteristic zero. There is a unique bijection $\mathcal{EE}^{\mathsf{sd}}_{2N}(F) \to \Psi^{\mathsf{sd}}_{2N+1}(F)$ which is compatible with the Langlands correspondence for cuspidal representations of G:*

$$\begin{tikzcd} \mathrm{Cusp}(G) \arrow[r, two heads] \arrow[d, two heads] & \Phi^{\mathsf{cusp}}(G) \arrow[d, two heads] \\ \mathcal{EE}^{\mathsf{sd}}_{2N}(F) \arrow[r, "\sim"] & \Psi^{\mathsf{sd}}_{2N+1}(F) \end{tikzcd}$$

The bijection here is not just that in the case of general linear groups (indeed, the degree has changed): one must first take the square of every endo-class in the support of the endo-parameter, then map across using the bijection for general linear groups, and finally add the trivial representation of $\mathscr{P}_F$.

0.4. The Ramification Theorem for G is in fact a consequence of rather more precise results, proved on the automorphic side of the Langlands correspondence. To explain the connection, we recall in more detail the structure of discrete Langlands parameters, and the results of Mœglin.

There is, up to isomorphism, exactly one irreducible m-dimensional representation St_m of $\mathrm{SL}_2(\mathbb{C})$, for each $m \geqslant 1$. Thus an irreducible representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ is a tensor product $\sigma \otimes \mathrm{St}_m$, where σ is an irreducible representation of W_F; moreover it is orthogonal if and only if either σ is self-dual symplectic and m is even, or σ is self-dual orthogonal and m is odd. By the Langlands correspondence for GL_n [24, 19, 20], such a σ is the Langlands parameter of a (single) cuspidal representation ρ of $\mathrm{GL}_n(F)$, where $n = \dim \sigma$. Saying that σ is self-dual is saying that ρ is self-dual (i.e. isomorphic to its contragredient), and σ is then symplectic (resp. orthogonal) if the Langlands–Shahidi L-function $L(s, \Lambda^2, \rho)$ (resp. $L(s, \mathrm{Sym}^2, \rho)$) has a pole at $s = 0$ [21], in which case we say that ρ is of symplectic (resp. orthogonal) type.

In conclusion, a discrete parameter ϕ for G can be given by a set of (distinct) pairs (ρ_i, m_i), where ρ_i is an isomorphism class of irreducible cuspidal representations of $\mathrm{GL}_{n_i}(F)$, with n_i and m_i positive integers, and

- $\sum_i n_i m_i = 2N+1$,
- each ρ_i is self-dual, of symplectic type if m_i is even and of orthogonal type if m_i is odd,

- if ω_i is the central character of ρ_i then $\prod_i \omega_i^{m_i} = 1$.

0.5. If π is an irreducible cuspidal representation of G and ϕ is its parameter, Mœglin [30] has given a criterion to determine the set attached to ϕ as above, i.e. the pairs (ρ_i, m_i) that she calls the "Jordan blocks" of π; we write $\mathrm{Jord}(\pi)$ for this set of pairs. Let us explain her results.

For any positive integer n, the group $\mathrm{GL}_n(F) \times G$ appears naturally as a standard maximal Levi subgroup of $\mathrm{Sp}_{2(N+n)}(F)$. If ρ is a cuspidal representation of $\mathrm{GL}_n(F)$ we can form the parabolically induced representation $\rho\nu^s \rtimes \pi$ (we use normalized induction and induce via the standard parabolic), where s is here a *real* parameter and ν is the character $g \mapsto |\det g|_F$ of $\mathrm{GL}_n(F)$. If no unramified twist of ρ is self-dual then $\rho\nu^s \rtimes \pi$ is always irreducible. On the other hand, if ρ is self-dual, there is a unique $s_\pi(\rho) \geqslant 0$ such that $\rho\nu^s \rtimes \pi$ is reducible if and only if $s = \pm s_\pi(\rho)$.

We define the *reducibility set* $\mathrm{Red}(\pi)$ to be the set of isomorphism classes of cuspidal representations ρ of some $\mathrm{GL}_n(F)$, with $n \geqslant 1$, for which $2s_\pi(\rho) - 1$ is a positive integer. Indeed, it is known that $2s_\pi(\rho)$ is an integer [31], so the condition for ρ to lie in $\mathrm{Red}(\pi)$ is that $s_\pi(\rho)$ is neither 0 nor 1/2. The *Jordan set* $\mathrm{Jord}(\pi)$ is then the set of pairs (ρ, m), where $\rho \in \mathrm{Red}(\pi)$ and $2s_\pi(\rho) - 1 = m + 2k$, for some integer $k \geqslant 0$.

From its construction, $\mathrm{Jord}(\pi)$ is "without holes" in the sense that, if it contains (ρ, m) then it also contains $(\rho, m-2)$ whenever $m - 2 > 0$. However there may be discrete series non-cuspidal representations of G with the same parameter as π; this happens as soon as $\mathrm{Jord}(\pi)$ contains a pair (ρ, m) with $m > 1$. For the number of cuspidal representations of G with a given parameter (without holes), see [28] (recalled in paragraph 7.4 below).

0.6. The results of Mœglin described in the previous paragraph now say that, in order to determine the Langlands parameter of an irreducible cuspidal representation π of G, we need only compute the reducibility points $s_\pi(\rho)$, for ρ an irreducible self-dual representation of some $\mathrm{GL}_n(F)$. Moreover, we need only find *enough* reducibility points $s_\pi(\rho) \geqslant 1$ to fill the parameter.

In order to compute these reducibility points, we use Bushnell–Kutzko's theory of types and covers [11]. The representation π takes the form $\text{c-Ind}_{J_\pi}^G \lambda_\pi$, for some irreducible representation λ_π of a compact open subgroup J_π; this pair (J_π, λ_π) is a *type* for π. Similarly, we have a Bushnell–Kutzko type $(\tilde{J}_\rho, \tilde{\lambda}_\rho)$ for ρ. Moreover, from [27] we have a *cover* (J, λ) in $\mathrm{Sp}_{2(N+n)}(F)$ of $(\tilde{J}_\rho \times J_\pi, \tilde{\lambda}_\rho \otimes \lambda_\pi)$.

The reducibility of the parabolically induced representation $\rho\nu^s \rtimes \pi$ for *complex* s is translated, via category equivalence, to the reducibility of induction from modules over the spherical Hecke algebra $\mathcal{H}(GL_n(F) \times G, \tilde{\lambda}_\rho \otimes \lambda_\pi)$ to $\mathcal{H}(\mathrm{Sp}_{2(N+n)}(F), \lambda)$. The former algebra is isomorphic to $\mathbb{C}[Z^{\pm 1}]$, while the latter is a Hecke algebra on an infinite dihedral group, with two generators each satisfying a quadratic relation of the form $(T+1)(T-q^r)$, with $r \geqslant 0$ an integer and q the cardinality of the residue field of F. The results of [4] then translate

the values of the parameters r for the two generators into the *real parts* of those $s \in \mathbb{C}$ for which $\rho\nu^s \rtimes \pi$ is reducible.

In the inertial class $[\rho] = \{\rho\nu^s \mid s \in \mathbb{C}\}$, there are precisely two inequivalent self-dual representations, and we write ρ' for the other one. Thus the method described above allows one to compute the set $\{s_\pi(\rho), s_\pi(\rho')\}$ but not to distinguish between the two values if they are distinct. Thus our method computes the *inertial Jordan set* $\mathrm{IJord}(\pi)$, which is the *multiset* of pairs $([\rho], m)$, such that $(\rho, m) \in \mathrm{Jord}(\pi)$.

0.7. According to the previous paragraph, computing $\mathrm{IJord}(\pi)$ explicitly comes down to computing the parameters in the quadratic relations for the spherical Hecke algebra of the cover. We do this in two steps.

First, we consider the special case when the semisimple character θ_π in π, from which the type (J_π, λ_π) is built, is in fact *simple*. In this case, it determines a self-dual endo-class $\mathbf{\Theta}$ and we consider only those irreducible cuspidal representations of some $\mathrm{GL}_n(F)$ which have endo-class $\mathbf{\Theta}^2$. We prove that just these representations already give us enough to fill the Jordan set (see 2.5 Theorem) and describe an algorithm to determine $\mathrm{IJord}(\pi)$ (see paragraph 5.10).

Here the computation of the parameters can be done using results of Lusztig on finite reductive groups: if $\mathbf{\Theta}$ is the trivial endo-class $\mathbf{\Theta}_0$, so that we are in depth zero, this was done already in [25]; otherwise, the groups in question are the reductive quotients of maximal parahoric subgroups in a unitary group (ramified or unramified). There is also an added subtlety which does not arise in the depth zero case: two signature characters of certain permutations (coming from a comparison of so-called *beta-extensions*) cause an extra twist which must be taken care of in the algorithm and counting.

In the second step, we consider an arbitrary irreducible cuspidal representation π and reduce to the first case. More precisely, the semisimple character θ_π determines by restriction its simple components θ_i, for $0 \leqslant i \leqslant l$, whence endo-classes $\mathbf{\Theta}_i$. From the construction of the type (J_π, λ_π), we define types (J_i, λ_i) in symplectic groups $\mathrm{Sp}_{2N_i}(F)$, with $\sum_{i=0}^{l} N_i = N$, which induce to irreducible cuspidal representations π_i containing a simple character of endo-class $\mathbf{\Theta}_i$. (See paragraph 2.6 for details.)

The reduction is obtained by showing that elements of $\mathrm{IJord}(\pi)$ with endo-class $\mathbf{\Theta}_i$ can be obtained from those of $\mathrm{IJord}(\pi_i)$ by a simple twisting process, by a character of order one or two (see 2.6 Theorem). This character arises as the comparison of pairs of signature characters as in the first case, for π and for π_i; the point that is both crucial and subtle is that, although we need to make two comparisons, they turn out to be equal. Now the first case, together with a dimension count, ensures that we have filled the expected size of $\mathrm{IJord}(\pi)$. If F is of characteristic zero then, by the results of Mœglin, this is indeed the entire inertial Jordan set (see 2.6 Corollary).

0.8. From our explicit description of the set $\mathrm{IJord}(\pi)$, we know the endo-class of every self-dual irreducible cuspidal representation of some $\mathrm{GL}_n(F)$ which appears in $\mathrm{Jord}(\pi)$. From this

we deduce the following result, which gives the compatibility of taking endo-parameters with the endoscopic transfer from G to $\mathrm{GL}_{2N+1}(F)$ and from which, via the results of Arthur, we deduce compatibility with the local Langlands correspondence. In the following, the map ι_{2N} sends a (self-dual) endo-parameter $\sum m_{\boldsymbol{\Theta}}\boldsymbol{\Theta}$ of degree $2N$ to the endo-parameter $\sum m_{\boldsymbol{\Theta}}\boldsymbol{\Theta}^2 + \boldsymbol{\Theta}_0$ of degree $2N+1$, where $\boldsymbol{\Theta}_0$ denotes the trivial endo-class.

Theorem (2.8 Theorem)**.** *Suppose F has characteristic 0. Then the following diagram commutes.*

$$\begin{tikzcd}
\mathrm{Cusp}(G) \arrow[r, "\text{transfer}"] \arrow[d] & \mathrm{Irr}(\mathrm{GL}_{2N+1}(F)) \arrow[d] \\
\mathcal{EE}^{\mathsf{sd}}_{2N}(F) \arrow[r, hook, "\iota_{2N}"'] & \mathcal{EE}_{2N+1}(F)
\end{tikzcd}$$

It is very tempting to think that this result could be an instance of a general theory of endo-parameters for arbitrary reductive groups, which would be in bijection with suitably-defined wild parameters and would be compatible with (twisted) endoscopy.

0.9. Let π be an irreducible cuspidal representation of G. Having given an explicit description of $\mathrm{IJord}(\pi)$, we can ask whether we can then determine $\mathrm{Jord}(\pi)$ precisely; that is, given $([\rho], m) \in \mathrm{IJord}(\pi)$, can we tell whether it is (ρ, m) or (ρ', m) in $\mathrm{Jord}(\pi)$, where ρ' is the self-dual unramified twist of ρ which is inequivalent to ρ. In certain cases the answer is yes: often the representations ρ, ρ' have opposite parities (that is, one is symplectic and the other orthogonal) and then we know that we must have the representation of symplectic type if m is even, and the one of orthogonal type if m is odd. In the exceptional case where ρ, ρ' have the same parity, we can only recover $\mathrm{Jord}(\pi)$ if it happens that *both* appear (that is, $([\rho], m)$ appears in $\mathrm{IJord}(\pi)$ with multiplicity two); otherwise, we are left with an ambiguity. (See 4.4 Remark for more on this.)

In Section 6, we explore this exceptional case on the Galois side – that is, we look at the self-dual irreducible representations of W_F which have the same parity as their self-dual unramified twist. It turns out that they have quite a special structure and that one can determine their parity (see 6.6 Proposition). This also translates to a criterion for determining the parity of a self-dual cuspidal representation ρ (such that ρ and its self-dual unramified twist ρ' have the same parity), in terms of the type it contains (see paragraph 6.8).

It is also possible, at least in certain cases, to be more precise in the analysis of the category equivalences and reducibility, in order to elucidate the ambiguity and recover $\mathrm{Jord}(\pi)$ completely. We hope to come back to this in the case of $\mathrm{Sp}_4(F)$, in a sequel to this paper.

0.10. As one of the referees has pointed out, given a *generic* cuspidal representation π of G, it follows from the results of Arthur and Mœglin that, for every (ρ, m) appearing in $\mathrm{Jord}(\pi)$, we have $m = 1$; we say that the Jordan set (or the corresponding L-packet) is *regular* in this case.

In general, determining the genericity of a cuspidal representation of G from the data used in its construction is difficult, as the example of $\mathrm{Sp}_4(F)$ shows (see [13]). However, the principal difficulties occur when trying to determine which cuspidal representations in a regular L-packet are generic, rather than in proving that the cuspidals in a non-regular L-packet are non-generic. Moreover, the case of depth zero representations is much simpler. Since the appearance of a pair (ρ, m) in $\mathrm{Jord}(\pi)$ with $m > 1$ arises from the "depth zero data" used in the construction of π, it may be possible to use the techniques of [13] to prove that any cuspidal in a non-regular L-packet is non-generic. We leave this as an interesting question to return to later.

A remark on characteristic. The bulk of our work is on the representation theory of symplectic groups; for this, while we require that the residual characteristic be odd, we have no further conditions on the characteristic – that is, we do *not* require F to be of characteristic zero. In particular, our description of the inertial Jordan set in 2.5 Theorem and 2.6 Theorem does not require characteristic zero. It is only when interpreting these results in terms of the Langlands correspondence (or the endoscopic transfer map) where, until these results have been proved with F of positive characteristic, we require characteristic zero.

Structure of the paper. In Section 1, we recall the basic structure of types for cuspidal representations, in particular semisimple characters and beta-extensions, including the choice of a base point for beta-extensions. Section 2 contains the statements of the main results on (inertial) Jordan sets, remaining entirely on the automorphic side, while the following three sections are devoted to their proofs: in Section 3, we recall the theory of covers and the results of [4, 27] on their Hecke algebras and reducibility of parabolic induction; in Section 4 we prove the reduction to the simple case which is at the heart of our method; and in Section 5 we prove the result in the simple case. The exploration of self-dual irreducible representations of W_F is given in Section 6 and finally, in Section 7, we interpret our results via the local Langlands correspondence.

Notation

Throughout the paper, F will be a locally compact nonarchimedean local field, with ring of integers $\mathfrak{o}_F$, maximal ideal $\mathfrak{p}_F$, and residue field $k_F = \mathfrak{o}_F/\mathfrak{p}_F$ of cardinality $q = q_F$ and *odd* characteristic p; similar notation will be used for extensions of F. The absolute value $|\cdot|_F$ on F is normalized to have image $q^{\mathbb{Z}}$ and we write ν for the character $g \mapsto |\det g|_F$ of $\mathrm{GL}_n(F)$.

All representations we consider here will be smooth and complex. By a *cuspidal* representation of the group of rational points of a connected reductive group over F, we mean a representation which is smooth, irreducible and cuspidal (i.e. killed by all proper Jacquet functors).

1. Cuspidal types and primary beta-extensions

In this section we fix notation following mostly [35]. We recall, in the first paragraphs, the main features of the construction of cuspidal representations of symplectic groups achieved in [35], to which we refer for relevant definitions. We do not give references for the by now classical definitions and constructions previously made for general linear groups by Bushnell and Kutzko. One of the key steps in the construction is the existence of a so-called *beta-extension*. We will have to compare such beta-extensions across different groups but, unfortunately, they are not uniquely defined. Here, following [7], we explain one way of picking out a particular beta-extension (which we call *p-primary*, see 1.8 Definition) in each case, giving a base point to make comparisons.

1.1. We recall the notation for skew semisimple strata and related objects. Let V be a finite dimensional symplectic space over F of dimension $2N$. We denote by h the symplectic form on V, by $x \mapsto \bar{x}$ the corresponding adjoint (anti-)involution on $\mathrm{End}_F(V)$ and by σ the corresponding involution on $\mathrm{GL}_F(V)$. We put $G = \mathrm{Sp}_F(V) \simeq \mathrm{Sp}_{2N}(F)$, the isometry group of h, which is the group of fixed points of σ in $\mathrm{GL}_F(V)$.

Let $[\Lambda, n, 0, \beta]$ be a skew semisimple stratum in $\mathrm{End}_F(V)$ [35, Definition 2.4, 2.5]. In particular Λ is a self-dual $\mathfrak{o}_F$-lattice sequence and $\beta = -\bar{\beta}$ belongs to the Lie algebra $\mathfrak{sp}_F(V)$. We write B for the commuting algebra of β in $\mathrm{End}_F(V)$.

Remark. Following [35] we always normalize self-dual lattice sequences such that their period over any relevant field is even and their duality invariant d is 1. With this convention, for any self-dual lattice sequence Λ and any multiple s of the period e of Λ, there is a unique self-dual lattice sequence of period s having the form $t \mapsto \Lambda(\frac{t+a}{s/e})$. There is thus a well defined way of summing two self-dual lattice sequences, by first transforming both so that they have the same period (see [12]). When performing such transformations, the valuation n of β relative to the lattice sequence Λ undergoes changes that are of no importance to us, since the associated groups H^1, J^1, J and characters (see paragraphs 1.2, 1.4 below) are left unchanged; we will thus ignore this parameter and write the stratum in the form $[\Lambda, -, 0, \beta]$.

The characteristic spaces of β determine a canonical orthogonal splitting $V = \perp_{i=0}^{l} V^i$ for the stratum $[\Lambda, -, 0, \beta]$ such that, letting $\Lambda^i = \Lambda \cap V^i$ (that is, $\Lambda^i(t) = \Lambda(t) \cap V^i$ for any $t \in \mathbb{Z}$) and $\beta^i = \beta_{|V^i}$, the strata $[\Lambda^i, -, 0, \beta^i]$, $0 \leqslant i \leqslant l$, are skew simple strata which are "sufficiently distant" in the sense of [35, Definition 2.4]. We put $E = F[\beta] = \bigoplus_{i=1}^{l} E^i$, where $E^i = F[\beta^i]$, and write $\mathfrak{o}_E^i$ for the ring of integers of E^i. We recall that Λ is an $\mathfrak{o}_E$-*lattice sequence*, by which we mean that each Λ^i is an $\mathfrak{o}_E^i$-lattice sequence in V^i.

Convention. In this paper we also take the convention that, for any skew semisimple stratum $[\Lambda, n, 0, \beta]$ with splitting $V = \perp_{i=0}^{l} V^i$, we have $\beta^0 = 0$. When 0 is not an eigenvalue of β, this can be achieved by taking V^0 to be the zero-dimensional space over F; since, in that case, $\dim_F V^0 = 0$, it does not affect any of the following constructions. The reason for this convention will become apparent later.

1.2. From the datum $[\Lambda, -, 0, \beta]$ are built open compact subrings:

- $\tilde{\mathfrak{H}}^1(\beta, \Lambda) \subseteq \tilde{\mathfrak{J}}^1(\beta, \Lambda)$ of $\mathrm{End}_F(V)$,
- $\mathfrak{H}^1(\beta, \Lambda) \subseteq \mathfrak{J}^1(\beta, \Lambda)$ of $\mathfrak{sp}_F(V)$, the fixed points of the former ones under the adjoint involution on $\mathrm{End}_F(V)$;

and open compact subgroups:

- $\tilde{H}^1(\beta, \Lambda) \subseteq \tilde{J}^1(\beta, \Lambda) \subset \tilde{J}(\beta, \Lambda)$ of $\mathrm{GL}_F(V)$,
- $H^1(\beta, \Lambda) \subseteq J^1(\beta, \Lambda) \subset J(\beta, \Lambda)$ of G, the subgroups of fixed points of the former ones under the adjoint involution on $\mathrm{GL}_F(V)$.

We will frequently write $H^1_\Lambda = H^1(\beta, \Lambda)$ and so on.

1.3. We introduce more notation relative to Λ. For $n \in \mathbb{Z}$ we write:

$$\mathfrak{a}_n(\Lambda) = \{x \in \mathrm{End}_F(V) \mid \forall t \in \mathbb{Z}, \quad x\Lambda(t) \subseteq \Lambda(t+n)\}, \qquad \mathfrak{b}_n(\Lambda) = \mathfrak{a}_n(\Lambda) \cap B.$$

In particular $\mathfrak{a}_0(\Lambda)$ is a hereditary $\mathfrak{o}_F$-order in $\mathrm{End}_F(V)$ with Jacobson radical $\mathfrak{a}_1(\Lambda)$. Let $\tilde{P}(\Lambda) = \mathfrak{a}_0(\Lambda)^\times$ and $\tilde{P}_1(\Lambda) = 1 + \mathfrak{a}_1(\Lambda)$. Then $P_1(\Lambda) = \tilde{P}_1(\Lambda) \cap G$ is the pro-p-radical of $P(\Lambda) = \tilde{P}(\Lambda) \cap G$. The quotient groups

$$\tilde{\mathcal{G}}(\Lambda) = \tilde{P}(\Lambda)/\tilde{P}_1(\Lambda) \quad \text{and} \quad \mathcal{G}(\Lambda) = P(\Lambda)/P_1(\Lambda)$$

are (the groups of rational points of) finite reductive groups over k_F. The latter may be disconnected so we let $\mathcal{G}^0(\Lambda)$ be (the group of rational points of) its neutral component and call $P^0(\Lambda)$ the inverse image of $\mathcal{G}^0(\Lambda)$ in $P(\Lambda)$; this is a parahoric subgroup of G.

Actually we will mainly work with $\mathfrak{b}_0(\Lambda) = \mathfrak{a}_0(\Lambda) \cap B$ and with $P(\Lambda_{\mathfrak{o}_E}) := P(\Lambda) \cap B$, with $\mathcal{G}(\Lambda_{\mathfrak{o}_E}) = P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E})$ and its neutral component $\mathcal{G}^0(\Lambda_{\mathfrak{o}_E})$, and with the parahoric subgroup $P^0(\Lambda_{\mathfrak{o}_E})$ of $G_E := B \cap G$, inverse image of $\mathcal{G}^0(\Lambda_{\mathfrak{o}_E})$ in $P(\Lambda_{\mathfrak{o}_E})$. Indeed we have the following:

$$J(\beta, \Lambda) = P(\Lambda_{\mathfrak{o}_E})J^1(\beta, \Lambda) \quad \text{and} \quad J(\beta, \Lambda)/J^1(\beta, \Lambda) \simeq P(\Lambda_{\mathfrak{o}_E})/P_1(\Lambda_{\mathfrak{o}_E}) = \mathcal{G}(\Lambda_{\mathfrak{o}_E}).$$

Moreover, we have natural isomorphisms

$$\mathcal{G}(\Lambda_{\mathfrak{o}_E}) \simeq \prod_{i=0}^{l} \mathcal{G}(\Lambda^i_{\mathfrak{o}^i_E}) \qquad \text{and} \qquad \mathcal{G}^0(\Lambda_{\mathfrak{o}_E}) \simeq \prod_{i=0}^{l} \mathcal{G}^0(\Lambda^i_{\mathfrak{o}^i_E}).$$

Note that, writing E^i_{o} for the field of fixed points of E^i under the adjoint involution $x \mapsto \overline{x}$ and k^i_{o} for its residue field, the groups on the right hand side here are reductive groups over k^i_{o}. We also have similar decompositions and isomorphisms for the group $\tilde{J}(\beta, \Lambda)$.

1.4. On the group $\tilde{H}^1(\beta,\Lambda)$ lives a family of one-dimensional representations endowed with very strong properties, called *semisimple characters* (*loc. cit.* §*3.1*), that restricts to a family of *skew semisimple characters* on $H^1(\beta,\Lambda)$. In particular, a skew semisimple character of $H^1(\beta,\Lambda)$, say θ, restricts to a skew *simple* character θ_i of $H^1(\beta,\Lambda)\cap \mathrm{Sp}_F(V^i) = H^1(\beta^i,\Lambda^i)$, for $0 \leqslant i \leqslant l$. Among the properties of these families the "transfer property" is specially important. It asserts that if $[\Lambda',-,0,\beta]$ is another skew semisimple stratum in $\mathrm{End}_F(V)$, then there is a canonical bijection between the sets of skew semisimple characters on $H^1(\beta,\Lambda)$ and $H^1(\beta,\Lambda')$ (*loc. cit. Proposition 3.2*). The image of θ under this bijection is called the *transfer* of θ.

To any semisimple character $\tilde{\theta}$ of $\tilde{H}^1(\beta,\Lambda)$ is associated the unique (up to equivalence) irreducible representation $\tilde{\eta}$ of $\tilde{J}^1(\beta,\Lambda)$ that contains $\tilde{\theta}$ upon restriction; actually $\tilde{\eta}$ restricts to a multiple of $\tilde{\theta}$ on $\tilde{H}^1(\beta,\Lambda)$. Now $\tilde{H}^1_\Lambda$ and $\tilde{J}^1_\Lambda$ are pro-p-groups with p odd, on which the adjoint involution σ acts. The Glauberman correspondence hence relates their representations to those of the fixed point subgroups H^1_Λ and J^1_Λ. Indeed if $\tilde{\theta}$ is fixed under the involution σ so is $\tilde{\eta}$ and its image η under the Glauberman correspondence is the unique (up to equivalence) irreducible representation of $J^1(\beta,\Lambda)$ that contains θ; it actually restricts to a multiple of θ on $H^1(\beta,\Lambda)$.

1.5. In turn the representation $\tilde{\eta}$ has special extensions to $\tilde{J}(\beta,\Lambda)$ called *beta-extensions* and denoted by $\tilde{\kappa}$. These beta-extensions in $\mathrm{GL}_F(V)$ are characterized by the fact that they are intertwined by $B^\times$ ([10, (5.2.1)]).

Remark. In the literature, these extensions are usually called β-extensions. However, the simple stratum $[\Lambda,-,0,\beta]$ giving rise to a particular simple character θ is not unique, while the notion of beta-extension turns out to be independent of the choice of β. It is thus convenient to write *beta-extension*, especially since we also have strata indexed by i so we would otherwise need to talk about β_i-extensions etc.

The definition of beta-extensions in classical groups is more delicate [35, §4]. A skew semisimple stratum as above is called *maximal* if $\mathfrak{b}_0(\Lambda)$ is a maximal self-dual $\mathfrak{o}_E$-order in B. If $[\Lambda,-,0,\beta]$ is a maximal skew semisimple stratum, a beta-extension of η is an extension κ of η to $J(\beta,\Lambda)$ such that the restriction of κ to any pro-p-Sylow subgroup is intertwined by G_E ([35, 3.11, 4.1]). In the general case, the notion of beta-extension is a relative one. Given a maximal skew semisimple stratum $[\mathfrak{M},-,0,\beta]$ in $\mathrm{End}_F(V)$ such that $\mathfrak{b}_0(\mathfrak{M}) \supset \mathfrak{b}_0(\Lambda)$, given the transfer $\theta_\mathfrak{M}$ of θ to $H^1_\mathfrak{M}$ and the representation $\eta_\mathfrak{M}$ of $J^1_\mathfrak{M}$ determined by $\theta_\mathfrak{M}$, there is a canonical way to associate to a beta-extension $\kappa_\mathfrak{M}$ of $\eta_\mathfrak{M}$, an extension κ of η, called *the beta-extension of η to J_Λ relative to $\mathfrak{M}$, compatible with $\kappa_\mathfrak{M}$* [35, 4.3, 4.5]. (We can also call κ a beta-extension of θ.)

Note that the groups $\tilde{J}(\beta,\Lambda)$ and $J(\beta,\Lambda)$ are not pro-p-groups: the notation κ here should not call to mind a Glauberman-like connection with the former $\tilde{\kappa}$.

1.6. Let $J = J(\beta, \Lambda)$, for a skew semisimple stratum $[\Lambda, -, 0, \beta]$ as above, let θ be a skew semisimple character of $H^1(\beta, \Lambda)$ and let λ be an irreducible representation of J of the form $\lambda = \kappa \otimes \tau$, with κ some beta-extension of θ, and τ the inflation of a cuspidal representation of $J/J^1 \simeq \mathcal{G}(\Lambda_{\mathfrak{o}_E})$. Under the additional assumptions that the group G_E has compact centre and that $P^0(\Lambda_{\mathfrak{o}_E})$ is a maximal parahoric subgroup of G_E, the pair (J, λ) is called a *cuspidal type* for G. Recall from [35] (see also [27] for complements):

Theorem ([35, Corollary 6.19, Theorem 7.14])**.** *A cuspidal type in G induces to a cuspidal representation of G and any cuspidal representation of G is thus obtained.*

1.7. There is of course a similar result for the group $\mathrm{GL}_F(V)$. Here we let $\tilde{J} = \tilde{J}(\beta, \Lambda)$ for a *simple* stratum $[\Lambda, -, 0, \beta]$ (so that $E = F[\beta]$ is a field) and let $\tilde{\lambda}$ be an irreducible representation of $\tilde{J}$ of the form $\tilde{\lambda} = \tilde{\kappa} \otimes \tilde{\tau}$, with $\tilde{\kappa}$ some beta-extension of $\tilde{\theta}$, and $\tilde{\tau}$ the inflation of a cuspidal representation of $\tilde{J}/\tilde{J}^1$. Under the additional assumptions that $\tilde{P}(\Lambda) \cap B$ is a maximal parahoric subgroup of $B^\times$, the pair $(\tilde{J}, \tilde{\lambda})$ is called a *maximal simple type* for $\mathrm{GL}_F(V)$.

Theorem ([10, 5.5.10, 6.2.4, 8.4.1])**.** *A maximal simple type in $\mathrm{GL}_F(V)$ extends to an irreducible representation of its normaliser, which then induces to a cuspidal representation of $\mathrm{GL}_F(V)$; any cuspidal representation ρ of $\mathrm{GL}_F(V)$ is thus obtained and the maximal simple type yielding ρ is unique up to conjugacy in $\mathrm{GL}_F(V)$.*

Remark. This theorem includes depth zero representations, by formally considering the null stratum $[\Lambda, -, 0, 0]$ to be simple.

1.8. In order to compare representations across different groups, we need a way to compare beta-extensions. (The transfer of semisimple characters already allows a comparison.) Two beta-extensions only differ by a character (of a specific shape), however we will need to choose beta-extensions in a unique way as in [7, §2.3 Lemma 1], which amounts to the GL-case in the following lemma.

Lemma.
(i) *Let $[\Lambda, n, 0, \beta]$ be a simple stratum in $\mathrm{End}_F(V)$, let $\tilde{\theta}$ be a simple character of $\tilde{H}^1(\beta, \Lambda)$, and let $\tilde{\eta}$ be the irreducible representation of $\tilde{J}^1(\beta, \Lambda)$ containing $\tilde{\theta}$. There exists one and only one beta-extension $\tilde{\kappa}$ of $\tilde{\eta}$ to $\tilde{J}(\beta, \Lambda)$ whose determinant has order a power of p.*
(ii) *With notation as in (i), assume the stratum and the simple character are skew so that the involution σ on $\mathrm{GL}_F(V)$ stabilizes $\tilde{H}^1(\beta, \Lambda)$, $\tilde{J}^1(\beta, \Lambda)$, $\tilde{J}(\beta, \Lambda)$ and $\tilde{\theta}$. The beta-extension $\tilde{\kappa}$ in (i) satisfies $\tilde{\kappa} \simeq \tilde{\kappa} \circ \sigma$.*
(iii) *Let $[\Lambda, n, 0, \beta]$ be a maximal skew semisimple stratum in $\mathrm{End}_F(V)$, let θ be a skew semisimple character of $H^1(\beta, \Lambda)$, and let η be the irreducible representation of $J^1(\beta, \Lambda)$ containing θ. There exists one and only one beta-extension of η to $J(\beta, \Lambda)$ whose determinant has order a power of p.*

Proof. (i) The reference is [10, (5.2.2)] which we imitate below to conclude the proof of (iii).

In (ii), self-duality with respect to σ follows from uniqueness. Indeed $\tilde{\eta}\circ\sigma$ is equivalent to η so there is an intertwining operator T such that $\tilde{\eta}(x) = T\,(\tilde{\eta}\circ\sigma(x))\,T^{-1}$, for $x \in \tilde{J}^1(\beta,\Lambda)$. Since σ stabilizes $\mathrm{GL}_{F[\beta]}(V)$, the representation $T(\tilde{\kappa}\circ\sigma(x))T^{-1}$, for $x \in \tilde{J}(\beta,\Lambda)$, is a beta-extension of $\tilde{\eta}$ by [10, Definition 5.2.1]; its determinant is a power of p, so it is equal to $\tilde{\kappa}$.

(iii) Let κ be a beta-extension of η and let $\Phi = \det(\kappa_{|P(\Lambda_{\mathfrak{o}_E})})$. The main point is to prove that the character Φ of $P(\Lambda_{\mathfrak{o}_E})$ factors through the determinant $\det_E$. By this we mean, as usual, that $\Phi_{|P(\Lambda_{\mathfrak{o}_E^i})}$ factors through $\det_{E^i}$, for $0 \leqslant i \leqslant l$; the remainder of the proof uses this convention.

Since θ is equal to $\chi\circ\det_E$ on $P^1(\Lambda_{\mathfrak{o}_E})$, for some character χ of $1+\mathfrak{p}_E$, we have that $\kappa_{|P^1(\Lambda_{\mathfrak{o}_E})}$ is the sum of $\dim(\eta)$ copies of $\chi\circ\det_E$. Now χ extends to a character $\tilde{\chi}$ of $\mathfrak{o}_E^\times$ and $\Phi' = (\tilde{\chi}\circ\det_E)^{-\dim(\eta)}\,\Phi$ is then a character of $P(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$. From [35, Lemma 3.10, Corollary 3.11 and Theorem 4.1], the character Φ' is trivial on all p-Sylow subgroups of $P(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$ so factors as $\Phi' = \psi\circ\det_E$ where ψ is a character of $\mathfrak{o}_E^\times$ trivial on $1+\mathfrak{p}_E$ (and depends on the choice of extension $\tilde{\chi}$).

Let us write $\mathfrak{o}_E^\times = \mu'_E(1+\mathfrak{p}_E)$, where μ'_E is the group of roots of unity in $E^\times$ of order prime to p, and, in the above, let us choose $\tilde{\chi}$ trivial on μ'_E so that the order of $\tilde{\chi}$ is a power of p. The corresponding character ψ has order prime to p, so prime to $\dim(\eta)$, and there is a character α of $\mathfrak{o}_E^\times$ (trivial on $1+\mathfrak{p}_E$) such that $\psi = \alpha^{\dim(\eta)}$.

The representation $\underline{\kappa} = (\alpha\circ\det_E)^{-1}\,\kappa$ satisfies the required condition. It is unique since any other beta-extension has the form $(\psi\circ\det_E)\underline{\kappa}$, with ψ as above, and if ψ is non-trivial then no p^i-th power of ψ can be trivial. □

Definition. With the notations of (i) above, we denote by $\underline{\tilde{\kappa}}$ the unique beta-extension of $\tilde{\eta}$ whose determinant has order a power of p. We call $\underline{\tilde{\kappa}}$ the *p-primary beta-extension* of $\tilde{\eta}$.

With the notations of (iii) above, we denote by $\underline{\kappa}$ the unique beta-extension of η whose determinant has order a power of p. We call $\underline{\kappa}$ the *p-primary beta-extension* of η.

We remark that, while the p-primary beta-extensions give a useful way of picking a base point amongst the beta-extensions, sufficient for our needs here, it is not clear whether this is the best choice of base point.

2. Inertial Jordan blocks

In this section, we state the main results on Jordan blocks and the consequences for the endoscopic transfer map. We continue with the notation from the previous section.

2.1. Let π be a cuspidal representation of $G \simeq \mathrm{Sp}_{2N}(F)$. We recall the *reducibility set* $\mathrm{Red}(\pi)$ and the *Jordan set* $\mathrm{Jord}(\pi)$ from the introduction. For any positive integer n, the group $\mathrm{GL}_n(F)\times G$ appears naturally as a standard maximal Levi subgroup of $\mathrm{Sp}_{2(N+n)}(F)$.

If ρ is a cuspidal representation of $\mathrm{GL}_n(F)$ we can form the normalized parabolically induced representation $\rho\nu^s \rtimes \pi$ (we use normalized induction and induce via the standard parabolic), where s is here a *real* parameter and ν is the character $g \mapsto |\det g|_F$ of $\mathrm{GL}_n(F)$. If no unramified twist of ρ is self-dual (i.e. isomorphic to its contragredient) then $\rho\nu^s \rtimes \pi$ is always irreducible. On the other hand, if ρ is self-dual, there is a unique $s_\pi(\rho) \geqslant 0$ such that $\rho\nu^s \rtimes \pi$ is reducible if and only if $s = \pm s_\pi(\rho)$.

Definition. Let π be a cuspidal representation of G.

- The *reducibility set* $\mathrm{Red}(\pi)$ is the set of isomorphism classes of self-dual cuspidal representations ρ of some $\mathrm{GL}_n(F)$, with $n \geqslant 1$, for which $s_\pi(\rho) \geqslant 1$.
- The *Jordan set* $\mathrm{Jord}(\pi)$ is the set of pairs (ρ, m), where $\rho \in \mathrm{Red}(\pi)$ and m is a positive integer such that $2s_\pi(\rho) - 1 - m$ is a non-negative even integer.

Note that, if $\rho \in \mathrm{Red}(\pi)$ then $2s_\pi(\rho) - 1$ is a positive integer by [31], so that there is a positive integer m such that $(\rho, m) \in \mathrm{Jord}(\pi)$.

2.2. For ρ an irreducible representation of some $\mathrm{GL}_n(F)$, we write $n = \deg(\rho)$. Recall that the *inertial class* $[\rho]$ of a cuspidal representation ρ of $\mathrm{GL}_n(F)$ is the equivalence class of ρ under the equivalence relation defined by twisting by an unramified character (that is, twisting by $\omega \circ \det$ where ω is a character of $F^\times$ trivial on $\mathfrak{o}_F^\times$). If ρ is self-dual then the inertial class $[\rho]$ contains precisely two self-dual representations: if $t(\rho)$ denotes the number of unramified characters χ of $\mathrm{GL}_n(F)$ such that $\rho \otimes \chi \simeq \rho$, and if χ' is an unramified character of order $2t(\rho)$, then $\rho' = \rho \otimes \chi'$ is the other self-dual representation in $[\rho]$.

Definition. Let π be a cuspidal representation of G. The *inertial Jordan set* of π is the *multiset* $\mathrm{IJord}(\pi)$ consisting of all pairs $([\rho], m)$ with $(\rho, m) \in \mathrm{Jord}(\pi)$.

Note that, if $([\rho], m) \in \mathrm{IJord}(\pi)$, with ρ a self-dual cuspidal representation of $\mathrm{GL}_n(F)$, then either $(\rho, m) \in \mathrm{Jord}(\pi)$ or $(\rho', m) \in \mathrm{Jord}(\pi)$, where ρ' as above is the second self-dual representation in the inertial class $[\rho]$. As discussed in the introduction, if one of ρ, ρ' is of symplectic type and the other of orthogonal type, then which occurs in $\mathrm{Jord}(\pi)$ is determined by the parity of m. On the other hand, if ρ, ρ' are both of the same parity then the inertial Jordan set $\mathrm{IJord}(\pi)$ does *not* distinguish them; of course, if $([\rho], m)$ occurs with multiplicity two in $\mathrm{IJord}(\pi)$, then both (ρ, m) and (ρ', m) occur in $\mathrm{Jord}(\pi)$ and there is no ambiguity; see 4.4 Remark for more on this.

2.3. In order to refine further the (inertial) Jordan set, we need to use the notion of the *endo-class* of a simple character, as defined in [5]. To any cuspidal representation ρ of $\mathrm{GL}_n(F)$ is attached in [6, §1.4] an endo-class of simple characters, denoted by $\mathbf{\Theta}(\rho)$, as follows. As recalled in 1.7 Theorem, there is a maximal simple type $(\tilde{J}, \tilde{\lambda})$ in $\mathrm{GL}_n(F)$ which occurs in ρ and ρ determines the $\mathrm{GL}_n(F)$-conjugacy class of $(\tilde{J}, \tilde{\lambda})$. This maximal simple type is built from a simple character $\tilde{\theta}$ and we define $\mathbf{\Theta}(\rho)$ to be the endo-class of $\tilde{\theta}$. (In fact, this is also the endo-class of *any* simple character contained in ρ.) Note that we are

allowing here the case of depth zero representations (where ρ contains the trivial character of $\tilde{P}^1(\Lambda)$, for some lattice sequence Λ), in which case $\mathbf{\Theta}(\rho) = \mathbf{\Theta}_0^F$ is the trivial endo-class over F.

Definition. Let π be a cuspidal representation of G and let $\mathbf{\Theta}$ be an endo-class of simple characters over F. The *inertial Jordan set of* π *relative to* $\mathbf{\Theta}$ is the *multiset* $\mathrm{IJord}(\pi, \mathbf{\Theta})$ consisting of all pairs $([\rho], m)$ with $(\rho, m) \in \mathrm{Jord}(\pi)$ and $\mathbf{\Theta}(\rho) = \mathbf{\Theta}$.

2.4. We will also need to twist inertial Jordan blocks as follows. With notation as in the previous paragraph, the $\mathrm{GL}_n(F)$-conjugacy class of $(\tilde{J}, \tilde{\lambda})$ depends only on the inertial class $[\rho]$; it also determines $[\rho]$ by [11, (5.5)]. The quotient group $\tilde{J}/\tilde{J}^1$ is a linear group over a finite field, say $\mathrm{GL}(m_{[\rho]}, k_{[\rho]})$. We define the twist of the inertial class $[\rho]$ by a character χ of $k_{[\rho]}^\times$ to be the inertial class $[\rho]_\chi$ determined by the maximal simple type $(\tilde{J}, \tilde{\lambda} \otimes \chi \circ \det)$ – that is, in the decomposition $\tilde{\lambda} = \tilde{\kappa} \otimes \tilde{\tau}$ with $\tilde{\kappa}$ a beta-extension, we replace the cuspidal representation $\tilde{\tau}$ by $\tilde{\tau} \otimes \chi \circ \det$.

Let $\mathbf{\Theta}$ be an endo-class of simple characters. By [5, Proposition 8.11], it determines a finite extension $k_{\mathbf{\Theta}}$ of k_F such that, for any cuspidal representation ρ of some $\mathrm{GL}_n(F)$ satisfying $\mathbf{\Theta}(\rho) = \mathbf{\Theta}$, if $(\tilde{J}, \tilde{\lambda})$ is a maximal simple type in ρ then the quotient group $\tilde{J}/\tilde{J}^1$ is a linear group over $k_{\mathbf{\Theta}}$ (that is, $k_{[\rho]} = k_{\mathbf{\Theta}}$ in the notation above). It is thus meaningful to give the following definition:

Definition. Let π be a cuspidal representation of G, let $\mathbf{\Theta}$ be an endo-class of simple characters, and let χ be a character of $k_{\mathbf{\Theta}}^\times$. The χ*-twisted inertial Jordan set of* π *relative to* $\mathbf{\Theta}$ is the multiset $\mathrm{IJord}(\pi, \mathbf{\Theta})_\chi$ consisting of all pairs $([\rho]_\chi, m)$ with $(\rho, m) \in \mathrm{Jord}(\pi)$ and $\mathbf{\Theta}(\rho) = \mathbf{\Theta}$.

The relevant case for us will be the case where χ is quadratic (that is, of order dividing 2).

Remark. Since p is odd, we have a squaring map $\mathbf{\Theta} \mapsto \mathbf{\Theta}^2$ on endo-classes: if θ is a simple character with endo-class $\mathbf{\Theta}$, associated to a simple stratum $[\Lambda, -, 0, \beta]$, then the character θ^2 is a simple character for the stratum $[\Lambda, -, 0, 2\beta]$ and $\mathbf{\Theta}^2$ is the endo-class corresponding to θ^2. This is well-defined and moreover gives a bijection on the set of endo-classes (again, since p is odd). We note also that the fields $k_{\mathbf{\Theta}}$ and $k_{\mathbf{\Theta}^2}$ coincide.

2.5. We begin the computation of the inertial Jordan set with a special case, to which we will reduce in the next paragraph. We call a cuspidal representation of G *simple* if it contains a simple character; that is, it contains a semisimple character θ of $H^1(\beta, \Lambda)$, associated to a skew semisimple stratum $[\Lambda, -, 0, \beta]$, such that $E = F[\beta]$ is a field. We allow the degenerate case $\beta = 0$, in which case π is of depth zero (and every depth zero representation is simple with $\beta = 0$); we also allow, in the case $\beta = 0$, the degenerate case that G is the trivial group, so that the trivial representation of the trivial group is regarded as being simple of depth zero.

Remark. Our use of the word *simple* here is consistent with, but not the same as, the use in [18] where, for symplectic groups, it means of minimal positive depth $1/2N$. More precisely, all cuspidal representations of depth $1/2N$ are simple in our sense, but the converse is false.

The following theorem tells us that, in the case of simple cuspidals, the Jordan set is filled by representations with the expected endo-class.

Theorem. *Let π be a simple cuspidal representation of G and let $\tilde{\theta}$ be a self-dual simple character whose restriction to G is contained in π. Let $\mathbf{\Theta}$ be the endo-class of the simple character $\tilde{\theta}$. Then*

$$\sum_{([\rho],m)\in \mathrm{IJord}(\pi,\Theta^2)} m\deg(\rho) = \begin{cases} 2N+1 & \text{if } \mathbf{\Theta} = \mathbf{\Theta}_0^F, \text{ the trivial endo-class;} \\ 2N & \text{otherwise.} \end{cases} \tag{2.1}$$

Note that we have $\mathbf{\Theta} = \mathbf{\Theta}_0^F$ if and only if π is of depth zero (which includes the degenerate case $N = 0$ where G is the trivial group). In this case, the theorem is a special case of the main result of [25].

Remark. Since the dual group of G is $\mathrm{SO}_{2N+1}(\mathbb{C})$, the reader may be surprised to see the sum in (2.1) being $2N$ rather than $2N+1$ in most cases. The reason is as follows. The Jordan set of π always contains a pair $(\chi, 1)$, with χ a quadratic character; since χ is tame, it has trivial endo-class so contributes to the sum in (2.1) if and only if $\Theta = \Theta_0^F$. The point then of Theorem 2.5 is that, apart from this quadratic character, every other cuspidal representation appearing in the Jordan set of π has endo-class Θ^2.

We will prove Theorem 2.5 in Section 5 by computing the real parts of the complex reducibility points of parabolically induced representations of the form $\rho\nu^s \rtimes \pi$, with ρ a self-dual cuspidal representation of some general linear group with endo-class $\mathbf{\Theta}^2$, using the theory of types and covers to reduce the calculation to computations of Lusztig for finite reductive groups. We note also that the proof not only gives the equality above but also gives an algorithm to compute the multiset $\mathrm{IJord}(\pi, \Theta^2)$ (see paragraph 5.10 for more detail).

2.6. Now let π be an arbitrary cuspidal representation of G. Recall from 1.6 Theorem that π can be constructed by induction, starting with a maximal skew semisimple stratum $[\Lambda, -, 0, \beta]$ and a skew semisimple character θ of $H^1(\beta, \Lambda)$, which decomposes into a family of skew simple characters θ_i of $H^1(\beta^i, \Lambda^i)$, for $i \in \{0, \ldots, l\}$. Let $\underline{\kappa}$ be the p-primary beta-extension of θ to J_Λ and, similarly, let $\underline{\kappa}_i$ be the p-primary beta-extension of θ_i to J_{Λ^i} (in $\mathrm{Sp}_F(V^i)$), for $0 \leqslant i \leqslant l$.

Let τ be the cuspidal representation of $\mathcal{G}(\Lambda_{\mathfrak{o}_E}) = P(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$ such that π is induced from $\lambda = \underline{\kappa} \otimes \tau$. Then we can uniquely decompose τ as $\tau = \otimes_{i=0}^{l} \tau_i$, with τ_i an irreducible

(cuspidal) representation of $\mathcal{G}(\Lambda_{\mathfrak{o}_E^i})$. We may then define, for each i, the cuspidal representation π_i of $\mathrm{Sp}_F(V^i)$ by

$$\pi_i = \text{c-Ind}_{J_{\Lambda^i}}^{\mathrm{Sp}_F(V^i)} \underline{\kappa}_i \otimes \tau_i.$$

Note that this representation is simple, in the sense of the previous paragraph.

Remark. Recall that we are using the notation of section 1.1, in particular 1.1 Convention so that we are assuming $\beta^0 = 0$. If the space V^0 is trivial then the representation π_0 is the trivial representation of the trivial group.

We can now state the crucial reduction theorem, which allows us to determine the inertial Jordan set of π from those of the simple cuspidals π_i.

Theorem. *With notation as above, for $0 \leqslant i \leqslant l$, let $\tilde{\theta}_i$ be the unique self-dual simple character of $\tilde{H}^1(\beta^i, \Lambda^i)$ restricting to θ_i on $H^1(\beta^i, \Lambda^i)$. Let $\mathbf{\Theta}_i$ be the endo-class of the simple character $\tilde{\theta}_i$ and let $k_{\mathbf{\Theta}_i}$ be the corresponding extension of k_F. Then there is a character χ_i of $k_{\mathbf{\Theta}_i}^\times$ of order at most two such that we have an equality of multisets*

$$\mathrm{IJord}(\pi, \mathbf{\Theta}_i^2) = \mathrm{IJord}(\pi_i, \mathbf{\Theta}_i^2)_{\chi_i}.$$

The character χ_i appearing here is in some sense explicit, coming from certain permutation characters (see 4.4 Theorem, 4.3 Proposition and 3.10 Proposition for more details). The proof of the theorem will be given in Section 4, following preparation in Section 3 (which is also needed for the proof of 2.5 Theorem). Again, the principle is to use the theory of types and covers to compare the real parts of the complex reducibility points of $\rho\nu^s \rtimes \pi$ with those of $\rho_i\nu^s \rtimes \pi_i$, for ρ a self-dual cuspidal representation of some general linear group with endo-class $\mathbf{\Theta}_i^2$ and ρ_i self-dual in the inertial class $[\rho]_{\chi_i}$.

For now, we put together the two previous theorems to get:

Corollary. *Suppose F is of characteristic zero. With the notation of the Theorem, we have*

$$\mathrm{IJord}(\pi) = \bigsqcup_{i=0}^{l} \mathrm{IJord}(\pi_i, \Theta_i^2)_{\chi_i}.$$

Since the proof of 2.5 Theorem gives us an algorithm to compute the multisets $\mathrm{IJord}(\pi_i, \Theta_i^2)$, we can then use this also to compute $\mathrm{IJord}(\pi)$, for any cuspidal representation π.

Proof. The Theorem says that $\mathrm{IJord}(\pi)$ contains the right hand side. On the other hand, by [30, Theorem 3.2.1] the multiset $\mathrm{IJord}(\pi)$ is finite and we have

$$\sum_{([\rho],m)\in\mathrm{IJord}(\pi)} m\deg(\rho) = \sum_{(\rho,m)\in\mathrm{Jord}(\pi)} m\deg(\rho) = 2N+1.$$

However, writing $\dim_F(V^i) = 2N_i$, we get from 2.5 Theorem that

$$\sum_{i=0}^{l} \sum_{([\rho],m)\in \mathrm{IJord}(\pi_i,\Theta_i^2)} m\deg(\rho) = (2N_0+1) + \sum_{i=1}^{l} 2N_i = 2N+1.$$

Thus we have equality, as required. □

We remark that the proof does not require the full strength of [30, Theorem 3.2.1]; indeed, it only uses the inequality

$$\sum_{(\rho,m)\in \mathrm{Jord}(\pi)} m\deg(\rho) \leqslant 2N+1,$$

which was proved previously in [29, §4 Corollaire]. Thus it does not in fact depend on Arthur's endoscopic classification of discrete series representations of G. One could also prove it (without the restriction on the characteristic of F) by checking that $\mathrm{IJord}(\pi,\boldsymbol{\Theta})$ is empty for any self-dual endo-class $\boldsymbol{\Theta} \neq \boldsymbol{\Theta}_i^2$; indeed, the methods of Section 4 together with results from [23] would allow this.

2.7. In this and the following paragraph, we interpret our results in terms of the endoscopic transfer map from cuspidal representations of G to $\mathrm{GL}_{2N+1}(F)$.

For $\boldsymbol{\Theta}$ an endo-class over F, we recall that the *degree* $\deg(\boldsymbol{\Theta})$ of $\boldsymbol{\Theta}$ is the degree of an extension $F[\beta]/F$ for which there are a simple stratum $[\Lambda,-,0,\beta]$ with a simple character of endo-class $\boldsymbol{\Theta}$. Although the stratum and the field extension are not uniquely determined by $\boldsymbol{\Theta}$, this degree is (see [5, Proposition 8.11]).

Let N' be a positive integer and write $\mathcal{E}(F)$ for the set of endo-classes of simple characters over F. An *endo-parameter of degree N' over F* is a formal sum

$$\sum_{\boldsymbol{\Theta}\in\mathcal{E}(F)} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta}, \qquad m_{\boldsymbol{\Theta}} \in \mathbb{Z}_{\geqslant 0},$$

such that

$$\sum_{\boldsymbol{\Theta}\in\mathcal{E}(F)} m_{\boldsymbol{\Theta}}\deg(\boldsymbol{\Theta}) = N'.$$

In particular, such a formal sum has finite support $\{\boldsymbol{\Theta}\in\mathcal{E}(F) \mid m_{\boldsymbol{\Theta}} \neq 0\}$. (In [32], these formal sums are called *semisimple endo-classes*; the nomenclature *endo-parameter* comes from [23].) We write $\mathcal{EE}_{N'}(F)$ for the set of endo-parameters of degree N' over F. We then have, for each positive integer N', a well-defined map

$$\mathsf{e}_{N'} : \mathrm{Irr}(\mathrm{GL}_{N'}(F)) \to \mathcal{EE}_{N'}(F)$$

given by mapping a cuspidal representation ρ to $\frac{N'}{\deg(\boldsymbol{\Theta}(\rho)}\boldsymbol{\Theta}(\rho)$, and mapping an arbitrary representation to the sum of the endo-parameters of its cuspidal support.

2.8. We call an endo-class $\boldsymbol{\Theta}$ over F *self-dual* if there is a self-dual simple character $\tilde{\theta}$ with endo-class $\boldsymbol{\Theta}$. We write $\mathcal{E}^{\mathsf{sd}}(F)$ for the set of self-dual endo-classes over F. An endo-parameter of degree N' over F is called *self-dual* if its support is contained in $\mathcal{E}^{\mathsf{sd}}(F)$, and we write $\mathcal{EE}^{\mathsf{sd}}_{N'}(F)$ for the set of self-dual endo-parameters of degree N' over F.

Since p is odd, the only self-dual endo-class over F of odd degree is the trivial endo-class $\boldsymbol{\Theta}^F_0$, which has degree 1. Indeed, if $\tilde{\theta}$ is a self-dual simple character which is not the trivial character, then [33, Theorem 6.3] implies that $\tilde{\theta}$ is associated to a skew simple stratum, whose associated field extension E/F is therefore of even degree. This implies, in particular, that there is a canonical bijection

$$\mathcal{EE}^{\mathsf{sd}}_{2N}(F) \to \mathcal{EE}^{\mathsf{sd}}_{2N+1}(F), \qquad \sum_{\boldsymbol{\Theta}\in\mathcal{E}^{\mathsf{sd}}} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta} \mapsto \sum_{\boldsymbol{\Theta}\in\mathcal{E}^{\mathsf{sd}}} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta} + \boldsymbol{\Theta}^F_0.$$

For any N', there is also the natural squaring map

$$\mathcal{EE}_{N'}(F) \to \mathcal{EE}_{N'}(F), \qquad \sum_{\boldsymbol{\Theta}\in\mathcal{E}} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta} \mapsto \sum_{\boldsymbol{\Theta}\in\mathcal{E}} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta}^2,$$

which is a bijection since p is odd. Combining these, we get a natural inclusion map

$$\iota_{2N} : \mathcal{EE}^{\mathsf{sd}}_{2N}(F) \hookrightarrow \mathcal{EE}_{2N+1}(F), \qquad \sum_{\boldsymbol{\Theta}\in\mathcal{E}^{\mathsf{sd}}} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta} \mapsto \sum_{\boldsymbol{\Theta}\in\mathcal{E}^{\mathsf{sd}}} m_{\boldsymbol{\Theta}}\boldsymbol{\Theta}^2 + \boldsymbol{\Theta}^F_0.$$

Given a maximal skew semisimple stratum $[\Lambda, n, 0, \beta]$ and a skew semisimple character θ of $H^1(\beta, \Lambda)$, which decomposes into a family of skew simple characters θ_i of $H^1(\beta^i, \Lambda^i)$, for $i \in \{0, \dots, l\}$, we define the self-dual endo-parameter of θ to be

$$\sum_{i=0}^{l} \frac{\dim_F(V^i)}{\deg(\boldsymbol{\Theta}_i)}\boldsymbol{\Theta}_i,$$

where $\boldsymbol{\Theta}_i$ is the endo-class of the unique self-dual simple character $\tilde{\theta}_i$ which restricts to θ_i. This is a self-dual endo-parameter of degree $2N$.

We write $\mathrm{Cusp}(G)$ for the set of equivalence classes of cuspidal representations of G. From 2.6 Theorem and 2.5 Theorem, we derive the following result.

Theorem. *Suppose that F is of characteristic zero. Let π be a cuspidal representation of G and let θ be a skew semisimple character contained in π. Then the self-dual endo-parameter of θ depends only on π. Moreover, the diagram*

$$\begin{tikzcd}
\mathrm{Cusp}(G) \arrow[r, "\text{transfer}"] \arrow[d, "\mathsf{e}_G"'] & \mathrm{Irr}(\mathrm{GL}_{2N+1}(F)) \arrow[d, "\mathsf{e}_{2N+1}"] \\
\mathcal{EE}^{\mathsf{sd}}_{2N}(F) \arrow[r, hook, "\iota_{2N}"'] & \mathcal{EE}_{2N+1}(F)
\end{tikzcd}$$

commutes, where $\mathsf{e}_G(\pi)$ denotes the endo-parameter of any skew semisimple character contained in π.

We remark that the fact that the map e_G is well-defined is also proved, in much greater generality and without the assumption that F has characteristic zero, in [23]; the proof here is quite different and long predates that in [23]. We also remark that we will see later (7.6 Theorem) that the map e_G is in fact surjective.

Proof. Let π be a cuspidal representation of G and let θ be a skew semisimple character contained in π, with all the notation from above. In particular, we have a family of skew simple characters θ_i, for $0 \leqslant i \leqslant l$, and, for each i, the unique self-dual simple character $\tilde{\theta}_i$ restricting to θ_i and the self-dual endo-class $\mathbf{\Theta}_i$ of $\tilde{\theta}_i$.

For $(\rho, m) \in \mathrm{Jord}(\pi)$, we write $\mathbf{\Theta}_\rho$ for the endo-class of any simple character in ρ. Then 2.6 Corollary implies that $\mathbf{\Theta}_\rho = \mathbf{\Theta}_i^2$, for some $0 \leqslant i \leqslant l$; moreover, together with 2.5 Theorem it implies

$$\sum_{(\rho,m)\in\mathrm{Jord}(\pi)} \frac{m\deg(\rho)}{\deg(\mathbf{\Theta}_\rho)}\mathbf{\Theta}_\rho = \sum_{i=0}^{l} \frac{\dim_F(V^i)}{\deg(\mathbf{\Theta}_i)}\mathbf{\Theta}_i^2 + \mathbf{\Theta}_0^F. \tag{2.2}$$

In particular, the right hand side here is $\mathbf{\Theta}_0^F$ plus the square of the endo-parameter of θ; since the squaring map is a bijection, this endo-parameter is therefore independent of the choice of θ in π since the left hand side is.

Now, according to the results of Mœglin [30, Theorem 3.2.1], the Jordan set exactly determines the endoscopic transfer of π to $\mathrm{GL}_{2N+1}(F)$; more precisely, the transfer of π is

$$\prod_{(\rho,m)\in\mathrm{Jord}(\pi)} \mathrm{St}(\rho, m)$$

where $\mathrm{St}(\rho, m)$ denotes the unique irreducible quotient of the normalised parabolically induced representation

$$\rho\nu^{\frac{1-m}{2}} \times \rho\nu^{\frac{3-m}{2}} \times \cdots \times \rho\nu^{\frac{m-1}{2}}$$

of $\mathrm{GL}_{m\deg(\rho)}(F)$. The endo-parameter of the transfer of π is thus

$$\sum_{(\rho,m)\in\mathrm{Jord}(\pi)} \frac{m\deg(\rho)}{\deg(\mathbf{\Theta}_\rho)}\mathbf{\Theta}_\rho,$$

where $\mathbf{\Theta}_\rho$ is the endo-class of (any simple character in) ρ. In particular, this lies in $\mathcal{EE}^{\mathsf{sd}}_{2N+1}(F)$ and (2.2) now implies that the diagram commutes. □

3. Types, covers and reducibility

In the following subsections we recall the main results about covers and their Hecke algebras, from [11] in the general situation and from [27] in the particular situation of interest to us: induction from a maximal parabolic subgroup of a symplectic group. One of the key features in [27] is the presence of quadratic characters arising from the comparison of beta-extensions. Using the notion of p-primary beta-extension, together with results from [4], we describe

these characters as signatures of permutations and recall the implications of the structure of the Hecke algebra (including its parameters) for the reducibility of parabolic induction from [4].

3.1. We briefly recall the general notion of a *type* as defined by Bushnell and Kutzko in [11]. Let for a moment G be the group of F-points of an arbitrary connected reductive group defined over F, let L be a Levi subgroup of G and let σ be a cuspidal representation of L. The pair (L, ρ) determines, through G-conjugacy and twisting by unramified characters of L, an inertial class $\mathfrak{s} = [L, \rho]_G$ in G. This class $\mathfrak{s}$ indexes the Bernstein block $\mathcal{R}^{\mathfrak{s}}(G)$ (in the category $\mathcal{R}(G)$ of smooth representations of G) which is the direct factor of $\mathcal{R}(G)$ consisting of representations all of whose irreducible subquotients are subquotients of a representation parabolically induced from an element of $\mathfrak{s}$.

Let (J, λ) be a pair made of an open compact subgroup J of G and an irreducible smooth representation λ of J, acting on the finite dimensional space V_λ. The Hecke algebra of the pair (J, λ) is the intertwining algebra of the representation $\text{c-Ind}_J^G \lambda$, traditionally viewed as:

$$\mathcal{H}(G, \lambda) = \{f : G \to \mathrm{End}(V_\lambda) \mid f \text{ compactly supported and } \forall g \in G,\ \forall j, k \in J,\ f(jgk) = \lambda(j) f(g) \lambda(k)\}$$

The pair (J, λ) is an $\mathfrak{s}$-type if the irreducible objects of $\mathcal{R}^{\mathfrak{s}}(G)$ are exactly the irreducible representations of G that contain λ upon restriction to J. In this case there is an equivalence of categories

$$\mathcal{M}_\lambda : \mathcal{R}^{\mathfrak{s}}(G) \longrightarrow \text{Mod–}\mathcal{H}(G, \lambda), \qquad \mathcal{M}_\lambda(\pi) = \mathrm{Hom}_J(\lambda, \pi).$$

3.2. There is a counterpart of parabolic induction for types: the notion of G-cover, also defined in [11] by Bushnell and Kutzko. Let M be a Levi subgroup of G, let J_M be a compact open subgroup of M and let λ_M be a smooth irreducible representation of J_M. A *G-cover* of the pair (J_M, λ_M) is an analogous pair (J, λ) in G satisfying the following conditions, for any parabolic subgroup P of G of Levi M, where we write N for the unipotent radical of P, and P^- for the parabolic subgroup opposite to P with respect to M, with unipotent radical N^-:

(i) J has an Iwahori decomposition with respect to $(M; P)$, i.e.
$$J = (J \cap N^-)(J \cap M)(J \cap N), \text{ and } J \cap M = J_M;$$
(ii) λ restricts to λ_M on J_M and to a multiple of the trivial representation on $J \cap N^-$ and $J \cap N$;
(iii) the Hecke algebra $\mathcal{H}(G, \lambda)$ contains an invertible element supported on the double coset of a strongly positive element of the centre of M [11, §7].

If the pair (J_M, λ_M) is an $\mathfrak{s}_M$-type in M for an inertial class $\mathfrak{s}_M = [L, \sigma]_M$ (so that L is a Levi subgroup of M) and if (J, λ) is a G-cover of (J_M, λ_M), then the pair (J, λ) is an $\mathfrak{s}_G$-type in G for the inertial class $s_G = [L, \sigma]_G$ [11, §8]. Furthermore, the third condition above

provides us with an injective morphism of algebras $t : \mathcal{H}(M, \lambda_M) \hookrightarrow \mathcal{H}(G, \lambda)$ that induces on modules a morphism t_* yielding a commutative diagram :

$$\begin{array}{ccc} \mathcal{R}^{\mathfrak{s}_G}(G) & \xrightarrow{\mathcal{M}_\lambda} & \text{Mod–}\,\mathcal{H}(G,\lambda) \\ {\scriptstyle \mathrm{Ind}_P^G}\uparrow & & \uparrow{\scriptstyle t_*} \\ \mathcal{R}^{\mathfrak{s}_M}(M) & \xrightarrow{\mathcal{M}_{\lambda_M}} & \text{Mod–}\,\mathcal{H}(M,\lambda_M) \end{array}$$

The reducibility of parabolically induced representations from P to G, on the left side, can thus be studied in terms of Hecke algebra modules, on the right side.

3.3. This is the tool we use in this paper, where the types will be cuspidal types as in paragraphs 1.6, 1.7, simple types and semisimple types. As for the relevant Levi and parabolic subgroups, they will come in most cases as follows – and now we come back to the symplectic group $G = \mathrm{Sp}_F(V)$ and the setting of paragraph 1.1. Thus we have a skew semisimple stratum $[\Lambda, -, 0, \beta]$ with associated orthogonal decomposition $V = \perp_{i=0}^{l} V^i$, as well as all the other notation from §1.

Let $V = \oplus_{j=-m}^{m} W^j$ be a *self-dual* decomposition of V (i.e. for which the orthogonal space of W^j is $\oplus_{k\neq -j} W^k$) such that:

(a) $W^j = \oplus_{i=0}^{l} W^j \cap V^i$ and $W^j \cap V^i$ is an E_i-subspace of V^i;
(b) $\Lambda(t) = \oplus_{j=-m}^{m} \Lambda(t) \cap W^{(j)}$, for all $t \in \mathbb{Z}$;
(c) for any $r \in \mathbb{Z}$ and i with $0 \leqslant i \leqslant l$, there is at most one j, with $-m \leqslant j \leqslant m$, such that $\Lambda(r) \cap V^i \cap W^{(j)} \supsetneq \Lambda(r+1) \cap V^i \cap W^{(j)}$;
(d) for $j \neq 0$ there exists $0 \leqslant i \leqslant l$ such that $W^j \subset V^i$, and $\tilde{P}((\Lambda \cap W^j)_{\mathfrak{o}_E^i})$ is a maximal parahoric subgroup of $\mathrm{GL}_{E_i}(W^j)$;
(e) $P^{\circ}((\Lambda \cap W^0)_{\mathfrak{o}_E})$ is a maximal parahoric subgroup of $G \cap \prod_{i=0}^{l} \mathrm{GL}_{E_i}(W^0 \cap V^i)$, which is a group with compact centre.

Such a decomposition is called *exactly subordinate* to the stratum $[\Lambda, -, 0, \beta]$ (compare to [35, Definition 6.5]).

Let then $V = \oplus_{j=-m}^{m} W^j$ be a self-dual decomposition of V exactly subordinate to the stratum $[\Lambda, -, 0, \beta]$, let M be the Levi subgroup of G stabilizing this decomposition and let P be a parabolic subgroup of G with Levi component M. Then the pairs $(H^1(\beta, \Lambda), \theta)$, $(J^1(\beta, \Lambda), \eta)$ and $(J(\beta, \Lambda), \kappa)$ all satisfy conditions (i) and (ii) of paragraph 3.2 above. In fact, for the first two pairs we need only conditions (a)–(b) and a self-dual decomposition satisfying these will be called *subordinate* to the stratum; for the final pair we need only (a)–(c).

3.4. In the next few paragraphs, we subsume the results of [35], in the form easier to refer to taken from [27] and in the case that we will focus on, that is, parabolic induction of self-dual cuspidal representations of a maximal Levi subgroup in a symplectic group. We

thus continue with the notation of §1 and fix a cuspidal representation π of $G = \mathrm{Sp}_F(V)$. We also fix a finite-dimensional vector space W over F and a self-dual cuspidal representation ρ of $\mathrm{GL}_F(W)$. We consider the symplectic space $X = V \perp (W \oplus W^*)$ over F, with form

$$h_X(v_1 + w_1 + w_1^*, v_2 + w_2 + w_2^*) = h(v_1, v_2) + \langle w_1, w_2^* \rangle - \langle w_2, w_1^* \rangle,$$

where h is the symplectic form on V and $\langle \cdot, \cdot \rangle$ is the pairing $W \times W^* \to F$. We put $M = \mathrm{GL}_F(W) \times G$, a maximal Levi subgroup of $\mathrm{Sp}_F(X)$. According to [15, Proposition 8.4] and [27, §4.1], one can find a type $(\tilde{J}_W \times J_V, \tilde{\lambda}_W \otimes \lambda_V)$ in M for the cuspidal representation $\rho \otimes \pi$ of M and a G-cover of this M-type as follows.

3.5. There exist a skew semisimple stratum $[\Lambda, -, 0, \beta]$ in $\mathrm{End}_F(X)$ and a skew semisimple character θ of $H^1(\beta, \Lambda)$ with the following properties.

- The decomposition $X = V \perp (W \oplus W^*)$ is exactly subordinate to the stratum $[\Lambda, -, 0, \beta]$. In particular, letting
$$\Lambda \cap V = \Lambda_V, \quad \beta_{|V} = \beta_V, \quad \Lambda \cap W = \Lambda_W \quad \text{and} \quad \beta_{|W} = \beta_W,$$
the stratum $[\Lambda_V, -, 0, \beta_V]$ in $\mathrm{End}_F(V)$ is skew semisimple maximal and the stratum $[\Lambda_W, -, 0, \beta_W]$ in $\mathrm{End}_F(W)$ is simple maximal. Moreover, the self-duality of ρ is reflected in the fact that the restriction of β to $W \oplus W^*$ generates a field (equivalently, the restricted stratum $[\Lambda \cap (W \oplus W^*), -, 0, \beta_{|W \oplus W^*}]$ is skew simple). We also have
$$H^1(\beta, \Lambda) \cap M \simeq \tilde{H}^1(\beta_W, \Lambda_W) \times H^1(\beta_V, \Lambda_V),$$
where the isomorphism is given by restriction, and similarly for $J^1(\beta, \Lambda)$ and for $J(\beta, \Lambda)$. We will abbreviate $H^1(\beta, \Lambda) = H^1_\Lambda$, $H^1(\beta_V, \Lambda_V) = H^1_V$ and $\tilde{H}^1(\beta_W, \Lambda_W) = \tilde{H}^1_W$, and similarly for J^1 and J.
- Let $\tilde{\vartheta}_W$ be the restriction of θ to $\tilde{H}^1_W$; this is a self-dual simple character. There are the p-primary beta-extension $\underline{\tilde{\kappa}}_W$ of $\tilde{\vartheta}_W$ and a self-dual cuspidal representation $\tilde{\tau}_W$ of $\tilde{J}_W/\tilde{J}^1_W$ such that ρ is induced by an extension of $\tilde{\lambda}_W = \underline{\tilde{\kappa}}_W \otimes \tilde{\tau}_W$ to the normalizer of $\tilde{J}_W$.
- Let θ_V be the restriction of θ to H^1_V; this is a skew semisimple character. There are the p-primary beta-extension $\underline{\kappa}_V$ of θ_V and a cuspidal representation τ_V of J_V/J^1_V such that π is induced by $\lambda_V = \underline{\kappa}_V \otimes \tau_V$.

3.6. Let P be the parabolic subgroup of $\mathrm{Sp}_F(X)$ which is the stabiliser of the subspace W (so stabilises the flag $W \subset W \perp V \subset X$), let U be the unipotent radical of P and let P^- be the parabolic subgroup opposite to P with respect to M (the stabiliser of W^*). Also set $J_P = H^1_\Lambda(J_\Lambda \cap P)$ and $J^1_P = H^1_\Lambda(J^1_\Lambda \cap P)$.

For any extension κ of η to J_Λ we denote by κ_P the natural representation of J_P in the space of $(J_\Lambda \cap U)$-fixed vectors under κ. In particular, there is a beta-extension κ_Λ of θ such that $\kappa_{\Lambda,P|J \cap M} = \underline{\tilde{\kappa}}_W \otimes \underline{\kappa}_V$. We can view $\tau = \tilde{\tau}_W \otimes \tau_V$ as a cuspidal representation of $J_P/J^1_P \simeq \tilde{J}_W/\tilde{J}^1_W \times J_V/J^1_V$. Then, letting $\lambda_P = \kappa_{\Lambda,P} \otimes \tau$, we have:

Theorem ([27, §4.1])**.** *(J_P, λ_P) is an $\mathrm{Sp}_F(X)$-cover of $(\tilde{J}_W \times J_V, \tilde{\lambda}_W \otimes \lambda_V)$.*

3.7. Furthermore precise information about the Hecke algebra of this cover is given in *loc.cit.*:

Theorem ([27, Theorem B])**.** *The Hecke algebra $\mathscr{H}(\mathrm{Sp}_F(X), \lambda_P)$ is a Hecke algebra on a dihedral group: it is generated by T_0 and T_1, each invertible and supported on a single double coset, with relations :*

$$(T_i - q^{r_i})\ (T_i + 1) = 0, \quad i = 0, 1, \quad r_0, r_1 \in \mathbb{Z}.$$

3.8. In fact, the parameters come from rank two Hecke algebras of finite reductive groups as follows [35, (7.3) and §7.2.2]. There are two self-dual $\mathfrak{o}_E$-lattice sequences $\mathfrak{M}_0$ and $\mathfrak{M}_1$ in X such that $[\mathfrak{M}_t, -, 0, \beta]$, for $t = 0, 1$, are semisimple strata and:

- the hereditary orders $\mathfrak{b}_0(\mathfrak{M}_0)$ and $\mathfrak{b}_0(\mathfrak{M}_1)$ are maximal self-dual $\mathfrak{o}_E$-orders containing $\mathfrak{b}_0(\Lambda)$;
- the decomposition $X = V \perp (W \oplus W^*)$ is subordinate to the strata $[\mathfrak{M}_t, -, 0, \beta]$, for $t = 0, 1$;
- we have $P(\Lambda_{\mathfrak{o}_E}) = (P(\mathfrak{M}_{1,\mathfrak{o}_E}) \cap P^-)\, P^1(\mathfrak{M}_{1,\mathfrak{o}_E}) = (P(\mathfrak{M}_{0,\mathfrak{o}_E}) \cap P)\, P^1(\mathfrak{M}_{0,\mathfrak{o}_E})$.

The representation $\tau = \tilde{\tau}_W \otimes \tau_V$ is a cuspidal representation of the Levi subgroup $\mathcal{G}(\Lambda_{\mathfrak{o}_E}) = P(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$ of $\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}) = P(\mathfrak{M}_{t,\mathfrak{o}_E})/P^1(\mathfrak{M}_{t,\mathfrak{o}_E})$, for $t = 0, 1$, that can be inflated to the parabolic subgroup $P(\Lambda_{\mathfrak{o}_E})/P^1(\mathfrak{M}_{t,\mathfrak{o}_E})$, then induced to the full group $\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E})$. A specific use of the notion of beta-extension relative to $\mathfrak{M}_{t,\mathfrak{o}_E}$ leads to self-dual characters χ_t of $\mathcal{G}(\Lambda_{\mathfrak{o}_E})$, for $t = 0, 1$, giving rise to injective homomorphisms of algebras:

$$\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \chi_t \otimes \tau) \ \hookrightarrow\ \mathscr{H}(G, \lambda_P) \qquad (t = 0, 1). \tag{3.1}$$

We will elaborate on this in paragraph 3.12 below.

3.9. In order to make use of this, we need some control on the characters χ_t and it is here that we really need to use the notion of p-primary beta-extension. We continue with the notation of the previous paragraphs but, for the moment, drop the subscript t on $\mathfrak{M}_t$. We will assume that $P(\Lambda_{\mathfrak{o}_E}) = (P(\mathfrak{M}_{\mathfrak{o}_E}) \cap P)\, P^1(\mathfrak{M}_{\mathfrak{o}_E})$ so that, in the notation above, we are doing the case $t = 0$; the case $t = 1$ is obtained by exchanging the parabolic P with its opposite P^-. Denote by $\theta_{\mathfrak{M}}$ the transfer of $\theta_\Lambda = \theta$ to $H^1_{\mathfrak{M}} = H^1(\beta, \mathfrak{M})$, and denote by $\eta_\Lambda, \eta_{\mathfrak{M}}$ the unique irreducible representations of $J^1_\Lambda, J^1_{\mathfrak{M}}$ which contain $\theta_\Lambda, \theta_{\mathfrak{M}}$ respectively. Similarly, we have the representations $\tilde{\eta}_W, \eta_V$ of $\tilde{J}^1_W, J^1_V$ which contain $\tilde{\vartheta}_W, \theta_V$ respectively.

For a moment, let (J, J^1, η) be either $(J_{\mathfrak{M}}, J^1_{\mathfrak{M}}, \eta_{\mathfrak{M}})$ or $(J_\Lambda, J^1_\Lambda, \eta_\Lambda)$, and let κ be any extension of η to J. We define $\mathbf{r}_P(\kappa)$, the *Jacquet restriction* of κ, as the natural representation of $J \cap M$ on the space of $J^1 \cap U$-invariants of κ [4, Corollaire 1.12, Lemme 1.18]; that is, $\mathbf{r}_P(\kappa)$ is the restriction to $J \cap M$ of κ_P, in the notation of paragraph 3.6.

3.10. In order to compute the character χ from (3.1), we need to compare the following two representations of J_Λ:

- the beta-extension $\underline{\kappa}_{\Lambda,\mathfrak{M}}$ of η_Λ to J_Λ which is compatible with the p-primary beta-extension $\underline{\kappa}_{\mathfrak{M}}$ of $\eta_{\mathfrak{M}}$ to $J_{\mathfrak{M}}$ (in the sense of [35, Definition 4.5]);
- the extension $\kappa_\Lambda = \kappa_\Lambda^P$ of η_Λ to J_Λ characterized by the property
$$\mathbf{r}_P(\kappa_\Lambda^P) \simeq \tilde{\underline{\kappa}}_W \otimes \underline{\kappa}_V,$$
where $\tilde{\underline{\kappa}}_W$ and $\underline{\kappa}_V$ are the p-primary beta-extensions of $\tilde{\eta}_W$ and η_V respectively, as above. (See [4, Lemme 1.16].)

We apply Jacquet restriction to $\underline{\kappa}_{\Lambda,\mathfrak{M}}$. The groups $J_{\mathfrak{M}} \cap M$ and $J_\Lambda \cap M$ are both equal to $\tilde{J}_W \times J_V$ and the representations $\mathbf{r}_P(\underline{\kappa}_{\mathfrak{M}})$ and $\mathbf{r}_P(\underline{\kappa}_{\Lambda,\mathfrak{M}})$ both extend $\tilde{\eta}_W \otimes \eta_V$. From [4, Proposition 1.20], the beta-extension $\underline{\kappa}_{\Lambda,\mathfrak{M}}$ is characterized by

$$\mathbf{r}_P(\underline{\kappa}_{\Lambda,\mathfrak{M}}) = \mathbf{r}_P(\underline{\kappa}_{\mathfrak{M}}). \tag{3.2}$$

Proposition. *For $m \in J_{\mathfrak{M}} \cap M$, define $\epsilon_{\mathfrak{M}}(m)$ as the signature of the permutation:*
$$\operatorname{Ad} m : \ u \mapsto m^{-1}um, \quad u \in J^1_{\mathfrak{M}} \cap U^- / H^1_{\mathfrak{M}} \cap U^-.$$
The p-primary beta-extension $\underline{\kappa}_{\mathfrak{M}}$ of $\eta_{\mathfrak{M}}$ to $J_{\mathfrak{M}}$ satisfies:

$$\mathbf{r}_P(\underline{\kappa}_{\mathfrak{M}}) \simeq \epsilon_{\mathfrak{M}}(\tilde{\underline{\kappa}}_W \otimes \underline{\kappa}_V). \tag{3.3}$$

This proposition and (3.2) immediately imply:

Corollary. *The extensions $\underline{\kappa}_{\Lambda,\mathfrak{M}}$ and κ_Λ^P of η_Λ to J_Λ are related by:*
$$\mathbf{r}_P(\kappa_\Lambda^P) = \epsilon_{\mathfrak{M}} \mathbf{r}_P(\underline{\kappa}_{\Lambda,\mathfrak{M}}).$$

Proof of Proposition. Let ϕ be an arbitrary extension of $\eta_{\mathfrak{M}}$ to $J_{\mathfrak{M}}$. By [4, Lemme 1.10] the restriction of ϕ to $(J_{\mathfrak{M}} \cap P)J^1_{\mathfrak{M}}$ is induced from the natural representation ϕ_P of $(J_{\mathfrak{M}} \cap P)H^1_{\mathfrak{M}}$ on the space of $J^1_{\mathfrak{M}} \cap U$-invariants of ϕ. Hence we can realize $\phi_{|J_{\mathfrak{M}} \cap M}$ as the action by right translation on functions taking values in the space of ϕ_P. We also have the representation $\eta_{\mathfrak{M},P}$ of $(J^1_{\mathfrak{M}} \cap P)H^1_{\mathfrak{M}}$ on the space of $J^1_{\mathfrak{M}} \cap U$-invariants of $\eta_{\mathfrak{M}}$.

Let $\widetilde{\mathcal{S}}$ be the space of $\tilde{\eta}_W$ and let $\mathcal{S}$ be the space of η_V, so that $\widetilde{\mathcal{S}} \otimes \mathcal{S}$ is the space of $\eta_{\mathfrak{M},P}$. The representation ϕ_P itself extends $\eta_{\mathfrak{M},P}$, so our representation $\phi_{|J_{\mathfrak{M}} \cap M}$ acts by right translation on the space of functions
$$f : \ (J_{\mathfrak{M}} \cap P)J^1_{\mathfrak{M}} \longrightarrow \widetilde{\mathcal{S}} \otimes \mathcal{S}$$
satisfying, for all $x \in (J_{\mathfrak{M}} \cap P)H^1_{\mathfrak{M}}$ and all $g \in (J_{\mathfrak{M}} \cap P)J^1_{\mathfrak{M}}$:
$$f(xg) = \phi_P(x)f(g).$$
Using Iwahori decompositions as in [4, §1.3] we identify this space with the space $\mathcal{T}$ of functions on $J^1_{\mathfrak{M}} \cap U^- / H^1_{\mathfrak{M}} \cap U^-$ with values in $\widetilde{\mathcal{S}} \otimes \mathcal{S}$. The action of $m \in J_{\mathfrak{M}} \cap M$ on $f \in \mathcal{T}$ is now given by:
$$\phi(m)f(u) = f(um) = f(m.m^{-1}um) = \mathbf{r}_P(\phi)(m)f(m^{-1}um),$$

for $u \in J^1_{\mathfrak{M}} \cap U^- / H^1_{\mathfrak{M}} \cap U^-$.

Let $\mathcal{T}_0$ be the space of complex functions on $J^1_{\mathfrak{M}} \cap U^- / H^1_{\mathfrak{M}} \cap U^-$ and $\mathcal{E}$ the permutation representation of $J_{\mathfrak{M}} \cap M$ on $\mathcal{T}_0$:

$$\mathcal{E}(m)f(u) = f(m^{-1}um), \qquad \text{for } f \in \mathcal{T}_0,\ m \in J_{\mathfrak{M}} \cap M,\ u \in J^1_{\mathfrak{M}} \cap U^- / H^1_{\mathfrak{M}} \cap U^-.$$

We can further identify $\mathcal{T}$ with $\mathcal{T}_0 \otimes (\widetilde{\mathcal{S}} \otimes \mathcal{S})$ to obtain $\phi_{|J_{\mathfrak{M}} \cap M} \ \simeq\ \mathcal{E} \otimes \mathbf{r}_P(\phi)$.

All of this applies to $\underline{\kappa}_{\mathfrak{M}}$, so

$$\underline{\kappa}_{\mathfrak{M}|J_{\mathfrak{M}} \cap M} \quad \simeq \mathcal{E} \otimes \mathbf{r}_P(\underline{\kappa}_{\mathfrak{M}}).$$

The determinant of this representation has order a power of p, a property that is unchanged by taking p^k-th powers. Recall that the determinant of some $x \otimes y$ acting on $X \otimes Y$ is $(\det x)^{\dim Y}(\det y)^{\dim X}$. The two spaces here, $\mathcal{T}_0$ and $\widetilde{\mathcal{S}} \otimes \mathcal{S}$, have dimension a power of p, which is odd, and the determinant of $\mathcal{E}$ acting on $\mathcal{T}_0$ is $\epsilon_{\mathfrak{M}}$.

We now write $\mathbf{r}_P(\underline{\kappa}_{\mathfrak{M}}) = \tilde{\kappa}_W \otimes \kappa_V$ where $\tilde{\kappa}_W$ is a beta-extension of $\tilde{\eta}_W$ and κ_V is a beta-extension of η_V (see (3.2) and [35, Proposition 6.3]). It is enough to prove

$$\epsilon_{\mathfrak{M}}\ \det(\tilde{\kappa}_W \otimes \kappa_V) \text{ has order a power of } p.$$

Writing $\epsilon_{\mathfrak{M}}$ for the restrictions of $\epsilon_{\mathfrak{M}}$ to $\tilde{J}_W$ and to J_V, this condition transforms into

$$\det(\epsilon_{\mathfrak{M}} \otimes \tilde{\kappa}_W) \text{ and } \det(\epsilon_{\mathfrak{M}} \otimes \kappa_V) \text{ have order a power of } p.$$

The character $\epsilon_{\mathfrak{M}}$ is trivial on pro-p-subgroups so $\epsilon_{\mathfrak{M}} \otimes \tilde{\kappa}_W$ and $\epsilon_{\mathfrak{M}} \otimes \kappa_V$ are beta-extensions of $\tilde{\eta}_W$ and η_V respectively [35, Theorem 4.1]. This last condition actually means that they are the p-primary beta-extensions of $\tilde{\eta}_W$ and η_V respectively, and (3.3) follows. □

3.11. Before returning to the implications on reducibility, we examine the character $\epsilon_{\mathfrak{M}}$ a little further. We begin with a general lemma.

Lemma. *Let Z be a finite dimensional vector space over a finite field $\mathbb{F}_q$ with odd cardinality q and let $g \in \mathrm{GL}_{\mathbb{F}_q}(Z)$. The signature of the permutation g of Z is equal to $(\det_{\mathbb{F}_q} g)^{\frac{q-1}{2}}$.*

Proof. As a character of $\mathrm{GL}_{\mathbb{F}_q}(Z)$, the signature is trivial on the derived subgroup, which is $\mathrm{SL}_{\mathbb{F}_q}(Z)$, as $q > 2$, hence factors through a character χ of the determinant over $\mathbb{F}_q$. We know χ^2 is trivial and it remains to show that χ is not identically trivial on $\mathrm{GL}_{\mathbb{F}_q}(Z)$.

Consider the permutation of $\mathbb{F}_q$ given by multiplication by an element ζ of $\mathbb{F}_q^\times$ of order 2^t, with $\frac{q-1}{2^t}$ an odd integer. This permutation fixes 0 and has $\frac{q-1}{2^t}$ cycles of length 2^t in $\mathbb{F}_q^\times$, so has odd signature. Then the element $g = \mathrm{diag}(\zeta, 1, \ldots, 1)$ has odd signature so χ is non-trivial. □

Proposition. *For $m \in J_{\mathfrak{M}} \cap M$, the permutation $\mathrm{Ad}\, m$ of $J^1_{\mathfrak{M}} \cap U^- / H^1_{\mathfrak{M}} \cap U^-$ is an $\mathbb{F}_p$-linear transformation of this $\mathbb{F}_p$-vector space and*

$$\epsilon_{\mathfrak{M}}(m) = [\det_{\mathbb{F}_p} \mathrm{Ad}\, m]^{\frac{p-1}{2}}.$$

Moreover, the permutation $u \mapsto m^{-1}um$ *of the space* $J^1_{\mathfrak{M}} \cap U/H^1_{\mathfrak{M}} \cap U$ *also has signature* $\epsilon_{\mathfrak{M}}(m)$.

Proof. The first part follows from the previous lemma. Since the decomposition $X = V \perp (W \oplus W^*)$ is subordinate to $[\mathfrak{M}, -, 0, \beta]$, the pairing

$$\langle x, y\rangle = \theta_{\mathfrak{M}}([x, y]), \qquad \text{for } x \in J^1_{\mathfrak{M}} \cap U^-/H^1_{\mathfrak{M}} \cap U^-, \quad y \in J^1_{\mathfrak{M}} \cap U/H^1_{\mathfrak{M}} \cap U,$$

identifies each of those $\mathbb{F}_p$-vector spaces to the dual of the other [35, Lemma 5.6], in such a way that, for $m \in J_{\mathfrak{M}} \cap M$, the transpose of the map $x \mapsto m^{-1}xm$, for $x \in J^1_{\mathfrak{M}} \cap U^-/H^1_{\mathfrak{M}} \cap U^-$, is $y \mapsto mym^{-1}$, for $y \in J^1_{\mathfrak{M}} \cap U/H^1_{\mathfrak{M}} \cap U$. The result follows. □

3.12. We return to the notation of paragraphs 3.4–3.8 and now put together the Hecke algebra homomorphisms (3.1) with 3.10 Proposition. Let $t = 0$ or 1. We recall from [35, (7.3)] (rephrased in the present framework in [4, Proposition 3.6]) that if $\kappa = \text{c-Ind}_{J_P}^{J_\Lambda} \kappa_P$ is a beta-extension of $\eta_\Lambda = \text{c-Ind}_{J^1_P}^{J^1_\Lambda} \eta_P$ **relative to** $\mathfrak{M}_t$, then there is an injective morphism of algebras

$$\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \tilde{\tau}_W \otimes \tau_V) \hookrightarrow \mathscr{H}(\mathrm{Sp}_F(X), \kappa_P \otimes (\tilde{\tau}_W \otimes \tau_V))$$

that preserves support. We want to express this with the fixed representation $\lambda_P = \kappa_{\Lambda,P}$ on the right, where $\kappa_{\Lambda,P_{|J\cap M}} = \underline{\tilde{\kappa}}_W \otimes \underline{\kappa}_V$, as in paragraph 3.6. We thus plug in 3.10 Proposition above and get:

Theorem. *Let* $t = 0$ *or* 1. *There is an injective morphism of algebras*

$$j_t : \mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_V)) \hookrightarrow \mathscr{H}(\mathrm{Sp}_F(X), \lambda_P)$$

that preserves support, i.e. $\mathrm{Supp}(j_t(\phi)) = J_P \,\mathrm{Supp}(\phi) J_P$, *for all* $\phi \in \mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_V))$.

3.13. We now focus on the finite-dimensional algebra $\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_V))$, a Hecke algebra on the finite reductive group $\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E})$ relative to a cuspidal representation of the parabolic subgroup $P(\Lambda_{\mathfrak{o}_E})/P^1(\mathfrak{M}_{t,\mathfrak{o}_E})$.

Let $X = \perp_{j=0}^{l} X^j$ be the splitting associated to the skew semisimple stratum $[\Lambda, -, 0, \beta]$. Since the stratum $[\Lambda_W, -, 0, \beta_W]$ is simple, there is a unique index i such that $W \subseteq X^i$, and then $W^* \subseteq X^i$ also. **This index** i **will be fixed until the end of the section.**

Writing $V^j = V \cap X^j$, the skew semisimple stratum $[\Lambda_V, -, 0, \beta_V]$ then has splitting consisting of the non-zero spaces in $V = \perp_{j=0}^{l} V^j$; the only spaces which may be zero here are V^0 (since we have the convention that $\beta^0 = 0$) and V^i (which is zero if and only if $X^i = W \oplus W^*$).

The ambient finite group $\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E})$ is a product over j, for $0 \leqslant j \leqslant l$, of analogous groups relative to X^j, but in all of them except X^i the parabolic subgroup considered is the full

group:

$$P(\mathfrak{M}_{t,\mathfrak{o}_E})/P^1(\mathfrak{M}_{t,\mathfrak{o}_E}) \simeq P(\mathfrak{M}^i_{t,\mathfrak{o}^i_E})/P^1(\mathfrak{M}^i_{t,\mathfrak{o}^i_E}) \times \prod_{j\neq i} P(\Lambda^j_{\mathfrak{o}^j_E})/P^1(\Lambda^j_{\mathfrak{o}^j_E})$$

$$P(\Lambda_{\mathfrak{o}_E})/P^1(\mathfrak{M}_{t,\mathfrak{o}_E}) \simeq P(\Lambda^i_{\mathfrak{o}^i_E})/P^1(\mathfrak{M}^i_{t,\mathfrak{o}^i_E}) \times \prod_{j\neq i} P(\Lambda^j_{\mathfrak{o}^j_E})/P^1(\Lambda^j_{\mathfrak{o}^j_E}).$$

The representation $\tilde{\tau}_W \otimes \tau_V$ decomposes accordingly using $\tau_V = \otimes_{j=0}^l \tau_j$ and we finally get an isomorphism of algebras:

$$\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_V)) \simeq \mathscr{H}(\mathcal{G}(\mathfrak{M}^i_{t,\mathfrak{o}^i_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_i)) \tag{3.4}$$

where $\tilde{\tau}_W \otimes \tau_i$ is a cuspidal representation of $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}^i_E}) \times \mathcal{G}(\Lambda^i_{V,\mathfrak{o}^i_E})$, identified with a maximal Levi subgroup of each finite reductive group $\mathcal{G}(\mathfrak{M}^i_{t,\mathfrak{o}^i_E})$, for $t = 0, 1$.

It follows from Lusztig's work [26], as recalled in section 5, that this algebra is two-dimensional, because $\tilde{\tau}_W$ and $\epsilon_{\mathfrak{M}_t}$ are self-dual. It has basis given by the identity element and an element $\mathcal{T}_t$ supported on the double coset of a certain Weyl group element, called s_i, if $t = 0$, or s_i^{ϖ} if $t = 1$, in [35, §7.2.2]; this only defines $\mathcal{T}_t$ up to a non-zero scalar, which will not matter to us at first. Lusztig gives an algorithm permitting the actual computation of the quadratic relation satisfied by $\mathcal{T}_t$. This relation always has the following shape, for some non-zero complex number ω_t:

$$(\mathcal{T}_t - \ q^{r_t}\ \omega_t)(\mathcal{T}_t + \ \omega_t) = 0, \qquad \text{where } r_t = r_t(\epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_i)) \geqslant 0. \tag{3.5}$$

We emphasise the dependency in the inducing cuspidal representation $\epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau_i)$.

3.14. Finally, we can restate [4, Proposition 3.12], describing the real parts of the reducibility points we wish to compute, in our notation. Recall that, for ρ a cuspidal representation of $\mathrm{GL}_F(W)$ as above, we write $t(\rho)$ for the number of unramified characters χ of $\mathrm{GL}_F(W)$ such that $\rho \otimes \chi \simeq \rho$. Recall also that, if ρ is self-dual, then there are precisely two representations ρ, ρ' in the inertial class of ρ which are self-dual.

Let π be a cuspidal representation of G. Recall that there is a real number $s_\pi(\rho) \geqslant 0$ such that, for *real* s, the normalised induced representation $\nu^s\rho \times \pi$ of $\mathrm{Sp}_F(X)$ is reducible if and only if $s = \pm s_\pi(\rho)$, and similarly we have $s_\pi(\rho')$. Then, for *complex* s, if $\nu^s\rho \times \pi$ is reducible then the real part of s must be $\pm s_\pi(\rho)$ or $\pm s_\pi(\rho')$; we say that these are the real parts of the reducibility points of $\nu^s\rho \times \pi$.

Proposition ([4, Proposition 3.12])**.** *Let π be a a cuspidal representation of G, let ρ be an irreducible self-dual cuspidal representation of $\mathrm{GL}_F(W)$, and take all the notation of the previous paragraphs. Then the real parts of the reducibility points of the normalized induced representation $\nu^s\rho \times \pi$ are the elements of the set*

$$\left\{\pm\frac{r_0 + r_1}{2t(\rho)}, \pm\frac{r_0 - r_1}{2t(\rho)}\right\}, \quad \textit{where } r_0 = r_0(\epsilon_{\mathfrak{M}_0}(\tilde{\tau}_W \otimes \tau_i)),\ r_1 = r_1(\epsilon_{\mathfrak{M}_1}(\tilde{\tau}_W \otimes \tau_i)). \tag{3.6}$$

Note that, by [10, Lemma 6.2.5], the unramified twist number $t(\rho)$ can also be computed from the formula $t(\rho) = \frac{\dim_F W}{e(F[\beta_W]/F)}$.

3.15. We can also apply the discussion of the previous paragraphs in the space $X^i = W \oplus V^i \oplus W^*$. From the splitting of our strata, we have the lattice sequence $\Lambda^i_V = \Lambda \cap V^i$ and the simple stratum $[\Lambda^i_V, -, 0, \beta^i_V]$ in V^i. We write $J_i = J(\beta^i_V, \Lambda^i_V)$, and similarly for J^1_i and H^1_i, and let $\underline{\kappa}_i$ be the p-primary beta-extension of the simple character $\theta_{|H^1_i}$. Then τ_i is a representation of the reductive quotient J_i/J^1_i and, putting $G_i = \mathrm{Sp}_F(V^i)$ we can define the cuspidal representation $\pi_i = \text{c-Ind}^{G_i}_{J_{\Lambda^i}} \underline{\kappa}_i \otimes \tau_i$ of G_i. (Note that, if $V^i = \{0\}$, then π_i is the trivial representation of the trivial group.)

Applying the discussion above to the representation π_i and the space X^i, we find that the real parts of the reducibility points of the normalized induced representation $\nu^s\rho \times \pi_i$ of $\mathrm{Sp}_F(X^i)$ are the elements of the set

$$\left\{\pm\frac{r'_0 + r'_1}{2t(\rho)}, \pm\frac{r'_0 - r'_1}{2t(\rho)}\right\}, \quad \text{where } r'_0 = r_0(\epsilon_{\mathfrak{M}^i_0}(\tilde{\tau}_W \otimes \tau_i)),\ r'_1 = r_1(\epsilon_{\mathfrak{M}^i_1}(\tilde{\tau}_W \otimes \tau_i)). \tag{3.7}$$

The comparison between (3.6) and (3.7) will be crucial.

3.16. We end this section with the simplest example of the computation of the parameters r_0, r_1 in (3.6), for positive depth representations. Continuing in the notation above, we assume that $i > 0$ and that β^i is *maximal* in the following sense: we have $[F[\beta^i] : F] = \dim_F V^i$, so that (the image of) $F[\beta^i]$ is a maximal extension of F in $\mathrm{End}_F(V^i)$. In particular, this implies that $V^i \neq \{0\}$. We assume moreover that $\dim_F W = \dim_F V^i$, the smallest example of the situation above. (It will turn out that this is in fact the only situation of interest, in this context.)

Let E^i_0 be the fixed field of the adjoint involution acting on $E^i = F[\beta^i]$. The centralizer of β^i in $\mathrm{Sp}_F(X^i)$ is thus isomorphic to the unitary group $\mathrm{U}(2,1)(E^i/E^i_0)$. In the latter group, there are two conjugacy classes of maximal compact subgroups, the reductive quotients of which are, for some a and b with $\{a, b\} = \{0, 1\}$ depending on the initial lattice sequence Λ^i:

- $\mathcal{G}(\mathfrak{M}_{a,\mathfrak{o}_E}) \simeq U(2,1)(k_{E^i}/k_{E^i_0})$ and $\mathcal{G}(\mathfrak{M}_{b,\mathfrak{o}_E}) \simeq U(1,1) \times U(1)(k_{E^i}/k_{E^i_0})$, if E^i/E^i_0 is unramified;
- $\mathcal{G}(\mathfrak{M}_{a,\mathfrak{o}_E}) \simeq \mathrm{SL}(2, k_{E^i}) \times \{\pm 1\}$ and $\mathcal{G}(\mathfrak{M}_{b,\mathfrak{o}_E}) \simeq \mathrm{O}(2,1)(k_{E^i})$, if E^i/E^i_0 is ramified.

We set $f = f(E^i_0/F)$. From the calculations in section 5 the possible values for r_a, r_b and the sets of real parts of reducibility points in (3.6) are:

- if E^i/E^i_0 is unramified: $r_a = 3f$ or f, and $r_b = f$; real parts $\{\pm 1, \pm\frac{1}{2}\}$ or $\{\pm\frac{1}{2}, 0\}$;
- if E^i/E^i_0 is ramified: $r_a = f$ or 0, and $r_b = f$; real parts $\{\pm 1, 0\}$ or $\{\pm\frac{1}{2}, \pm\frac{1}{2}\}$.

In both cases the value of $r_b = r_b(\epsilon_{\mathfrak{M}_b}\tilde{\tau}_W \otimes \tau_i)$ is independent of the representation. We choose $\tilde{\tau}_W$ such that $r_a(\epsilon_{\mathfrak{M}_a}(\tilde{\tau}_W \otimes \tau_i)) = 3f$ if E^i/E^i_0 is unramified, or $r_a(\epsilon_{\mathfrak{M}_a}(\tilde{\tau}_W \otimes \tau_i)) = f$

if E^i/E^i_0 is ramified. This choice, which we denote by $\tilde{\underline{\tau}}_W$, is unique and provides us with a reducibility with real part 1.

We conclude that there exists one and only one self-dual cuspidal representation ρ of $\mathrm{GL}_F(W)$ containing the simple character $\tilde{\vartheta}_W$ such that the parabolically induced representation $\nu^1\rho\otimes\pi$ is reducible. The representation ρ contains the type $(\tilde{J}_W, \tilde{\underline{\kappa}}_W \otimes \tilde{\underline{\tau}}_W)$. However, as discussed previously, this does not give us a full description of the self-dual representation ρ: we know its inertial class but this still leaves two possibilities. This situation is explored more fully in Section 6.

3.17. Applying the previous paragraph again to the representation π_i of $G_i = \mathrm{Sp}_F(V^i)$ and comparing (3.6) and (3.7), we remark that the relevant choice of $\tilde{\rho}_W$ for the situation in X, with the cuspidal representation π of $G = \mathrm{Sp}_F(V)$, differs from the analogous choice relative to the situation in X^i, with the cuspidal representation π_i of G_i, by a simple twist by the character $\epsilon_{\mathfrak{M}_a}\epsilon_{\mathfrak{M}^i_a}$. Indeed, in our example, the value of r_b is independent of the representation. In the next section we will study the general case, when r_a and r_b may both depend on the representation.

4. Reduction to the simple case

In this section, we make the reduction to the simple case, proving 2.6 Theorem. As intimated at the end of the last chapter, the key point to prove is that the character $\epsilon_{\mathfrak{M}_t}\epsilon_{\mathfrak{M}^i_t}$ is independent of t (see 4.3 Proposition). Note that the character $\epsilon_{\mathfrak{M}_t}\epsilon_{\mathfrak{M}^i_t}$ is the character χ_i appearing in the statement of 2.6 Theorem. While we have a description of it as a permutation character and, through careful analysis of this permutation, give a recipe by which one could compute it, we do not here compute it precisely; we only check that it is independent of t.

There is one further subtlety which should be remarked upon. In Section 3, we began with a pair of cuspidal representations (ρ, π) and built from them a cover of a type, without starting from types for ρ and π. In this section, we begin just with a cuspidal representation π of G and a cuspidal type λ for it, and use this to define certain cuspidal representations ρ of general linear groups, and maximal simple types $\tilde{\lambda}$ for them. The cover obtained in Section 3 is then indeed a cover of $\tilde{\lambda}\otimes\lambda$ but this is only clear because the (semi)simple characters in λ and $\tilde{\lambda}$ are suitably related. Thus we take great care to set up the notation in this section.

4.1. We first review the notation that we need. This is the notation as in paragraph 2.6 so that it differs slightly from the notation of the previous chapter. In particular, objects in the symplectic space V do not have the subscript V; instead, the corresponding objects in X (which we have yet to define) will have the subscript X.

Throughout this and the following paragraphs, we fix a cuspidal representation π of $G = \mathrm{Sp}_F(V)$. We have the following data.

In the symplectic space V.

- A maximal skew semisimple stratum $[\Lambda, -, 0, \beta]$ in $\mathrm{End}_F(V)$ and a skew semisimple character θ of $H^1 = H^1(\beta, \Lambda)$ such that θ occurs in π.
- The irreducible representation η of $J^1 = J^1(\beta, \Lambda)$ containing θ and the p-primary beta-extension $\underline{\kappa}$ of η to $J = J(\beta, \Lambda)$.
- A cuspidal representation τ of $\mathcal{G}(\Lambda_{\mathfrak{o}_E}) = P(\Lambda_{\mathfrak{o}_E})/P^1(\Lambda_{\mathfrak{o}_E})$ such that π is induced from $\lambda = \underline{\kappa} \otimes \tau$.

The stratum $[\Lambda, -, 0, \beta]$ can be written (uniquely) as an orthogonal direct sum of skew simple strata $[\Lambda^j, -, 0, \beta^j]$ in $\mathrm{End}_F(V^j)$, for $j = 0, \ldots, l$, with the convention that $\beta^0 = 0$. The data above then give us the following data in the spaces V^j.

- Skew simple characters θ_j of $H^1_j = H^1(\beta^j, \Lambda^j)$, which are the restriction of θ.
- The irreducible representation η_j of $J^1_j = J^1(\beta^j, \Lambda^j)$ containing θ_j and the p-primary beta-extension $\underline{\kappa}_j$ of η_j to $J_j = J(\beta^j, \Lambda^j)$.
- The cuspidal representations τ_j of $\mathcal{G}(\Lambda^j_{\mathfrak{o}^j_E})$ such that, via the isomorphism $\mathcal{G}(\Lambda_{\mathfrak{o}_E}) \simeq \prod_{j=0}^l \mathcal{G}(\Lambda^j_{\mathfrak{o}^j_E})$, we have $\tau = \otimes_{j=0}^l \tau_j$.
- The representation $\lambda_j = \underline{\kappa}_j \otimes \tau_j$ of J_j.

Note that, writing $G_j = \mathrm{Sp}_F(V^j)$, the representation $\pi_j = \text{c-Ind}_{J_j}^{G_j} \lambda_j$ is a cuspidal representation. A priori, it is not determined uniquely by the representation π, but it is determined by our choice of data $([\Lambda, -, 0, \beta], \theta)$ such that π contains θ.

We now fix $i \in \{0, \cdots, l\}$ and choose an F-vector space W whose dimension is divisible by the degree $[E^i : F]$. We then have the following data.

In the vector space W.

- A maximal simple stratum $[\Lambda_W, -, 0, \beta_W]$ in $\mathrm{End}_F(W)$, together with a field isomorphism $E^i = F[\beta^i] \to F[\beta_W] = E_W$ fixing F and taking β^i to β_W.
- The simple character $\tilde{\vartheta}_W$ of $\tilde{H}^1_W = \tilde{H}^1(\beta_W, \Lambda_W)$ which is the transfer of the square $(\tilde{\theta}_i)^2$ of the unique self-dual simple character of $\tilde{H}^1(\beta^i, \Lambda^i)$ restricting to θ_i.
- The p-primary beta-extension $\underline{\tilde{\kappa}}_W$ of $\tilde{\vartheta}_W$ to $\tilde{J}_W = \tilde{J}(\beta_W, \Lambda_W)$ and an irreducible self-dual cuspidal representation $\tilde{\tau}_W$ of $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_W})$, inflated to $\tilde{J}_W$, where we have written $\mathfrak{o}_W$ for the ring of integers of E_W.
- A self-dual cuspidal representation ρ of $\mathrm{GL}_F(W)$ containing $\tilde{\lambda}_W = \underline{\tilde{\kappa}}_W \otimes \tilde{\tau}_W$.

These data also induce data in the dual space W^* as follows. By duplicating if necessary, we assume that Λ_W has period divisible by 4 and that $\Lambda_W(-1) \neq \Lambda_W(0)$. (The reason for doing this is to ensure that the self-dual lattice sequence we will obtain conforms to our standard normalization – see 1.1 Remark.) Writing $\langle \cdot, \cdot \rangle$ for the pairing $W \times W^* \to F$, we

define Λ_W^* by

$$\Lambda_W^*(r) = \{w^* \in W^* \mid \langle \Lambda_W(1-r), w^* \rangle \subseteq \mathfrak{p}_F\}, \qquad \text{for } r \in \mathbb{Z};$$

then the lattice sequence $\Lambda_W \oplus \Lambda_W^*$ is self-dual with respect to the natural symplectic structure on $W \oplus W^*$. We also define β_W^* in $\mathrm{End}_F(W^*)$ by

$$\langle w, \beta_W^*(w^*) \rangle = -\langle \beta_W(w), w^* \rangle, \qquad \text{for all } w \in W,\ w^* \in W^*.$$

Note that, by the fact that $[\Lambda^i, -, 0, \beta^i]$ is skew, there is a unique isomorphism $E^i \to F[\beta_W^*]$ which takes β^i to β_W^*.

We now use these data to define corresponding data in the larger spaces on which we will have covers (as in Section 3). We define the symplectic space $X^i = (W \oplus W^*) \perp V^i$, for which we have the following.

In the symplectic space X^i.

- The maximal Levi subgroup $M_i \simeq \mathrm{GL}_F(W) \times \mathrm{Sp}_F(V^i)$ of $\mathrm{Sp}_F(X^i)$ which stabilizes the decomposition $X^i = (W \oplus W^*) \perp V^i$, and the maximal parabolic subgroup $P_i = M_i U_i$ which stabilizes the subspace W (so stabilizes the flag $W \subseteq W \perp V^i \subset X^i$).
- The skew simple stratum $[\Lambda_X^i, -, 0, \beta_X^i]$ in $\mathrm{End}_F(X^i)$, where $\Lambda_X^i = (\Lambda_W \oplus \Lambda_W^*) \perp \Lambda$ and β_X^i is the unique skew simple element which stabilizes the decomposition $X^i = (W \oplus W^*) \perp V^i$ and acts as β^i on V^i and as β_W on W; it then acts as β_W^* on W^*. We identify E^i with $F[\beta_X^i]$ via the isomorphism which takes β^i to β_X^i.
- Two further skew simple strata in $\mathrm{End}_F(X^i)$,
$$[\mathfrak{M}_0^i, -, 0, \beta_X^i],\ [\mathfrak{M}_1^i, -, 0, \beta_X^i],$$
such that $\mathfrak{b}_0(\mathfrak{M}_t^i)$, for $t = 0, 1$, are the two maximal self-dual $\mathfrak{o}_E^i$-orders in the commuting algebra of β_X^i which contain $\mathfrak{b}_0(\Lambda_X^i)$.
- The unique skew simple character θ_X^i of $H_{X^i}^1 = H^1(\beta_X^i, \Lambda_X^i)$ that restricts to θ_i on H_i^1 and to $\tilde{\vartheta}_W$ on $\tilde{H}_W^1$; this is the transfer to Λ_X^i of the skew simple character θ_i.
- For $t = 0, 1$, the skew simple character $\theta_{\mathfrak{M}_t^i}$ of $H_{\mathfrak{M}_t^i}^1$ that is transferred from θ_X^i; the corresponding irreducible representation $\eta_{\mathfrak{M}_t^i}$ of $J_{\mathfrak{M}_t^i}^1$; and the p-primary beta-extension $\underline{\kappa}_{\mathfrak{M}_t^i}$ of $\eta_{\mathfrak{M}_t^i}$ to $J_{\mathfrak{M}_t^i}$.
- An $\mathrm{Sp}_F(X^i)$-cover (J_P^i, λ_P^i) of the pair $(\tilde{J}_W \times J_i, \tilde{\lambda}_W \otimes \lambda_i)$ in M_i.

Finally, we define the symplectic space $X = (W \oplus W^*) \perp V = X^i \perp V^{\vee i}$, where $V^{\vee i} = \underset{j \neq i}{\perp} V^j$, for which we have the following.

In the symplectic space X.

- The maximal Levi subgroup $M \simeq \mathrm{GL}_F(W) \times \mathrm{Sp}_F(V)$ of $\mathrm{Sp}_F(X)$ which stabilizes the decomposition $X = (W \oplus W^*) \perp V$, and the maximal parabolic subgroup $P = MU$ which stabilizes the subspace W.

- The skew semisimple stratum $[\Lambda_X, -, 0, \beta_X]$, where $\Lambda_X = \Lambda_X^i \perp \Lambda^{\vee i}$, with $\Lambda^{\vee i} = \perp_{j \neq i} \Lambda^j$, and β the unique skew semisimple element which stabilises the decomposition $X = (W \oplus W^*) \perp V$ and acts as β on V and β_W on W (or, equivalently, acts as β_X^i on X^i and as $\beta^{\vee i} = \bigoplus_{j \neq i} \beta^j$ on $V^{\vee i}$, from which it is clear that the resulting stratum is indeed semisimple). We identify E with $F[\beta_X]$ via the isomorphism which takes β^i to β_X^i and β^j to itself, for $j \neq i$.
- Two further skew semisimple strata
$$[\mathfrak{M}_0, -, 0, \beta_X],\ [\mathfrak{M}_1, -, 0, \beta_X],$$
where $\mathfrak{M}_t = \mathfrak{M}_t^i \perp \Lambda^{\vee i}$, for $t = 0, 1$; then $\mathfrak{b}_0(\mathfrak{M}_t)$ are the two maximal self-dual $\mathfrak{o}_E$-orders in the commuting algebra of β_X which contain $\mathfrak{b}_0(\Lambda_X)$.
- The unique skew semisimple character θ_X of $H_X^1 = H^1(\beta_X, \Lambda_X)$ which restricts to θ on H^1 and to $\tilde{\vartheta}_W$ on $\tilde{H}_W^1$; it is the transfer to H_X^1 of the skew semisimple character θ, and restricts to θ_X^i on $H_{X^i}^1$.
- For $t = 0, 1$, the skew semisimple character $\theta_{\mathfrak{M}_t}$ of $H_{\mathfrak{M}_t}^1$ that is transferred from θ_X; the corresponding irreducible representation $\eta_{\mathfrak{M}_t}$ of $J_{\mathfrak{M}_t}^1$; and the p-primary beta-extension $\underline{\kappa}_{\mathfrak{M}_t}$ of $\eta_{\mathfrak{M}_t}$ to $J_{\mathfrak{M}_t}$.
- An $\mathrm{Sp}_F(X)$-cover (J_P, λ_P) of the pair $(\tilde{J}_W \times J, \tilde{\lambda}_W \otimes \lambda)$ in M.

4.2. We use the setup in the previous paragraph and come back to the comparison of real parts of reducibility points, as in paragraph 3.17. The comparison of beta-extensions yields, as in 3.10 Proposition, characters $\epsilon_{\mathfrak{M}_t^i}$ and $\epsilon_{\mathfrak{M}_t}$ for $t = 0, 1$.

We fix $t = 0, 1$ and temporarily drop the subscript t. By definition $\epsilon_{\mathfrak{M}}(m)$, for $m \in J_{\mathfrak{M}} \cap M$, is the signature of the permutation $\mathrm{Ad}\, m : u \mapsto m^{-1}um$ of the quotient $J_{\mathfrak{M}}^1 \cap U / H_{\mathfrak{M}}^1 \cap U$, isomorphic to the $\mathbb{F}_p$-vector space $\mathfrak{J}_{\mathfrak{M}}^1 \cap \mathbb{U} / \mathfrak{H}_{\mathfrak{M}}^1 \cap \mathbb{U}$, where $\mathbb{U}$ is the Lie algebra of U (see 3.10 Proposition and 3.11 Proposition). The same holds with $\epsilon_{\mathfrak{M}^i}(m)$, for $m \in J_{\mathfrak{M}^i} \cap M^i$: it is the signature of the same permutation on $\mathfrak{J}_{\mathfrak{M}^i}^1 \cap \mathbb{U}_i / \mathfrak{H}_{\mathfrak{M}^i}^1 \cap \mathbb{U}_i$. On the other hand $\mathbb{U}$ is isomorphic to $\mathbb{U}_i \oplus \mathrm{Hom}_F(V^{\vee i}, W)$ in an M_i-equivariant way, and the action of $(m, y) \in \mathrm{GL}_F(W) \times \mathrm{Sp}_F(V^i)$ on $\phi \in \mathrm{Hom}_F(V^{\vee i}, W)$ is given by $\phi \mapsto m\phi$. The associated decompositions of the lattices $\mathfrak{J}^1$ and $\mathfrak{H}^1$ (as in [10, Proposition 7.1.12]) lead to:

Lemma. *Let $(m, y) \in \tilde{P}(\Lambda_{W, \mathfrak{o}_W}) \times P(\Lambda^i_{\mathfrak{o}_E^i})$. Then $(\epsilon_{\mathfrak{M}^i} \epsilon_{\mathfrak{M}})((m, y))$ is the signature of the permutation $\phi \mapsto m\phi$ of*
$$\mathfrak{X} := \mathfrak{J}_{\mathfrak{M}}^1 \cap \mathrm{Hom}_F(V^{\vee i}, W) / \mathfrak{H}_{\mathfrak{M}}^1 \cap \mathrm{Hom}_F(V^{\vee i}, W).$$

Now the quotient group $\tilde{P}(\Lambda_{W, \mathfrak{o}_W}) / \tilde{P}^1(\Lambda_{W, \mathfrak{o}_W})$ is a general linear group $\mathrm{GL}_{m_W}(k_W)$ over the finite extension $k_W = k_{E^i}$ of k_F; this extension depends only on the endo-class of the simple character $\tilde{\vartheta}_W$. The lemma actually asserts that the character $\epsilon_{\mathfrak{M}} \epsilon_{\mathfrak{M}^i}$ is trivial on $P(\Lambda^i_{\mathfrak{o}_E^i})$ and factors through the signature of the natural left action of $\mathrm{GL}_{m_W}(k_W)$ on $\mathfrak{X}$.

4.3. Retrieving the subscripts t, our main tool is the following comparison of characters:

Proposition. *With notation as above, we have*

$$\epsilon_{\mathfrak{M}_0^i}\epsilon_{\mathfrak{M}_0} = \epsilon_{\mathfrak{M}_1^i}\epsilon_{\mathfrak{M}_1}.$$

This character, as a character of $\mathrm{GL}_{m_W}(k_W)$*, can be written as* $\chi_i \circ \det_{k_W}$*, where* χ_i *is a quadratic character of* $k_W^\times$ *which is independent of the choice of the space* W*.*

The independence on the space W (for a fixed choice of i) is particularly important. We postpone the proof of the Proposition for now and, taking it for granted, deduce 2.6 Theorem.

4.4. Recall that we have written $\pi = \text{c-Ind}_J^G \lambda$ and, for $j = 0, \dots, l$, we have the cuspidal representation $\pi_j = \text{c-Ind}_{J_j}^{G_j} \lambda_j$ of $G_j = \mathrm{Sp}_F(V^j)$. We have θ_j, the simple character of H_j^1 contained in λ_j, and we write $\tilde{\theta}_j$ for the self-dual simple character of $\tilde{H}_j^1$ which restricts to θ_j. Let $\mathbf{\Theta}_j$ be the endo-class of the simple character $(\tilde{\theta}_j)^2$, which is a simple character for the stratum $[\Lambda^j, -, 0, 2\beta^j]$, and $k_{\mathbf{\Theta}_j}$ for the corresponding extension of k_F.

Recall that, for an endo-class $\mathbf{\Theta}$ and a character χ of the multiplicative group of the corresponding finite field $k_{\mathbf{\Theta}}$, we have:

- $\mathrm{Jord}(\pi)$, the Jordan set of π (see paragraph 2.1);
- $\mathrm{IJord}(\pi, \mathbf{\Theta})$, the inertial Jordan set of π relative to $\mathbf{\Theta}$, which is the multiset of pairs $([\rho], m)$, for $(\rho, m) \in \mathrm{Jord}(\pi)$ such that ρ has endo-class $\mathbf{\Theta}$;
- $\mathrm{IJord}(\pi, \mathbf{\Theta})_\chi$, the χ-twisted inertial Jordan set of π relative to $\mathbf{\Theta}$, which is the multiset of pairs $([\rho]_\chi, m)$ with $(\rho, m) \in \mathrm{Jord}(\pi, \mathbf{\Theta})$.

Recall here that, if ρ contains a maximal simple type $(\tilde{J}, \tilde{\lambda})$, then $[\rho]_\chi$ denotes the inertial class of cuspidal representations containing $(\tilde{J}, \tilde{\lambda} \otimes \chi \circ \det)$ (see paragraph 2.4). Also, when χ is the trivial character we just write $\mathrm{IJord}(\pi, \mathbf{\Theta})$.

We restate 2.6 Theorem in a refined form:

Theorem. *Fix* i *with* $0 \leqslant i \leqslant l$*, and let* χ_i *be the character of* $k_{\mathbf{\Theta}_i}^\times$ *such that* $\chi_i \circ \det_{k_{\mathbf{\Theta}_i}}$ *is the twisting character in 4.3 Proposition. We have an equality of multisets*

$$\mathrm{IJord}(\pi, \mathbf{\Theta}_i) = \mathrm{IJord}(\pi_i, \mathbf{\Theta}_i)_{\chi_i}.$$

Proof. This is now just a matter of putting together the previous results. Let ρ be a cuspidal representation with endo-class $\mathbf{\Theta}_i$ and use the notation of paragraph 4.1 so that ρ is a representation of $\mathrm{GL}_F(W)$ containing the maximal simple type $\tilde{\lambda}_W = \underline{\tilde{\kappa}}_W \otimes \tilde{\tau}_W$. The values of m, if any, for which $([\rho], m) \in \mathrm{IJord}(\pi_i, \mathbf{\Theta}_i)$ can then be computed from (3.7): more precisely, they are

$$\left| \frac{r_0(\epsilon_{\mathfrak{M}_0^i}(\tilde{\tau}_W \otimes \tau_i)) \pm r_1(\epsilon_{\mathfrak{M}_1^i}(\tilde{\tau}_W \otimes \tau_i))}{t(\rho)} \right| - 1, \tag{4.1}$$

whenever these integers are strictly positive, together with positive integers less than this and of the same parity.

Now we consider the inertial class $[\rho]_{\chi_i}$. The cuspidal representations in this class contain the maximal simple type $\tilde{\lambda}_W \otimes \chi_i \circ \det = \underline{\tilde{\kappa}}_W \otimes (\tilde{\tau}_W \otimes \chi_i \circ \det)$. Then, using (3.6), we see that the values of m for which $([\rho]_{\chi_i}, m) \in \mathrm{IJord}(\pi, \mathbf{\Theta}_i)$ are

$$\left| \frac{r_0(\epsilon_{\mathfrak{M}_0}((\tilde{\tau}_W \otimes \chi_i \circ \det) \otimes \tau_i)) \pm r_1(\epsilon_{\mathfrak{M}_1}((\tilde{\tau}_W \otimes \chi_i \circ \det) \otimes \tau_i))}{t(\rho)} \right| - 1,$$

whenever these integers are strictly positive, together with positive integers less than this and of the same parity. But 4.3 Proposition says that these are precisely the same integers as those in (4.1) (recall that all the characters here are quadratic), and the result follows. □

Remark. As we have seen in the proof, the pairs $([\rho], m)$ which appear in $\mathrm{IJord}(\pi)$ are determined by the values of $r_t = r_t(\epsilon_{\mathfrak{M}^i}(\tilde{\tau}_W \otimes \tau_i))$. Denote by ρ' the other self-dual cuspidal in the inertial class $[\rho]$. If ρ and ρ' are of opposite parity, say ρ is of symplectic type and ρ' is of orthogonal type, then we also recover this part of the full Jordan set $\mathrm{Jord}(\pi)$: if m is even then it is (ρ, m) which appears in $\mathrm{Jord}(\pi)$, while if m is odd then it is (ρ', m).

Suppose now that ρ, ρ' are of the same parity and $([\rho], m)$ appears in $\mathrm{IJord}(\pi)$. Then ρ and ρ' both appear *with the same multiplicities* in $\mathrm{Jord}(\pi)$ if and only if $r_0 r_1 = 0$. Thus in this case we also recover this part of the full Jordan set. Both ρ and ρ' appear with *some* multiplicity in $\mathrm{Jord}(\pi)$ if and only if $|r_0 - r_1| > t(\rho)$; when ρ, ρ' are both of orthogonal type, this condition simplifies to $r_0 \neq r_1$, since the reducibility points must be integers in this case.

The situations in which ρ, ρ' have the same parity are examined more closely from the Galois point of view in Section 6.

It remains now to prove 4.3 Proposition, which will take up the remainder of this section.

4.5. In this and the next few paragraphs, we define and study an auxiliary lattice sequence which will be needed for the calculations. Let Λ_W and Λ_Y be $\mathfrak{o}_F$-lattice sequences in finite dimensional F-vector spaces W and Y respectively, with the same $\mathfrak{o}_F$-period e. We define an $\mathfrak{o}_F$-lattice sequence $\mathcal{C} = \mathcal{C}(\Lambda_Y, \Lambda_W)$ in the vector space $C = \mathrm{Hom}_F(Y, W)$ by

$$\mathcal{C}(t) = \{g \in C \mid g\Lambda_Y(i) \subseteq \Lambda_W(i+t) \text{ for all } i \in \mathbb{Z}\}, \qquad \text{for } t \in \mathbb{Z}.$$

We call the *jumps* of Λ_Y those integers i such that $\Lambda_Y(i) \neq \Lambda_Y(i+1)$ (and similarly for any lattice sequence). The set of jumps of Λ_Y is also the image of $Y \backslash \{0\}$ by the valuation map attached to Λ_Y, given by $\mathrm{val}_Y(y) = \max\{k \in \mathbb{Z} \mid y \in \Lambda_Y(k)\}$, for $y \in Y \backslash \{0\}$.

We make the following assumptions:

(i) The set of jumps of Λ_W is equal to $a_W + s_W\mathbb{Z}$ and the set of jumps of Λ_Y is equal to $a_Y + s_Y\mathbb{Z}$.

(ii) The orders $\mathfrak{a}(\Lambda_W)$ and $\mathfrak{a}(\Lambda_Y)$ are principal orders, in other words non-zero quotients $\Lambda_W(i)/\Lambda_W(i+1)$ are all isomorphic, and the same for Λ_Y. In particular there is an element $\Pi_W \in \mathfrak{a}(\Lambda_W)$ (resp. $\Pi_Y \in \mathfrak{a}(\Lambda_Y)$) such that $\Pi_W(\Lambda_W(i)) = \Lambda_W(i+1)$ (resp. $\Pi_Y(\Lambda_Y(i)) = \Lambda_Y(i+1)$) whenever i is a jump of Λ_W (resp. of Λ_Y) [10, §5.5].

Lemma. *The set of jumps of $\mathcal{C}$ is equal to $(a_W - a_Y) + \gcd(s_W, s_Y)\mathbb{Z}$. Moreover, the quotient spaces $\mathcal{C}(i)/\mathcal{C}(i+1)$ that are non-zero are all isomorphic as k_F-vector spaces, and their common dimension is*

$$c = c(\Lambda_Y, \Lambda_W) = \frac{\gcd(s_W, s_Y)}{e} \dim_F W \dim_F Y.$$

Proof. Proving that the set of jumps is contained in the given $\mathbb{Z}$-coset is straightforward using only assumption (i). Now we use assumption (ii) and remark that Π_W and Π_Y satisfy $\Pi_W(\Lambda_W(i)) = \Lambda_W(i+s_W)$ and $\Pi_Y(\Lambda_Y(i)) = \Lambda_Y(i+s_Y)$, for any integer i. For any $\phi \in C$ we check that:

$$\mathrm{val}_{\mathcal{C}}(\Pi_W\phi) = \mathrm{val}_{\mathcal{C}}(\phi) + s_W, \quad \mathrm{val}_{\mathcal{C}}(\phi\Pi_Y) = \mathrm{val}_{\mathcal{C}}(\phi) + s_Y.$$

The left (resp. right) multiplication by Π_W (resp. Π_Y) is thus an isomorphism of $\mathfrak{o}_F$-modules from $\mathcal{C}(t)$ onto $\mathcal{C}(t+s_W)$ (resp. $\mathcal{C}(t+s_Y)$) whence the isomorphy.

To compute the dimension we use the generalized index notation $[A : B]$ for two lattices A and B in a same finite dimensional vector space: $[A : B]$ is just the ordinary quotient of $[A : X]$ and $[B : X]$ for any lattice X contained in A and B.

The common $\mathfrak{o}_F$-period e is a multiple of s_W and s_Y, say $e = r_W s_W = r_Y s_Y$. Write $s = \gcd(s_W, s_Y)$ and pick integers n, m such that $s = ns_W + ms_Y$. We have, for any integer k,

$$\begin{aligned}
[\mathcal{C}(k) : \mathcal{C}(k+s)] &= [\mathcal{C}(k) : \mathcal{C}(k+ns_W)]\ [\mathcal{C}(k+ns_W) : \mathcal{C}(k+ns_W+ms_Y)] \\
&= [\mathcal{C}(k) : \mathcal{C}(k+s_W)]^n\ [\mathcal{C}(k) : \mathcal{C}(k+s_Y)]^m \\
&= [\mathcal{C}(k) : \varpi_F\mathcal{C}(k)]^{\frac{n}{r_W}}\ [\mathcal{C}(k) : \varpi_F\mathcal{C}(k)]^{\frac{m}{r_Y}} \\
&= [\mathcal{C}(k) : \varpi_F\mathcal{C}(k)]^{\frac{s}{e}}
\end{aligned}$$

whence the result. □

4.6. We will need to determine the effect on $\mathcal{C}(\Lambda_Y, \Lambda_W)$ of a shift in indices on Λ_W. We further assume the following.

Notation. (i) The space W is an E_W-vector space for some finite extension E_W of F, with ramification index e_W and residue field k_W of cardinality q_W.

(ii) We fix two $\mathfrak{o}_W$-lattice sequences $\Lambda_{W,0}$ and $\Lambda_{W,1}$ in W with the same underlying lattice chain of period 1 over E_W (so that $s_{W,0} = s_{W,1} = e/e_W$) and with jumps at $a_{W,0} = 0$ and $a_{W,1} = \frac{e}{2e_W}$ respectively.

We write $s_Y = \frac{e}{r_Y}$ and put $\mathcal{C}_t = \mathcal{C}(\Lambda_Y, \Lambda_{W,t})$, for $t = 0, 1$. The sets of jumps of $\mathcal{C}_0$, $\mathcal{C}_1$ are respectively

$$-a_Y + \gcd\left(\frac{e}{r_Y}, \frac{e}{e_W}\right)\mathbb{Z} \quad \text{and} \quad \frac{e}{2e_W} - a_Y + \gcd\left(\frac{e}{r_Y}, \frac{e}{e_W}\right)\mathbb{Z};$$

they are the same when $\frac{e}{2e_W}$ divides $\gcd\left(\frac{e}{r_Y}, \frac{e}{e_W}\right)$. We get the following, where val_2 is the 2-adic valuation of an integer.

Lemma. *$\mathcal{C}_0$ and $\mathcal{C}_1$ have the same jumps if and only if $\mathrm{val}_2(e_W) < \mathrm{val}_2(r_Y)$. Otherwise the jumps of $\mathcal{C}_0$ and $\mathcal{C}_1$ are shifted by $\frac{1}{2}\gcd(\frac{e}{r_Y}, \frac{e}{e_W})$.*

4.7. We now observe that the group $\mathrm{GL}_{m_W}(k_W)$ acts on the quotients $\mathcal{C}_t(i)/\mathcal{C}_t(i+1)$ by left multiplication, where $m_W = \dim_{E_W} W$. These actions commute with the left action of $E_W^\times$ and with the right action of Π_Y so, on the non-zero quotients, they are all equivalent and the corresponding permutations of the non-zero sets $\mathcal{C}_t(i)/\mathcal{C}_t(i+1)$ all have the same signature.

In the same fashion the non-zero quotients $\mathcal{C}_t(i)/\mathcal{C}_t(i+1)$ are isomorphic left modules over $\mathfrak{a}_0(\Lambda_{W,t,\mathfrak{o}_W})/\mathfrak{a}_1(\Lambda_{W,t,\mathfrak{o}_W}) \simeq M_{m_W}(k_W)$. The latter is a simple algebra hence those modules have composition series with d simple quotients all isomorphic to the natural module $k_W^{m_W}$. The determinant of the action of $g \in \mathrm{GL}_{m_W}(k_W)$ on any such module is thus $(\det_{k_W} g)^d$ and the signature of the corresponding permutation is $\left((\det_{k_W} g)^{\frac{q_W-1}{2}}\right)^d$ by 3.11 Lemma. The associated character of $\mathrm{GL}_{m_W}(k_W)$ is then trivial if and only if d is even. Now 4.5 Lemma gives us:

$$d = \frac{1}{m_W[k_W : k_F]} \frac{\gcd\left(\frac{e}{r_Y}, \frac{e}{e_W}\right)}{e} \dim_F W \dim_F Y$$

Since $\dim_F W = e_W[k_W : k_F]m_W$, we conclude:

Lemma. *The signature of the natural left action of $\mathrm{GL}_{m_W}(k_W)$ on the non trivial quotients $\mathcal{C}_t(i)/\mathcal{C}_t(i+1)$ is the trivial character if and only if*

$$d = \frac{e_W}{\mathrm{lcm}(r_Y, e_W)} \dim_F Y$$

is even*; otherwise it is the unique character of $\mathrm{GL}_{m_W}(k_W)$ of order two. In particular:*

- *this signature only depends on e_W, not on W itself;*
- *when $\mathcal{C}_0$ and $\mathcal{C}_1$ do not have the same jumps, we have $d \equiv \dim_F Y \pmod 2$.*

4.8. We return to the notation of paragraphs 4.1, 4.2 but, for now, we drop the subscript t so that $\mathfrak{M}$ denotes either of the orders $\mathfrak{M}_0$ or $\mathfrak{M}_1$. We first detail the structure of the $\mathfrak{b}_0(\mathfrak{M})$-bimodule $\mathfrak{J}^1_{\mathfrak{M}} \cap \mathbb{U}/\mathfrak{H}^1_{\mathfrak{M}} \cap \mathbb{U}$, isomorphic to $J^1_{\mathfrak{M}} \cap U/H^1_{\mathfrak{M}} \cap U$ by the Cayley map, or equivalently

by $Y \mapsto 1+Y$. (Recall that $\mathbb{U}$ denotes the Lie algebra of U.) We use the inductive definition of the orders $\mathfrak{J}_{\mathfrak{M}}$ and $\mathfrak{H}_{\mathfrak{M}}$ given in [34, §3.2].

We have, for some $u \geqslant 1$, a sequence $(\gamma_0 = \beta, \gamma_1, \cdots, \gamma_u = 0)$ and a strictly increasing sequence of integers $0 < r_0 < \cdots < r_{u-1} = n := v_{\mathfrak{M}}(\beta)$ such that, for $0 \leqslant v \leqslant u-1$, the stratum $[\mathfrak{M}, -, r_v - 1, \gamma_v]$ is semi-simple and the stratum $[\mathfrak{M}, -, r_v, \gamma_v]$ is equivalent to $[\mathfrak{M}, -, r_v, \gamma_{v+1}]$. Using the inductive definition and writing $[Z]$ for the image in the Grothendieck group of a $\mathfrak{b}_0(\mathfrak{M})$-bimodule Z, we find that:

(4.2)

$$[\mathfrak{J}^1_{\mathfrak{M}}/\mathfrak{H}^1_{\mathfrak{M}}] = [\mathfrak{a}^{n/2}_{\mathfrak{M}}/\mathfrak{a}^{(n/2)+}_{\mathfrak{M}}] - [\mathfrak{b}^{r_0/2}_{\gamma_0,\mathfrak{M}}/\mathfrak{b}^{(r_0/2)+}_{\gamma_0,\mathfrak{M}}] + \sum_{v=1}^{u-1} \left([\mathfrak{b}^{r_{v-1}/2}_{\gamma_v,\mathfrak{M}}/\mathfrak{b}^{(r_{v-1}/2)+}_{\gamma_v,\mathfrak{M}}] - [\mathfrak{b}^{r_v/2}_{\gamma_v,\mathfrak{M}}/\mathfrak{b}^{(r_v/2)+}_{\gamma_v,\mathfrak{M}}] \right),$$

where $\mathfrak{b}^{r/2}_{\gamma,\mathfrak{M}}$ is shorthand for the intersection of $\mathfrak{a}_{r/2}(\mathfrak{M})$ with the centraliser of γ.

From [34, Proposition 3.4], we may choose the elements γ_v so that the decomposition $X = V \perp (W \oplus W^*)$ is subordinate to all strata considered above; in particular we can take intersections with $\mathbb{U}$ in every term in the above equality. Then the value $\epsilon_{\mathfrak{M}}(m)$ of the quadratic character $\epsilon_{\mathfrak{M}}$ can be calculated as the product of the signatures of the permutation $\operatorname{Ad} m$ on each resulting quotient.

4.9. We now begin the proof of 4.3 Proposition. Recall that, by 4.2 Lemma, the character $\epsilon_{\mathfrak{M}^i}\epsilon_{\mathfrak{M}}$ is given by the signature of the permutation $\phi \mapsto m\phi$ on

$$\mathfrak{X} := \mathfrak{J}^1_{\mathfrak{M}} \cap \operatorname{Hom}_F(V^{\vee i}, W)/\mathfrak{H}^1_{\mathfrak{M}} \cap \operatorname{Hom}_F(V^{\vee i}, W),$$

for $m \in \tilde{P}(\Lambda_{W,\mathfrak{o}_W})$.

The space $\operatorname{Hom}_F(V^{\vee i}, W)$ decomposes as a direct sum $\oplus_{j \neq i} \operatorname{Hom}_F(V^j, W)$. Moreover, each V^j in turn decomposes as a direct sum $V^j = \perp_{s=1}^{s_j} Y^{j,s}$ of subspaces $Y^{j,s}$ for which the assumptions of §4.5–4.7 are satisfied, and such that the resulting decomposition of V is subordinate to $[\Lambda, -, 0, \beta]$. Precisely:

- if β_j is non-zero, we take a direct sum of lines over E_j that splits the lattice sequence Λ^j as in [12, §5.3 Lemma].
- If $\beta_j = 0$, the reductive quotient of the maximal parahoric subgroup $P(\Lambda^j)$ is isomorphic to the direct product of at most two symplectic groups over k_F, whence a decomposition of V^j as an orthogonal sum of at most two symplectic spaces satisfying the conditions required.

The action of $\mathrm{GL}_{m_W}(k_W)$ on $\mathfrak{X}$ then decomposes as a direct sum over j, s of actions on

$$\mathfrak{X}^{j,s} = \mathfrak{J}^1_{\mathfrak{M}} \cap \mathbf{Y}^{j,s}/\mathfrak{H}^1_{\mathfrak{M}} \cap \mathbf{Y}^{j,s},$$

where $\mathbf{Y}^{j,s} = \operatorname{Hom}_F(Y^{j,s}, W)$.

Using [34, Proposition 3.4] and [12, §5.3 Corollary], we may choose the elements γ_v for (4.2) so that the decomposition $X = \perp_{j,s} Y^{j,s} \perp (W \oplus W^*)$ is subordinate to all strata considered. The action of $\mathrm{GL}_{m_W}(k_W)$ then decomposes further along (4.2) into pieces that fit the

hypotheses of 4.7 Lemma, namely pieces of the following forms:

$$Q_1 = [\mathfrak{a}_{\mathfrak{M}}^{n/2} \cap \mathbf{Y}^{j,s}/\mathfrak{a}_{\mathfrak{M}}^{(n/2)+} \cap \mathbf{Y}^{j,s}],$$
$$Q_2 = [\mathfrak{b}_{\gamma_0,\mathfrak{M}}^{r_0/2} \cap \mathbf{Y}^{j,s}/\mathfrak{b}_{\gamma_0,\mathfrak{M}}^{(r_0/2)+} \cap \mathbf{Y}^{j,s}],$$
$$Q_3 = [\mathfrak{b}_{\gamma_v,\mathfrak{M}}^{r_{v-1}/2} \cap \mathbf{Y}^{j,s}/\mathfrak{b}_{\gamma_v,\mathfrak{M}}^{(r_{v-1}/2)+} \cap \mathbf{Y}^{j,s}] - [\mathfrak{b}_{\gamma_v,\mathfrak{M}}^{r_v/2} \cap \mathbf{Y}^{j,s}/\mathfrak{b}_{\gamma_v,\mathfrak{M}}^{(r_v/2)+} \cap \mathbf{Y}^{j,s}].$$

4.10. At last we come to the point, which is not actually to compute the character $\epsilon_{\mathfrak{M}^i}\epsilon_{\mathfrak{M}}$, but rather to prove that this character does not depend on the maximal self-dual order $\mathfrak{M}$. In our setting there are exactly two choices for $\mathfrak{M}$, with a given period e and duality invariant $d = 1$. Indeed, the lattice chain underlying the self-dual lattice sequence $\Lambda_X \cap (W \oplus W^*)$ is the disjoint union of two self-dual lattice chains, one containing a self-dual lattice and its multiples, the other containing a non-self-dual lattice (whose dual is $\mathfrak{p}_W$ times it) and its multiples. Let $\mathfrak{M}_0$ and $\mathfrak{M}_1$ be the two possible choices and write $\Lambda_{W,0} = \mathfrak{M}_0 \cap W$ and $\Lambda_{W,1} = \mathfrak{M}_1 \cap W$. According to [35, Lemma 6.7], the sets of jumps of $\Lambda_{W,0}$ and $\Lambda_{W,1}$ are $\frac{e}{e_W}\mathbb{Z}$ and $\frac{e}{2e_W} + \frac{e}{e_W}\mathbb{Z}$ respectively, and all results in paragraphs 4.5–4.7 apply.

We can thus compare $\epsilon_{\mathfrak{M}_0^i}\epsilon_{\mathfrak{M}_0}$ and $\epsilon_{\mathfrak{M}_1^i}\epsilon_{\mathfrak{M}_1}$ term by term.

Term Q_1. We apply paragraphs 4.5–4.7, replacing Y by $Y^{j,s}$, Λ_Y by $\Lambda \cap Y^{j,s}$ and using $\Lambda_{W,t} = \mathfrak{M}_t \cap W$ as above, for $t = 0, 1$. We remark that $\dim_F Y^{j,s}$ is always even. Hence, by 4.7 Lemma, the signature on Q_1 is trivial unless $\mathcal{C}_0$ and $\mathcal{C}_1$ have the same jumps, so give the same signature.

Term Q_2. We actually have $\gamma_0 = \beta$, hence this term is zero if the centralizer of β does not intersect $\mathrm{Hom}_F(V^j, W)$. This condition holds under the assumptions of 4.3 Proposition because $j \neq i$.

Term Q_3. Since $\mathfrak{M}_0$ and $\mathfrak{M}_1$ have the same intersection with V, we may and do choose the same sequence (γ_v, r_v) for both. We may also scale all our lattice sequences to make the period big enough so that all numbers $r_v/2$ are integers. Now Q_3 is zero unless the centralizer of γ_v intersects $\mathrm{Hom}_F(V^j, W)$, which we now assume. We then apply paragraphs 4.5–4.7 over $F[\gamma_v]$.

If the lattice sequences $\mathcal{C}_0$ and $\mathcal{C}_1$ have the same jumps we have the equality we want. Otherwise, they are shifted by half a period (4.6 Lemma) and the integer d given by 4.7 Lemma is equal to $\dim_{F[\gamma_v]} Y^{j,s}$. If d is even we are also done. Otherwise we have $\beta_j \neq 0$ and $s_\Lambda = e/e_j$, and the period of $\mathcal{C}_0$ and $\mathcal{C}_1$ is $e/\mathrm{lcm}(e_W, e_j)$.

Since Q_3 is the difference of two terms $[\mathcal{C}_t(a)] - [\mathcal{C}_t(b)]$ in the lattice sequence $\mathcal{C}_t$, for $t = 0, 1$, over $F[\gamma_v]$, the values of Q_3 for $t = 0$ and $t = 1$ will be the same on condition that the difference $a - b$ is a multiple of half the period. This is what we will now prove.

In the notation of (4.2), we let $h \geqslant 1$ be the smallest integer such that the centralizer of γ_h intersects $\mathrm{Hom}_F(V^j, W)$, so that we only need to consider terms with $v \geqslant h$. If $h = u$ there

is nothing to do. Otherwise, we need to examine the values of r_h and r_{h-1} more closely, in terms of the *normalized critical exponents* $k_0^F[\gamma_v]$ (see [34, pp. 129,141–2]). We use [34, p. 141], case (ii) for $r_{h-1} = -k_0(\gamma_{h-1}, \mathfrak{M})$ (the unnormalized critical exponent relative to $\mathfrak{M}$) and case (i) for $r_h = -k_0(\gamma_h, \mathfrak{M})$, to get

$$r_{h-1} = v_{F[\gamma_h]}(c)\frac{e(\mathfrak{M}|\mathfrak{o}_F)}{e(F[\gamma_h]/F)}, \qquad r_h = -k_0^F(\gamma_h)\frac{e(\mathfrak{M}|\mathfrak{o}_F)}{e(F[\gamma_h]/F)},$$

for some element c in $F[\gamma_h]$, so that

$$\frac{r_{h-1}}{2} - \frac{r_h}{2} = \frac{e(\mathfrak{M}|\mathfrak{o}_{F[\gamma_h]})}{2}\left(v_{F[\gamma_h]}(c) + k_0^F(\gamma_h)\right).$$

This is indeed an integer multiple of the half-period of jumps

$$\frac{e(\mathfrak{M}|\mathfrak{o}_{F[\gamma_h]})}{2}\left(\frac{\mathrm{lcm}(e_W, e_j)}{e(F[\gamma_h]/F)}\right)^{-1}$$

since the last term is the inverse of an integer.

For $v > h$ we use [34, p. 141] case (i) again and get:

$$\frac{r_{v-1}}{2} - \frac{r_v}{2} = \frac{e(\mathfrak{M}|\mathfrak{o}_{F[\gamma_v]})}{2}\left(-k_0^F(\gamma_{v-1})\frac{e(F[\gamma_v]/F)}{e(F[\gamma_{v-1}]/F)} + k_0^F(\gamma_v)\right)$$

This is a multiple of the half-period of jumps if and only if

$$\begin{aligned}\frac{-k_0^F(\gamma_{v-1})e(F[\gamma_v]/F) + k_0^F(\gamma_v)e(F[\gamma_{v-1}]/F)}{e(F[\gamma_{v-1}]/F)}\ \frac{\mathrm{lcm}(e_W, e_j)}{e(F[\gamma_v]/F)}\\ = \left(-k_0^F(\gamma_{v-1}) + k_0^F(\gamma_v)\frac{e(F[\gamma_{v-1}]/F)}{e(F[\gamma_v]/F)}\right)\ \frac{\mathrm{lcm}(e_W, e_j)}{e(F[\gamma_{v-1}]/F)}\end{aligned}$$

is an integer, which is the case because $e(F[\gamma_v]/F)$ divides $e(F[\gamma_{v-1}]/F)$ (see [11, 2.4.1]).

Putting this together, we obtain the character $\epsilon_{\mathfrak{M}^i}\epsilon_{\mathfrak{M}}$ as a product of signatures, each of them only depending on e_W by 4.7 Lemma, hence our character only depends on e_W, not on W. Furthermore, the extension E_W is isomorphic to $F[\beta_i]$, hence e_W is equal to $e(F[\beta_i]/F)$, independent of the choice of W. This completes the proof of 4.3 Proposition, whence of 4.4 Theorem.

5. The simple case

In this section we prove 2.5 Theorem. Recalling that, by 3.12 Theorem, the parameters of the Hecke algebra of our cover are those in the Hecke algebra of a finite reductive group, we are required to analyse these Hecke algebras. Fortunately, these are described by the work of Lusztig [26] and have been computed in our cases in [25]. One subtlety is that the twisting characters $\epsilon_{\mathfrak{M}_t}$ give rise to involutions which we have not computed explicitly so remain unknown. Fortunately, the numerics are such that an exact description of these involutions is not needed.

5.1. Let π be a simple cuspidal representation of G in the sense of paragraph 2.5. Since the case of depth zero representations is already dealt with in [25], we assume moreover that π has positive depth. Thus π contains a skew *simple* character θ of $H^1 = H^1(\beta, \Lambda)$, for some maximal skew simple stratum $[\Lambda, -, 0, \beta]$, with $\beta \neq 0$, and $E = F[\beta]$ is a field. We write $\mathbf{\Theta}$ for the endo-class of the unique self-dual simple character $\tilde{\theta}$ which restricts to θ. We retain all the notation of paragraph 4.1 so interpret simplicity as meaning that $l = 1$ and drop the index 1 for notation. We will be considering the space $X = X^1$, while varying the self-dual cuspidal representation ρ of $\mathrm{GL}_F(W)$ (and the space W). Note that we have $E_W \simeq E$ so we will identify them.

For a self-dual cuspidal representation ρ of some $\mathrm{GL}_F(W)$, recall that we write $\deg(\rho) = \dim_F(W)$ and $s_\pi(\rho)$ for the unique non-negative real number such that the normalized induced representation $\nu^s\rho \times \pi$ is reducible. Then the description of the Jordan set in paragraph 2.1 shows that, in order to prove 2.5 Theorem, the equality we must prove is

$$\sum_\rho \lfloor s_\pi(\rho)^2 \rfloor \deg(\rho) = 2N, \tag{5.1}$$

where the sum runs over all self-dual cuspidal representations ρ with endo-class $\mathbf{\Theta}(\rho) = \mathbf{\Theta}^2$.

5.2. Recall that we have $\pi = \text{c-Ind}_J^G \lambda$, with $\lambda = \underline{\kappa} \otimes \tau$ and that ρ contains the maximal simple type $\tilde{\lambda}_W = \tilde{\underline{\kappa}}_W \otimes \tilde{\tau}_W$ and has unramified twist number $t(\rho) = \frac{\dim_F W}{e(\mathbf{\Theta})}$, where we have written $e(\mathbf{\Theta}) = e(\mathbf{\Theta}^2) = e(E/F)$ since it depends only on the endo-class. Moreover, by 3.14 Proposition, we have that the real parts of the reducibility points of the normalized induced representation $\nu^s\rho \times \pi$ are the elements of the set

$$\left\{ \pm\frac{r_0 + r_1}{2t(\rho)}, \pm\frac{r_0 - r_1}{2t(\rho)} \right\},$$

where, for $t = 0, 1$, the integers $r_t = r_t(\epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau))$ comes from the quadratic relations in the finite Hecke algebra $\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau))$ as in (3.5).

Remark. It will be crucial to note that the character $\epsilon_{\mathfrak{M}_t}$ depends only on the dimension $\deg(\rho) = \dim_F(W)$, and not on the representation ρ itself.

The contribution to the sum (5.1) of the inertial class $[\rho]$ (that is, writing $\rho' = \nu^{\pi\mathrm{i}/t(\rho)\log(q)}\rho$ for the other self-dual representation in the inertial class, the combined contributions of ρ and ρ') is

$$\left\lfloor \left(\frac{r_0 + r_1}{2t(\rho)} \right)^2 \right\rfloor + \left\lfloor \left(\frac{r_0 - r_1}{2t(\rho)} \right)^2 \right\rfloor.$$

From results of Lusztig (see [25, §8] and also paragraph 5.6 below), the numbers $r_t/t(\rho)$ are either both integers or both half-integers so that this simplifies to

$$\left\lfloor \frac{r_0^2 + r_1^2}{2t(\rho)^2} \right\rfloor. \tag{5.2}$$

5.3. In order to prove (5.1) we will need to recall Lusztig's parametrisation of cuspidal representations of classical groups, and the computation of the parameter r_t in the Hecke algebra $\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau))$. We follow the description in [25, §§2, 3, 6 and, especially, 7], to which we refer for details and references for the assertions made here.

In almost all cases, we have

$$\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau)) \simeq \mathscr{H}(\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau^\circ)),$$

where τ° is an irreducible component of the restriction $\tau_{|\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})}$, and it is here that we will perform our calculations. In the exceptional cases we have $r_t = 0$ and it will turn out that this matches the formula one would obtain by following the recipe for computing the parameters in the connected component $\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E})$. Thus we will assume first that the calculation is to be done in $\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E})$ and then, in paragraph 5.7, we will treat the exceptional cases.

5.4. Since $P(\Lambda_{\mathfrak{o}_E})$ is the normaliser of a maximal parahoric subgroup of the centraliser G_E, we can decompose

$$\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E}) = \mathcal{G}^{(0)}(\Lambda_{\mathfrak{o}_E}) \times \mathcal{G}^{(1)}(\Lambda_{\mathfrak{o}_E})$$

as a product of two connected classical groups over k_E° (the residue field of the fixed points $E_\circ$ in E under the involution on A). We have a similar decomposition of $\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E})$ with, moreover,

$$\mathcal{G}^{(1)}(\mathfrak{M}_{0,\mathfrak{o}_E}) = \mathcal{G}^{(1)}(\Lambda_{\mathfrak{o}_E}) \text{ and } \mathcal{G}^{(0)}(\mathfrak{M}_{1,\mathfrak{o}_E}) = \mathcal{G}^{(0)}(\Lambda_{\mathfrak{o}_E}),$$

and the Levi subgroup

$$\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E}) \times \mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E}) \subseteq \mathcal{G}^{(t)}(\mathfrak{M}_{t,\mathfrak{o}_E}).$$

We choose an irreducible component τ° of the restriction $\tau_{|\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})}$ and write it as $\tau^{(0)} \otimes \tau^{(1)}$. Writing the character $\epsilon_{\mathfrak{M}_t}$ as $\epsilon^W_{\mathfrak{M}_t} \otimes \epsilon^{(0)}_{\mathfrak{M}_t} \otimes \epsilon^{(1)}_{\mathfrak{M}_t}$, we have isomorphisms of Hecke algebras

$$\mathscr{H}(\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau^\circ)) \simeq \mathscr{H}(\mathcal{G}^{(t)}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon^W_{\mathfrak{M}_t}\tilde{\tau}_W \otimes \epsilon^{(t)}_{\mathfrak{M}_t}\tau^{(t)}),$$

and it is in this Hecke algebra that we compute the parameter r_t.

5.5. We now fix $t = 0$ or 1 so drop the sub/superscript t from our notations for now. Thus we have:

- a connected classical group $\mathcal{G}^\circ(\mathfrak{M}_{\mathfrak{o}_E})$ over k_E°, with Levi subgroup $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E}) \times \mathcal{G}(\Lambda_{\mathfrak{o}_E})$ and $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E}) \simeq \mathrm{GL}_m(k_E)$, where $m = \dim_E(W)$;
- a self-dual cuspidal representation $\tilde{\tau}_W \otimes \tau$ of $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E}) \times \mathcal{G}(\Lambda_{\mathfrak{o}_E})$;
- a character $\epsilon^W_{\mathfrak{M}} \otimes \epsilon_{\mathfrak{M}}$ of $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E}) \times \mathcal{G}(\Lambda_{\mathfrak{o}_E})$ of order at most two, which depends on $m = \dim_E(W)$ but not on $\tilde{\tau}_W$.

By Green's parametrization (and after fixing an isomorphism $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E}) \simeq \mathrm{GL}_m(k_E)$), the cuspidal representation $\tilde{\tau}_W$ corresponds to an irreducible monic polynomial $Q \in k_E[X]$ of degree m; moreover, this polynomial is k_E/k_E°-self-dual, that is

$$Q(X) = (\overline{Q(0)})^{-1} X^{\deg Q} \overline{Q}(1/X),$$

where $x \mapsto \overline{x}$ is the automorphism of k_E with fixed field k_E°, extended to $k_E[X]$ coefficientwise (see [25, §7.1]). Since a cuspidal representation $\tilde{\tau}_W$ of $\tilde{\mathcal{G}}(\Lambda_{W,\mathfrak{o}_E})$ is self-dual if and only if $\tilde{\tau}_W \epsilon_{\mathfrak{M}}^W$ is cuspidal self-dual, twisting by $\epsilon_{\mathfrak{M}}^W$ induces an involution on the set of irreducible k_E/k_E°-self-dual monic polynomials of degree m. We denote this involution by $\sigma_{m,W}$; it is either trivial, or given by $Q(X) \mapsto (-1)^{\deg Q} Q(-X)$.

Similarly, by Lusztig's parametrization, the cuspidal representation τ lies in a *rational Lusztig series* $\mathcal{E}(\mathbf{s})$ corresponding to (the rational conjugacy class of) a semisimple element $\mathbf{s}$ of the dual group of $\mathcal{G}(\Lambda_{\mathfrak{o}_E})$. Since its series contains a cuspidal representation, this semisimple element $\mathbf{s}$ has characteristic polynomial of a particular form, namely

$$P_{\mathbf{s}}(X) = \prod_P P(X)^{a_P},$$

where the product is over all irreducible k_E/k_E°-self-dual monic polynomials and the integers a_P satisfy certain combinatorial constraints (see [25, (7.2) and §7.7]); more precisely, we have:

- $\sum_P a_P \deg(P)$ is the dimension of the space $\mathcal{V}$ on which the dual group of $\mathcal{G}(\Lambda_{\mathfrak{o}_E})$ naturally acts;
- if either $k_E \neq k_E^\circ$ or $P(X) \neq (X \pm 1)$, then $a_P = \frac{1}{2}(b_P^2 + b_P)$, for some non-negative integers b_P;
- if $P(X) = X \pm 1$ then, writing $a_+ := a_{(X-1)}$ and $a_- := a_{(X+1)}$, there are integers $b_+, b_- \geqslant 0$ such that
  (i) if $\mathcal{G}(\Lambda_{\mathfrak{o}_E})$ is odd special orthogonal then $a_+ = 2(b_+^2 + b_+)$ and $a_- = 2(b_-^2 + b_-)$,
  (ii) if $\mathcal{G}(\Lambda_{\mathfrak{o}_E})$ is symplectic then $a_+ = 2(b_+^2 + b_+) + 1$ and $a_- = 2b_-^2$,
  (iii) if $\mathcal{G}(\Lambda_{\mathfrak{o}_E})$ is even special orthogonal then $a_+ = 2b_+^2$ and $a_- = 2b_-^2$,
  and, in case (iii), the (± 1)-eigenspace in $\mathcal{V}$ is an even-dimensional orthogonal space of type $(-1)^{b_\pm}$, and the same in case (ii) for the (-1)-eigenspace only.

As above, twisting by the character $\epsilon_{\mathfrak{M}}$ will induce a degree-preserving involution on the set of irreducible k_E/k_E°-self-dual monic polynomials. If the character $\epsilon_{\mathfrak{M}}$ is trivial then this involution is trivial. If the character $\epsilon_{\mathfrak{M}}$ is non-trivial quadratic then, by [14, Proposition 8.26], twisting by $\epsilon_{\mathfrak{M}}$ induces a bijection between rational Lusztig series

$$\mathcal{E}(\mathbf{s}) \xrightarrow{\sim} \mathcal{E}(-\mathbf{s}),$$

and the involution is given by $P(X) \mapsto (-1)^{\deg P} P(-X)$. In either case, we denote by $\sigma_{m,G}$ the involution induced by twisting by $\epsilon_{\mathfrak{M}}$. (Note that this is a degree-preserving involution on the set of *all* irreducible k_E/k_E°-self-dual monic polynomials; the subscript m is included to indicate that the involution depends on m.) The characteristic polynomial corresponding to the cuspidal representation $\tau\epsilon_{\mathfrak{M}}$ is then

$$\prod_P P(X)^{a_{\sigma_{m,G}(P)}}.$$

Putting together our two involutions, we get an involution on the set of irreducible k_E/k_E°-self-dual monic polynomials of degree m given by

$$\sigma_m = \sigma_{m,G} \circ \sigma_{m,W}.$$

5.6. Recall that the Hecke algebra $\mathscr{H}(\mathcal{G}(\mathfrak{M}_{\mathfrak{o}_E}), \epsilon_{\mathfrak{M}}^W \tilde{\tau}_W \otimes \epsilon_{\mathfrak{M}} \tau)$ is generated by an element $\mathcal{T}$ satisfying a quadratic relation

$$(\mathcal{T} - q^r \omega)(\mathcal{T} + \omega) = 0,$$

where q is the cardinality of the residue field of k_F. The work of Lusztig, explicated in [25, §7], allows one to write down explicitly the parameter r in terms of the characteristic polynomials of the previous paragraph, as follows.

Let $Q(X)$ be the irreducible k_E/k_E°-self-dual monic polynomial of degree m corresponding to $\tilde{\tau}_W$, and let $P_s(X) = \prod_P P(X)^{a_P}$ be the monic polynomial corresponding to τ, where the a_P are as described in the previous paragraph. Writing f for the degree of the extension k_E/k_F, one gets the following values:

- if $k_E = k_E^\circ$ and $\sigma_1(Q) = X - 1$ then
$$\frac{r}{f} = \begin{cases} 2b_+ & \text{if } \mathcal{G} \text{ is even special orthogonal,} \\ 2b_+ + 1 & \text{otherwise;} \end{cases}$$
- if $k_E = k_E^\circ$ and $\sigma_1(Q) = X + 1$ then
$$\frac{r}{f} = \begin{cases} 2b_- + 1 & \text{if } \mathcal{G} \text{ is odd special orthogonal,} \\ 2b_- & \text{otherwise;} \end{cases}$$
- if $k_E \neq k_E^\circ$ or m is even then
$$\frac{r}{f} = (2b_{\sigma_m(Q)} + 1)\frac{m}{2}.$$

Note that, since $t(\rho) = mf$, the number $r/t(\rho)$ is a half-integer, as asserted above. Moreover, $r/t(\rho)$ is an integer precisely when $E/E_\mathfrak{o}$ is ramified and E/F is a maximal extension (i.e. of degree $\dim_F(W)$); in particular, this depends only on the polynomial Q (that is, on $\tilde{\tau}_W$, so on the representation ρ) and not on either the representation τ or on the involution σ_1.

5.7. In this paragraph, we treat the exceptional cases, where we do *not* have an isomorphism

$$\mathscr{H}(\mathcal{G}(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau)) \simeq \mathscr{H}(\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E}), \epsilon_{\mathfrak{M}_t}(\tilde{\tau}_W \otimes \tau^\circ)), \tag{5.3}$$

According to the description in [27, §6.3], this occurs precisely when:

- $E/E_\mathfrak{o}$ is ramified;
- $\dim_E(W) = 1$, so that $\tilde{\tau}_W$ is a character of order at most 2;
- and either $\mathcal{G}(\Lambda_{\mathfrak{o}_E}) = \mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})$ or $\epsilon_{\mathfrak{M}_t}\tau_{|\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})}$ is reducible.

We remark that $\epsilon_{\mathfrak{M}_t}\tau_{|\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})}$ is reducible if and only if $\tau_{|\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})}$ is reducible.

In these cases, writing $\mathcal{G}^\circ(\mathfrak{M}_{t,\mathfrak{o}_E}) = \mathcal{G}^{(0)}(\Lambda_{\mathfrak{o}_E}) \times \mathcal{G}^{(1)}(\Lambda_{\mathfrak{o}_E})$, there is one value of t for which $\mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E})$ is an even special orthogonal group (for the other it is a symplectic group) and it is precisely for this value of t that we do not have an isomorphism (5.3) and we get parameter $r_t = 1$.

As above, we write $\tau^{(0)}\otimes\tau^{(1)}$ for an irreducible component of $\tau_{|\mathcal{G}^\circ(\Lambda_{\mathfrak{o}_E})}$, and write $\epsilon_{\mathfrak{M}_t}$ as $\epsilon^W_{\mathfrak{M}_t}\otimes \epsilon^{(0)}_{\mathfrak{M}_t}\otimes\epsilon^{(1)}_{\mathfrak{M}_t}$. Writing $P_s(X) = \prod P(X)^{a_P}$ for the polynomial corresponding to the cuspidal representation $\tau^{(t)}$, the fact that it does *not* extend to the full even orthogonal group implies, by [25, Proposition 7.9], that ± 1 are *not* roots of P_s, that is, $a_+ = a_- = 0$.

Since $\tilde{\tau}_W$ is a character of order at most 2, the corresponding polynomial is $Q(X) = X \pm 1$. In particular, since we have $b_+ = b_- = 0$, the formulae of paragraph 5.6 are still valid, since they too give $r_t = 0$. Thus those formulae are valid in every case.

5.8. Finally, using the formulae of paragraph 5.6, we return to computing the contribution (5.2), so we retrieve the sub/superscripts t. We have:

- an irreducible k_E/k_E°-self-dual monic polynomial $Q(X)$, corresponding to the cuspidal representation $\tilde{\tau}_W$;
- for $t = 0, 1$, a polynomial $\prod_P P(X)^{a_P^{(t)}}$ corresponding to the cuspidal representation $\tau^{(t)}$;
- for $t = 0, 1$, an involution $\sigma_m^{(t)}$ on the set of irreducible k_E/k_E°-self-dual monic polynomials of degree m.

Suppose first that either $k_E \neq k_E^\circ$ or m is even; then we get

$$\begin{aligned}\left\lfloor \frac{r_0^2 + r_1^2}{2t(\rho)^2}\right\rfloor &= \left\lfloor \frac{(2b^{(0)}_{\sigma_m^{(0)}(Q)} + 1)^2 + (2b^{(1)}_{\sigma_m^{(1)}(Q)} + 1)^2}{8}\right\rfloor \\ &= \left\lfloor \tfrac{1}{2}b^{(0)}_{\sigma_m^{(0)}(Q)}\left(b^{(0)}_{\sigma_m^{(0)}(Q)} + 1\right) + \tfrac{1}{2}b^{(1)}_{\sigma_m^{(1)}(Q)}\left(b^{(1)}_{\sigma_m^{(1)}(Q)} + 1\right) + \tfrac{1}{2}\right\rfloor \\ &= a^{(0)}_{\sigma_m^{(0)}(Q)} + a^{(1)}_{\sigma_m^{(1)}(Q)}.\end{aligned}$$

If $k_E = k_E^\circ$ then one of the groups $\mathcal{G}^{(t)}(\mathfrak{M}_{t,\mathfrak{o}_E})$ is symplectic while the other is orthogonal. Here we can treat each case, each polynomial $X \pm 1$, and each possibility for the involutions $\sigma_1^{(t)}$, separately. Up to permuting $\{0, 1\}$ we are in one of the following two cases:

If $\mathcal{G}^{(0)}(\Lambda_{\mathfrak{o}_E})$ is odd special orthogonal and $\mathcal{G}^{(1)}(\Lambda_{\mathfrak{o}_E})$ is symplectic, then the contribution of $\sigma_1^{(1)}(X-1)$ is

$$\left\lfloor \frac{\left(2b_\zeta^{(0)}+1\right)^2+\left(2b_+^{(1)}+1\right)^2}{2} \right\rfloor = 2b_\zeta^{(0)}\left(b_\zeta^{(0)}+1\right)+2b_+^{(1)}\left(b_+^{(1)}+1\right)+1 = a_\zeta^{(0)}+a_+^{(1)},$$

where ζ is the sign defined by $\sigma_1^{(0)}\sigma_1^{(1)}(X-1)=X-\zeta$; and the contribution of $\sigma_1^{(1)}(X+1)$ is

$$\left\lfloor \frac{\left(2b_{-\zeta}^{(0)}+1\right)^2+\left(2b_-^{(1)}\right)^2}{2} \right\rfloor = 2b_{-\zeta}^{(0)}\left(b_{-\zeta}^{(0)}+1\right)+2\left(b_-^{(1)}\right)^2 = a_{-\zeta}^{(0)}+a_-^{(1)}.$$

In particular, the sum of the contributions of $X\pm 1$ is

$$a_+^{(0)}+a_-^{(0)}+a_+^{(1)}+a_-^{(1)}.$$

If $\mathcal{G}^{(0)}(\Lambda_{\mathfrak{o}_E})$ is even special orthogonal and $\mathcal{G}^{(1)}(\Lambda_{\mathfrak{o}_E})$ is symplectic, then the contribution of $\sigma_1^{(1)}(X-1)$ is

$$\left\lfloor \frac{\left(2b_\zeta^{(0)}\right)^2+\left(2b_+^{(1)}+1\right)^2}{2} \right\rfloor = 2\left(b_\zeta^{(0)}\right)^2+2b_+^{(1)}\left(b_+^{(1)}+1\right) = a_\zeta^{(0)}+a_+^{(1)}-1,$$

where ζ is again the sign defined by $\sigma_1^{(0)}\sigma_1^{(1)}(X-1)=X-\zeta$; and the contribution of $\sigma_1^{(1)}(X+1)$ is

$$\left\lfloor \frac{\left(2b_{-\zeta}^{(0)}\right)^2+\left(2b_-^{(1)}\right)^2}{2} \right\rfloor = 2\left(b_{-\zeta}^{(0)}\right)^2+2\left(b_-^{(1)}\right)^2 = a_{-\zeta}^{(0)}+a_-^{(1)}.$$

In this second case, the sum of the contributions of $X\pm 1$ is

$$a_+^{(0)}+a_-^{(0)}+a_+^{(1)}+a_-^{(1)}-1;$$

the term -1 reflects the fact that the sum of the dimensions of the spaces on which the dual groups of $\mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E})$ act naturally is 1 more than the sum of the dimensions of the spaces on which the groups $\mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E})$ act naturally. Note also that this latter sum of dimensions is precisely $\dim_E(V)$, where we recall that V is the symplectic space on which our group G acts.

5.9. Having computed all the contributions to the sum (5.1) in the previous paragraph, we can now sum them over all possible Q, noting that, if the cuspidal representation ρ

corresponds to the polynomial Q, then $\deg(\rho) = [E : F]\deg(Q)$. If $k_E \neq k_E^\circ$ this is straightforward and we obtain

$$\sum_\rho \lfloor s_\pi(\rho)^2 \rfloor \deg(\rho) = [E:F]\sum_m \left(m \sum_{\deg(Q)=m} \left(a^{(0)}_{\sigma_m^{(0)}(Q)} + a^{(1)}_{\sigma_m^{(1)}(Q)} \right) \right)$$
$$= [E:F]\left(\sum_P a_P^{(0)} \deg(P) + \sum_P a_P^{(1)} \deg(P) \right) = [E:F]\dim_E(V) = 2N,$$

as required. Here the penultimate equality occurs because each group $\mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E})$ is a unitary group (whose dual group is then a unitary group acting naturally on a space of the same dimension), and the sum of the dimensions of the spaces on which they act is $\dim_E(V)$.

If $k_E = k_E^\circ$ then we need to be a little more careful with the polynomials $X \pm 1$ (that is, the k_E/k_E°-self-dual monic polynomials of degree 1), as described at the end of the previous paragraph. If one of the $\mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E})$ is *even* special orthogonal (and the other symplectic) then we get that $\sum_\rho \lfloor s_\pi(\rho)^2 \rfloor \deg(\rho)$ is

$$[E:F]\left(\sum_{m\geqslant 2} \left(m \sum_{\deg(Q)=m} \left(a^{(0)}_{\sigma_m^{(0)}(Q)} + a^{(1)}_{\sigma_m^{(1)}(Q)} \right) \right) + a_+^{(0)} + a_-^{(0)} + a_+^{(1)} + a_-^{(1)} - 1 \right)$$
$$= [E:F]\left(\sum_P a_P^{(0)} \deg(P) + \sum_P a_P^{(1)} \deg(P) - 1 \right) = [E:F]\dim_E(V) = 2N,$$

where the penultimate equality uses the fact that the dual of a symplectic group acts naturally on a space of dimension 1 greater, while the dual of an even special orthogonal group acts naturally on a space of the same dimension.

On the other hand, if one of the $\mathcal{G}^{(t)}(\Lambda_{\mathfrak{o}_E})$ is *odd* special orthogonal (and the other symplectic) then we get the same sum except without the term -1, and the penultimate equality uses the fact that the dual of an odd special orthogonal group acts naturally on a space of dimension 1 smaller, while the dual of a symplectic group acts naturally on a space of dimension 1 greater.

This completes the proof of (5.1), whence of 2.5 Theorem.

5.10. The results in this section not only prove 2.5 Theorem but also give an algorithm to compute the inertial Jordan set of a positive depth simple cuspidal representation of G. (The case of depth zero is treated already in [25].) Moreover, 2.6 Corollary then gives the inertial Jordan set for any cuspidal representation of G.

Indeed, suppose π is a simple cuspidal representation of G, induced from a cuspidal type $\lambda = \underline{\kappa} \otimes \tau$. With the usual notation, let τ° be any irreducible component of the restriction of τ to the maximal parahoric subgroup $P^\circ(\Lambda_{\mathfrak{o}_E})$. Then τ° is the inflation of a representation $\tau^{(0)} \otimes \tau^{(1)}$, with each $\tau^{(t)}$ a cuspidal representation of a finite reductive group over k_E°. These each appear in some rational Lusztig series and we consider the set $\mathcal{Q}^{(t)}$ of monic irreducible

polynomials dividing the characteristic polynomial (over k_E) of the corresponding semisimple conjugacy class, for $t = 0, 1$, all of which are k_E/k_E°-self-dual. For each $m \in \deg(\mathcal{Q}^{(t)})$, we compute the signature character $\epsilon_{\mathfrak{M}_t}$, and thus deduce the involution $\sigma_m^{(t)}$ as in paragraph 5.5. We set

$$\mathcal{Q} = \left\{\sigma_m^{(t)}(Q) \mid Q \in \mathcal{Q}^{(t)},\ \deg(Q) = m,\ t = 0, 1\right\}.$$

Now let $\mathbf{\Theta}$ be the endo-class of the self-dual simple character lifting any skew simple character in π and let $Q \in \mathcal{Q}$. We put $n = \deg(Q)\deg(\mathbf{\Theta})$ and let $\tilde{\theta}$ be the unique (up to conjugacy) *m-simple* character in $\mathrm{GL}_n(F)$ with endo-class $\mathbf{\Theta}^2$ (in the language of [9], for example). Let $\underline{\tilde{\kappa}}$ be the p-primary extension of $\tilde{\theta}$, a representation of a group $\tilde{J}$. The group $\tilde{J}/\tilde{J}^1$ is then a finite general linear group of rank $\deg(Q)$ over k_E, and we let $\tilde{\tau}_Q$ be the unique cuspidal representation in the Lusztig series corresponding to a semisimple conjugacy class with characteristic polynomial Q. Write $[\rho_Q]$ for the inertial class of cuspidal representations of $\mathrm{GL}_n(F)$ containing $\underline{\tilde{\kappa}} \otimes \tilde{\tau}_Q$.

The inertial classes in $\{[\rho_Q] \mid Q \in \mathcal{Q}\}$ are precisely the inertial classes which will appear in $\mathrm{IJord}(\pi)$. In order to compute the multiplicities with which $[\rho_Q]$ appears, we follow the recipe of paragraph 5.6 to compute the corresponding Hecke algebra parameters r_0 and r_1, whence the real parts of the reducibility points $|r_0 \pm r_1|/(\deg(Q)[k_E : k_F])$ and the multiplicities from Mœglin's criterion. In the case that $k_E \neq k_E^\circ$ or $m = \deg(Q) > 1$, this is straightforward, with the real parts of the reducibility points given by

$$\frac{b^{(0)}_{\sigma_m^{(0)}(Q)} + b^{(1)}_{\sigma_m^{(1)}(Q)} + 1}{2} \quad \text{and} \quad \frac{b^{(0)}_{\sigma_m^{(0)}(Q)} - b^{(1)}_{\sigma_m^{(1)}(Q)}}{2},$$

where $a_P^{(t)} = \frac{1}{2}b_P^{(t)}(b_P^{(t)} + 1)$ is the power to which P divides the characteristic polynomial corresponding to $\tau^{(t)}$. By construction of $\mathcal{Q}$, the first of these is certainly greater than $\frac{1}{2}$. In the case $k_E = k_E^\circ$ and $\deg(Q) = 1$ (so that Q is $X \pm 1$) there is no such simple universal formula, and instead one must proceed in a case-by-case analysis as in paragraph 5.8. We leave this as an exercise to the reader; a similar calculation is done in [25, section 8].

6. Galois parameters

In this section we study self-duality in terms of Galois parameters with a view, in particular, to understanding the ambiguities in our results in terms of the local Langlands correspondence.

6.1. We denote by $\bar{F}$ a fixed separable closure of F and by W_F the absolute Weil group of F (with similar notation for intermediate fields). We would like to explore a self-dual irreducible representation σ of W_F, with a view to determining its parity (that is, whether it is symplectic or orthogonal); in particular, we would like to know when the self-dual irreducible representation σ' which is an unramified twist of (and not isomorphic to) σ has

the same parity as σ, since it is in this case that we have ambiguity. For now, we do not require p to be odd.

Let χ be an unramified character of W_F. Then $(\chi\sigma)^\vee$ is isomorphic to $\chi^{-1}\sigma^\vee$, so, σ being self-dual, $\chi\sigma$ is self-dual if and only if $\chi^2\sigma \simeq \sigma$.

We let $t(\sigma)$ be the number of unramified characters η of W_F such that $\eta\sigma \simeq \sigma$ – such characters form a cyclic group. We deduce that the only unramified character twist σ' of σ which is self-dual but not isomorphic to σ is obtained as $\chi\sigma$, where χ is an unramified character of order $2t(\sigma)$. (If $r = \mathrm{val}_2(t(\sigma))$, any unramified character of order 2^{r+1} would do equally well.)

Let E be the unramified extension of F in $\bar{F}$ of degree $t(\sigma)$. Then σ is induced from a representation τ of W_E; the restriction of σ to W_E is the direct sum of the conjugates of τ under $\mathrm{Gal}(E/F)$, which are pairwise inequivalent. As σ is self-dual, $\tau^\vee$ is one of those conjugates.

Assume first that τ is self-dual – which, we remark, is necessarily true if $t(\sigma)$ is odd. Since $t(\tau) = 1$, the unramified twist τ' of τ which is self-dual but not isomorphic to τ has the form $\chi\tau$, where χ is the order 2 unramified character of W_E, and it has the same parity as τ. Since induction for self-dual representations preserves the parity, we deduce that σ and σ' share the same parity too.

Assume then that τ is not self-dual. Then $\tau^\vee$ is necessarily isomorphic to τ^γ, where γ is the order 2 element of $\mathrm{Gal}(E/F)$. Let $\tilde{E} = E^\gamma$, so that $E/\tilde{E}$ is quadratic, and let T be the (irreducible) representation of $W_{\tilde{E}}$ induced from τ. As $\tau^\vee \simeq \tau^\gamma$, we see that T is self-dual. Its restriction to W_E is $\tau \oplus \tau^\vee$, with τ not isomorphic to $\tau^\vee$, so the W_E-invariant bilinear forms on the space of T form a space of dimension 2, with a line of alternating forms and a line of symmetric ones. Each of these lines is invariant under $\mathrm{Gal}(E/F)$, one offering the trivial representation, the other the order 2 character ω of $\mathrm{Gal}(E/F)$. The self-dual unramified twist T' of T which is not isomorphic to T is $T' = \eta T$ where η is unramified of order 4, so that $T' \otimes T' \simeq \omega T \otimes T$. From the previous analysis, we deduce that if T is symplectic then T' is orthogonal and conversely: T and T' have different parities. By induction again we see that σ and σ' have different parities.

6.2. Let us look at some special cases. Assume first that σ is tame. Then $t(\sigma) = \dim(\sigma)$. Introducing E and τ as in paragraph 6.1, we have that τ is a character, regular under the action of $\mathrm{Gal}(E/F)$. If τ were self-dual it would have order 1 or 2, but any character of $E^\times$ of order 1 or 2 factors through $N_{E/F}$, hence can be regular under the action of $\mathrm{Gal}(E/F)$ only if $t(\sigma) = \dim(\sigma) = 1$, so $E = F$. Thus, apart from quadratic characters of W_F, *tame* self-dual irreducible representations σ of W_F have even dimension, and we can apply the discussion of paragraph 6.1 to them, concluding that σ and σ' have different parities.

6.3. We now assume that σ is *not tame*, but we concentrate on our case of interest: that is, we assume from now on that *p is odd*. We want in that case to spot when σ and σ' have the same parity, and then try to say whether they are orthogonal or symplectic.

Let us first analyse σ. Its restriction to the wild ramification subgroup $\mathscr{P}_F$ of W_F is non-trivial, since σ is not tame. Let γ be an irreducible component of this restriction – so that γ is not the trivial character of $\mathscr{P}_F$ – and $S = S_\gamma$ its stabilizer in W_F. Then by Clifford theory σ is induced from the representation of S on the isotypical component $\mathcal{V}(\gamma)$ of γ in the space $\mathcal{V}$ of σ.

Now by assumption σ is self-dual, and so is its restriction to $\mathscr{P}_F$. But $\mathscr{P}_F$ is a pro-p-group and p is odd, so no non-trivial irreducible representation of $\mathscr{P}_F$ is self-dual, and we see that $\gamma^\vee$ is not isomorphic to γ. Thus there is g in $W_F\backslash S$ with ${}^g\gamma$ isomorphic to $\gamma^\vee$; the coset gS is the same for all possible choices of g, and g^2 belongs to S, so $\tilde{S} = S \cup gS$ is a subgroup of W_F containing S as an index 2 subgroup.

To get σ, we can first induce $\mathcal{V}(\gamma)$ from S to $\tilde{S}$, and then from $\tilde{S}$ to W_F. We shall prove now that $\mathrm{Ind}_S^{\tilde{S}}\,\mathcal{V}(\gamma)$ is self-dual; its parity is then inherited by σ. This reduces the problem to understanding the parity of $\mathrm{Ind}_S^{\tilde{S}}\,\mathcal{V}(\gamma)$.

6.4. To prove that $\mathrm{Ind}_S^{\tilde{S}}\,\mathcal{V}(\gamma)$ is self-dual, we take an abstract viewpoint:

Proposition. *Let $\mathscr{G}$ be a group with a subgroup $\mathscr{H}$ of index 2, and let $g \in \mathscr{G}\backslash\mathscr{H}$. Let $(\rho, \mathcal{V})$ be an irreducible representation of $\mathscr{H}$. Assume that ρ is not self-dual, but that $\rho^\vee$ is equivalent to ${}^g\rho$. Then $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho$ is irreducible and self-dual. If $\dim \mathcal{V}$ is odd, then $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho$ is symplectic if and only if its determinant is trivial.*

Proof. Since ${}^g\rho$ is not isomorphic to ρ, the induced representation $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho$ is irreducible, and it is self-dual because $(\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho)^\vee$ is isomorphic to $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho^\vee$ hence to $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,{}^g\rho$, itself isomorphic to $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho$. If $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho$ is symplectic, then clearly its determinant is trivial. To prove the converse statement when $\dim \mathcal{V}$ is odd, we need to analyse the situation carefully.

Since $\rho^\vee$ is equivalent to ${}^g\rho$, there is a non-degenerate bilinear form $\Phi : \mathcal{V} \times \mathcal{V} \to \mathbb{C}$ such that

$$\Phi(hv, ghg^{-1}v') = \Phi(v, v') \quad \text{for all } h \in \mathscr{H}, v, v' \in \mathcal{V}.$$

It is unique up to scalar. We claim that the form Ψ, defined by $\Psi(v, v') = \Phi(v', g^2 v)$ for v, v' in $\mathcal{V}$, is proportional to Φ. Indeed, for $v, v' \in \mathcal{V}$ and $h \in \mathscr{H}$, we find

$$\Psi(hv, ghg^{-1}v') = \Phi(ghg^{-1}v', g^2 hv) = \Phi(v', g^2 v) = \Psi(v, v').$$

Writing $\Psi = \lambda\Phi$ with $\lambda \in \mathbb{C}^\times$, we compute

$$\Phi(v', v) = \Phi(g^2 v', g^2 v) = \Psi(v, g^2 v') = \lambda\Phi(v, g^2 v') = \lambda\Psi(v', v) = \lambda^2\Phi(v', v)$$

for v, v' in $\mathcal{V}$ so that $\lambda^2 = 1$. We shall see that the parity of $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}}\,\rho$ is governed by the scalar λ.

On the space $\mathcal{V} \oplus \mathcal{V}$ equipped with the representation $\rho \oplus {}^g\rho$, there is an $\mathscr{H}$-invariant symplectic form f, unique up to scalar, which we can take to be

$$f : ((v_1, v_2), (w_1, w_2)) \longmapsto \Phi(v_1, w_2) - \Phi(w_1, v_2).$$

The space of $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}} \rho$ can be taken as $\mathcal{V} \oplus \mathcal{V}$ where $\mathscr{H}$ acts as $\rho \oplus {}^g\rho$ and g acts via

$$g(v_1, v_2) = (v_2, g^2 v_1).$$

Since $\Phi(v_2, g^2 v_1) = \Psi(v_1, v_2) = \lambda\Phi(v_1, v_2)$, we get that g acts on f by multiplication by $-\lambda$, so $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}} \rho$ is symplectic if and only if $\lambda = -1$.

Let us choose a basis $(e_1, \ldots, e_d)$ of $\mathcal{V}$, where $d = \dim \rho$. Then $\Phi(v_1, v_2) = ({}^t\mathbf{x}_1)H\mathbf{x}_2$, for $v_1, v_2 \in \mathcal{V}$ with coordinates given by $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{C}^d$ respectively, and H the $d \times d$ Gram matrix of Φ in the basis. If M_{g^2} is the matrix of $\rho(g^2)$ we get $\Phi(v_2, g^2 v_1) = ({}^t\mathbf{x}_2)HM_{g^2}\mathbf{x}_1$, from which we deduce that $HM_{g^2} = \lambda({}^tH)$, which implies that $\det \rho(g^2) = \lambda^d$.

Now $\det(\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}} \rho)$ is an order 2 character of $\mathscr{G}$ which is trivial on $\mathscr{H}$; in fact it is given by

$$(\det \rho \circ \mathrm{Ver})\ \omega^{\dim \rho}$$

where $\mathrm{Ver} : \mathscr{G} \longmapsto \mathscr{G}^{\mathrm{ab}} \longmapsto \mathscr{H}^{\mathrm{ab}}$ is the transfer and ω is the non-trivial character of $\mathscr{G}$ trivial on $\mathscr{H}$. In this special case where $\mathscr{H}$ has index 2 in $\mathscr{G}$, the transfer map Ver is trivial on $\mathscr{H}$ and sends g to g^2, so $\det(\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}} \rho(g)) = (-\lambda)^d$.

When d is odd, we find that $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}} \rho$ is symplectic if and only if its determinant is trivial, as desired. □

Remark. When d is even, $\mathrm{Ind}_{\mathscr{H}}^{\mathscr{G}} \rho$ always has trivial determinant, regardless of its parity. Determining the parity amounts to computing the scalar λ.

6.5. We revert to the context of paragraphs 6.1–6.3. We want to spot the cases where σ and σ' (in the notation of paragraph 6.1) have the same parity, and in those cases possibly apply 6.4 Proposition to determine that parity. For that we have to analyse the situation further.

It is known (see [9, 1.3 Proposition]) that γ extends to a representation Γ of $S = S_\gamma$, and we can even impose that $\det \Gamma$ have order a power of p; then Γ is unique up to twist by an unramified character of S, of order a power of p. Since ${}^g\gamma$ is equivalent to $\gamma^\vee$, we see that ${}^g\Gamma$ is equivalent to $\chi\Gamma^\vee$ where χ is an unramified character of S of order a power of p. Such a χ has a unique square root η with order a power of p and replacing Γ with $\eta^{-1}\Gamma$, we may – and do – assume that ${}^g\Gamma \simeq \Gamma^\vee$. This now specifies Γ completely.

As a representation of S, the space $\mathcal{V}(\gamma)$ is a tensor product $\Gamma \otimes \delta$, where δ is an irreducible representation of S trivial on $\mathscr{P}_F$, well-defined up to isomorphism. Since ${}^g\mathcal{V}(\gamma) \simeq \mathcal{V}(\gamma)^\vee$ as representations of S, we get that ${}^g\delta \simeq \delta^\vee$.

Let K be the fixed field of S, and $\tilde{K}$ that of $\tilde{S}$; thus the extension $K/\tilde{K}$ is quadratic, in particular tame. Writing $d = \dim \delta$, the representation δ is induced from a character α of

the unramified degree d extension K_d of K in $\bar{F}$, with α tamely ramified and regular under the action of $\mathrm{Gal}(K_d/K)$; this character α is determined up to the action of $\mathrm{Gal}(K_d/K)$.

In those terms, we try to see when σ and σ' have the same parity; that is, writing $\sigma = \mathrm{Ind}\,\tau$ as in paragraph 6.1, where τ is a representation of W_E with E/F unramified of degree $t(\sigma)$, we want to know if τ is self-dual. Note that $t(\mathcal{V}(\gamma)) = d$, so $t(\sigma) = df(K/F)$, where $f(K/F)$ is the inertia degree of K/F. The extension K_d/E is totally tamely ramified, and we can take τ to be $\mathrm{Ind}(\Gamma \otimes \alpha)$ where the induction is from W_{K_d} to W_E (and we first restrict Γ from S to W_{K_d}).

6.6. The following result describes when σ, σ' have the same parity.

Proposition. *Let σ be a self-dual irreducible representation of W_F. Assume σ is not tame, and adopt the above notation. Then the following are equivalent:*

(i) *σ and σ' have the same parity;*
(ii) *$K/\tilde{K}$ is ramified and $d = 1$.*

When these conditions are satisfied, σ and σ' are symplectic if and only if the character α is ramified.

Remark. When $d = 1$, we see that α is a tame character of $K^\times$ which satisfies ${}^g\alpha = \alpha^{-1}$. If $K/\tilde{K}$ is ramified, g acts trivially on the residue field of K, and $\alpha_{|U_K}$ has order 1 or 2. In that case, let ϖ be a uniformizer of K with $\varpi^2 \in \tilde{K}$; then the condition ${}^g\alpha = \alpha^{-1}$ translates into $\alpha(-\varpi^2) = 1$: either α is unramified of order 1 or 2 or $\alpha_{|\tilde{K}^\times}$ is the quadratic character $\omega_{K/\tilde{K}}$ defining K.

Proof. To prove the proposition, we need to see when $\tau = \mathrm{Ind}_{W_{K_d}}^{W_E}(\Gamma \otimes \alpha)$ is self-dual. The restriction of $\Gamma \otimes \alpha$ to $\mathscr{P}_F$ is γ, so τ can be self-dual only if there is h in W_E such that ${}^h\gamma \simeq {}^g\gamma$ – that is $h \in gS$, or equivalently $W_E \cap S \neq W_E \cap \tilde{S}$. Recalling that E is the maximal unramified extension of F in K_d, we see that the fixed field of $W_E \cap S$ is K_d; if $K/\tilde{K}$ were unramified, the fixed field of $W_E \cap \tilde{S}$ would also be K_d, so that τ could not be self-dual.

Thus, if τ is self-dual then $K/\tilde{K}$ is ramified and we take g in $W_E \cap \tilde{S}$. Reasoning as in paragraph 6.3 and using 6.4 Proposition, we see that τ is self-dual if and only if $\Gamma \otimes \alpha$ induces to a self-dual representation of $W_{\tilde{K}_d}$, where $\tilde{K}_d$ is the fixed field of g in K_d (so that $K_d/\tilde{K}_d$ is quadratic ramified); in particular we then have ${}^g(\Gamma \otimes \alpha) \simeq (\Gamma \otimes \alpha)^\vee$. Since ${}^g\Gamma \simeq \Gamma^\vee$ by construction, this implies ${}^g\alpha = \alpha^{-1}$ and since $K_d/\tilde{K}_d$ is ramified, g acts trivially on the residue field of K_d so $\alpha_{|U_{K_d}}$ has order 1 or 2 and regularity with respect to $\mathrm{Gal}(K/K_d)$ implies $d = 1$. Thus if τ is self-dual then $d = 1$, which proves (i) $\Rightarrow$ (ii).

Conversely if (ii) is satisfied then τ is self-dual if and only if ${}^g\alpha = \alpha^{-1}$ by the above analysis, which gives (ii) $\Rightarrow$ (i).

Assume finally that conditions (i) and (ii) are satisfied. Using again 6.4 Proposition, we have to check whether the determinant of $\mathrm{Ind}_{W_K}^{W_{\tilde{K}}}(\Gamma\otimes\alpha)$ – which by self-duality has order 1 or 2 – is trivial. Seeing that determinant as a character of $\tilde{K}^\times$ (via class field theory), it is equal to

$$\nu = \det(\Gamma\otimes\alpha)_{|\tilde{K}^\times}(\omega_{K/\tilde{K}})^{\dim\gamma}.$$

But $\det\Gamma$ has order a power of p and p is odd, and α has order at most 4 (cf. the remark above) so we find $\nu = (\alpha_{|\tilde{K}^\times}\omega_{K/\tilde{K}})^{\dim\gamma}$. If α is unramified then $\nu = \omega_{K/\tilde{K}}$ (since $\dim\gamma$ is odd) is non-trivial; if α is ramified then $\alpha_{|\tilde{K}^\times} = \omega_{K/\tilde{K}}$ by the remark and ν is trivial. The final claim of the proposition now follows from 6.4 Proposition. □

6.7. Now we interpret the conditions of 6.4 Proposition in terms of the cuspidal representation ρ of $\mathrm{GL}_n(F)$, with $n = \dim\sigma$, which corresponds to σ under the Langlands correspondence. To describe this representation ρ we will use the machinery of the construction of cuspidal representations as in §1.

Assume σ is not tame, i.e. ρ is not of depth zero. Then ρ contains a simple character $\tilde{\theta}$, belonging to a set of simple characters built using an element $\beta\in\mathrm{GL}_n(F)$ which generates a field $F[\beta]$. We have $n = d[F[\beta]:F]$, so that d is determined by ρ. Moreover the extension K/F which appears above in the discussion on the construction of σ is isomorphic to the maximal tame subextension L/F of $F[\beta]/F$ (see [9, Tame Parameter Theorem]).

When ρ – equivalently σ – is self-dual, we can choose β such that the self duality comes from an automorphism $x\mapsto\bar{x}$ of $F[\beta]$, sending β to $-\beta$, and $\tilde{\theta}$ to $\tilde{\theta}^{-1}$ (see [3, Theorem 1]). That automorphism induces an order 2 automorphism of L; let $\tilde{L}$ be its fixed field.

Proposition. *The extensions $K/\tilde{K}$ and $L/\tilde{L}$ are isomorphic.*

Thus condition (ii) in 6.6 Proposition can be translated in terms of ρ. See below (paragraph 6.8) for a translation of the last assertion of *loc. cit.*.

Proof. The proof relies on the compatibility of tame lifting of simple characters with the induction process for Weil group representations [7, 5, 9]. Choose an isomorphism ι of K/F onto L/F.

The representation σ_K of W_K on $\mathcal{V}(\sigma)$ corresponds to a (cuspidal) representation ρ_L of $\mathrm{GL}_m(L)$, where $m = d[F[\beta]:L]$; the simple character $\tilde{\theta}_L$ appearing in ρ_L is an L/F-lift of $\tilde{\theta}$ and L/F is the maximal tame extension such that $\tilde{\theta}$ has a lift to $\mathrm{GL}_{[F[\beta]:L]}(L)$.

If K' is intermediate between F and K, and $L' = \iota(K')$, then $\sigma_{K'} = \mathrm{Ind}_{W_K}^{W_{K'}}\mathcal{V}(\sigma)$ corresponds to a (cuspidal) representation $\rho_{L'}$ of $\mathrm{GL}_{m'}(L')$, with $m' = d[F[\beta]:L']$, and the simple character $\tilde{\theta}_{L'}$ appearing in $\rho_{L'}$ is an L'/F-lift of $\tilde{\theta}$ and lifts to $\tilde{\theta}_L$ in L/L'. But $\tilde{K}$ is the maximal intermediate subfield K' such that $\sigma_{K'}$ is self-dual. Because the Langlands correspondence

is compatible with taking contragredients, the field $\iota(\tilde{K})$ is the maximal field L' intermediate between F and L such that $\tilde{\theta}_{L'}$ is self-dual (i.e. conjugate to $\tilde{\theta}_{L'}^{-1}$ in $\mathrm{GL}_{[F[\beta]:L']}(L')$). Thus $\iota(\tilde{K}) = \tilde{L}$. □

6.8. Now assume that $d = 1$ and $L/\tilde{L}$ (or equivalently $K/\tilde{K}$) is ramified. We want to express the condition that α is ramified in 6.6 Proposition in terms of ρ. For that we have to review a little bit the construction of ρ from $\tilde{\theta}$ from Section 1, whose notation we use.

We also continue with the notation for ρ introduced in the previous paragraph. Recall that, since $d = 1$, we have $n = [F[\beta] : F]$. The simple character $\tilde{\theta}$ is a character of $\tilde{H}^1$ and we have the open subgroups $\tilde{J}^1$, $\tilde{J}$ of $\mathrm{GL}_n(F)$. We write $\tilde{\eta}$ for the unique irreducible representation of $\tilde{J}^1$ containing $\tilde{\theta}$. Then $\tilde{J} = U_{F[\beta]}\tilde{J}^1$ and, by the Types Theorem [9, 7.6 Theorem], there is a unique beta-extension $\tilde{\kappa}$ such that $\operatorname{tr}\tilde{\kappa}$ is constant on the roots of unity of $F[\beta]$ of order prime to p which are regular for the action of $\mathrm{Gal}(K_{nr}/F)$, where K_{nr} is the maximal unramified extension of F in K. Moreover, the same result gives that ρ contains the representation $\tilde{\kappa} \otimes \omega\alpha$ of J, where α is seen as a character of $J/J^1 \simeq U_{F[\beta]}/U^1_{F[\beta]} \simeq U_K/U^1_K$ and ω is the order 2 character of $U_{F[\beta]}$.

Thus we conclude that ρ is symplectic when ρ contains $\tilde{\kappa}$, and is orthogonal when ρ contains $\omega\tilde{\kappa}$.

6.9. We have discussed at length above the ambiguity between σ and σ' inherent to our method – of course when σ and σ' have different parities it is the orthogonal one that features.

Let us now briefly mention a few favourable circumstances when our methods do allow us to determine completely the parameter of a cuspidal representation π of $\mathrm{Sp}_{2N}(F)$.

Since the parameter ϕ of π is orthogonal of dimension $2N + 1$, one irreducible component must have odd dimension. But in our case where p is odd, the only irreducible orthogonal representations of W_F with odd dimension are the four quadratic characters of W_F. Thus at least one of them, say ω, has to occur in the parameter, and if the Jordan block it belongs to is (ω, m) then m has to be congruent to 1 (mod 4), to yield an odd-dimensional contribution to ϕ; the contribution to the determinant is then ω. We then see that if we know all other components, then we can decide between ω and ω' by taking into account the condition $\det\phi = 1$. To know all the other components σ, it is necessary that for each of them, σ and σ' have different parities. We conclude that it will be rather rare that we determine ϕ without ambiguity.

Let us give just a few examples in low dimension. See [25] for a discussion of depth zero cases.

<u>$N = 1$, $\mathrm{SL}_2(F)$</u>

The parameter is either $\rho \oplus \omega$ with ρ irreducible orthogonal of dimension 2 and $\omega = \det\rho$, or $\omega_1 \oplus \omega_2 \oplus \omega_3$ where the ω_i's are the non-trivial quadratic characters of W_F. In terms

of homomorphisms $W_F \longrightarrow \mathrm{SO}_3(\mathbb{C}) \simeq \mathrm{PGL}_2(\mathbb{C})$, the second case corresponds to a triply imprimitive representation of W_F, the first case to a simply imprimitive one [8]. In the first case, our methods allow us to determine ρ only if it is induced from the quadratic unramified extension of F (i.e., in fact, when ω is unramified of order 2).

$N = 2$, $\mathrm{Sp}_4(F)$

There has to be a quadratic character ω of W_F occurring with Jordan block $(\omega, 1)$ only. If another quadratic character η occurs, the Jordan block can be $(\eta, 1)$ or $(\eta, 3)$. In the latter case $\phi = \omega \oplus \eta \oplus \eta \otimes \mathrm{St}_3$ and the determinant condition implies that ω is trivial and consequently that η is not trivial. If our computation shows that both 1 and the non-trivial quadratic unramified character ω_{nr} occur, then the parameter is necessarily $\phi = 1 \oplus \omega_{nr} \oplus \omega_{nr} \otimes \mathrm{St}_3$; if, on the contrary, our method gives that a ramified quadratic character η occurs, then we cannot distinguish between η and $\eta' = \eta\omega_{nr}$.

Let us look at the case where two distinct characters ω, η occur with Jordan blocks $(\omega, 1)$ and $(\eta, 1)$ only. Then a third character, ν say, must also occur and $\phi = \omega \oplus \eta \oplus \nu \oplus \rho$ where ρ is irreducible orthogonal of dimension two. The determinant of ρ is the quadratic character $\omega_{E/F}$ defining the extension from which ρ is induced so that the determinant condition on ϕ is $\omega\eta\nu\omega_{E/F} = 1$.

When E/F is unramified, there is no ambiguity in ρ in our computation, and the parameter is

$$\phi = 1 \oplus \mu \oplus \mu' \oplus \rho$$

where μ, μ' are the two ramified quadratic characters of W_F.

When E/F is ramified, the parameter could be

$$\phi = 1 \oplus \omega_{nr} \oplus \omega_{nr}\omega_{E/F} \oplus \rho, \qquad \text{or} \qquad \phi = 1 \oplus \omega_{nr} \oplus \omega_{nr}\omega_{E/F} \oplus \rho'$$

and we cannot resolve the ambiguity between ρ and ρ'.

Finally if there is only one quadratic character ω of W_F occurring in ϕ, we can compute ω, and thus determine ϕ completely, only if the other components (necessarily even-dimensional) offer no ambiguity.

We hope to come back to the case of $\mathrm{Sp}_4(F)$ in a sequel to this paper, where a refinement of our methods will allow a more complete determination of ϕ.

7. Langlands correspondence and ramification

In this final section we interpret our results on the endoscopic transfer map in terms of the Langlands correspondence for G. In particular, we prove a Ramification Theorem for the symplectic group G, giving a bijection between self-dual endo-classes and self-dual orbits of irreducible representations of the wild inertia group $\mathscr{P}_F$ which is simultaneously compatible (in a suitable sense) with the Langlands correspondence for symplectic groups over F in all dimensions.

7.1. We first recall the Ramification Theorem for general linear groups, from [6, 8.2 Theorem] (see also [9, 6.3 Theorem]). Recall that $\mathcal{E}(F)$ denotes the set of endo-classes over F. We write $W_F\backslash \operatorname{Irr}(\mathscr{P}_F)$ for the set of W_F-orbits of irreducible representations of $\mathscr{P}_F$. By abuse of notation, we will identify such an orbit with the direct sum of the inequivalent irreducible representations in the orbit; thus, for γ an irreducible representation of $\mathscr{P}_F$ with stabiliser S, we identify its W_F-orbit $[\gamma]$ with $\bigoplus_{W_F/S} {}^g\gamma$. In particular, we can then talk of the dimension of an orbit.

Given an irreducible representation of W_F, by Mackey theory its restriction to $\mathscr{P}_F$ is a multiple of a single W_F-orbit of irreducible representations, so we get a natural map $\operatorname{Irr}(W_F) \to W_F\backslash \operatorname{Irr}(\mathscr{P}_F)$, which is surjective.

Theorem. *There is a unique bijection $\mathcal{E}(F) \to W_F\backslash \operatorname{Irr}(\mathscr{P}_F)$, $\boldsymbol{\Theta} \mapsto [\gamma(\boldsymbol{\Theta})]$, which is compatible with the local Langlands correspondence:*

$$\begin{tikzcd} \bigcup_{n\geqslant 1} \operatorname{Cusp}(\mathrm{GL}_n(F)) \arrow[r, "\sim"] \arrow[d, two heads] & \operatorname{Irr}(W_F) \arrow[d, two heads] \\ \mathcal{E}(F) \arrow[r, "\sim"] & \mathcal{W}_F\backslash \operatorname{Irr}(\mathscr{P}_F) \end{tikzcd}$$

Moreover we have $\deg(\boldsymbol{\Theta}) = \dim[\gamma(\boldsymbol{\Theta})]$.

7.2. Now we consider how this bijection behaves with respect to duality. Recall that we write $\mathcal{E}^{\mathsf{sd}}(F)$ for the set of self-dual endo-classes; that is, those endo-classes $\boldsymbol{\Theta}$ for which there is a self-dual simple character $\tilde{\theta}$ with endo-class $\boldsymbol{\Theta}$. If the endo-class is non-trivial then $\tilde{\theta}$ is associated to a skew simple stratum $[\Lambda, -, 0, \beta]$ and the associated field $E = F[\beta]$ has degree n over F and is equipped with a Galois involution with fixed field E_{o}. If $\boldsymbol{\Theta}$ is the trivial endo-class then we have $E = E_{\mathsf{o}} = F$.

It will be useful to have the following result, which guarantees the existence of self-dual cuspidal representations of general linear groups with given (self-dual) endo-class.

Lemma. *Let $\boldsymbol{\Theta}$ be a self-dual endo-class and E/E_{o} as above. Let m be an integer which is*

(i) *odd, if E/E_{o} is unramified quadratic,*
(ii) *1 or even, if E/E_{o} is ramified quadratic,*
(iii) *even, if $E = F$,*

and put $n = m\deg(\boldsymbol{\Theta})$. Then there are (at least) two inequivalent orthogonal self-dual cuspidal representations of $\mathrm{GL}_n(F)$ with endo-class $\boldsymbol{\Theta}$, and two inequivalent symplectic self-dual cuspidal representations of $\mathrm{GL}_n(F)$ with endo-class $\boldsymbol{\Theta}$.

Note that, in the case that $E = F$ (so $\boldsymbol{\Theta}$ is trivial) and $m = 1$, there are four inequivalent self-dual (cuspidal) representations of $\mathrm{GL}_1(F)$ with endo-class $\boldsymbol{\Theta}$ but all four are orthogonal; they are the four quadratic characters.

Proof. Suppose first that $\mathbf{\Theta}$ is non-trivial. Let $\widetilde{\theta}$ be a self-dual simple character with endo-class $\mathbf{\Theta}$, as above, with associated skew simple stratum $[\Lambda, -, 0, \beta]$ and $E = F[\beta]$. Then any transfer (in the sense of simple characters) of $\tilde{\theta}$ is also self-dual, by [34, Corollary 2.13].

Let m be an integer as in the hypotheses of the lemma and let f be a non-degenerate skew-hermitian form on an m-dimensional E-vector space V such that the associated unitary group (a group over E_o) is quasi-split. We write $\mathfrak{o}_E^\mathsf{o}$ for the ring of integers of E_o and $\mathfrak{p}_E^\mathsf{o}$ for its unique maximal ideal, with k_E^o the residue field. Fix λ_o an F-linear form on E_o such that $\{e \in E_\mathsf{o} \mid \lambda_\mathsf{o}(e\mathfrak{o}_E^\mathsf{o}) \subseteq \mathfrak{p}_F\} = \mathfrak{p}_E^\mathsf{o}$, and consider the form $h = \lambda_\mathsf{o} \circ \mathrm{tr}_{E/E_\mathsf{o}} \circ f$ on V. Thinking of V as an n-dimensional F-vector space, this is a nondegenerate alternating form. We take the transfer $\tilde{\theta}_m$ of $\tilde{\theta}$ to the unique (up to conjugacy) self-dual $\mathfrak{o}_E$-lattice chain Λ_m on V such that $\Lambda_m(0) \neq \Lambda_m(1)$. Thus $\tilde{\theta}$ is a self-dual simple character of endo-class $\mathbf{\Theta}$.

Denote by $\underline{\tilde{\kappa}}_m$ the unique p-primary extension of $\tilde{\theta}_m$, and denote by $\tilde{J}_m$ the group on which it lives; then, by uniqueness, $\underline{\tilde{\kappa}}_m$ is self-dual (that is, invariant under the involution σ defining the symplectic group $\mathrm{Sp}_F(V)$). Now $\underline{\tilde{\kappa}}_m$ extends to a representation $\underline{\tilde{\mathcal{K}}}_m$ of $E^\times \tilde{J}_m$ with determinant a power of p and any two such extensions differ by an unramified character of order a power of p. In particular, $\underline{\tilde{\mathcal{K}}}_m \circ \sigma$ is another such extension so has the form $\underline{\tilde{\mathcal{K}}}_m \otimes \chi$, for χ unramified of order a power of p. Since p is odd, χ has a unique square root χ' of order a power of p, and then we can replace $\underline{\tilde{\mathcal{K}}}_m$ by $\underline{\tilde{\mathcal{K}}}_m \otimes \chi'$, which is self-dual.

Now we consider the quotient $\tilde{J}_m/\tilde{J}_m^1 \simeq \tilde{P}(\Lambda_{m,\mathfrak{o}_E})/\tilde{P}^1(\Lambda_{m,\mathfrak{o}_E}) \simeq \mathrm{GL}_m(k_E)$. The involution σ also acts here, with fixed points a unitary group if E/E_o is unramified and a symplectic group if E/E_o is ramified (in the latter case, it is symplectic rather than orthogonal because $\Lambda_m(0) \neq \Lambda_m(1)$); the action of σ is conjugate to the map transpose-inverse-$\mathrm{Gal}(k_E/k_E^\mathsf{o})$-conjugate. The conditions on m are then precisely those required for the existence of a $\mathrm{Gal}(k_E/k_E^\mathsf{o})$-self-dual cuspidal representation $\tilde{\tau}$ of $\mathrm{GL}_m(k_E)$ (that is, such that the Galois conjugate of $\tilde{\tau}$ is equivalent to $\tilde{\tau}^\vee$) – see [1, Theorem 7.1] in the case $k_E = k_E^\mathsf{o}$ and [22, Corollary 5.8] in the case $k_E \neq k_E^\mathsf{o}$.

Let ω be a quadratic character of E, necessarily tame since p is odd. We also write ω for the character of $k_E^\times$ induced by restricting ω; then the representation $\tilde{\tau}\omega$ is also $\mathrm{Gal}(k_E/k_E^\mathsf{o})$-self-dual. We inflate $\tilde{\tau}\omega$ to $\tilde{J}_m$ and extend to a representation $\tilde{\mathcal{T}}_\omega$ of $E^\times \tilde{J}_m$ by setting $\tilde{\mathcal{T}}_\omega(\varpi_E) = \omega(\varpi_E) Id_{\tilde{\tau}\omega}$, for ϖ_E a fixed uniformizer of E such that $\overline{\varpi}_E = (-1)^{e(E/E_\mathsf{o})}\varpi_E$, where $x \mapsto \overline{x}$ denotes the generator of $\mathrm{Gal}(E/E_\mathsf{o})$. This representation $\tilde{\mathcal{T}}_\omega$ is then self-dual, that is, equivalent to $\tilde{\mathcal{T}}_\omega \circ \sigma$.

Finally, the representation $\rho_\omega = \text{c-Ind}_{E^\times \tilde{J}}^{\mathrm{GL}_n(F)} \underline{\tilde{\mathcal{K}}}_m \otimes \tilde{\mathcal{T}}_\omega$ is then irreducible and cuspidal, and equivalent to $\rho_\omega \circ \sigma$. Since the involution σ is a conjugate of the involution transpose-inverse, by a theorem of Gelfand–Kazhdan [16, Theorem 2], the representation $\rho_\omega \circ \sigma$ is equivalent to $\rho_\omega^\vee$.

Thus we have constructed four self-dual cuspidal representations ρ_ω of $\mathrm{GL}_m(F)$ with endo-class $\mathbf{\Theta}$, and it remains only to see that two are orthogonal and two symplectic. Note that ρ_ω

and $\rho_{\omega'}$ are unramified twists of each other if and only if $\omega^{-1}\omega'$ is the non-trivial unramified quadratic character ω_{nr}. If either $m > 1$ or E/E_{o} is unramified then, by 6.6 Proposition and 6.7 Proposition, the representations ρ_ω and its self-dual unramified twist $\rho_{\omega\omega_{nr}}$ have opposite parities so we are done. (Note that, writing L for the maximal tame subextension of E/F and L_{o} for that of E_{o}/F, we have that L/L_{o} is ramified if and only if E/E_{o} is ramified, since p is odd.)

On the other hand, if $m = 1$ and E/E_{o} is ramified then we are in the situation of paragraph 6.8, and the argument there explains that one pair $\rho_\omega, \rho_{\omega\omega_{nr}}$ consists of two orthogonal representations, while the other pair consists of two symplectic representations, as required.

We are left with the case that $\mathbf{\Theta}$ is the trivial endo-class and m is even. The existence of self-dual cuspidal depth zero representations is [1, Theorem 7.1] and the argument that there are (at least) two orthogonal and two symplectic is formally exactly as in the previous case, with $\underline{\tilde{\mathcal{K}}}$ the trivial representation. □

We say that an orbit $[\gamma]$ in $W_F\backslash \operatorname{Irr}(\mathscr{P}_F)$ is self-dual if it is self-dual when considered as a representation of $\mathscr{P}_F$; that is, if there is $g \in W_F$ such that $\gamma^\vee \simeq {}^g\gamma$. We write $(W_F\backslash \operatorname{Irr}(\mathscr{P}_F))^{\mathsf{sd}}$ for the set of self-dual orbits. Then we have:

Proposition. *The bijection of 7.1 Theorem restricts to a bijection*

$$\mathcal{E}^{\mathsf{sd}}(F) \to (W_F\backslash \operatorname{Irr}(\mathscr{P}_F))^{\mathsf{sd}}\,. \tag{7.1}$$

Proof. Let γ be an irreducible representation of $\mathscr{P}_F$ and put $n = \dim[\gamma]$. Suppose that $[\gamma]$ is a self-dual orbit and let $g \in W_F$ be such that ${}^g\gamma \simeq \gamma^\vee$. Then, as in paragraph 6.5, there is a unique irreducible representation Γ of the stabilizer S of γ such that $\det\Gamma$ has order a power of p and ${}^g\Gamma \simeq \Gamma^\vee$. Then the representation $\operatorname{Ind}_S^{W_F}\Gamma$ is irreducible self-dual so the corresponding cuspidal representation ρ of $\mathrm{GL}_n(F)$ is also self-dual. By [3, 2.2 Corollary] (see also [17, p.10]), ρ contains a simple character with self-dual transfer to $\mathrm{GL}_{2n}(F)$, so the endo-class $\mathbf{\Theta}(\rho)$, which corresponds to $[\gamma]$ by 7.1 Theorem, is self-dual.

Conversely, let $\mathbf{\Theta}$ be a self-dual endo-class and put $n = \deg(\mathbf{\Theta})$. By the lemma, there is a self-dual cuspidal representation ρ of $\mathrm{GL}_n(F)$ with endo-class $\mathbf{\Theta}$. Then the corresponding irreducible representation of W_F is self-dual so the orbit in its restriction to $\mathscr{P}_F$ is also self-dual, as required. □

7.3. We now introduce the notion of *wild parameter*.

Definition. A *wild parameter (over F)* is a finite-dimensional semisimple complex representation $\mathcal{V}$ of $\mathscr{P}_F$ such that ${}^g\mathcal{V} \simeq \mathcal{V}$, for all $g \in W_F$. We write $\Psi(F)$ for the set of equivalence classes of wild parameters over F, and $\Psi_n(F)$ for the set of equivalence classes of n-dimensional wild parameters over F.

Equivalently, we can think of an element of $\Psi_n(F)$ as the $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of a homomorphism $\psi : \mathscr{P}_F \to \mathrm{GL}_n(\mathbb{C})$ for which there exists $A \in \mathrm{GL}_n(\mathbb{C})$ such that $\psi \circ \operatorname{Ad} g = \operatorname{Ad} A \circ \psi$, for all $g \in W_F$.

Thus a finite-dimensional semisimple complex representation $\mathcal{V}$ of $\mathscr{P}_F$ is a wild parameter if and only if, when we decompose it into its isotypic components $\mathcal{V} = \bigoplus_{\gamma \in \mathrm{Irr}(\mathscr{P}_F)} \mathcal{V}(\gamma)$, we have

$$\dim \mathcal{V}(\gamma) = \dim \mathcal{V}({}^g\gamma), \text{ for all } g \in W_F.$$

Therefore a wild parameter is equivalent to

$$\bigoplus_{W_F \backslash \mathrm{Irr}(\mathscr{P}_F)} m_{[\gamma]}[\gamma],$$

where we are thinking of the orbit $[\gamma]$ as the sum over the W_F-conjugates of $\gamma \in \mathrm{Irr}(\mathscr{P}_F)$, and $m_{[\gamma]} \in \mathbb{Z}_{\geqslant 0}$.

Equivalently, the n-dimensional wild parameters are precisely the restrictions to $\mathscr{P}_F$ of the Langlands parameters for $\mathrm{GL}_n(F)$; that is, writing $\Phi_n(F)$ for the set of admissible homomorphisms $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ up to conjugacy, and $\Phi(F) = \bigcup_{n \geqslant 1} \Phi_n(F)$, the natural map

$$\Phi(F) \to \Psi(F)$$

induced by $\phi \mapsto \phi_{|\mathscr{P}_F}$ is surjective. Indeed, by taking direct sums one need only check that, for any $\gamma \in \mathscr{P}_F$, there is a Langlands parameter ϕ whose restriction to $\mathscr{P}_F$ is isomorphic to $[\gamma]$. This, however, follows from the discussion in paragraph 6.5: γ extends to a representation Γ of its stabiliser S_γ by [9, 1.3 Proposition], and then $\mathrm{Ind}_{S_\gamma}^{W_F} \Gamma$ is the required Langlands parameter (with trivial $\mathrm{SL}_2(\mathbb{C})$ action).

Recall from paragraph 2.7 that an endo-parameter of degree n over F is a formal sum

$$\sum_{\boldsymbol{\Theta} \in \mathcal{E}} m_{\boldsymbol{\Theta}} \boldsymbol{\Theta}, \quad m_{\boldsymbol{\Theta}} \in \mathbb{Z}_{\geqslant 0}, \quad \text{such that} \quad \sum_{\boldsymbol{\Theta} \in \mathcal{E}} m_{\boldsymbol{\Theta}} \deg(\boldsymbol{\Theta}) = n.$$

We write $\mathcal{EE}_n(F)$ for the set of endo-parameters of degree n over F. Then the Ramification Theorem for GL_n (7.1 Theorem) together with the compatibility of the Langlands correspondence with parabolic induction immediately give:

Theorem. *The bijection of 7.1 Theorem induces, for each n, a bijection $\mathcal{EE}_n(F) \to \Psi_n(F)$ which is compatible with the Langlands correspondence:*

$$\begin{array}{ccc} \mathrm{Irr}(\mathrm{GL}_n(F)) & \xrightarrow{\sim} & \Phi_n(F) \\ \downarrow & & \downarrow \\ \mathcal{EE}_n(F) & \xrightarrow{\sim} & \Psi_n(F) \end{array}$$

7.4. Now we turn to the case of the symplectic group G and recall Arthur's local Langlands correspondence in this case.

We denote by $\Phi(G)$ the set of Langlands parameters for G, that is, the set of conjugacy classes of homomorphisms $\phi : W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2N+1}(\mathbb{C})$ such that the representation obtained by composing with the natural inclusion map $\iota : \mathrm{SO}_{2N+1}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2N+1}(\mathbb{C})$ is semisimple.

We denote by $\Phi^{\mathsf{disc}}(G)$ the set of *discrete* Langlands parameters, that is, those whose image is not contained in a proper parabolic subgroup of $\mathrm{SO}_{2N+1}(\mathbb{C})$; equivalently, $\iota\circ\phi$ is a direct sum of inequivalent irreducible orthogonal representations of $W_F\times\mathrm{SL}_2(\mathbb{C})$ and has determinant 1. Thus, given ϕ a discrete Langlands parameter, the representation $\iota\circ\phi$ decomposes as a multiplicity-free direct sum

$$\bigoplus_{i\in I}\sigma_i\otimes\mathrm{St}_{m_i}, \tag{7.2}$$

where St_m denotes the unique m-dimensional irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$, for $m\geqslant 1$, and the σ_i are irreducible self-dual representations of W_F, such that

- $\sum_{i\in I} m_i\dim(\sigma_i)=2N+1$,
- σ_i is symplectic if m_i is even and orthogonal if m_i is odd,
- $\prod_{i\in I}\det(\sigma_i)^{m_i}=1$.

We say that a discrete Langlands parameter ϕ is *cuspidal* if, whenever $\sigma\otimes\mathrm{St}_m$ is a subrepresentation of $\iota\circ\phi$ and $m>2$, the representation $\sigma\otimes\mathrm{St}_{m-2}$ is also a subrepresentation of $\iota\circ\phi$. We denote by $\Phi^{\mathsf{cusp}}(G)$ the set of cuspidal Langlands parameters.

As usual, for ϕ a discrete Langlands parameter, we denote by $\mathscr{S}_\phi$ the group of connected components of the centralizer in $\mathrm{SO}_{2N+1}(\mathbb{C})$ of the image of ϕ. This is a finite product of copies of the cyclic group of order 2; if $\iota\circ\phi$ decomposes as in (7.2), then $\mathscr{S}_\phi$ has order $2^{\#I-1}$.

Theorem ([2, Theorems 1.5.1 and 2.2.1], [28, Theorem 1.5.1])**.** *Suppose that F is of characteristic zero. There is a natural surjective map from the set of discrete series representations of G to $\Phi^{\mathsf{disc}}(G)$ with finite fibres, characterised by an equality of stable distributions via transfer to $\mathrm{GL}_{2N+1}(\mathbb{C})$. Moreover:*

- *the fibre of $\phi\in\Phi^{\mathsf{disc}}(G)$ is in bijection with the set of characters of $\mathscr{S}_\phi$;*
- *the fibre Π_ϕ of $\phi\in\Phi^{\mathsf{disc}}(G)$ contains a cuspidal representation of G if and only if ϕ is cuspidal, in which case $\Pi_\phi\cap\mathrm{Cusp}(G)$ is in bijection with the set of* alternating *characters of $\mathscr{S}_\phi$.*

We do not recall the definition of *alternating* character (see [28, §1.5]) but only recall that if, for ϕ a cuspidal Langlands parameter as in (7.2), we set $I_0=\{\sigma_i\mid\sigma_i\text{ is orthogonal}\}$, then there are $2^{\#I_0-1}$ alternating characters of $\mathscr{S}_\phi$. (Note that I_0 is non-empty, since one of the σ_i must be a quadratic character.) In particular, the L-packet of a cuspidal Langlands parameter ϕ consists only of cuspidal representations if and only if $m_i=1$, for all $i\in I$ (in the description (7.2); that is, each self-dual irreducible representation of W_F which appears in ϕ is orthogonal and appears with multiplicity one. In this case, we say that ϕ is *regular*.

7.5. We say that a wild parameter $\mathcal{V}$ is *self-dual* if it is self-dual as a representation of $\mathscr{P}_F$, in which case $\det(\mathcal{V})$ is trivial (since p is odd).

Given a self-dual wild parameter, we would like to see that there is a unique choice of orthogonal structure on it. This is indeed a special case of the following result on the

existence and uniqueness of orthogonal structures on self-dual representations of groups of odd order.

Proposition. *Let $\mathscr{G}$ be a finite group of odd order and let $\mathcal{V}$ be a finite dimensional complex representation of $\mathscr{G}$. If $\mathcal{V}$ is self-dual, then $\mathcal{V}$ is orthogonal: there is on $\mathcal{V}$ a $\mathscr{G}$-invariant non-degenerate symmetric bilinear form; moreover such a form is unique up to the action of* $\mathrm{Aut}_{\mathscr{G}}(\mathcal{V})$.

In other words, a self-dual representation of $\mathscr{G}$ is underlying a unique (up to isomorphism) orthogonal representation.

Proof. As $\mathscr{G}$ has odd order, the only self-dual irreducible representation of $\mathscr{G}$ is the trivial representation $1_{\mathscr{G}}$. For an irreducible representation γ of $\mathscr{G}$, let $\mathcal{V}(\gamma)$ be the γ-isotypic component of $\mathcal{V}$, and put $\mathcal{V}_\gamma = \mathrm{Hom}_{\mathscr{G}}(\gamma, \mathcal{V})$, so that $\mathcal{V}(\gamma)$ decomposes canonically as $\gamma \otimes \mathcal{V}_\gamma$. Then $\mathcal{V}$ is self-dual if and only if $\mathcal{V}_\gamma$ and $\mathcal{V}_{\gamma^\vee}$ have the same dimension for all γ.

Assume $\mathcal{V}$ is self-dual. For any $\mathscr{G}$-invariant non-degenerate symmetric bilinear form on $\mathcal{V}$, we can write $\mathcal{V}$ as the orthogonal direct sum of its subspaces $\mathcal{V}(1_{\mathscr{G}})$ and $\mathcal{V}(\gamma) \oplus \mathcal{V}(\gamma^\vee)$, for γ running through a set of representatives of the non-trivial irreducible representations up to contragredient. On $\mathcal{V}(1_{\mathscr{G}})$, where $\mathscr{G}$ acts trivially, there is a non-degenerate symmetric bilinear form, unique up to the action of $\mathrm{Aut}(\mathcal{V}(1_{\mathscr{G}}))$.

Therefore, for existence and uniqueness, it is enough to consider the case where $\mathcal{V} = \mathcal{V}(\gamma) \oplus \mathcal{V}(\gamma^\vee)$, for some non-trivial γ. Then the dual of $\mathcal{V}(\gamma)$ is $\gamma^\vee \otimes (\mathcal{V}_\gamma)^*$, whereas the dual of $\mathcal{V}(\gamma^\vee)$ is $\gamma \otimes (\mathcal{V}_{\gamma^\vee})^*$. An isomorphism $j : \mathcal{V} \to \mathcal{V}^\vee$ (that is, a self-duality on $\mathcal{V}$) is the direct sum of $Id_\gamma \otimes i$ and $Id_{\gamma^\vee} \otimes i'$, where i is an isomorphism of $\mathcal{V}_\gamma$ onto $(\mathcal{V}_{\gamma^\vee})^*$, and i' an isomorphism of $\mathcal{V}_{\gamma^\vee}$ onto $(\mathcal{V}_\gamma)^*$. The self-duality is orthogonal if and only if i and i' are transpose to each other.

Obviously there exists then an orthogonal structure on $\mathcal{V}$, and moreover all such structures are given by the choice of i (with i' its transpose). Since $\mathrm{Aut}_{\mathscr{G}}(\mathcal{V})$, which is the product $\mathrm{Aut}(\mathcal{V}_\gamma) \times \mathrm{Aut}(\mathcal{V}_{\gamma^\vee})$, acts transitively on the set of i, we have uniqueness too. □

7.6. Now let $\mathcal{V}$ be an n-dimensional self-dual wild parameter over F. By 7.5 Proposition, $\mathcal{V}$ then carries a $\mathscr{P}_F$-invariant nondegenerate symmetric bilinear form, unique up to the action of $\mathrm{Aut}_{\mathscr{P}_F}(\mathcal{V})$. Thus we can regard $\mathcal{V}$ as a homomorphism $\psi : \mathscr{P}_F \to \mathrm{SO}(\mathcal{V}) \simeq \mathrm{SO}_n(\mathbb{C})$.

For $\gamma \in \mathrm{Irr}(\mathscr{P}_F)$, we write $\mathcal{V}[\gamma]$ for the component of $\mathcal{V}$ corresponding to the orbit of γ under W_F; that is $\mathcal{V}[\gamma] = \sum_{g \in W_F} \mathcal{V}({}^g\gamma)$. We consider the stabiliser in $\mathrm{SO}(\mathcal{V})$ of the self-dual decomposition

$$\mathcal{V} = \bigoplus_{W_F \backslash \mathrm{Irr}(\mathscr{P}_F)} \mathcal{V}[\gamma]$$

and say that $\mathcal{V}$ is *discrete* if this stabiliser is contained in no proper Levi subgroup of $SO(\mathcal{V})$. Equivalently, the self-dual parameter $\mathcal{V}$ is discrete if and only if every orbit $[\gamma]$ in the support of $\mathcal{V}$ (that is, such that $\mathcal{V}[\gamma]$ is non-zero) is self-dual.

We write $\Psi_n^{\mathsf{sd}}(F)$ for the set of equivalence classes of discrete self-dual n-dimensional wild parameters over F. Note that the restriction to $\mathscr{P}_F$ of any discrete Langlands parameter for G is a discrete self-dual wild parameter of dimension $2N+1$, which explains the nomenclature.

Recall also that we have the set $\mathcal{EE}_n^{\mathsf{sd}}(F)$ of self-dual endo-parameters of degree n over F, which consists of those endo-parameters of degree n with support in the set $\mathcal{E}^{\mathsf{sd}}(F)$ of self-dual endo-classes. Then we have the following Ramification Theorem for G.

Theorem. *The bijection (7.1) induces, for each $N \geqslant 1$, a bijection $\mathcal{EE}_{2N}^{\mathsf{sd}}(F) \to \Psi_{2N+1}^{\mathsf{sd}}(F)$ which, when F is of characteristic zero, is compatible with the Langlands correspondence for cuspidal representations of G:*

$$\begin{tikzcd} \mathrm{Cusp}(G) \arrow[r, two heads] \arrow[d, two heads] & \Phi^{\mathsf{cusp}}(G) \arrow[d, two heads] \\ \mathcal{EE}_{2N}^{\mathsf{sd}}(F) \arrow[r, "\sim"] & \Psi_{2N+1}^{\mathsf{sd}}(F) \end{tikzcd}$$

The induced bijection $\mathcal{EE}_{2N}^{\mathsf{sd}}(F) \to \Psi_{2N+1}^{\mathsf{sd}}(F)$ is not as obvious as in the case of general linear groups. If we denote the bijection (7.1) by $\boldsymbol{\Theta} \mapsto [\gamma(\boldsymbol{\Theta})]$ then the induced map is

$$\sum_{\boldsymbol{\Theta}} m_{\boldsymbol{\Theta}} \boldsymbol{\Theta} \mapsto 1_{\mathscr{P}_F} \oplus \bigoplus_{\boldsymbol{\Theta}} m_{\boldsymbol{\Theta}} [\gamma(\boldsymbol{\Theta}^2)]. \tag{7.3}$$

We remark also that this Theorem asserts that the restriction map $\Phi^{\mathsf{cusp}}(G) \to \Psi_{2N+1}^{\mathsf{sd}}(F)$ is surjective, so that every discrete self-dual wild parameter of dimension $2N+1$ occurs as the restriction of not only some discrete Langlands parameter for G but of some cuspidal parameter. In fact, we show that it occurs as the restriction of a *regular* parameter (i.e. one whose L-packet consists only of cuspidal representations).

Proof. Since the only irreducible self-dual representation of $\mathscr{P}_F$ is the trivial representation (so the only odd-dimensional self-dual class $[\gamma]$ is that of the trivial representation), while the squaring map on endo-classes is a bijection (since p is odd), it is clear that (7.3) defines a bijection. Its compatibility with the Langlands correspondence is now just a reinterpretation of 2.8 Theorem, using 7.3 Theorem.

It remains to prove that the vertical maps are surjective. We prove that the map on the right is surjective, and then surjectivity on the left follows. So let $\mathcal{V} = \bigoplus m_{[\gamma]}[\gamma]$ be a $(2N+1)$-dimensional self-dual wild parameter (where the sum is over the W_F orbits in $\mathrm{Irr}(\mathscr{P}_F)$ as usual). We will define a regular Langlands parameter $\sigma = \bigoplus \sigma[\gamma]$ for G such that $\sigma[\gamma]$ restricts to $\mathcal{V}[\gamma] = m_{[\gamma]}[\gamma]$.

Let $\gamma \in \mathrm{Irr}(\mathscr{P}_F)$ be a non-trivial representation. If $m_{[\gamma]} = 0$ then we put $\sigma[\gamma] = \{0\}$ so assume $m_{[\gamma]} > 0$, in which case the orbit $[\gamma]$ is self-dual. Let $\boldsymbol{\Theta}$ be corresponding (self-dual) endo-class and let E/E_{o} be the quadratic extension associated to a skew simple stratum which has a simple character with endo-class $\boldsymbol{\Theta}$. We pick non-negative integers m_1, m_2 with $m_1 + m_2 = m_{[\gamma]}$ such that:

(i) m_1, m_2 are odd or 0 if E/E_o is unramified;
(ii) m_1, m_2 are even or 1 if E/E_o is ramified.

For $i = 1, 2$ we put $n_i = m_i \deg(\mathbf{\Theta})$. Then, by 7.2 Lemma, there exist inequivalent orthogonal self-dual cuspidal representations ρ_1, ρ_2 of $\mathrm{GL}_{n_1}(F), \mathrm{GL}_{n_2}(F)$ respectively, both with endo-class $\mathbf{\Theta}$. Let σ_1, σ_2 denote the corresponding Langlands parameters, which are orthogonal and, put $\sigma[\gamma] = \sigma_1 \oplus \sigma_2$; then the restriction of $\sigma[\gamma]$ to $\mathscr{P}_F$ is $\mathcal{V}[\gamma]$, as required, by 7.1 Theorem.

Finally, put $m = m_{[1_{\mathscr{P}_F}]} - 1$, which is even. By 7.2 Lemma, there is an orthogonal self-dual depth zero cuspidal representation of $\mathrm{GL}_m(F)$, and let δ be the corresponding representation of W_F. We put $\sigma[1_{\mathscr{P}_F}] = \delta \oplus \omega$, where $\omega = \det(\delta) \prod_{[\gamma] \neq [1_{\mathscr{P}_F}]} \det \sigma[\gamma]$.

Then $\sigma = \bigoplus_{[\gamma]} \sigma[\gamma]$ is a regular cuspidal Langlands parameter for G which restricts to $\mathcal{V}$. □

7.7. In the proof of 7.6 Theorem we saw that, for any self-dual wild parameter $\mathcal{V}$ of odd dimension, there is a regular Langlands parameter for G which restricts to $\mathcal{V}$. As well as this, one can (in general) cook up other examples of Langlands parameters which restrict to $\mathcal{V}$ and are highly irregular. Since we find it amusing, we include here a description of how to find a highly irregular Langlands parameter which restricts to $\mathcal{V}$.

We begin with the following observation, which is just the translation of 7.2 Lemma (with $m = 1$) to Galois representations. Suppose $\gamma \in \mathrm{Irr}(\mathscr{P}_F)$ is non-trivial with self-dual W_F-orbit. Then there are four self-dual representations of W_F whose restriction to $\mathscr{P}_F$ is $[\gamma]$, two of which are orthogonal and two of which are symplectic. We write $\sigma_{\gamma,1}, \sigma_{\gamma,2}$ for the two orthogonal ones, and $\sigma_{\gamma,3}, \sigma_{\gamma,4}$ for the two symplectic ones.

Now we decompose $\mathcal{V} = \bigoplus m_{[\gamma]}[\gamma]$ as above. As before, we will define a Langlands parameter $\sigma = \bigoplus \sigma[\gamma]$, with $\sigma[\gamma]_{|\mathscr{P}_F} = m_{[\gamma]}[\gamma]$. We will obtain a parameter which is not regular whenever either $m_{[1_{\mathscr{P}_F}]} > 3$ or $m_{[\gamma]} > 1$, for some non-trivial self-dual $[\gamma]$.

By Lagrange's 4-squares theorem, we can find non-negative integers such that

$$4m_{[\gamma]} + 2 = a_1^2 + a_2^2 + a_3^2 + a_4^2.$$

Moreover, two of the a_i are even and the other two odd. We label them so that a_1, a_2 are even and a_3, a_4 are odd and, when $m_{[\gamma]} = 2$, we take the solution with $a_1 = a_2 = 0$. Then we set

$$\sigma[\gamma] = \bigoplus_{i=1}^{4} \sigma_{\gamma,i} \otimes (\mathrm{St}_{a_i - 1} \oplus \mathrm{St}_{a_i - 3} \oplus \cdots),$$

where we understand that we ignore the terms on the right where $a_i \leqslant 1$.

Finally, write $\omega_1 = \prod_{[\gamma] \neq [1_{\mathscr{P}_F}]} \det \sigma[\gamma]$, which is a quadratic character, and let $\omega_2, \omega_3, \omega_4$ denote the other three quadratic characters. Again, there are non-negative integers such that

$$m_{[1_{\mathscr{P}_F}]} = a_1^2 + a_2^2 + a_3^2 + a_4^2.$$

Since $m_{[1_{\mathscr{P}_F}]}$ is odd, there is exactly one a_i which has opposite parity to the other three, and we choose our numbering so that this is a_1. Then we take

$$\sigma[1_{\mathscr{P}_F}] = \bigoplus_{i=1}^{4} \omega_i \otimes \left(\mathrm{St}_{2a_i-1} \oplus \mathrm{St}_{2a_i-3} \oplus \cdots \oplus \mathrm{St}_1\right),$$

where, again, we ignore the terms for which $a_i = 0$.

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