On Decombination of Belief Function

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***Abstract*—The evidence combination is a kind of decision-level information fusion in the theory of belief functions. Given two basic belief assignments (BBAs) originated from different sources, one can combine them using some combination rule, e.g., Dempster's rule to expect a better decision result. If one only has a combined BBA, how to determine the original two BBAs to combine? This can be considered as a defusion of information. This is useful, e.g., one can analyze the difference or dissimilarity between two different information sources based on the BBAs obtained using evidence decombination. Therefore, in this paper, we research on such a defusion in the theory of belief functions. We find that it is a well-posed problem if one original BBA and the combined BBA are both available, and it is an under-determined problem if both BBAs to combine are unknown. We propose an optimization-based approach for the evidence decombination according to the criteria of divergence maximization. Numerical examples are provided illustrate and verify our proposed decombination approach, which is expected to be used in applications such the difference analysis between information sources in information fusion systems when the original BBAs are discarded, and performance evaluation of combination rules.**

***Index Terms*—information fusion, decombination, belief functions, combination, divergence maximization**

I. Introduction

The theory of belief functions, which is also known as the Dempster-Shafer evidence theory [1], has been widely used in many information fusion based applications including the pattern classification [2], [3], multi criteria decision making (MCDM) [4], fault diagnosis [5] and image processing [6].

The information fusion in the theory of belief functions is implemented by evidence combination based on some combination rule, e.g., the well-known Demspter's rule. There have also emerged various alternative combination rules including Yager's rule [7], Dubois & Prade's rule [8], Smets' rule [9], Murphy's rule [10], Florea's rule [11], proportional conflict redistribution 5 (PCR5), and PCR6 [12], [13], etc.

The inverse process of the information fusion, which can also be called as information "defusion" or "decombination", is also meaningful in information processing and analysis. Like the blind source separation (BSS) [14] and independent component analysis [15], which aim to recover independent sources given only the observations that are unknown linear mixtures of the unobserved independent source signals, can be considered as a process of information decombination. One can analyze the original information sources and judge their relationship based on the results obtained using decombination. The community of belief functions theory seldom research on the information decombination problem, which means that given a combined BBA, how to determine the original BBAs for the combination. In Smets' work [16], the concept of decomposition of evidence was proposed, which focuses on decomposing any BBA (not always assumed to a combined BBA) into many simple support function of BBAs. He also proposed the inverse operation of evidence combination, which only focus on the following case: given a combined BBA and one BBA participating the combination, how to restore another BBA participating the combination. In this paper, we focus on the information decombination (separation) or evidence decombination in the theory of belief functions. For simplicity, here we only concern the evidence decombination for two information sources. We find that given the combined BBA together with one original BBA, it is well-posed, that is, the other BBA can be uniquely determined. However, it turns out to be an under-determined problem (with multiple solutions) if both BBAs participating the combination are unknown and the combined BBA is given. The optimization (maximization) based decombination method is proposed accordingly, where the objective function is the distance between the two original BBAs (unknown variables to determine). Examples and experiments are provided to illustrate and verify our proposed information decombination method for the belief function.

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II. Basics of Belief Functions Theory

The basic concept in the theory of belief functions [1] is the frame of discernment (FOD), which is determined by what we want to know and what we know. Elements in an FOD are mutually exclusive and exhaustive. $m : 2^{\Theta} \rightarrow [0, 1]$ is defined as a basic belief assignment (BBA, also called a mass function) defined on the FOD Θ satisfying

$$\sum\nolimits_{A \subseteq \Theta} m(A) = 1,\ m(\emptyset) = 0 \tag{1}$$

where 2^Θ denotes the powerset of Θ. if $\forall m(A) > 0$, then A is called a focal element of $m(\cdot)$. If a BBA only has singleton focal elements, then it is called a Bayesian BBA.

Given a BBA $m(\cdot)$, its corresponding belief function (Bel) and plausibility function (Pl) are respectively defined as

$$Bel(A) = \sum_{B \subseteq A} m(B) \tag{2}$$

$$Pl(A) = \sum_{A \cap B \neq \emptyset} m(B) \tag{3}$$

The belief $Bel(A)$ represents the justified specific support for the focal element (or proposition) A, while the plausibility $Pl(A)$ represents the potential specific support for A. The length of the belief interval $[Bel(A), Pl(A)]$ represents the imprecision degree of A.

The evidence combination is the fusion of the BBAs originated from different sources. Two independent BBAs $m_1(\cdot)$ and $m_2(\cdot)$ can be combined using Dempster's rule of combination [1] defined by

$$m(A) = \begin{cases} 0, & A = \emptyset \\ \frac{\sum_{A_i \cap B_j = A} m_1(A_i) m_2(B_j)}{1 - \sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)}, & A \neq \emptyset \end{cases} \tag{4}$$

Dempster's rule in general can be considered as a multiplicative and conjunctive fusion rule. Dempster's rule of combination has been criticized for its counter-intuitive behaviors [17], [18], especially in high conflict cases. Many alternative combination rules have been proposed accordingly. See [12], [19], [20] for details. Other researchers like Haenni [21] think that the conflict results from a fault in the framing of problem.

Distance of evidence is for measuring the dissimilarity between BBAs. The most commonly used and strict distance of evidence is Jousselme's distance [22] defined as follows.

$$d_J(m_1, m_2) \triangleq \sqrt{0.5 \cdot (m_1 - m_2)^T \mathbf{Jac} \ (m_1 - m_2)} \tag{5}$$

where the elements $Jac(A, B)$ of Jaccard's weighting matrix $\mathbf{Jac}$ are defined as

$$\mathbf{Jac}(A, B) = |A \cap B| / |A \cup B| \tag{6}$$

Here A, B are focal elements of m_1 and m_2, respectively. Jaccard's matrix has been proved to be positive-definite [23], therefore, Jousselme's distance is a strict metric satisfying four requirements of the distance metric including the non-negativity, non-degeneracy, symmetry, and triangular inequality.

III. Evidence Decombination in Belief Functions Theory

The evidence combination can be considered as a procedure of information fusion[1] as shown in Fig. 1.

[1]or information compression because from two BBAs we get one.

Fig. 1. Evidence combination - Information Fusion.

Given a BBA obtained after the combination, if one wants to know the possible original BBAs, then the evidence decombination is needed, which can be considered as a procedure of information decombination as shown in Fig. 2.

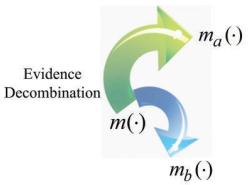

Fig. 2. Evidence decombination - Information Decombination or "Defusion".

In this paper, we focus on determining the original BBAs given a combined BBA. First, we analyze the relationship between the combined BBA and the original ones. For simplicity, we only suppose that there are two original BBAs in this paper.

A. Relation between Combined BBA and Original Ones according to Dempster's Rule

According to the Dempster's rule in Eq.(4), one can obtain the following equations. Suppose that $m_1(\cdot)$ and $m_2(\cdot)$ are two BBAs defined on the FOD $\Theta = \{\theta_1, ..., \theta_n\}$. For each BBA, there are at most $2^n - 1$ focal elements as shown below.

$$\begin{matrix} m_1 \\ \begin{bmatrix} \{\theta_1\} \\ \{\theta_2\} \\ \{\theta_1, \theta_2\} \\ \vdots \\ \Theta \end{bmatrix} \end{matrix} \quad \begin{matrix} m_2 \\ \begin{bmatrix} \{\theta_1\} \\ \{\theta_2\} \\ \{\theta_1, \theta_2\} \\ \vdots \\ \Theta \end{bmatrix} \end{matrix}$$

Define a matrix $R^{(k)}$ for each $k = 1, ..., 2^n - 1$ where

$$R^{(k)}(i, j) = \begin{cases} 1, & \text{if} \quad C_k = A_i \cap B_j \\ 0, & \text{if} \quad C_k \neq A_i \cap B_j \end{cases} \tag{7}$$

where A_i is the focal element of $m_1(\cdot)$, and where B_j is the focal element of $m_2(\cdot)$. The combined BBA is $m(\cdot) = m_1(\cdot) \oplus m_2(\cdot)$, and C_k is the focal element of $m(\cdot)$. Note that $i, j, k = 1, ..., 2^n - 1$. According to Dempster's rule, the mass assignment of focal element C_k in the combined BBA is

$$m(C_k) = \frac{\begin{bmatrix} m_1(\{\theta_1\}) \\ m_1(\{\theta_2\}) \\ \vdots \\ m_1(\Theta) \end{bmatrix}^T R^{(k)} \begin{bmatrix} m_2(\{\theta_1\}) \\ m_2(\{\theta_2\}) \\ \vdots \\ m_2(\Theta) \end{bmatrix}}{1 - K} \tag{8}$$

where $k = 1, ..., 2^n - 1$ and $K = \sum_{A_i \cap B_j = \emptyset} m_1(A_i) m_2(B_j)$ denotes the conflict coefficient. For simplicity in the sequel, we denote the mass value vector as

$$\boldsymbol{m}_1 = \begin{bmatrix} m_1(\{\theta_1\}) \\ m_1(\{\theta_2\}) \\ m_1(\{\theta_1, \theta_2\}) \\ m_1(\{\theta_3\}) \\ m_1(\{\theta_1, \theta_3\}) \\ m_1(\{\theta_2, \theta_3\}) \\ m_1(\Theta) \end{bmatrix}^T, \boldsymbol{m}_2 = \begin{bmatrix} m_2(\{\theta_1\}) \\ m_2(\{\theta_2\}) \\ m_2(\{\theta_1, \theta_2\}) \\ m_2(\{\theta_3\}) \\ m_2(\{\theta_1, \theta_3\}) \\ m_2(\{\theta_2, \theta_3\}) \\ m_2(\Theta) \end{bmatrix}^T$$

Then, Eq. (8) can be rewritten as

$$m(C_k) = \frac{\boldsymbol{m}_1^T R^{(k)} \boldsymbol{m}_2}{1 - K}$$

1) Case I: In this case, the combined BBA $m(\cdot)$ is available, and both original BBAs are unknown. That is, $m_1(A_i)$ $(i = 1, ..., 2^n - 1)$ and $m_2(B_j)$ $(j = 1, ..., 2^n - 1)$ are unknown variables to determine, then the quantity of the unknown variable is $2^n - 1 \times 2 = 2^{n+1} - 2$. For the BBA, there exists

$$\sum_{i=1}^{2^n - 1} m_1(A_i) = 1 \tag{9}$$

$$\sum_{j=1}^{2^n - 1} m_2(B_j) = 1 \tag{10}$$

Considering Eqs. (8)-(10), we have $2^n - 1 + 2 = 2^n + 1$ simultaneous equations. As aforementioned, to determine all the mass values of $m_1(\cdot)$ and $m_2(\cdot)$, we have $2^{n+1} - 2$ unknown variables. That is, the quantity of the unknown variables is larger than that of the equations. Therefore, this is an under-determined problem with multiple solutions in general.

2) Case II: In this case, the combined BBA $m(\cdot)$ and one original BBA (e.g., $m_1(\cdot)$) are available, while another original BBA (e.g., $m_2(\cdot)$) is unknown. To determine $m_2(\cdot)$, we have $2^n - 1$ unknown variables. By considering Eqs. (8) and (10), we have 2^n simultaneous equations. That is, the quantity of the unknown variables is less than that of the equations. Therefore, this is an over-determined problem, and then $m_2(\cdot)$ can be determined uniquely.

B. Optimization Based Evidence Decombination

As aforementioned, given a combined BBA, to determine the two original BBAs is an under-determined problem, for which, the optimization-based approach is feasible. Then, the key issue is to select an appropriate criterion to establish the objective function for the optimization.

In fact, the evidence decombination is like the blind source separation (BSS), where the divergence between different sources are used for the optimization based source separation, e.g, minimization of the mutual information (MMI) [24], which represents the largest divergence. Therefore, in this paper, we use for reference the criterion in BSS to design the objective function in optimization based evidence decombination. Here we use the distance of evidence to describe the divergence between BBAs. Furthermore, we use the simultaneous equations including the Eqs (8)-(10) together with inequalities (to assure a legal BBA[2] with the mass value lies in [0,1]) as the constraints for the distance maximization to implement the evidence decombination as illustrated in Eq. (11).

$$\max_{m_1, m_2} d_J(m_1, m_2) = \sqrt{0.5 \cdot (m_1 - m_2)^T \mathbf{Jac} (m_1 - m_2)}$$

$$s.t. \begin{cases} m(C_k) = \frac{\boldsymbol{m}_1{}^T R^{(k)} \boldsymbol{m}_2}{1 - K} \\ \sum_{i=1}^{2^n - 1} m_1(A_i) = 1 \\ \sum_{j=1}^{2^n - 1} m_2(B_j) = 1 \\ 0 \le m_1(A_i) \le 1, \forall i = 1, ..., 2^n - 1 \\ 0 \le m_2(B_j) \le 1, \forall j = 1, ..., 2^n - 1 \end{cases} \tag{11}$$

By solving[3] the constrained maximization problem in Eq. (11), one can obtain a pair of BBAs that are farthest to each other, and that provide the combined BBA when fusioned with Demspter's rule.

IV. Numerical Examples of Evidence Decombination Based on Optimization

In this section we give different examples illustrating how BBAs decombination can be obtained based on optimization of evidence decombination.

A. Example 1

Suppose that the FOD is $\{\theta_1, \theta_2, \theta_3\}$. A BBA obtained after the combination of two unknown BBAs is

$$m(\{\theta_1\}) = 0.1, m(\{\theta_2\}) = 0.2, m(\{\theta_1, \theta_2\}) = 0.1,$$
$$m(\{\theta_3\}) = 0.1, m(\{\theta_1, \theta_3\}) = 0.1,$$
$$m(\{\theta_2, \theta_3\}) = 0.3, m(\Theta) = 0.1.$$

The equality constraints for the maximization problem include

$$m(\{\theta_1\}) = 0.1 = \frac{\begin{bmatrix} m_1(\{\theta_1\}) \\ m_1(\{\theta_2\}) \\ m_1(\{\theta_1, \theta_2\}) \\ m_1(\{\theta_3\}) \\ m_1(\{\theta_1, \theta_3\}) \\ m_1(\{\theta_2, \theta_3\}) \\ m_1(\Theta) \end{bmatrix}^T R^{(1)} \begin{bmatrix} m_2(\{\theta_1\}) \\ m_2(\{\theta_2\}) \\ m_2(\{\theta_1, \theta_2\}) \\ m_2(\{\theta_3\}) \\ m_2(\{\theta_1, \theta_3\}) \\ m_2(\{\theta_2, \theta_3\}) \\ m_2(\Theta) \end{bmatrix}}{1 - K}$$

where

$$R^{(1)} = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It can be rewritten to a simpler form as

$$m(\{\theta_1\}) = 0.1 = \frac{\boldsymbol{m}_1{}^T R^{(1)} \boldsymbol{m}_2}{1 - K}$$

For other focal elements,

$$m(\{\theta_2\}) = 0.2 = \frac{\boldsymbol{m}_1{}^T R^{(2)} \boldsymbol{m}_2}{1 - K}$$

[2]to obtain admissible BBAs with values in [0,1] and their sum equals to one.

[3]Here we use the global optimization toolbox in Matlab™.

where

$$R^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$m(\{\theta_1, \theta_2\}) = 0.1 = \frac{\boldsymbol{m}_1{}^T R^{(3)} \boldsymbol{m}_2}{1-K}$$

where

$$R^{(3)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$m(\{\theta_3\}) = 0.1 = \frac{\boldsymbol{m}_1{}^T R^{(4)} \boldsymbol{m}_2}{1-K}$$

where

$$R^{(4)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

$$m(\{\theta_1, \theta_3\}) = 0.1 = \frac{\boldsymbol{m}_1{}^T R^{(5)} \boldsymbol{m}_2}{1-K}$$

where

$$R^{(5)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

$$m(\{\theta_2, \theta_3\}) = 0.3 = \frac{\boldsymbol{m}_1{}^T R^{(6)} \boldsymbol{m}_2}{1-K}$$

where

$$R^{(6)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$m(\Theta) = 0.1 = \frac{\boldsymbol{m}_1{}^T R^{(7)} \boldsymbol{m}_2}{1-K}$$

where

$$R^{(7)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the two equations in Eqs.(9) and Eqs.(10). The inequality constraints are

$$\begin{aligned} &0 \leq m_1(\{\theta_1\}) \leq 1 \\ &0 \leq m_1(\{\theta_2\}) \leq 1 \\ &0 \leq m_1(\{\theta_1, \theta_2\}) \leq 1 \\ &0 \leq m_1(\{\theta_3\}) \leq 1 \\ &0 \leq m_1(\{\theta_1, \theta_3\}) \leq 1 \\ &0 \leq m_1(\{\theta_2, \theta_3\}) \leq 1 \\ &0 \leq m_1(\Theta) \leq 1 \end{aligned}$$

and

$$\begin{aligned} &0 \leq m_2(\{\theta_1\}) \leq 1 \\ &0 \leq m_2(\{\theta_2\}) \leq 1 \\ &0 \leq m_2(\{\theta_1, \theta_2\}) \leq 1 \\ &0 \leq m_2(\{\theta_3\}) \leq 1 \\ &0 \leq m_2(\{\theta_1, \theta_3\}) \leq 1 \\ &0 \leq m_2(\{\theta_2, \theta_3\}) \leq 1 \\ &0 \leq m_2(\Theta) \leq 1 \end{aligned}$$

According to the constrained maximization in Eq. (11), one can obtain two BBAs as follows:

$$\begin{aligned} &m_a(\{\theta_1\}) = 0, m_a(\{\theta_2\}) = 0, m_a(\{\theta_1, \theta_2\}) = 0.0323, \\ &m_a(\{\theta_3\}) = 0.1612, m_a(\{\theta_1, \theta_3\}) = 0.1612, \\ &m_a(\{\theta_2, \theta_3\}) = 0.4840, m_a(\Theta) = 0.1613. \end{aligned}$$

and

$$\begin{aligned} &m_b(\{\theta_1\}) = 0.0834, m_a(\{\theta_2\}) = 0, m_b(\{\theta_1, \theta_2\}) = 0.3666, \\ &m_b(\{\theta_3\}) = 0, m_b(\{\theta_1, \theta_3\}) = 0.0001, \\ &m_b(\{\theta_2, \theta_3\}) = 0, m_b(\Theta) = 0.5499. \end{aligned}$$

It is easy to verify that the combination result $m_a(\cdot) \oplus m_b(\cdot)$ is the same as the given BBA $m(\cdot)$.

B. Example 2

Suppose that there are two BBAs defined on the FOD $\Theta = \{\theta_1, \theta_2, \theta_3\}$:

$$\begin{aligned} &m_1(\{\theta_1\}) = 0.6, m_1(\{\theta_2\}) = 0.2, \\ &m_1(\{\theta_2, \theta_3\}) = 0.1, m_1(\Theta) = 0.1. \end{aligned}$$

and

$$\begin{aligned} &m_2(\{\theta_1\}) = 0.2, m_2(\{\theta_2\}) = 0.6, \\ &m_2(\{\theta_2, \theta_3\}) = 0.1, m_2(\Theta) = 0.1. \end{aligned}$$

By calculating the Jousselme's distance in Eq. (5), one obtains that

$$d_J(m_1, m_2) = 0.4.$$

With Dempster's rule of combination, one obtains that $m(\cdot) = m_1(\cdot) \oplus m_2(\cdot)$ with

$$\begin{aligned} &m(\{\theta_1\}) = 0.3846, m(\{\theta_2\}) = 0.5385, \\ &m(\{\theta_2, \theta_3\}) = 0.0577, m(\Theta) = 0.0192. \end{aligned}$$

According to the evidence decombination approach in Eq (11), one obtains that

$$\begin{aligned} &m_a(\{\theta_1\}) = 0, m_a(\{\theta_2\}) = 0.8750, \\ &m_a(\{\theta_2, \theta_3\}) = 0.0851, m_a(\Theta) = 0.0400. \end{aligned}$$

and

$$\begin{aligned} &m_b(\{\theta_1\}) = 0.9399, m_b(\{\theta_2\}) = 0, \\ &m_b(\{\theta_2, \theta_3\}) = 0.0131, m_b(\Theta) = 0.0470. \end{aligned}$$

It is easy to verify that the combination result $m_a(\cdot) \oplus m_b(\cdot) = m(\cdot)$, which is the same as $m_1(\cdot) \oplus m_2(\cdot)$.

By calculating the Jousselme's distance given by Eq. (5), one can verify that

$$d_J(m_a, m_b) = 0.9265 > d_J(m_1, m_2) = 0.4.$$

C. Example 3

A given combined BBA is the same as that in Example 2.

$$\begin{aligned} &m(\{\theta_1\}) = 0.3846, m(\{\theta_2\}) = 0.5385, \\ &m(\{\theta_2, \theta_3\}) = 0.0577, m(\Theta) = 0.0192. \end{aligned}$$

Moreover, suppose that we have additional information and we also know $m_1(\cdot)$:

$$\begin{aligned} &m_1(\{\theta_1\}) = 0.6, m_1(\{\theta_2\}) = 0.2, \\ &m_1(\{\theta_2, \theta_3\}) = 0.1, m_1(\Theta) = 0.1. \end{aligned}$$

Then, we try to use the BBA decombination to calculate the $\hat{m}_2(\cdot)$ and to check whether it is the same as $m_2(\cdot)$ in

Example 2. Here is just the case II as aforementioned in Sect III.A. Therefore, $\hat{m}_2(\cdot)$ should be unique. So, there should exist $m_2(\cdot) = \hat{m}_2(\cdot)$. It is an over-determined problem, and we can still use the optimization to solve $\hat{m}_2(\cdot)$ by modifying the optimization problem to

$$\max_{\hat{m}_2} d_J(m_1, \hat{m}_2) = \sqrt{0.5 \cdot (m_1 - \hat{m}_2)^T \mathbf{Jac} (m_1 - \hat{m}_2)}$$

$$s.t. \begin{cases} m(C_k) = \frac{\boldsymbol{m}_1{}^T R^{(k)} \hat{\boldsymbol{m}}_2}{1-K} \\ \sum_{j=1}^{2^n-1} \hat{m}_2(B_j) = 1 \\ 0 \leq \hat{m}_2(B_j) \leq 1, \forall j = 1, ..., 2^n - 1 \end{cases} \tag{12}$$

where

$$\hat{\boldsymbol{m}}_2 = \begin{bmatrix} \hat{m}_2(\{\theta_1\}) \\ \hat{m}_2(\{\theta_2\}) \\ \vdots \\ \hat{m}_2(\Theta) \end{bmatrix}$$

By solving Eq. (12), one obtains

$$\hat{m}_2(\{\theta_1\}) = 0.2, \hat{m}_2(\{\theta_2\}) = 0.6,$$
$$\hat{m}_2(\{\theta_2, \theta_3\}) = 0.1, \hat{m}_2(\Theta) = 0.1.$$

That is, given a combined BBA and one original BBA, another original one can be determined uniquely.

V. Further Analysis on Evidence Decombination

A. Divergence Minimization or Maximization?

In the evidence decombination shown in Eq. (11), distance maximization is adopted. This is inspired by the minimization of mutual information (i.e., the maximization of divergence) between sources in Blind Source Separation (BSS), which aims to bring out more independent components [24]. One can also try to implement the evidence decombination based on the distance minimization. Based on our analysis, we find that if the distance minimization is used, the minimum distance will be zero and the BBAs of two sources are identical.

Suppose that $m_1(\cdot) = m_2(\cdot) = m_0(\cdot)$, one can rewrite the constraints in Eq. (11) as

$$\begin{cases} m(C_k) = \frac{\boldsymbol{m}_0{}^T R^{(k)} \boldsymbol{m}_0}{1-K} \\ \sum_{i=1}^{2^n-1} m_0(A_i) = 1 \\ 0 \leq m_0(B_j) \leq 1, \forall j = 1, ..., 2^n - 1 \end{cases} \tag{13}$$

where

$$\boldsymbol{m}_0 = \begin{bmatrix} m_0(\{\theta_1\}) \\ m_0(\{\theta_2\}) \\ \vdots \\ m_0(\Theta) \end{bmatrix}$$

As we see in Eq. (13), there are $2^n - 1$ unknown variables (mass values for $2^n - 1$ focal elements in $m_0(\cdot)$) to determine. There are $2^n - 1 + 1 = 2^n$ simultaneous equations in total. Therefore, if the solution exists, in general this is an over-determined problem which has the unique solution.

Here we provide an example to verify this, where the combined BBA is still as chosen in Example 2, which is

$$m(\{\theta_1\}) = 0.3846, m(\{\theta_2\}) = 0.5385,$$
$$m(\{\theta_2, \theta_3\}) = 0.0577, m(\Theta) = 0.0192.$$

According to Eq. (11) and change the maximization to minimization, we obtain that $m_1(\cdot) = m_2(\cdot) = m_0(\cdot)$, which is

$$m_0(\{\theta_1\}) = 0.3877, m_0(\{\theta_2\}) = 0.3958,$$
$$m_0(\{\theta_2, \theta_3\}) = 0.1082, m_0(\Theta) = 0.1082.$$

It is easy to verify that $m_0(\cdot) \oplus m_0(\cdot) = m(\cdot)$.

We prefer the criterion of distance maximization, since it can bring out more distinct (likely to be more independent) evidences.

Note that since we select the maximization, to assure to find the unique global optimal, the objective should be upper-convex. However, the objective function, i.e., the distance of evidence cannot satisfy this. Therefore, in our work in this paper, intelligent optimization algorithms [25] (e.g., the particle swarm algorithm and genetic algorithm) are adopted for the maximization to achieve a better solution.

B. Possible Applications

Note that given a combined BBA $m(\cdot)$, $m_a(\cdot)$ and $m_b(\cdot)$ after the evidence decombination. However, we do not know the specific correspondence between $\{m_a(\cdot), m_b(\cdot)\}$ and $\{m_1(\cdot), m_2(\cdot)\}$. That is, $m_a(\cdot)$ could correspond to $m_1(\cdot)$ or $m_2(\cdot)$, and $m_b(\cdot)$ could also correspond to $m_1(\cdot)$ or $m_2(\cdot)$. Therefore, it cannot be used for analyzing or evaluating specific single sensor; however, the evidence decombination is expected to be used in applications like divergence evaluation between sensors, which is helpful for the sensor management. Given a BBA, if one can decombine it into two BBAs, then the maximum difference between corresponding information sources can be evaluated by calculating the distance between the two BBAs.

Another possible application is the evaluation of different combination rules. Here, we only use the Dempster's rule to construct the evidence decombination. In fact, other alternative combination rules can also be used for finding evidence decombination, where the difference between most of existing rules of combinations available in the literature lies in the choice of matrix $R^{(k)}$ in Eq. (7). Then, given a BBA, one can use different decombination methods corresponding to different combination rule to bring out different pairs of BBAs. One can calculate the distance between two BBAs in each pair to represent the aggregation capability of the corresponding combination rule. That is, an evidence decombination approach can bring out a more divergent BBA pair, then the decombination method's corresponding combination rule can aggregate (combine) a more divergent BBA pair to the same BBA compared with other rules. So we say that it has a better aggregation capability.

VI. Conclusions

In this paper, an evidence decombination approach is proposed, where the distance maximization criterion is adopted in the evidence decombination. Some numerical examples and related analysis are provided to illustrate our proposed method and the possible applications are forecasted.

In this paper, the distance of evidence used in the optimization is Jousselme's distance. In our future work, we will try other strict distance metric [26], [27] in the theory of belief functions for comparison. Currently, the objective function is the distance of evidence. In future work, we will try to use the difference between BBAs' uncertainty measure values [28], [29]. Furthermore, we only consider two sources of evidence for the evidence decombination for simplicity. In our future work, we will try to design more sources (larger than two) for the evidence decombination. This paper is only a preliminary work on the evidence decombination, in future research work, we will try to apply the proposed method in various appropriate applications.

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