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User-specified Optimization Based Transformation of Fuzzy Membership into Basic Belief Assignment

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Abstract—To combine different types of uncertain information from different sources under different frameworks, we need transformations between different frameworks. For the transformation of a fuzzy membership function (FMF) into a basic belief assignment (BBA), several approaches have been proposed. Among these approaches, the uncertainty optimization based transformations can provide BBAs without predefining focal elements. However, these two transformations, which respectively use the uncertainty maximization and minimization criteria, emphasize the extreme cases of uncertainty. We expect to obtain a BBA, which is the trade-off between the two BBAs obtained by solving the uncertainty maximization and minimization, to avoid extreme attitudinal bias. In this paper, we propose two transformations of an FMF into a BBA by using a user-specified weighting factor to obtain such a trade-off (or balanced) BBA. Some examples and related analyses are provided to show the rationality and effectiveness of the proposed transformations.

Index Terms—evidence theory, basic belief assignment, fuzzy membership function, optimization, transformation

I. INTRODUCTION

In the information fusion, we need to deal with a large amount of uncertain information. Various types of uncertainty theories have been proposed to deal with different types of uncertainty, e.g., the probability theory, fuzzy set theory [1], possibility theory [2], rough set theory [3] and Dempster-Shafer evidence theory (DST) [4] etc. When we fuse the information from different sources under different theoretical frameworks, we need the transformation between different frameworks.

For the information represented by the FMF and BBA, we can transform an FMF into a BBA. Then, we can combine the BBAs to implement the information fusion. There have been proposed many transformations of an FMF into a BBA [5]–[9]. In [5], Bi et al. proposed a transformation that normalizes a given FMF to generate a BBA with singleton focal elements only. By using the α -cut approach, Florea et al. [6] transformed an FMF into a BBA with focal elements nested in order. However, these two approaches above have to predefine the focal elements, which lack of intuitiveness and

objectiveness. Han et al. [7] proposed two approaches without predefining focal elements. These two approaches can provide BBAs by solving constrained uncertainty maximization and minimization.

For the two transformations of Han et al. [7], both two objective functions are the ambiguity measure (AM) and their constraints are mainly constructed based on the given FMF. Their rationality and effectiveness are both justified in [7]. During the process of solving optimization problems, these two transformations emphasize on the minimum and maximal uncertainties of the BBA, respectively. We think that the BBA being the trade-off (or balanced) between the two BBAs obtained by solving the uncertainty maximization and minimization is more preferred, which might avoid being “one-sided” on the uncertainty degree. In this paper, we propose two approaches by using a user-specified weighting factor to determine BBAs. One transformation is the weighted average by using the user-specified weighting factor with the two BBAs obtained by optimization based transformations [7]. The other transformation brings out a trade-off BBA by solving a constrained minimization problem. The objective function is based on the user-specified weighting factor, the distance of evidence and the two BBAs obtained by uncertainty optimization. The constraints are mainly based on the given FMF. That is, each of our proposed transformations can transform an FMF into a BBA, which can be considered as the trade-off between the two BBAs obtained with uncertainty optimization. Some examples and related analyses are provided to justify the proposed transformations.

II. PRELIMINARY

A. Basics of the Theory of Belief Functions

The theory of belief functions [4], introduced historically by Shafer in DST, is a powerful framework for uncertainty modeling and reasoning. Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be the frame of discernment (FOD). Under the closed world assumption, the FOD is mutually exclusive and exhaustive. The BBA (also called a mass function) is defined on the power set of Θ , which can be denoted by a function $m : 2^\Theta \rightarrow [0, 1]$ satisfying

$$\sum_{A \subseteq \Theta} m(A) = 1, m(\emptyset) = 0 \quad (1)$$

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where \emptyset denotes the empty set. $\forall A$, if $m(A) > 0$, then A is called a focal element. $m(A)$ denotes the evidence support to the proposition A .

The belief function Bel for all $A \subseteq \Theta$, as:

$$Bel(A) = \sum_{B \subseteq A} m(B) \quad (2)$$

The plausibility function Pl for all $A \subseteq \Theta$, as:

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B) \quad (3)$$

Suppose there are two independent BBAs m_1 and m_2 on the same FOD. Historically Shafer proposed Dempster's rule to combine two (or more) BBAs. Dempster's rule of combination is

$$m(A) = \begin{cases} 0, & A = \emptyset \\ \frac{\sum_{B \cap C = A} m_1(B)m_2(C)}{1 - K}, & A \neq \emptyset \end{cases} \quad (4)$$

where $K = \sum_{B \cap C = \emptyset} m_1(B)m_2(C)$ represents the conflict coefficient between two BBAs. There exist other alternative combination rules [10], [11].

B. Uncertainty Measure of a BBA

The uncertainty of a BBA includes two types: the discord and the non-specificity. Different measures of uncertainty [12]–[16] have been proposed, e.g., the non-specificity measure [14], the ambiguity measure (AM) [15] and the aggregated uncertainty (AU) [16]. The definition of AM is as follows:

$$AM(m) = - \sum_{\theta \in \Theta} BetP_m(\theta) \log_2(BetP_m(\theta)) \quad (5)$$

where $BetP_m(\theta) = \sum_{\theta \in A \subseteq \Theta} m(A)/|A|$ is the pignistic probability [17]. $|A|$ denotes the cardinality of the set A .

III. TRANSFORMATION OF FMF INTO BBA

A. Concept of Fuzzy Set

Fuzzy sets [1] were proposed by Zadeh to describe the concepts without precise definitions. Let Θ be the universe of discourse (equivalent to FOD in the belief functions). A fuzzy membership function is denoted by $\mathbf{u} = \mu(\theta)$, $\theta \in \Theta$. For $\mu : \Theta \rightarrow [0, 1]$, $\mu(\theta) \in [0, 1]$ is called the degree of membership for θ .

B. Traditional Transformations of FMF into BBA

a) *Transformations with the predefinition of focal elements:* For a given FMF, two available types of transformations below can provide a BBA, which have to predefine the focal elements. Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ and the given FMF is $\boldsymbol{\mu} = [\mu(\theta_1), \mu(\theta_2), \dots, \mu(\theta_n)]$. The obtained BBA is represented by m .

In the work of Bi et al. [5], the BBA is determined as follows:

$$m(\{\theta_i\}) = \mu(\theta_i) / \sum_{j=1}^n \mu(\theta_j) \quad (6)$$

This approach predefines all focal elements as singletons, and it is the result of normalization for the given FMF.

Another transformation with the predefinition of focal elements is the work of Florea et al. [6] by using the α -cut approach. Suppose that $\mu(\theta_1), \mu(\theta_2), \dots, \mu(\theta_n)$ are sorted into ascending order as $0 = \alpha_0 < \alpha_1 < \alpha_2 < \dots < \alpha_M \leq 1$, where $M \leq |\Theta|$. The BBA is determined by using the transformation [6] as follows:

$$m(A_j) = \frac{\alpha_j - \alpha_{j-1}}{\alpha_M} \quad (7)$$

where $A_j = \{\theta_i \in \Theta | \mu(\theta_i) \geq \alpha_j\}$, $i = 1, 2, \dots, n$, $j = 1, 2, \dots, M$. This transformation predefines the focal elements nested in order for the given FMF.

Both two approaches can transform an FMF into a BBA. However, the transformations with the predefinition of focal elements lack of intuitiveness and objectiveness. For a given FMF, the optimization based transformations can obtain a BBA without predefining the focal elements.

b) *Transformations based on the uncertainty optimization:* In the work of Han et al. [7], the two transformations that have no predefinition of focal elements are obtained by solving the uncertainty maximization and minimization. Suppose that the FOD is $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ and the given FMF is $\boldsymbol{\mu} = [\mu(\theta_1), \mu(\theta_2), \dots, \mu(\theta_n)]$. The obtained BBA is represented by m .

There exists a relationship [18] between the FMF and the belief function or plausibility function. When $\sum_{i=1}^n \mu(\theta_i) \geq 1$, the FMF is equivalent to a singleton plausibility function, which is denoted by

$$Pl(\{\theta_i\}) = \sum_{\{\theta_i\} \cap A \neq \emptyset} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \quad (8)$$

It is the necessary and sufficient condition for the FMF to be a singleton plausibility function.

When $\sum_{i=1}^n \mu(\theta_i) \leq 1$, the FMF is equivalent to a singleton belief function, which is denoted by

$$Bel(\{\theta_i\}) = \sum_{A \subseteq \{\theta_i\}} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \quad (9)$$

Similarly, it is the necessary and sufficient condition for the FMF to be a singleton belief function.

The detailed proof of the above relationships are given in [18].

There is a BBA transformed from a given FMF, and the FMF and BBA satisfy Eq. (8) or Eq. (9). Then, n linear equations for the corresponding relations can be obtained. In addition, one has $\sum_{A \subseteq \Theta} m(A) = 1$. There exist $n + 1$ linear equations. However, except for $m(\emptyset) = 0$, in the worst case there are $2^n - 1$ focal elements to assign the belief. The $n + 1$ linear equations with respect to $2^n - 1$ undetermined variables, which is an under-determined problem, i.e., it usually has multiple solutions.

Therefore, to obtain a unique BBA, Han et al. [7] established two uncertainty optimization based transformations for the given FMF. Both two objective functions are AM and the constraints are mainly based on the given FMF.

The objective function of the uncertainty maximization problem and the corresponding constraints are as follows:

When $\sum_{i=1}^n \mu(\theta_i) \geq 1$,

$$\begin{aligned} \max_m & \left\{ - \sum_{i=1}^n [\text{Bet}P_m(\theta_i) \log_2(\text{Bet}P_m(\theta_i))] \right\} \\ \text{s.t.} & \begin{cases} \sum_{\{\theta_i\} \cap A \neq \emptyset} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \\ \sum_{A \subseteq \Theta} m(A) = 1 \\ 0 \leq m(A) \leq 1 \end{cases} \end{aligned} \quad (10)$$

When $\sum_{i=1}^n \mu(\theta_i) \leq 1$,

$$\begin{aligned} \max_m & \left\{ - \sum_{i=1}^n [\text{Bet}P_m(\theta_i) \log_2(\text{Bet}P_m(\theta_i))] \right\} \\ \text{s.t.} & \begin{cases} \sum_{A \subseteq \{\theta_i\}} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \\ \sum_{A \subseteq \Theta} m(A) = 1 \\ 0 \leq m(A) \leq 1 \end{cases} \end{aligned} \quad (11)$$

In the sequel, this transformation is represented by “ T_{\max} ” for convenience.

The objective function of the uncertainty minimization problem and the corresponding constraints are as follows:

When $\sum_{i=1}^n \mu(\theta_i) \geq 1$,

$$\begin{aligned} \min_m & \left\{ - \sum_{i=1}^n [\text{Bet}P_m(\theta_i) \log_2(\text{Bet}P_m(\theta_i))] \right\} \\ \text{s.t.} & \begin{cases} \sum_{\{\theta_i\} \cap A \neq \emptyset} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \\ \sum_{A \subseteq \Theta} m(A) = 1 \\ 0 \leq m(A) \leq 1 \end{cases} \end{aligned} \quad (12)$$

When $\sum_{i=1}^n \mu(\theta_i) \leq 1$,

$$\begin{aligned} \min_m & \left\{ - \sum_{i=1}^n [\text{Bet}P_m(\theta_i) \log_2(\text{Bet}P_m(\theta_i))] \right\} \\ \text{s.t.} & \begin{cases} \sum_{A \subseteq \{\theta_i\}} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \\ \sum_{A \subseteq \Theta} m(A) = 1 \\ 0 \leq m(A) \leq 1 \end{cases} \end{aligned} \quad (13)$$

In the sequel, this transformation is represented by “ T_{\min} ” for convenience.

The unique BBA can be determined without predefining focal elements by using “ T_{\max} ” or “ T_{\min} ”. The obtained BBA is the optimal solution of the uncertainty maximization or minimization. During the process of transforming an FMF into a BBA, “ T_{\max} ” and “ T_{\min} ” emphasize on the maximal and minimum uncertainty cases of the obtained BBA, respectively. We think that the BBA being the trade-off between the two BBAs obtained by using “ T_{\max} ” and “ T_{\min} ” is more preferred, which might avoid bias in terms of uncertainty degree.

IV. TRANSFORMATIONS WITH USER-SPECIFIED WEIGHTING FACTOR

As aforementioned, we can obtain two BBAs by using “ T_{\max} ” and “ T_{\min} ”, respectively. Based on these two BBAs,

we aim to construct a transformation to determine a trade-off BBA. The trade-off BBA which satisfies the relationship between the FMF and the singleton plausibility or singleton belief. We use a user-specified weighting factor to influence how close the trade-off BBA is to each of the two BBAs above. Suppose that the user-specified weighting factor is represented by α and $0 \leq \alpha \leq 1$. When $\alpha \rightarrow 0$, the trade-off BBA is close to the BBA obtained by using “ T_{\min} ”. When $\alpha \rightarrow 1$, the trade-off BBA is close to the BBA obtained by using “ T_{\max} ”. To meet the requirements above, we propose two different approaches to determine the trade-off BBAs.

A. Weighted Average based Transformation

Let $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$ be the FOD. The given FMF is represented by $\boldsymbol{\mu} = [\mu(\theta_1), \mu(\theta_2), \dots, \mu(\theta_n)]$. Suppose that the BBA obtained by using “ T_{\min} ” is denoted by m_{\min} , and the BBA obtained by using “ T_{\max} ” is denoted by m_{\max} . The user-specified weighting factor is denoted by α ($0 \leq \alpha \leq 1$). The trade-off BBA is denoted by m . The Weighted Average of m_{\min} and m_{\max} can bring out a trade-off BBA as follows:

$$m(A) = (1 - \alpha) \cdot m_{\min}(A) + \alpha \cdot m_{\max}(A) \quad (14)$$

where $A \subseteq \Theta$. In the sequel, the transformation based on the weighted average (WA) is denoted by “ T_{WA} ” for convenience.

The BBA obtained in (14) is an admissible BBA and it satisfies the constraints established based on FMF.

According to Eq. (14), the following conditions can be satisfied:

$$\sum_{A \subseteq \Theta} m(A) = 1, \quad 0 \leq m(A) \leq 1 \quad (15)$$

For the transformation of an FMF into a BBA, it is necessary that the obtained BBA satisfies the relationship between the FMF and the singleton plausibility or singleton belief. Although “ T_{WA} ” is a simple and direct transformation of an FMF into a trade-off BBA, it also satisfies the relationship. The proof is provided below.

When $\sum_{i=1}^n \mu(\theta_i) \geq 1$, m_{\min} and m_{\max} satisfy Eq. (6), respectively, i.e., $Pl_{\min}(\{\theta_i\}) = Pl_{\max}(\{\theta_i\}) = \mu(\theta_i)$, $i = 1, 2, \dots, n$. According to Eq. (14),

$$\begin{aligned} Pl(\{\theta_i\}) &= \sum_{\{\theta_i\} \cap A \neq \emptyset} m(A) \\ &= \sum_{\{\theta_i\} \cap A \neq \emptyset} [(1 - \alpha) \cdot m_{\min}(A) + \alpha \cdot m_{\max}(A)] \\ &= (1 - \alpha) \cdot \sum_{\{\theta_i\} \cap A \neq \emptyset} m_{\min}(A) + \alpha \cdot \sum_{\{\theta_i\} \cap A \neq \emptyset} m_{\max}(A) \quad (16) \\ &= (1 - \alpha) \cdot Pl_{\min}(\{\theta_i\}) + \alpha \cdot Pl_{\max}(\{\theta_i\}) \\ &= (1 - \alpha) \cdot \mu(\theta_i) + \alpha \cdot \mu(\theta_i) \\ &= \mu(\theta_i) \end{aligned}$$

Similarly, when $\sum_{i=1}^n \mu(\theta_i) \leq 1$, m_{\min} and m_{\max} satisfy Eq. (7), respectively, i.e., $Bel_{\min}(\{\theta_i\}) = Bel_{\max}(\{\theta_i\}) =$

$\mu(\theta_i)$, $i = 1, 2, \dots, n$. According to Eq. (14),

$$\begin{aligned} Bel(\{\theta_i\}) &= m(A) \\ &= (1 - \alpha) \cdot m_{min}(A) + \alpha \cdot m_{max}(A) \\ &= (1 - \alpha) \cdot Bel_{min}(\{\theta_i\}) + \alpha \cdot Bel_{max}(\{\theta_i\}) \quad (17) \\ &= (1 - \alpha) \cdot \mu(\theta_i) + \alpha \cdot \mu(\theta_i) \\ &= \mu(\theta_i) \end{aligned}$$

For the trade-off BBA, when $\sum_{i=1}^n \mu(\theta_i) \geq 1$, the given FMF is equivalent to the corresponding singleton plausibility. When $\sum_{i=1}^n \mu(\theta_i) \leq 1$, the given FMF is equivalent to the corresponding singleton belief. That is, “T_{WA}” can transform the given FMF into the trade-off BBA, which satisfies the relationship between the FMF and the BBA.

B. User-specified Optimization based Transformation

Let the FOD be $\Theta = \{\theta_1, \theta_2, \dots, \theta_n\}$. The given FMF is denoted by $\mu = [\mu(\theta_1), \mu(\theta_2), \dots, \mu(\theta_n)]$. Suppose that m_{min} and m_{max} denote the BBAs obtained by “T_{min}” and “T_{max}”, respectively. The user-specified weighting factor is denoted by α ($0 \leq \alpha \leq 1$). The trade-off BBA is represented by m .

The user-specified weighting factor is used to influence the similarity between the trade-off BBA and m_{min} (or m_{max}). The degree of similarity between two BBAs is represented by the distance of evidence. We can use the Jousselme’s distance [19], which is a strict metric defined as

$$d_J(m_a, m_b) = \sqrt{\frac{1}{2}(m_a - m_b)D(m_a - m_b)} \quad (18)$$

where $D(A, B) = |A \cap B| / |A \cup B|$, $A \subseteq \Theta$, $B \subseteq \Theta$. According to Eq. (18), the obtained BBA is more similar to m_{min} , if $d_J(m, m_{min})$ is smaller. If $d_J(m, m_{max})$ is smaller, the obtained BBA is more similar to m_{max} .

To obtain the trade-off BBA between m_{min} and m_{max} , a relationship between the user-specified weighting factor and the distance of evidence can be constructed. When α is given from 0 to 1, with the decreasing of $d_J(m, m_{min})$, the value of $d_J(m, m_{max})$ is increasing. Then we can establish the following equation:

$$\frac{d_J(m, m_{min})}{d_J(m, m_{max})} = \frac{\alpha}{1 - \alpha} \quad (19)$$

The BBA satisfies Eq. (19) may not always exist. If the following function (equivalent to Eq. (19)) achieves the minimum value, then the trade-off BBA is obtained.

$$obj(m) = [(1 - \alpha) \cdot d_J(m, m_{min}) - \alpha \cdot d_J(m, m_{max})]^2 \quad (20)$$

When α is given, we can establish a constrained minimization problem to transform an FMF into a BBA. The objective function is Eq. (20) and the constraints are mainly based on Eq. (6) or Eq. (7). The transformation of an FMF into a trade-off BBA is obtained by solving the user-specified optimization

problem as follows:

When $\sum_{i=1}^n \mu(\theta_i) \geq 1$,

$$\begin{aligned} \min_m & \left\{ [(1 - \alpha) \cdot d_J(m, m_{min}) - \alpha \cdot d_J(m, m_{max})]^2 \right\} \\ \text{s.t.} & \begin{cases} \sum_{\substack{A \subseteq \Theta \\ \{\theta_i\} \cap A \neq \emptyset}} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \\ \sum_{A \subseteq \Theta} m(A) = 1 \\ 0 \leq m(A) \leq 1 \end{cases} \quad (21) \end{aligned}$$

When $\sum_{i=1}^n \mu(\theta_i) \leq 1$,

$$\begin{aligned} \min_m & \left\{ [(1 - \alpha) \cdot d_J(m, m_{min}) - \alpha \cdot d_J(m, m_{max})]^2 \right\} \\ \text{s.t.} & \begin{cases} \sum_{A \subseteq \{\theta_i\}} m(A) = \mu(\theta_i), \forall \{\theta_i\} \subseteq \Theta \\ \sum_{A \subseteq \Theta} m(A) = 1 \\ 0 \leq m(A) \leq 1 \end{cases} \quad (22) \end{aligned}$$

In the sequel, the transformation based on the user-specified optimization (USO) is denoted by “T_{USO}” for convenience.

For a given FMF, the trade-off BBA can be obtained by using “T_{USO}”, which is a user-specified optimization based transformation.

V. EXPERIMENTS

In this section, we provide some examples to illustrate how to transform an FMF into a trade-off BBA using our approaches. Here, we use the optimization toolbox in the MatlabTM to solve the optimization problems under constraints.

A. Example 1

Let the FOD be $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The given FMF is $\mu(\theta_1) = 0.9$, $\mu(\theta_2) = 0.7$, $\mu(\theta_3) = 0.3$. Suppose that m_{min} and m_{max} are the BBAs obtained by using “T_{min}” and “T_{max}”, respectively. We just list the corresponding BBAs for $\alpha = 0, 0.3, 0.7$ and 1.

This FMF satisfies $\sum_{i=1}^3 \mu(\theta_i) = 1.9 > 1$. Therefore, the given FMF is equivalent to the singleton plausibility. The BBAs obtained by using “T_{WA}” and “T_{USO}” are listed in the Table I and Table II, respectively.

By using “T_{WA}” and “T_{USO}”, when $\alpha = 0$, the obtained BBAs are identical to m_{min} , and the values of AM are the minimum uncertainty. When $\alpha \rightarrow 0$, the obtained BBA is similar to m_{min} , and its uncertainty is close to the minimum uncertainty.

Similarly, when $\alpha = 1$, the obtained BBAs are identical to m_{max} , and the values of AM are the maximal uncertainty. When $\alpha \rightarrow 1$, the obtained BBA is similar to m_{max} , and its uncertainty is close to the maximal uncertainty.

B. Example 2

Let the FOD be $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. The given FMF is $\mu(\theta_1) = 1$, $\mu(\theta_2) = 0.2$, $\mu(\theta_3) = 0.3$, $\mu(\theta_4) = 0.3$. Suppose that m_{min} and m_{max} are the BBAs obtained by using “T_{min}” and “T_{max}”, respectively. We just list the corresponding BBAs for $\alpha = 0, 0.2, 0.8$ and 1.

According to $\sum_{i=1}^4 \mu(\theta_i) = 1.8 > 1$, the FMF is equivalent to the singleton plausibility. In the Table III and Table IV,

TABLE I
USING “T_{WA}” TO OBTAIN BBAS IN EXAMPLE 1.

| α | BBA | AM |
|----------------|--|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.3, m(\{\theta_1, \theta_2\}) = 0.3$ $m(\{\theta_2\}) = 0.1, m(\Theta) = 0.3$ | 1.3367 |
| $\alpha = 0.3$ | $m(\{\theta_1\}) = 0.21, m(\{\theta_1, \theta_2\}) = 0.42$ $m(\{\theta_2\}) = 0.07, m(\{\theta_1, \theta_3\}) = 0.06$ $m(\{\theta_3\}) = 0.03, m(\Theta) = 0.21$ | 1.4003 |
| $\alpha = 0.7$ | $m(\{\theta_1\}) = 0.09, m(\{\theta_1, \theta_2\}) = 0.58$ $m(\{\theta_2\}) = 0.03, m(\{\theta_1, \theta_3\}) = 0.14$ $m(\{\theta_3\}) = 0.07, m(\Theta) = 0.09$ | 1.4730 |
| $\alpha = 1$ | $m(\{\theta_1, \theta_2\}) = 0.7, m(\{\theta_3\}) = 0.1$ $m(\{\theta_1, \theta_3\}) = 0.2$ | 1.5129 |

TABLE II
USING “T_{USO}” TO OBTAIN BBAS IN EXAMPLE 1.

| α | BBA | AM |
|----------------|--|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.3, m(\{\theta_1, \theta_2\}) = 0.3$ $m(\{\theta_2\}) = 0.1, m(\Theta) = 0.3$ | 1.3367 |
| $\alpha = 0.3$ | $m(\{\theta_1\}) = 0.2027, m(\{\theta_1, \theta_2\}) = 0.4145$ $m(\{\theta_2\}) = 0.0828, m(\{\theta_1, \theta_3\}) = 0.0802$ $m(\{\theta_3\}) = 0.0172, m(\Theta) = 0.2026$ | 1.3969 |
| $\alpha = 0.7$ | $m(\{\theta_1\}) = 0.1037, m(\{\theta_1, \theta_2\}) = 0.5962$ $m(\{\theta_2\}) = 0.0001, m(\{\theta_1, \theta_3\}) = 0.0965$ $m(\{\theta_3\}) = 0.0999, m(\Theta) = 0.1036$ | 1.4828 |
| $\alpha = 1$ | $m(\{\theta_1, \theta_2\}) = 0.7, m(\{\theta_3\}) = 0.1$ $m(\{\theta_1, \theta_3\}) = 0.2$ | 1.5129 |

the BBAs obtained by using “T_{WA}” and “T_{USO}” are listed, respectively.

When $\alpha = 0$, the obtained BBAs are identical to m_{min} , and the values of AM are the minimum uncertainty. When $\alpha = 1$, the obtained BBAs are identical to m_{max} , and the values of AM are the maximal uncertainty.

In the Table III and Table IV, when $\alpha \rightarrow 0$, the obtained BBA is similar to m_{min} , and its uncertainty is close to the minimum uncertainty. When $\alpha \rightarrow 1$, the obtained BBA is similar to m_{max} , and its uncertainty is close to the maximal uncertainty.

TABLE III
USING “T_{WA}” TO OBTAIN BBAS IN EXAMPLE 2.

| α | BBA | AM |
|----------------|--|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.7, m(\Theta) = 0.2$ $m(\{\theta_1, \theta_3, \theta_4\}) = 0.1$ | 1.0896 |
| $\alpha = 0.2$ | $m(\{\theta_1\}) = 0.60, m(\{\theta_1, \theta_2\}) = 0.04$ $m(\{\theta_1, \theta_3\}) = 0.06, m(\{\theta_1, \theta_4\}) = 0.06$ $m(\{\theta_1, \theta_3, \theta_4\}) = 0.08, m(\Theta) = 0.16$ | 1.2099 |
| $\alpha = 0.8$ | $m(\{\theta_1\}) = 0.3, m(\{\theta_1, \theta_2\}) = 0.16$ $m(\{\theta_1, \theta_3\}) = 0.24, m(\{\theta_1, \theta_4\}) = 0.24$ $m(\{\theta_1, \theta_3, \theta_4\}) = 0.02, m(\Theta) = 0.04$ | 1.5122 |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.2, m(\{\theta_1, \theta_2\}) = 0.2$ $m(\{\theta_1, \theta_3\}) = 0.3, m(\{\theta_1, \theta_4\}) = 0.3$ | 1.5955 |

TABLE IV
USING “T_{USO}” TO OBTAIN BBAS IN EXAMPLE 2.

| α | BBA | AM |
|----------------|--|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.7, m(\Theta) = 0.2$ $m(\{\theta_1, \theta_3, \theta_4\}) = 0.1$ | 1.0896 |
| $\alpha = 0.2$ | $m(\{\theta_1\}) = 0.6008, m(\{\theta_1, \theta_2\}) = 0.0471$ $m(\{\theta_1, \theta_3\}) = 0.0484$ $m(\{\theta_1, \theta_2, \theta_3\}) = 0.0037$ $m(\{\theta_1, \theta_4\}) = 0.0515$ $m(\{\theta_1, \theta_2, \theta_4\}) = 0.0006$ $m(\{\theta_1, \theta_3, \theta_4\}) = 0.0993, m(\Theta) = 0.1486$ | 1.2133 |
| $\alpha = 0.8$ | $m(\{\theta_1\}) = 0.2990, m(\{\theta_1, \theta_2\}) = 0.1665$ $m(\{\theta_1, \theta_3\}) = 0.2345, m(\{\theta_1, \theta_4\}) = 0.2010$ $m(\{\theta_1, \theta_2, \theta_4\}) = 0.0335$ $m(\{\theta_1, \theta_3, \theta_4\}) = 0.0655$ | 1.5227 |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.2, m(\{\theta_1, \theta_2\}) = 0.2$ $m(\{\theta_1, \theta_3\}) = 0.3, m(\{\theta_1, \theta_4\}) = 0.3$ | 1.5955 |

C. Example 3

Let the FOD be $\Theta = \{\theta_1, \theta_2, \theta_3, \theta_4\}$. The given FMF is $\mu(\theta_1) = 0.6, \mu(\theta_2) = 0.1, \mu(\theta_3) = 0.2, \mu(\theta_4) = 0.1$. We just list the corresponding BBAs for $\alpha = 0, 0.4, 0.9$ and 1.

This FMF satisfies $\sum_{i=1}^4 \mu(\theta_i) = 1$. The FMF is equivalent to the singleton plausibility or singleton belief. In the Table V and Table VI, the BBAs obtained by using “T_{WA}” and “T_{USO}” are listed, respectively.

In the Table V and Table VI, the BBAs obtained when $\alpha = 0$ are identical to the BBAs obtained when $\alpha = 1$, i.e., the two BBAs obtained by using “T_{min}” and “T_{max}” are the same. Therefore, $\forall \alpha \in [0, 1]$, the obtained BBAs are without the influence of α . When α is given from 0 to 1, all the obtained BBAs are Bayesian belief functions and are identical.

TABLE V
USING “T_{WA}” TO OBTAIN BBAS IN EXAMPLE 3.

| α | BBA | AM |
|----------------|--|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |
| $\alpha = 0.4$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |
| $\alpha = 0.9$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |

D. Example 4

Let the FOD be $\Theta = \{\theta_1, \theta_2, \theta_3\}$. The given FMF is $\mu(\theta_1) = 0.6, \mu(\theta_2) = 0.2, \mu(\theta_3) = 0.1$. Suppose that m_{min} and m_{max} are the BBAs obtained by using “T_{min}” and “T_{max}”, respectively. We just list the corresponding BBAs for $\alpha = 0, 0.3, 0.8$ and 1.

This FMF satisfies $\sum_{i=1}^3 \mu(\theta_i) = 0.9 < 1$. The FMF is equivalent to the singleton belief. The BBAs obtained by using “T_{WA}” and “T_{USO}” are listed in the Table VII and Table VIII, respectively.

TABLE VI
USING "T_{USO}" TO OBTAIN BBAs IN EXAMPLE 3.

| α | BBA | AM |
|----------------|--|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |
| $\alpha = 0.4$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |
| $\alpha = 0.9$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.1$ $m(\{\theta_3\}) = 0.2, m(\{\theta_4\}) = 0.1$ | 1.5710 |

In the Table VII and Table VIII, when $\alpha = 0$, the obtained BBAs are identical to m_{min} , and the values of AM are the minimum uncertainty. When $\alpha = 1$, the obtained BBAs are identical to m_{max} , and the values of AM are the maximal uncertainty. When $\alpha \rightarrow 0$, the obtained BBA is similar to m_{min} , and its uncertainty is close to the minimum uncertainty. When $\alpha \rightarrow 1$, the obtained BBA is similar to m_{max} , and its uncertainty is close to the maximal uncertainty.

TABLE VII
USING "T_{WA}" TO OBTAIN BBAs IN EXAMPLE 4.

| α | BBA | AM |
|----------------|---|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_3\}) = 0.1$ $m(\{\theta_2\}) = 0.2, m(\{\theta_1, \theta_2\}) = 0.1$ | 1.2362 |
| $\alpha = 0.3$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_1, \theta_2\}) = 0.07$ $m(\{\theta_2\}) = 0.2, m(\{\theta_2, \theta_3\}) = 0.03$ $m(\{\theta_3\}) = 0.1$ | 1.2749 |
| $\alpha = 0.8$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_1, \theta_2\}) = 0.02$ $m(\{\theta_2\}) = 0.2, m(\{\theta_2, \theta_3\}) = 0.08$ $m(\{\theta_3\}) = 0.1$ | 1.3321 |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.2$ $m(\{\theta_3\}) = 0.1, m(\{\theta_2, \theta_3\}) = 0.1$ | 1.3527 |

TABLE VIII
USING "T_{USO}" TO OBTAIN BBAs IN EXAMPLE 4.

| α | BBA | AM |
|----------------|---|--------|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_3\}) = 0.1$ $m(\{\theta_2\}) = 0.2, m(\{\theta_1, \theta_2\}) = 0.1$ | 1.2362 |
| $\alpha = 0.3$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_1, \theta_2\}) = 0.0697$ $m(\{\theta_2\}) = 0.2, m(\{\theta_1, \theta_3\}) = 0.0006$ $m(\{\theta_3\}) = 0.1, m(\{\theta_2, \theta_3\}) = 0.0297$ | 1.2748 |
| $\alpha = 0.8$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_1, \theta_2\}) = 0.02$ $m(\{\theta_2\}) = 0.2, m(\{\theta_2, \theta_3\}) = 0.08$ $m(\{\theta_3\}) = 0.1$ | 1.3321 |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.6, m(\{\theta_2\}) = 0.2$ $m(\{\theta_3\}) = 0.1, m(\{\theta_2, \theta_3\}) = 0.1$ | 1.3527 |

E. Example 5

Suppose that a system of classification with three sensors, including displacement sensor S_1 , pressure sensor S_2 and image sensor S_3 . Let the FOD be $\Theta = \{\theta_1, \theta_2, \theta_3\}$. Three sensors are used for measuring the size, weight and state of the sample, respectively. The measurements of sensors are used to

obtain two FMFs and a BBA. According to the parameters and the measurements of the sensor, the FMF is defined as

$$\mu(\theta_i) = \begin{cases} \frac{x - \min_i}{\text{ave}_i - \min_i}, & x \in [\min_i, \text{ave}_i] \\ \frac{x - \max_i}{\text{ave}_i - \max_i}, & x \in [\text{ave}_i, \max_i] \\ 0, & \text{others} \end{cases} \quad (23)$$

where $i = 1, 2$. \min_i and \max_i are the minimum and maximal values of the class θ_i , respectively. ave_i is the average value of the class θ_i .

TABLE IX
THE PARAMETERS AND THE MEASUREMENTS OF SENSORS.

| Class | S_1 | | | S_2 | | |
|------------|----------|----------|----------------|----------|----------|----------------|
| | \min_i | \max_i | ave_i | \min_i | \max_i | ave_i |
| θ_1 | 43.3 | 58.4 | 50.8 | 2.9 | 4.1 | 3.4 |
| θ_2 | 50.9 | 70.1 | 59.5 | 2.0 | 3.4 | 2.8 |
| θ_3 | 49.4 | 79.3 | 65.7 | 2.2 | 3.8 | 2.9 |
| Sample | 56 | | | 3.2 | | |

In the Table IX, the parameters of S_1 and S_2 and the measurements of a sample are listed. The class of this sample is θ_2 . According to (23), two FMFs are as follows:

$$S_1 : \mu(\theta_1) = 0.3158, \mu(\theta_2) = 0.5930, \mu(\theta_3) = 0.4049;$$

$$S_2 : \mu(\theta_1) = 0.6, \mu(\theta_2) = 0.6667, \mu(\theta_3) = 0.3333.$$

According to the image of S_3 , the expert determined the BBA directly as follows:

$$m_{S_3}(\{\theta_3\})=0.51, m_{S_3}(\{\theta_2, \theta_3\})=0.38, m_{S_3}(\Theta)=0.11.$$

TABLE X
USING "T_{WA}" TO OBTAIN BBAs IN EXAMPLE 5.

| α | m_{S_1} | m_{S_2} |
|----------------|--|--|
| $\alpha = 0$ | $m_{S_1}(\{\theta_1\}) = 0.0021$ $m_{S_1}(\{\theta_2\}) = 0.5930$ $m_{S_1}(\{\theta_3\}) = 0.0912$ $m_{S_1}(\{\theta_1, \theta_3\}) = 0.3137$ | $m_{S_2}(\{\theta_1\}) = 0.2667$ $m_{S_2}(\{\theta_2\}) = 0.4$ $m_{S_2}(\{\theta_1, \theta_3\}) = 0.0666$ $m_{S_2}(\{\Theta\}) = 0.2667$ |
| $\alpha = 0.3$ | $m_{S_1}(\{\theta_1\}) = 0.0759$ $m_{S_1}(\{\theta_2\}) = 0.4989$ $m_{S_1}(\{\theta_1, \theta_2\}) = 0.0203$ $m_{S_1}(\{\theta_3\}) = 0.1115$ $m_{S_1}(\{\theta_1, \theta_3\}) = 0.2196$ $m_{S_1}(\{\theta_2, \theta_3\}) = 0.0738$ | $m_{S_2}(\{\theta_1\}) = 0.1967$ $m_{S_2}(\{\theta_2\}) = 0.3$ $m_{S_2}(\{\theta_1, \theta_2\}) = 0.17$ $m_{S_2}(\{\theta_3\}) = 0.09$ $m_{S_2}(\{\theta_1, \theta_3\}) = 0.0466$ $m_{S_2}(\{\theta_2, \theta_3\}) = 0.01$ $m_{S_2}(\{\Theta\}) = 0.1867$ |
| $\alpha = 0.8$ | $m_{S_1}(\{\theta_1\}) = 0.1989$ $m_{S_1}(\{\theta_2\}) = 0.3420$ $m_{S_1}(\{\theta_1, \theta_2\}) = 0.0542$ $m_{S_1}(\{\theta_3\}) = 0.1454$ $m_{S_1}(\{\theta_1, \theta_3\}) = 0.0627$ $m_{S_1}(\{\theta_2, \theta_3\}) = 0.1968$ | $m_{S_2}(\{\theta_1\}) = 0.08$ $m_{S_2}(\{\theta_2\}) = 0.1334$ $m_{S_2}(\{\theta_1, \theta_2\}) = 0.4534$ $m_{S_2}(\{\theta_3\}) = 0.24$ $m_{S_2}(\{\theta_1, \theta_3\}) = 0.0133$ $m_{S_2}(\{\theta_2, \theta_3\}) = 0.0266$ $m_{S_2}(\{\Theta\}) = 0.0533$ |
| $\alpha = 1$ | $m_{S_1}(\{\theta_1\}) = 0.2480$ $m_{S_1}(\{\theta_2\}) = 0.2793$ $m_{S_1}(\{\theta_1, \theta_2\}) = 0.0687$ $m_{S_1}(\{\theta_3\}) = 0.1590$ $m_{S_1}(\{\theta_2, \theta_3\}) = 0.2459$ | $m_{S_2}(\{\theta_1\}) = 0.0333$ $m_{S_2}(\{\theta_2\}) = 0.0667$ $m_{S_2}(\{\theta_1, \theta_2\}) = 0.5667$ $m_{S_2}(\{\theta_3\}) = 0.3$ $m_{S_2}(\{\theta_2, \theta_3\}) = 0.0333$ |

Suppose that m_{S_1} and m_{S_2} denote the obtained BBAs transformed from the two FMFs of S_1 and S_2 , respectively.

TABLE XIII
THE PIGNISTIC PROBABILITIES IN EXAMPLE 5.

| α | “T _{WA} ” | “T _{USO} ” |
|----------------|---|---|
| $\alpha = 0$ | $BetP(\{\theta_1\}) = 0.0446$ $BetP(\{\theta_2\}) = 0.5731$ $BetP(\{\theta_3\}) = 0.3822$ | $BetP(\{\theta_1\}) = 0.0446$ $BetP(\{\theta_2\}) = 0.5731$ $BetP(\{\theta_3\}) = 0.3822$ |
| $\alpha = 0.3$ | $BetP(\{\theta_1\}) = 0.0525$ $BetP(\{\theta_2\}) = 0.5648$ $BetP(\{\theta_3\}) = 0.3827$ | $BetP(\{\theta_1\}) = 0.0511$ $BetP(\{\theta_2\}) = 0.5627$ $BetP(\{\theta_3\}) = 0.3863$ |
| $\alpha = 0.8$ | $BetP(\{\theta_1\}) = 0.0565$ $BetP(\{\theta_2\}) = 0.5567$ $BetP(\{\theta_3\}) = 0.3868$ | $BetP(\{\theta_1\}) = 0.0520$ $BetP(\{\theta_2\}) = 0.5570$ $BetP(\{\theta_3\}) = 0.3910$ |
| $\alpha = 1$ | $BetP(\{\theta_1\}) = 0.0549$ $BetP(\{\theta_2\}) = 0.5556$ $BetP(\{\theta_3\}) = 0.3895$ | $BetP(\{\theta_1\}) = 0.0549$ $BetP(\{\theta_2\}) = 0.5556$ $BetP(\{\theta_3\}) = 0.3895$ |

TABLE XI
USING “T_{USO}” TO OBTAIN BBAs IN EXAMPLE 5.

| α | m_{S_1} | m_{S_2} |
|----------------|--|--|
| $\alpha = 0$ | $m_{S_1}(\{\theta_1\}) = 0.0021$ $m_{S_1}(\{\theta_2\}) = 0.5930$ $m_{S_1}(\{\theta_3\}) = 0.0912$ $m_{S_1}(\{\theta_1, \theta_3\}) = 0.3137$ | $m_{S_2}(\{\theta_1\}) = 0.2667$ $m_{S_2}(\{\theta_2\}) = 0.4$ $m_{S_2}(\{\theta_1, \theta_3\}) = 0.0666$ $m_{S_2}(\{\Theta\}) = 0.2667$ |
| $\alpha = 0.3$ | $m_{S_1}(\{\theta_1\}) = 0.0354$ $m_{S_1}(\{\theta_2\}) = 0.4927$ $m_{S_1}(\{\theta_1, \theta_2\}) = 0.0670$ $m_{S_1}(\{\theta_3\}) = 0.1654$ $m_{S_1}(\{\theta_1, \theta_3\}) = 0.2062$ $m_{S_1}(\{\theta_2, \theta_3\}) = 0.0260$ $m_{S_1}(\{\Theta\}) = 0.0073$ | $m_{S_2}(\{\theta_1\}) = 0.1902$ $m_{S_2}(\{\theta_2\}) = 0.2641$ $m_{S_2}(\{\theta_1, \theta_2\}) = 0.2124$ $m_{S_2}(\{\theta_3\}) = 0.0291$ $m_{S_2}(\{\theta_1, \theta_3\}) = 0.1140$ $m_{S_2}(\{\theta_2, \theta_3\}) = 0.1068$ $m_{S_2}(\{\Theta\}) = 0.0834$ |
| $\alpha = 0.8$ | $m_{S_1}(\{\theta_1\}) = 0.1918$ $m_{S_1}(\{\theta_2\}) = 0.3190$ $m_{S_1}(\{\theta_1, \theta_2\}) = 0.0844$ $m_{S_1}(\{\theta_3\}) = 0.2152$ $m_{S_1}(\{\theta_2, \theta_3\}) = 0.15$ $m_{S_1}(\{\Theta\}) = 0.0396$ | $m_{S_2}(\{\theta_1\}) = 0.0942$ $m_{S_2}(\{\theta_2\}) = 0.1193$ $m_{S_2}(\{\theta_1, \theta_2\}) = 0.4532$ $m_{S_2}(\{\theta_3\}) = 0.2379$ $m_{S_2}(\{\theta_1, \theta_3\}) = 0.0012$ $m_{S_2}(\{\theta_2, \theta_3\}) = 0.0428$ $m_{S_2}(\{\Theta\}) = 0.0514$ |
| $\alpha = 1$ | $m_{S_1}(\{\theta_1\}) = 0.2480$ $m_{S_1}(\{\theta_2\}) = 0.2793$ $m_{S_1}(\{\theta_1, \theta_2\}) = 0.0687$ $m_{S_1}(\{\theta_3\}) = 0.1590$ $m_{S_1}(\{\theta_2, \theta_3\}) = 0.2459$ | $m_{S_2}(\{\theta_1\}) = 0.0333$ $m_{S_2}(\{\theta_2\}) = 0.0667$ $m_{S_2}(\{\theta_1, \theta_2\}) = 0.5667$ $m_{S_2}(\{\theta_3\}) = 0.3$ $m_{S_2}(\{\theta_2, \theta_3\}) = 0.0333$ |

The BBAs obtained by using “T_{WA}” and “T_{USO}” are listed in the Table X and Table XI, respectively. Then, we combine these three BBAs (i.e., m_{S_1} , m_{S_2} and m_{S_3}). The combined BBA is represented by m . The combined BBAs and the pignistic probabilities are listed in the Table XII and Table XIII, respectively. We just list the corresponding BBAs for $\alpha = 0, 0.3, 0.8$ and 1 .

In the Table XIII, all the classification results are θ_2 and are correct. When α is given from 0 to 1 , $m_{S_1}(\{\theta_2\})$ and $m_{S_2}(\{\theta_2\})$ are decreasing in the Table X and Table XI. With the increasing value of α , m_{S_1} and m_{S_2} are more close to m_{max} (i.e., the BBA obtained by using “T_{max}” or the BBA obtained when $\alpha = 1$), which is the reason of the decreasing value of $BetP(\{\theta_2\})$ in the Table XIII.

TABLE XII
THE COMBINED BBAs IN EXAMPLE 5.

| α | “T _{WA} ” | “T _{USO} ” |
|----------------|--|---|
| $\alpha = 0$ | $m(\{\theta_1\}) = 0.0276$ $m(\{\theta_2\}) = 0.5731$ $m(\{\theta_3\}) = 0.3652$ $m(\{\theta_1, \theta_3\}) = 0.0340$ | $m(\{\theta_1\}) = 0.0276$ $m(\{\theta_2\}) = 0.5731$ $m(\{\theta_3\}) = 0.3652$ $m(\{\theta_1, \theta_3\}) = 0.0340$ |
| $\alpha = 0.3$ | $m(\{\theta_1\}) = 0.0429$ $m(\{\theta_2\}) = 0.5531$ $m(\{\theta_1, \theta_2\}) = 0.0024$ $m(\{\theta_3\}) = 0.3637$ $m(\{\theta_1, \theta_3\}) = 0.0168$ $m(\{\theta_2, \theta_3\}) = 0.0212$ | $m(\{\theta_1\}) = 0.0408$ $m(\{\theta_2\}) = 0.5547$ $m(\{\theta_1, \theta_2\}) = 0.0069$ $m(\{\theta_3\}) = 0.375$ $m(\{\theta_1, \theta_3\}) = 0.0135$ $m(\{\theta_2, \theta_3\}) = 0.0089$ $m(\{\Theta\}) = 0.0002$ |
| $\alpha = 0.8$ | $m(\{\theta_1\}) = 0.0513$ $m(\{\theta_2\}) = 0.5409$ $m(\{\theta_1, \theta_2\}) = 0.0089$ $m(\{\theta_3\}) = 0.3747$ $m(\{\theta_1, \theta_3\}) = 0.0014$ $m(\{\theta_2, \theta_3\}) = 0.0228$ | $m(\{\theta_1\}) = 0.0418$ $m(\{\theta_2\}) = 0.5341$ $m(\{\theta_1, \theta_2\}) = 0.0199$ $m(\{\theta_3\}) = 0.3780$ $m(\{\theta_2, \theta_3\}) = 0.0255$ $m(\{\Theta\}) = 0.0007$ |
| $\alpha = 1$ | $m(\{\theta_1\}) = 0.0487$ $m(\{\theta_2\}) = 0.5435$ $m(\{\theta_1, \theta_2\}) = 0.0124$ $m(\{\theta_3\}) = 0.3837$ $m(\{\theta_2, \theta_3\}) = 0.0118$ | $m(\{\theta_1\}) = 0.0487$ $m(\{\theta_2\}) = 0.5435$ $m(\{\theta_1, \theta_2\}) = 0.0124$ $m(\{\theta_3\}) = 0.3837$ $m(\{\theta_2, \theta_3\}) = 0.0118$ |

VI. CONCLUSIONS

In this paper, we have proposed two approaches with a user-specified weighting factor to transform a given FMF into a trade-off BBA. These two approaches are both effective approaches for obtaining a trade-off BBA. The users can transform a given FMF into a BBA by their preferred approach. With the cardinality of FOD increasing, the computational complexity of the optimization will become exponential growth. The reason for this is the structure of the belief functions. By using the user-specified weighting factor to influence how close the trade-off BBA is to each of the two BBAs obtained by solving the uncertainty maximization and minimization. The example of using our transformations in the practical application is provided. The numerical examples indicate that the uncertainty of the obtained BBA is between the minimum and maximal uncertainties. In a future work, we will try to use and compare different types of the distance of evidence as objective function to expect a better trade-off BBA.

REFERENCES

- [1] L. A. Zadeh, “Fuzzy Sets,” *Information and Control*, vol. 8, pp. 338-353, June 1965.
- [2] L. A. Zadeh, “Fuzzy sets as a basis for a theory of possibility,” *Fuzzy Sets and Systems*, vol. 1, pp. 3-28, January 1978.
- [3] Z. Pawlak, “Rough sets,” *International Journal of Computer & Information Sciences*, vol. 11, pp. 341-356, October 1982.

- [4] G. Shafer, *A Mathematical Theory of Evidence*, Princeton University Press Princeton, 1976.
- [5] Y. Bi, D. Bell, H. Wang, G. Guo and K. Greer, "Combining Multiple Classifiers Using Dempsters Rule of Combination for Text Categorization," *Modeling Decisions for Artificial Intelligence*, vol. 3131, pp. 127-138, 2004.
- [6] M. C. Florea, A. L. Jousselme, D. Grenier and É. Bossé, "Approximation techniques for the transformation of fuzzy sets into random sets," *Fuzzy Sets and Systems*, vol. 159, pp. 270-288, February 2008.
- [7] D. Han, C. Han and Y. Deng, "Novel Approaches for the Transformation of Fuzzy Membership Function into Basic Probability Assignment based on Uncertainty Optimization," *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems*, vol. 21, pp. 289-322, 2013.
- [8] Y. Yang, X. R. Li and D. Han, "An improved α -cut approach to transforming fuzzy membership function into basic belief assignment," *Chinese Journal of Aeronautics*, vol. 29, pp.1042-1051, August 2016.
- [9] T. Ma, F. Xiao, "An Improved Method to Transform Triangular Fuzzy Number Into Basic Belief Assignment in Evidence Theory," *IEEE Access*, vol 7, pp. 25308-25322, February 2019.
- [10] R. Ilin, E. Blasch, "Information Fusion with Belief Functions: a Comparison of Proportional Conflict Redistribution PCR5 and PCR6 Rules for Networked Sensors," *International Conference on Information Fusion IEEE: Washington, DC, USA 2015: 2084-2091*.
- [11] F. Smarandache, J. Dezert, "On the consistency of PCR6 with the averaging rule and its application to probability estimation," *Proceedings of the 16th International Conference on Information Fusion IEEE: Istanbul, Turkey, 2013: 1119-1126*.
- [12] U. Höhle, "Entropy with respect to plausibility measures," *Proceedings of the 12th IEEE International Symposium on Multiple-Valued Logic*, pp. 167-169, 1982.
- [13] R. R. Yager, "Entropy and specificity in a mathematical theory of evidence," *International Journal of General Systems*, vol. 9, pp. 249-260, 1983.
- [14] D. Dubois, and H. Prade, "A note on measures of specificity for fuzzy sets," *International Journal of General Systems*, vol. 10, pp. 279-283, 1985.
- [15] A. L. Jousselme, C. Liu, D. Grenier, and É. Bossé, "Measuring ambiguity in the evidence theory," *IEEE Transactions on Systems, Man, and Cybernetics Part A: Systems and Humans*, vol. 36, pp. 890-903, August 2006.
- [16] D. Harmanec, G. J. Klir, "Measuring total uncertainty in Dempster-Shafer theory: a novel approach," *International Journal of General Systems*, vol. 22, pp. 405-419, 1994.
- [17] P. Smets, R. Kennes, "The transferable belief model," *Artificial Intelligence*, vol. 66, pp. 191-234, April 1994.
- [18] J. Kampe de Fariet, "Interpretation of fuzzy membership functions of fuzzy sets in terms of plausibility and belief," *Fuzzy Information and Decision Processes*, pp. 93-98, January 1982.
- [19] A. L. Jousselme, D. Grenier and É. Bossé, "A new distance between two bodies of evidence," *Information Fusion*, vol. 2, pp. 91-101, June 2001.