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Abstract

This paper aims to assess the stability of a polytope of linear systems by their vertices. These results are based on the Hermite-Biehler and Edge theorems. A sufficient condition satisfying a constraint on the even (or odd) part of the closed-loop characteristic polynomials associated with the stabilization of its vertices, is proved. Finally, a constructive method to stabilize a polytope of plants with simultaneous and robust linear time-invariant controllers is established.

Keywords: polytope of systems, simultaneous stabilization, robust stabilization, common interlacing property.

1 Introduction

This paper concerns the stabilization of polytopes of SISO plants with a fixed order controller. Until now the problem of deciding of the stability or stabilizability of a polytope of linear systems with a linear time-invariant (LTI) controller stays an open question. We do not know which are the conditions so that any polytope of systems is stabilizable by a LTI controller. In this article, we show that this question is equivalent to that of the stability of the set of segments of systems linking the vertices of the polytope with a LTI controller. The issue is therefore about the stability of this set of segments with a LTI controller. Remark that this problem is also an open question. This is even an undecidable mathematical issue that may be classified as an NP-hard problem, in the sense that it is not possible to find necessary and sufficient conditions to stabilize this family of systems with a LTI controller when the number of plants to stabilize, is greater than two, see [1]. Accordingly, there exist only sufficient or necessary conditions for the robust stabilization of a polytope of systems. This paper proposes some sufficient conditions based on the Edge theorem [2]. More precisely, the question of stabilization of a polytope of systems is treated as a simple simultaneous control design of a family of LTI systems submitted to an additional constraint on the closed-loop characteristic polynomials associated with the vertices of this polytope. Furthermore, the controller is shown to not only simultaneously stabilize the vertices of the polytope of systems, i.e. a simultaneous control purpose, but also the set of systems belonging to the polytope, i.e. a robust control purpose. That is hence a simultaneous and robust controller for the polytopes of systems. Let us review briefly the main results in simultaneous and robust control.

Concerning the simultaneous control, [3] and [4] have proved that there exits a necessary and sufficient condition to stabilize simultaneously two LTI systems. Moreover, [3] and [4] have shown

that this question is equivalent to the stabilization of one plant by a stable compensator. This issue called "strong stabilization" has been originally treated in [5]. The problem of the simultaneous stabilization of three or more LTI plants was initially tackled by [6] and [7]. Otherwise [8] has shown that there exists no tractable and complete conditions to check the simultaneous stabilizability of three or more systems. About the simultaneous stabilization of a segment of LTI systems, i.e. a continuum of LTI systems described by a parameter and its two endpoints, [9] has provided existence conditions which depend on the choice of the initial compensator stabilizing the extremities of the segment. For this class of system, transcendental conditions have been given by [10] and [11]. In [12] and [13] the authors address the question of the simultaneous strong stabilizability of a segment of systems. They are conditions that have been stated to stabilize each element of this family with a stable compensator. That does not imply existence conditions of a single controller stabilizing the whole set of systems belonging to this segment. Conditions for the simultaneous stabilizability of a segment of systems have been given in [14].

Concerning the robust control, Kharitonov's theorem and its generalization, [15] and [16] have been at the origin of many results. Others analysis methods have been studied in the literature but those dedicated to the synthesis of robust control with polynomial methods are only a few. In [17], the zero-exclusion principle has given conditions that are used to test the stability and allow to design robust controllers. More recently, [18], [19], [20] and [21] propose fixed-order controllers for systems with polytopic uncertainty. The methods described in these papers relied on a criterion of stability which necessitates exactly proper and strict positive realness (EP-SPR) functions. These rational functions are particular units in H_∞ which also verify the properties of the SPR functions, see [9], [22] and [23]. Nevertheless, derived from [24], it may be easily deduced that to stabilize a polytope of systems, it is only required to synthesize units in H_∞ whose Nyquist plots do not cut the negative real axis. Then the conditions of the EP-SPR functions are not necessary. Consequently, the constraint to make EP-SPR functions introduces sufficiency and conservatism in regard to the necessary and sufficient condition to stabilize a polytope of systems. In this paper, we propose a new approach to synthesize units in H_∞ for stabilizing the polytopes of systems which are not SPR functions and which remain units in H_∞ if one interchanges even (or odd) part of the numerator and denominator polynomials. The Nyquist plots of these units in H_∞ do not cut the negative real axis, see [24].

In this manuscript, we focus on the stabilization of polytope of single input single output LTI systems. Based on the Hermite-Biehler theorem and the Edge theorem, we prove that to stabilize a polytope of systems it is sufficient to stabilize all its vertices with a simultaneous controller giving characteristic polynomials that have a common even (or odd) part. Moreover, we show that the synthesized units in H_∞ do not cut the real negative axis of the Nyquist plot. In addition a controller design for stabilizing the polytopes of systems is provided mixing polynomial techniques and linear optimization.

The paper is organized as it follows. After preliminaries in section 2, the problem of the robust stabilization of a polytope of systems is stated in section 3 by applying the Edge theorem. Then an approach to stabilize the vertices of a polytope of systems with a simultaneous compensator is given in section 4 that leads to a simple linear programming (LP) problem. The issue of the stabilization of a polytope of systems by a simultaneous controller is formulated as positivity conditions if particular pole placement constraints hold. Finally in section 5, examples are given to illustrate this framework. In the appendix, our approach is extended to the case where characteristic polynomials in closed-loop have a common odd part.

2 Preliminaries

Notations: The degree of a real polynomial X is denoted $\delta(X)$. $\mathbb{P}_m$ denotes the set of real polynomials of degree m. The set of Hurwitz-stable polynomials is denoted $\mathcal{H}$. $\mathbb{R}$ is the real field. $\mathbb{C}$ is the complex field. $\mathbb{R}^n_{++} = \{\, x \in \mathbb{R}^n : x_i > 0,\ \forall i \,\}$ and $\mathbb{R}_{-\{0\}} = \{\, x \in \mathbb{R} :\ x < 0 \text{ or } x > 0\}$. The notation $x > 0$ with $x \in \mathbb{R}^n$ means that each component of the vector x is positive. If and only if is defined as *iff*. The exactly proper and strict positive realness functions are denoted as EP-SPR functions. H_∞ is the set of proper stable rational functions. $\|\|\|$ denotes the operator norm. $\mathbb{U}$ is the set of units in H_∞ i.e. $U \in \mathbb{U}$ *iff* $U \in H_\infty$ and $U^{-1} \in H_\infty$.

In the sequel of this section, concepts on Hurwitz-stability of a polytope of real polynomials are presented. These results are used in section 3 to study the stability of a polytope of systems. The notion of Hurwitz-stability highlighted by [25] is approached hereafter under the angle of interlacing of real zeros.

2.1 The Hermite-Biehler's stability.

Let $f \in \mathbb{P}_m$ expanded as

$$f(s) = f^e(s^2) + sf^o(s^2) = \sum_{i=0}^{m} \sigma_i s^i$$

where σ_i are real coefficients and f^e and f^o denote the even and odd parts of f respectively. The polynomials f^e and f^o are given by

- if $\delta(f) = m = 2\ell + 1$

$$f^e(u) = \sigma_0 + \sigma_2 u + \ldots + \sigma_{(m-1)} u^\ell, \tag{1a}$$
$$f^o(u) = \sigma_1 + \sigma_3 u + \ldots + \sigma_m u^\ell. \tag{1b}$$

- if $\delta(f) = m = 2\ell$

$$f^e(u) = \sigma_0 + \sigma_2 u + \ldots + \sigma_m u^\ell, \tag{2a}$$
$$f^o(u) = \sigma_1 + \sigma_3 u + \ldots + \sigma_{(m-1)} u^{(\ell-1)}. \tag{2b}$$

Let us recall the Zeros Interlacing Property.

Definition 1 ([25], *The Zeros Interlacing Property*) Let $f(s) = f^e(s^2) + sf^o(s^2)$ be a real polynomial with $\delta(f) = m$. Assume that the roots of f^e and f^o are defined by the following sets

- if $\delta(f) = m = 2\ell + 1$ and $\delta(f^e) = \ell$, $\delta(f^o) = \ell$

$$roots(f^e) = \{a_1, \ldots, a_\ell\},$$
$$roots(f^o) = \{b_1, \ldots, b_\ell\},$$

- if $\delta(f) = m = 2\ell$ and $\delta(f^e) = \ell$, $\delta(f^o) = \ell - 1$

$$roots(f^e) = \{a_1, \ldots, a_\ell\},$$
$$roots(f^o) = \{b_1, \ldots, b_{\ell-1}\}.$$

Then f^e and f^o interlace iff

- the roots of f^e and f^o are real, negative, distinct and simple,
- the leading coefficients of f^e and f^o have the same sign,
- the ℓ roots of f^e alternate with the ℓ or $\ell-1$ roots of f^o as it follows.

  If $\delta(f) = m = 2\ell + 1$ then

  $$b_1 < a_1 < b_2 \ldots < b_\ell < a_\ell < 0.$$

  If $\delta(f) = m = 2\ell$ then

  $$a_1 < b_1 < \ldots < b_{\ell-1} < a_\ell < 0.$$

Let us note that the cases $m = 2\ell + 1$ or $m = 2\ell$ are considered indistinctly in this paper but to simplify the writing, the theoretical results are presented for only $m = 2\ell + 1$ and an example is given section 5.1 for the two cases $m = 2\ell$ and $m = 2\ell + 1$. It should be noticed that the case $m = 2\ell$ may be directly deduced from the case $m = 2\ell + 1$.

The relationship between the zeros interlacing property of a real polynomial and the Hurwitz-stability is emphasized by the Hermite-Biehler's theorem that we remind.

Lemma 1 ([25], *Hermite-Biehler's theorem*) The real polynomial $f(s) = f^e(s^2) + sf^o(s^2)$ is Hurwitz-stable (i.e. all the real parts of the roots of f are strictly negative) iff f^e and f^o verify the Zeros Interlacing Property.

Let us write the partial fraction expansion of the real rational function f^o/f^e with f^e and f^o satisfying (1). Consequently, from lemma 1, simple conditions for testing the Hurwitz-stability of f are deduced.

Lemma 2 ([26], [27]) The polynomial $f(s) = f^e(s^2) + sf^o(s^2)$ with $f \in \mathbb{P}_m$ is Hurwitz-stable iff the two following conditions hold.

1. f^e has real, simple and negative roots $(a_1, \ldots, a_\ell)$.
2. The real coefficients c_k $(k = 0, \ldots, \ell)$ given by the expansion of f^o as the following sum of products of factors of f^e

   $$f^o(s^2) = c_0 f^e(s^2) + \sum_{k=1}^{\ell} c_k \frac{f^e(s^2)}{s^2 - a_k}. \tag{3}$$

   are strictly positive.

Note that $c_0 = 0$ in the case where $m = 2\ell$.

Mention that all the next developments are made in function of the writing (3) with f^e given a priori. The developments in function of f^o are presented in the appendix section.

Let us consider n real polynomials f_i, $(i = 1, \ldots, n)$ with a a same even part f^e. Then we have $f_i(s) = f^e(s^2) + sf_i^o(s^2)$, $(i = 1, \ldots, n)$.

In association with f_i, a generalization of the *Zeros Interlacing Property* is now given.

Definition 2 (*Common even interlacing property*) Consider a finite collection of polynomials f_i^o, $(i = 1, \ldots, n)$ and a polynomial f^e with their leading coefficients of same sign. Assume that for each pair (f_i^o, f^e), $(i = 1, \ldots, n)$, the roots of f_i^o and f^e are all real, negative, distinct and simple defined by the following sets

$$roots(f_i^o) = \{b_1^i, \ldots, b_\ell^i\},$$
$$roots(f^e) = \{a_1, \ldots, a_\ell\}.$$

The polynomials f_i^o have a common even interlacing f^e iff the ℓ roots of each f_i^o alternate with the ℓ roots of f^e as below

$$b_1^i < a_1 < b_2^i < \ldots < b_\ell^i < a_\ell < 0.$$

Remark that if the polynomials f_i^o, $(i = 1, \ldots, n)$ have a common interlacing f^e this implies that for any jth root, we have

$$\max_{i \in \{1,\ldots,n\}} b_j^i < a_j < \min_{i \in \{1,\ldots,n\}} b_{j+1}^i$$

By using the common even interlacing property given above, an extension of lemma 2 is deduced for testing the Hurwitz-stability of polynomials f_i, $(i = 1, \ldots, n)$.

Corollary 1 The polynomials $f_i(s) = f^e(s^2) + sf_i^o(s^2)$, $(i = 1, \ldots, n)$ with $f_i \in \mathbb{P}_m$ are Hurwitz-stable iff the two following conditions hold.

1. f^e has real, simple and negative roots $(a_1, \ldots, a_\ell)$.
2. The real coefficients $c_{i,k}$ $(i = 1, \ldots, n)$ and $(k = 0, \ldots, \ell)$ given by the expansions of f_i^o $(i = 1, \ldots, n)$ as the following sums of products of factors of f^e

$$f_i^o(s^2) = c_{i,0} f^e(s^2) + \sum_{k=1}^{\ell} c_{i,k} \frac{f^e(s^2)}{s^2 - a_k}. \tag{4}$$

are strictly positive.

Proof: That is a consequence of lemma 2. □

2.2 Matrix formulation of f_i^o and f^e.

Let us set $f^e(s^2) = \gamma^e \prod_{j=1}^{\ell}(s^2 - a_j)$ with $\gamma^e \in \mathbb{R}_{-\{0\}}$. Consequently relation (4) can be simplified as it follows

$$\begin{aligned} f_i^o(s^2) &= \gamma^e c_{i,0} \prod_{j=1}^{\ell}(s^2 - a_j) + \gamma^e \sum_{k=1}^{\ell} c_{i,k} \prod_{\substack{j=1 \\ j\neq k}}^{\ell}(s^2 - a_j) \\ &= \sum_{k=0}^{\ell} c_{i,k} \sum_{j=0}^{\ell-1} \upsilon_{j,k} s^{2j}, \\ &= V A^e \Psi_i^e, \qquad i = 1, \ldots, n. \end{aligned} \tag{5a}$$

$$f^e(s^2) = V H^e, \tag{5b}$$

where

$$\Psi_i^e = \begin{bmatrix} c_{i,0} & \ldots & c_{i,\ell} \end{bmatrix}^T \tag{6a}$$

$$V = \begin{bmatrix} 1 & s^2 & \ldots & s^{2\ell} \end{bmatrix} \tag{6b}$$

$$A^e = \begin{bmatrix} \upsilon_{0,0} & \upsilon_{0,1} & \ldots & \upsilon_{0,\ell} \\ \upsilon_{1,0} & \upsilon_{1,1} & \ldots & \upsilon_{1,\ell} \\ \vdots & \vdots & \vdots & \vdots \\ \upsilon_{\ell,0} & \upsilon_{\ell,1} & \ldots & \upsilon_{\ell,\ell} \end{bmatrix} \tag{6c}$$

$$H^e = \begin{bmatrix} h_0^e & h_1^e & \ldots & h_\ell^e \end{bmatrix}^T \tag{6d}$$

Note that matrix A^e only depends on the even part f^e.
By using the above notations, a reformulation of corollary 1 is now deduced.

Corollary 2 The polynomials $f_i \in \mathbb{P}_m$, $(i = 1, \ldots, n)$ are Hurwitz-stable iff the two following conditions hold

1. f^e is a polynomial with real, simple and negative roots $(a_1, \ldots, a_\ell)$.
2. There exist $\Psi_i^e \in \mathbb{R}_{++}^{\ell+1}$ for $i = 1, \ldots, n$, given by (6a).

Proof: That is a consequence of the writing (4) as (5a). □

2.3 Hurwitz-stability of a polytope of real polynomials.

Let S_f be a convex polytope of real polynomials described by the set of n vertices $f_i \in \mathbb{P}_m$ as it follows

$$S_f \triangleq \left\{ f : f = \sum_{i=1}^{n} \lambda_i f_i,\ \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\} \tag{7}$$

The set of the exposed edges[1] connecting two vertices f_i and f_j in $\mathbb{P}_m$ of the polytope of polynomials S_f is defined by $S_{f_{i,j}}$ where

$$S_{f_{i,j}} \triangleq \left\{ f_{i,j} : f_{i,j} = \alpha f_i + (1-\alpha) f_j,\ \alpha \in [0,1],\ i = 1,\ldots,n-1,\ \ j = i+1,\ldots,n \right\} \tag{8}$$

Recall theorem 1 in [2] that shows that the stability of a polytope of real polynomials can be inferred from its exposed edges.

Theorem 1 ([2]) The polytope of real polynomials S_f is Hurwitz-stable iff the set of exposed edges $S_{f_{i,j}}$ are Hurwitz-stable.

The Hurwitz-stability issue of S_f may be simplified by considering corollary 3 in [28].

Lemma 3 ([28]) Let $f_i \in \mathbb{P}_m$ and $f_j \in \mathbb{P}_m$ be two vertices of S_f with their leading coefficients of same sign and with a same even part f^e, then the exposed edge $f_{i,j}$ defined by (8) is Hurwitz-stable iff f_i and f_j are Hurwitz-stable.

Corollary 3 If the vertices $f_i \in \mathbb{P}_m$, $(i = 1,\ldots,n)$ of a polytope of real polynomials S_f are Hurwitz-stable with a same even part f^e and if the leading coefficients of f_i, $(i = 1,\ldots,n)$ have a same sign then the polytope of real polynomials S_f is Hurwitz-stable.

Proof: A consequence of theorem 1 and lemma 3. □

In connection with corollary 2, a reformulation of corollary 3 is now deduced.

Corollary 4 Consider a convex polytope of real polynomials S_f defined by the set of vertices $f_i \in \mathbb{P}_m$ verifying (7) with a same even part f^e. If the two following conditions hold then S_f is Hurwitz-stable.

1. f^e is a polynomial with real, simple and negative roots $(a_1,\ldots,a_\ell)$.
2. There exist $\Psi_i^e \in \mathbb{R}_{++}^{\ell+1}$ $(i = 1\ldots n)$ given by (6a).

Proof: As a consequence of corollaries 2 and 3. □

Lemma 4 Let S_f be a convex polytope of real polynomials described by a set of n vertices $f_i \in \mathbb{P}_m$ verifying (7). The number of exposed edges of S_f is $\overline{n} = \sum_{i=1}^{n-1}(n-i)$.

Proof: Consider f_i, $(i = 1,\ldots,n)$, vertices of a convex polytope S_f. Count the number of different exposed edges connected to each vertex with $f_{i,j} = f_{j,i}$. Each vertex f_i, $(i = 1,\ldots,n-1)$, is connected by $n-i$ different exposed edges to $n-i$ vertices. In consequence, the total number of exposed edges of S_f is $\overline{n} = \sum_{i=1}^{n-1}(n-i)$. □

Note that the problem of Hurwitz-stability of a polytope of real polynomials S_f is significantly reduced by considering corollary 3 and lemma 4, with which we only need to verify the Hurwitz-stability of n vertices of S_f instead to verify the Hurwitz-stability of $\overline{n}$ exposed edges of S_f, $n < \overline{n}$ if $n > 3$.

[1] See [2] for the definition of the exposed edges.

3 Robust stabilization of a polytope of plants.

Now a robust stability analysis for a polytope of systems may be carried out.

Definition 3 (*A polytope of plants*) The convex polytope of plants Ω_G generated by n SISO LTI systems $G_i = N_i D_i^{-1}$ $(i = 1 \dots n)$ is defined as

$$\Omega_G \triangleq \left\{ G : G = ND^{-1},\ N = \sum_{i=1}^{n} \lambda_i N_i,\ D = \sum_{i=1}^{n} \lambda_i D_i,\ \lambda_i \geqslant 0, \sum_{i=1}^{n} \lambda_i = 1 \right\} \tag{9}$$

where N_i and D_i are real polynomials with $\delta(N_i) \leqslant \delta(D_i)$ and $D_i \in \mathbb{P}_d$.

All pairs of vertices (G_i, G_j), with $i = 1, \dots, n-1$ and $j = i+1, \dots, n$, of a polytope of systems Ω_G are connected by an exposed edge $G_{i,j}$. Each exposed edge is a segment of systems. The set of all segments of systems of Ω_G is defined as the set $\Omega_{G_{i,j}}$ given below.

Definition 4 (*The set of segments of systems* $\Omega_{G_{i,j}}$) The set of segments of systems $\Omega_{G_{i,j}}$ connecting any two vertices of the polytope of systems Ω_G is defined as

$$\begin{aligned} \Omega_{G_{i,j}} \triangleq \Big\{ & G_{i,j} : G_{i,j} = N_{i,j} D_{i,j}^{-1},\ N_{i,j} = \alpha N_i + (1-\alpha) N_j, D_{i,j} = \alpha D_i + (1-\alpha) D_j, \\ & \alpha \in [0,\ 1],\ i = 1, \dots, n-1,\ \ j = i+1, \dots, n \Big\}. \end{aligned} \tag{10}$$

The objective of this paper is to propose a proper controller $C = XY^{-1}$ that stabilizes the polytope of systems Ω_G where X and Y are real polynomials. Consequently, the existence conditions of a compensator C satisfying

$$\Phi(G, C) = (NX + DY) \in \mathcal{H} \tag{11}$$

are studied.

Lemma 5 The compensator C stabilizes the polytope of systems Ω_G iff C stabilizes the set of segments of systems $\Omega_{G_{i,j}}$.

Proof: The compensator C stabilizes the polytope of systems Ω_G iff

$$\Phi(G, C) = NX + YD = \Big(\sum_{i=1}^{n} \lambda_i (N_i X + D_i Y) \Big) \in \mathcal{H}$$

or iff

$$\Big(\sum_{i=1}^{n} \lambda_i \Phi(G_i, C) \Big) \in \mathcal{H} \text{ with } \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1.$$

Hence, the compensator C stabilizes the polytope of systems Ω_G iff the polytope of polynomials ϕ is Hurwitz-stable where

$$\phi \triangleq \left\{ \Phi(G, C) : \Phi(G, C) = \sum_{i=1}^{n} \lambda_i \Phi(G_i, C),\ \lambda_i \geq 0,\ \sum_{i=1}^{n} \lambda_i = 1 \right\} \tag{12}$$

By applying theorem 1, we know that $\Phi(G,C) \in \mathcal{H}$ iff $\Phi(G_{i,j},C) \in \mathcal{H}$. This implies that ϕ is Hurwitz-stable iff $\phi_{i,j}$ is Hurwitz-stable with $\phi_{i,j}$ defined as the set of exposed edges of ϕ where

$$\phi_{i,j} \triangleq \Big\{ \Phi(G_{i,j},C) : \Phi(G_{i,j},C) = \alpha\Phi(G_i,C) + (1-\alpha)\Phi(G_j,C), \quad \alpha \in [0,\,1],\ i = 1,\ldots,n-1,\ \ j = i+1,\ldots,n \Big\}. \tag{13}$$

Consequently, the controller C stabilizes the polytope of plants Ω_G iff C stabilizes the set of segments of systems $\Omega_{G_{i,j}}$ defined by (10). □

Now to conclude this section, we can observe that by applying corollary 3 to the polytope of polynomials ϕ, we can simplify the issue of the Hurwitzness of ϕ. That is, if the vertices $\Phi(G_i,C)$ $(i = 1,\ldots n)$ of the polytope ϕ are Hurwitz-stable with a same degree and with their leading coefficients of same sign and a same even part $\Phi(G_i,C)^e$ where

$$\Phi(G_i,C)(s) = \Phi(G_i,C)^e(s^2) + s\Phi(G_i,C)^o(s^2)$$

then the polytope of polynomials ϕ is Hurwitz-stable.

4 Computational conditions for robust stabilization of a polytope of plants with a simultaneous compensator.

In this section, the problem of the stabilization of the polytope of systems Ω_G is posed as that of the stabilization of its vertices G_i with a simultaneous compensator C that guarantees a same even part $\Phi(G_i,C)^e$ for the closed-loop characteristic polynomials associated with G_i for $i = 1,\ldots,n$.

Theorem 2 Consider a convex polytope of plants Ω_G defined by (9). If there exists a simultaneous compensator C satisfying the following relations

$$\Phi(G_i,C) \in \mathcal{H}, \tag{14a}$$
$$\Phi(G_i,C)^e = \Phi(G_j,C)^e,\ i = 1,\ldots,n-1,\ j = i+1,\ldots,n. \tag{14b}$$

with the leading coefficients of $\Phi(G_i,C)$ of same sign, then C stabilizes Ω_G.

Proof: This theorem is a consequence of lemma 5 and corollary 3. □

Henceforth, the issue of the stabilization of the polytope of systems Ω_G is reduced to the design of a simultaneous compensator that must satisfy the particular poles placements required by theorem 2. This yields to the following lemma.

Lemma 6 Consider a convex polytope of plants Ω_G defined by (9). Let V, A^e, H^e and Ψ_i^e be given by (6b), (6c), (6d) and (6a) respectively. If there exist $\Psi_i^e \in \mathbb{R}_{++}^{\ell+1}$ $(i = 1,\ldots,n)$ and a compensator C verifying the following relations

$$\Phi(G_i,C)^e = VH^e, \qquad i = 1,\ldots,n \tag{15a}$$
$$\Phi(G_i,C)^o = VA^e\Psi_i^e \tag{15b}$$

then C stabilizes Ω_G.

Proof: This lemma is a consequence of corollary 4 and theorem 2. □

Notice that the simultaneous controller C proposed in lemma 6 has a fixed order since the degree of the closed-loop characteristic polynomials $\Phi(G_i, C)$ associated with the vertices of the polytope of polynomials ϕ given in (12), are chosen a priori.

Consider the simultaneous compensator $C = XY^{-1}$ and the vertices $G_i = N_i D_i^{-1}$ $(i = 1, \ldots, n)$ of a polytope of systems Ω_G where X, Y, N_i and D_i be real polynomials expressed in terms of their even and odd parts as follows

$$X(s) = X^e(s^2) + sX^o(s^2), \quad Y(s) = Y^e(s^2) + sY^o(s^2)$$
$$N_i(s) = N_i^e(s^2) + sN_i^o(s^2), \quad D_i(s) = D_i^e(s^2) + sD_i^o(s^2)$$

Then we obtain

$$\Phi(G_i, C)^e = \Gamma_i^e C_p, \quad i = 1, \ldots, n \tag{16a}$$
$$\Phi(G_i, C)^o = \Gamma_i^o C_p \tag{16b}$$

with

$$C_p(s^2) = [X^e(s^2) \quad Y^e(s^2) \quad X^o(s^2) \quad Y^o(s^2)]^T$$
$$\Gamma_i^e(s^2) = [N_i^e(s^2) \quad D_i^e(s^2) \quad s^2 N_i^o(s^2) \quad s^2 D_i^o(s^2)]$$
$$\Gamma_i^o(s^2) = [N_i^o(s^2) \quad D_i^o(s^2) \quad N_i^e(s^2) \quad D_i^e(s^2)]$$

Reformulate $\Phi(G_i, C)^e$ and $\Phi(G_i, C)^o$ in a matrix form. For that consider

$$C_\upsilon = \left[x_0^e \ \ldots \ x_{p^e}^e \ y_0^e \ \ldots \ y_{q^e}^e \ x_0^o \ \ldots \ x_{p^o}^o \ y_0^o \ \ldots \ y_{q^o}^o\right]^T \tag{17}$$

with $p^e = \delta(X^e)$, $p^o = \delta(X^o)$, $q^e = \delta(Y^e)$ and $q^o = \delta(Y^o)$.

Define two matrices Λ_i^e and Λ_i^o such that

$$\Phi(G_i, C)^e = V\Lambda_i^e C_\upsilon, \quad i = 1, \ldots, n \tag{18a}$$
$$\Phi(G_i, C)^o = V\Lambda_i^o C_\upsilon \tag{18b}$$

where Λ_i^e and Λ_i^o are a rewriting of Γ_i^e and Γ_i^o according to the coefficients of polynomials N_i^e, D_i^e, N_i^o and D_i^o.

Lemma 6 and relations (18) permit to deduce the following lemma.

Lemma 7 Consider a convex polytope of plants Ω_G defined by (9). Let A^e, H^e and Ψ_i^e given by (6c), (6d) and (6a) respectively. If there exist C_υ and $\Psi_i^e \in \mathbb{R}_{++}^{\ell+1}$, $(i = 1, \ldots, n)$ satisfying the following relations

$$\Lambda_i^e C_\upsilon = H^e \qquad i = 1, \ldots, n, \tag{19a}$$
$$\Lambda_i^o C_\upsilon = A^e \Psi_i^e \tag{19b}$$

then C stabilizes Ω_G.

Proof: This result is a consequence of lemma 6 and relations (18a) and (18b). □

Now rewrite relations (19)

$$B^e = \Theta^e \chi^e \tag{20}$$

where

$$B^e = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ H^e \\ H^e \\ \vdots \\ H^e \end{bmatrix}, \; \Theta^e = \begin{bmatrix} A^e & 0 & \dots & 0 & -\Lambda_1^o \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & A^e & -\Lambda_n^o \\ 0 & 0 & \dots & 0 & \Lambda_1^e \\ 0 & 0 & \dots & 0 & \Lambda_2^e \\ \vdots & \vdots & \dots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \Lambda_n^e \end{bmatrix},$$

$$\chi^e = \begin{bmatrix} \psi^{eT} & C_v^T \end{bmatrix}^T,$$
$$\psi^e = \begin{bmatrix} \Psi_1^{eT} & \Psi_2^{eT} & \Psi_3^{eT} & \dots & \Psi_n^{eT} \end{bmatrix}^T \in \mathbb{R}_{++}^{n(\ell+1)}.$$

There is solutions $\psi^e \in \mathbb{R}^{n(\ell+1)}$ and C_v to the equation (20) iff [29].

$$\operatorname{rank}\left(\begin{bmatrix} B^e & \Theta^e \end{bmatrix}\right) = \operatorname{rank}(\Theta^e). \tag{21}$$

By considering equation (19), the robust stabilization of the polytope of systems Ω_G given by lemma 7 is now set as the issue of feasibility of a linear programming problem (LP).

Theorem 3 Let Ω_G be a convex polytope of plants defined by (9). Assume that (21) holds then the controller C stabilizes Ω_G if there exists a solution χ^e to the linear programming (LP) problem given by (20).

Proof: Since (20) is a rewritten of the relations (19), then theorem 3 is equivalent to lemma 7. □

This LP problem can be expressed as a LP computational procedure. Readily available, linear programming software allows to solve this type of system (20) in an efficient way. This possibility is demonstrated through some numerical examples that are reported in the section 5.

After focusing initially on the controller design for stabilizing a polytope of systems, we are again going to review some literature results. Derived from [24], an equivalent condition to the one given in lemma 5 may be formulated.

Corollary 5 To stabilize a polytope of plants Ω_G with a LTI compensator, it is necessary and sufficient that the units in H_∞ formed by the ratio of every two closed-loop characteristic polynomials associated with the vertices of the polytope of systems have the modulus of their argument less than 180°.

Proof: This necessary and sufficient condition is a direct consequence deduced from lemma 5 and theorem 2.3 in [24]. □

The condition given in corollary 5 may be rewritten as

$$\left| \arg\left(\frac{\Phi(G_i, C)(jw)}{\Phi(G_j, C)(jw)}\right)\right| < 180^o, \quad i = 1, \ldots, n-1,\ j = i+1, \ldots, n \tag{22}$$

with

$$\frac{\Phi(G_i, C)}{\Phi(G_j, C)} \in \mathbb{U} \tag{23}$$

The approach outlined here, permits to synthesize units in H_∞ which remain units in H_∞ if one interchanges even (or odd) part of the numerator and denominator polynomials. The rational functions $\frac{\Phi(G_i, C)}{\Phi(G_j, C)}$ may have the modulus of their argument more than 90°, see [24]. Consequently the framework described in this article for the stabilization of polytopes of systems allows to envisage other solutions than those allowed by [18].

5 Illustrative examples

Several examples are now provided to illustrate the applicability of our approach. Since the interval plants are an important class of systems considered in many applications, we begin in the sections 5.1 and 5.2 to apply our approach on these particular uncertain systems. In the last part, section 5.3, an application is given for a polytope of systems that does not belong to the class of interval systems. That is a system whose the polynomial coefficients are not bound by some minimum and maximum values. On the other hand these coefficients can be dependant.

Although our analysis and our conditions of robust stability given in the sections 3 and 4 do not fall within the framework of the so-called "16-plant theorem", see [30], [31], we show by these examples that we can use our approach to stabilize these polytopes of systems.

5.1 A detailed example

To illustrate this design as defined in section 4, we propose to stabilize an interval plant $\widetilde{G}$ with a proper LTI compensator where

$$\widetilde{G}(s) = \frac{s^2 + n_0}{2s^2 + d_1 s + 1} \tag{24}$$

with $n_0 \in [\epsilon_{\min}, \epsilon_{\max}]$ and $d_1 \in [\eta_{\min}, \eta_{\max}]$.

First of all, if we consider a simple gain K as compensator, the characteristic equation of the closed-loop system is

$$\Phi(\widetilde{G}, K)(s) = (2+K)s^2 + d_1 s + (1 + K n_0)$$

We observe that a constant gain K cannot stabilize a such system $\widetilde{G}$ if d_1 has not always the same sign.

Before developing our approach to synthesize a robust controller stabilizing $\widetilde{G}$, the following result is stated.

Proposition 1: Consider the interval plant $\widetilde{G}$ given by (24). For all $n_0 \in [\epsilon_{\min}, \epsilon_{\max}]$ and $d_1 \in [\eta_{\min}, \eta_{\max}]$, there exists a polytope of plants Ω_G defined by its vertices $G_i = N_i D_i^{-1}$ $(i = 1 \dots 4)$ where

$$G_1(s) = \frac{s^2 + \epsilon_{\min}}{2s^2 + \eta_{\min} s + 1}, \quad G_2(s) = \frac{s^2 + \epsilon_{\min}}{2s^2 + \eta_{\max} s + 1} \tag{25a}$$

$$G_3(s) = \frac{s^2 + \epsilon_{\max}}{2s^2 + \eta_{\min} s + 1}, \quad G_4(s) = \frac{s^2 + \epsilon_{\max}}{2s^2 + \eta_{\max} s + 1} \tag{25b}$$

such that $\widetilde{G} \in \Omega_G$.

Proof: Let $G = ND^{-1}$ with

$$N = \sum_{i=1}^{4} \lambda_i N_i,\ D = \sum_{i=1}^{4} \lambda_i D_i,\ \sum_{i=1}^{4} \lambda_i = 1 \text{ and } \lambda_i \geq 0,$$

Or equivalently, we have

$$\begin{aligned} N(s) &= s^2 + (\lambda_1 + \lambda_2)\epsilon_{\min} + (\lambda_3 + \lambda_4)\epsilon_{\max} \\ D(s) &= 2s^2 + (\lambda_1 + \lambda_3)\eta_{\min} s + (\lambda_2 + \lambda_4)\eta_{\max} s + 1 \\ \text{with } &\sum_{1}^{4} \lambda_i = 1 \quad \text{and} \quad \lambda_i \geq 0,\ i = 1 \dots 4 \end{aligned}$$

Using the above relations, we can define the scalars $\alpha_j \in [0,\ 1]$ $(j = 1, 2)$ that satisfy the following relations

$$\begin{cases} \alpha_1 &= \lambda_1 + \lambda_2 \\ 1 - \alpha_1 &= \lambda_3 + \lambda_4 \\ \alpha_2 &= \lambda_1 + \lambda_3 \\ 1 - \alpha_2 &= \lambda_2 + \lambda_4 \end{cases} \tag{26}$$

Now let us show that for any $\alpha_j \in [0,\ 1]$ $(j = 1, 2)$, there exist λ_i $(i = 1 \dots, 4)$ with $0 \leq \lambda_i$ and $\sum_{i=1}^{4} \lambda_i = 1$. In terms of (26), let us remark that for any $\alpha_j \in [0,\ 1]$ $(j = 1, 2)$, we can always choose $0 \leq \lambda_1 \leq 1$ such that the following inequalities hold.

$$\left\{\begin{array}{lllllll} 0 & \leq & \lambda_1 & \leq & \alpha_1 & \leq & 1 \\ 0 & \leq & \lambda_1 & \leq & \alpha_2 & \leq & 1 \\ -1 & \leq & \alpha_1+\alpha_2-1 & \leq & \lambda_1 & \leq & 1 \end{array}\right. \tag{27}$$

From relations (26) and (27), we deduce that for any $\alpha_j \in [0,\ 1]$ $(j = 1, 2)$, there exist $\lambda_2 \geq 0$, $\lambda_3 \geq 0$ and $\lambda_4 \geq 0$ with $\sum_1^4 \lambda_i = 1$ that the following relationships hold.

$$\left\{\begin{array}{lllll} 0 & \leq & \lambda_2 = \alpha_1 - \lambda_1 & \leq & 1 \\ 0 & \leq & \lambda_3 = \alpha_2 - \lambda_1 & \leq & 1 \\ 0 & \leq & \lambda_4 = 1 - \alpha_1 - \alpha_2 + \lambda_1 & \leq & 1 \end{array}\right. \tag{28}$$

Then we can rewrite $\widetilde{G} = \widetilde{N}\widetilde{D}^{-1}$ with

$$\begin{gathered} \widetilde{N}(s) = s^2 + \left(\alpha_1\ \epsilon_{\min} + (1-\alpha_1)\ \epsilon_{\max}\right) \\ \widetilde{D}(s) = 2s^2 + \left(\alpha_2\ \eta_{\min} + (1-\alpha_2)\ \eta_{\max}\right)s + 1 \\ \alpha_1 \in [0,\ 1], \quad \alpha_2 \in [0,\ 1] \end{gathered}$$

Notice that for any $\alpha_j \in [0,\ 1]$, $(j = 1, 2)$, there exist always scalars λ_i, $(i = 1, 4)$ such that $0 \leq \lambda_i \leq 1$, $(i = 1, \ldots, 4)$ that satisfy (27) and (28) by taking $\lambda_1 = \min(\alpha_1, \alpha_2)$. □

Now consider $\widetilde{G}$ defined by (24) with $\epsilon_{\min} = 1$, $\epsilon_{\max} = 2$, $\eta_{\min} = -1$, $\eta_{\max} = 1$. By applying Proposition 1, we know that if a compensator C exists that stabilizes the polytope of plants Ω_G defined by its four vertices (25) then this compensator stabilizes $\widetilde{G}$. We can note that for stabilizing the polytope of systems Ω_G, it is sufficient to stabilize simultaneously only the four vertices (25) instead of six segments of systems as in the case of lemma 5. Consequently, to synthesize a robust controller that stabilizes Ω_G, theorem 3 and theorem 5 will be considered successively.

Firstly, let us solve this robust stabilization problem guaranteeing that the closed-loop characteristic polynomials associated with the four vertices of Ω_G have the same even part. To that goal, we can take a controller of the following form

$$C^e(s) = \frac{x_1 s}{y_2 s^2 + y_0}$$

We choose two negative distinct real roots: $a_1 = -0.5$, $a_2 = -4$. Then we find

$$\psi = [3.06\ \ 0.01\ \ 2.04\ \ 1.03\ \ 3.06\ \ 1.01\ \ 2.04\ \ 2.03]$$

and we get

$$C^e(s) = \frac{3.57s}{0.5s^2 + 2}$$

The closed-loop characteristic polynomials associated with the four vertices of Ω_G are Hurwitz-stable where

$$\Phi(G_1, C^e)(s) = s^4 + 3.07s^3 + 4.5s^2 + 1.57s + 2$$
$$\Phi(G_2, C^e)(s) = s^4 + 3.07s^3 + 4.5s^2 + 5.14s + 2$$
$$\Phi(G_3, C^e)(s) = s^4 + 4.07s^3 + 4.5s^2 + 5.57s + 2$$
$$\Phi(G_4, C^e)(s) = s^4 + 4.07s^3 + 4.5s^2 + 9.14s + 2$$

Now solve the same problem that the one that we have just treated but by considering theorem 5. With this formulation detailed in the appendix section, we can compute a controller C^o guaranteeing that the closed-loop characteristic polynomials associated with the four vertices of Ω_G have the same odd part. To that goal, we can consider a controller of the form

$$C^o(s) = \frac{x_2 s^2 + x_0}{y_3 s^3 + y_1 s}$$

The odd part of the characteristic polynomials can be fixed by the two negative distinct real roots $b_1 = -2.3704$, $b_2 = -0.5$. This yields to

$$\psi = [3.5926 \;\; 0.198 \;\; 6.8363 \;\; 3.5926 \;\; 1.0941 \;\; 1.8476 \;\; 4.5926 \;\; 0.6980 \;\; 6.8363 \;\; 4.5926 \;\; 1.5941 \;\; 1.8476]$$

and

$$C^o(s) = \frac{221s^2 + 20}{27s^3 + 64s}$$

The closed-loop characteristic polynomials associated with the four vertices of the polytope Ω_G are Hurwitz-stable where

$$\Phi(G_1, C^o)(s) = 54s^5 + 194s^4 + 155s^3 + 177s^2 + 64s + 20$$
$$\Phi(G_2, C^o)(s) = 54s^5 + 194s^4 + 155s^3 + 398s^2 + 64s + 40$$
$$\Phi(G_3, C^o)(s) = 54s^5 + 248s^4 + 155s^3 + 305s^2 + 64s + 20$$
$$\Phi(G_4, C^o)(s) = 54s^5 + 248s^4 + 155s^3 + 526s^2 + 64s + 40$$

Consider $U^o_{i,j} = \dfrac{\Phi(G_i, C^o)}{\Phi(G_j, C^o)}$ and $U^e_{i,j} = \dfrac{\Phi(G_i, C^e)}{\Phi(G_j, C^e)}$ with $i = 1, \ldots, 3$, $j = i+1, \ldots, 4$. Let us plot the modulus of the argument of the rational functions $U_{i,j}(jw)^e$ and $U_{i,j}(jw)^o$, see figures 1 and 2. We observe that all these units have the modulus of their argument less than 180° and some of these units have modulus of their argument more than 90°. Hence, they are not all EP-SPR functions.

The conditions given in the theorems 2 and 4 are all satisfied then C^e and C^o stabilize the polytope of systems Ω_G. Accordingly, C^e and C^o stabilize $\widetilde{G}$.

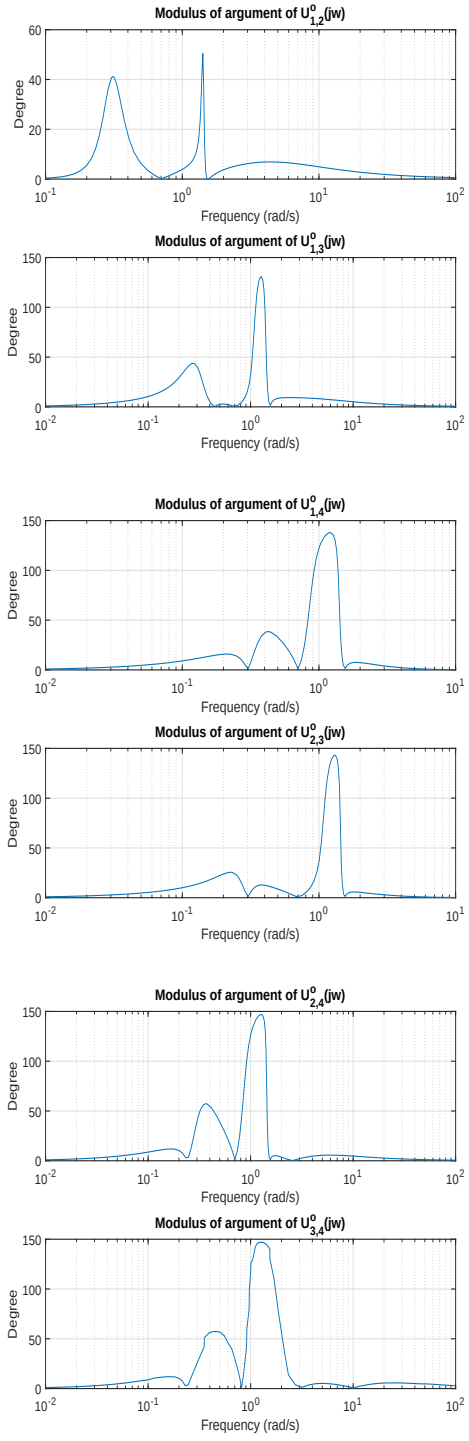

Fig 1: Plot of $\left|\arg U^o_{i,j}(jw)\right|$ in regard to C_o

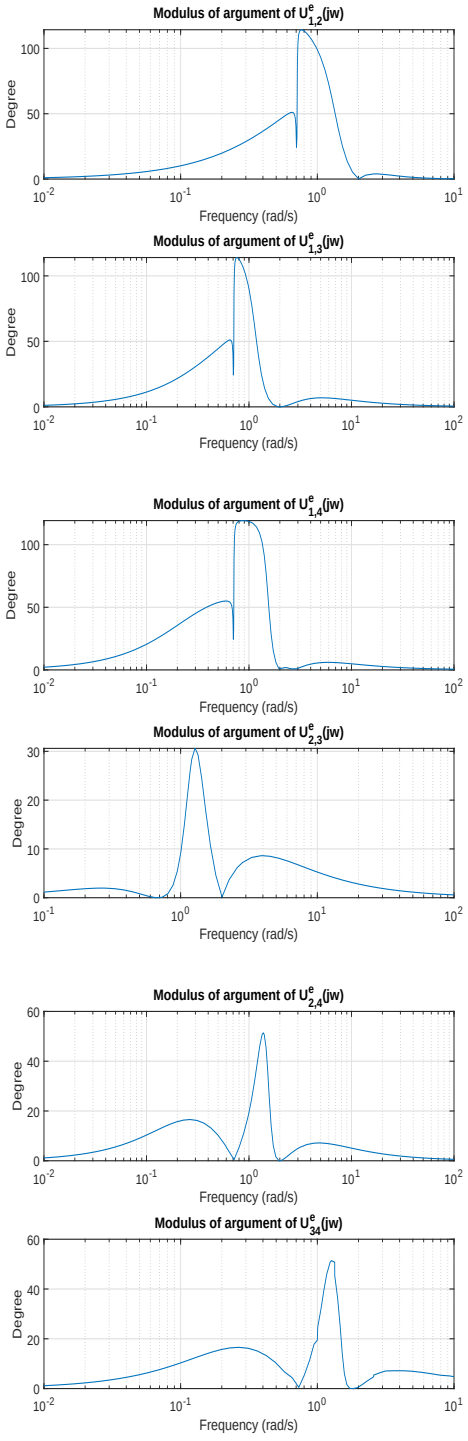

Fig 2: Plot of $\left|\arg U^e_{i,j}(jw)\right|$ in regard to C_e

5.2 A numerical benchmark

In this part, a collection of interval plants are now presented showing the efficiency of our numerical design when the order of these systems grows. Consequently, the number of constraints treated in these configurations by the linear programming procedure becomes important. It is about pointing out that in spite of this numerical complexity, the algorithm can find feasible solutions. The allowed maximum uncertainty is defined by a parameter τ_{max} which is linked to the size of the uncertainty intervals with $0 \leq \tau_{max}$. To simplify the presentation, only simultaneous compensators guaranteeing a same even part for the characteristic polynomials are given.

5.2.1 Example 1

A similar example to the one given previously in section 5.1 but by considering a polytope of systems of 8 vertices instead of 4.

<table>
<tr><td rowspan="2">System</td><td>$$\widetilde{G}(s) = \frac{n_2 s^2 + n_0}{2s^2 + d_1 s + 1}$$
$n_0 \in [1.25 - 0.25\tau_0, 1.25 + 0.25\tau_0]$, $n_2 \in [1.25 - 0.25\tau_2, 1.25 + 0.25\tau_2]$
$d_1 \in [-\tau_1, \tau_1]$ with $\tau_{i,\ (i=0,\ldots,2)} \in [0,\ \tau_{max}]$</td></tr>
<tr><td>From $\widetilde{G}$ is deduced a polytope of plants Ω_G defined by the set of its vertices $G_i, (i = 1 \ldots 8)$ associated with the 8 combinations of the bounds of the uncertainty intervals.</td></tr>
<tr><td>Controller 1</td><td>A stabilizing controller C^e allowing a maximum uncertainty $\tau_{max} = 1$
$$C^e(s) = \frac{81s}{1.85s^2 + 8}$$</td></tr>
<tr><td colspan="2">The roots of the even part of the 8 characteristic polynomials are $a_i = \{-4.3243,\ -0.5\}$.</td></tr>
<tr><td>Controller 2</td><td>A stabilizing controller C^e allowing a maximum uncertainty $\tau_{max} = 1.5$
$$C^e(s) = \frac{13.4277s^3 + 1.1379s}{0.28s^4 + 0.65s^2 + 0.05}$$</td></tr>
<tr><td colspan="2">The roots of the even part of the 8 characteristic polynomials are $a_i = \{-2.2418, -0.5, -0.0797\}$.</td></tr>
</table>

5.2.2 Example 2

A 3rd order unstable system is considered with four uncertain coefficients.

System	$\widetilde{G}(s) = \dfrac{s^3 + n_2 s^2 + n_0}{d_3 s^3 - 2s^2 + d_1 s + 1}$ $n_0 \in [1 - 0.5\tau_0, 1 + 0.5\tau_0]$, $n_2 \in [3 - 0.5\tau_2, 3 + 0.5\tau_2]$ $d_1 \in [-\tau_1, \tau_1]$, $d_3 \in [1 - 0.5\tau_3, 1 + 0.5\tau_3]$ with $\tau_{i,\ (i=0,\ldots,3)} \in [0,\ \tau_{max}]$
	From $\widetilde{G}$ is deduced a polytope of plants Ω_G defined by the set of its vertices $G_i, (i = 1 \ldots 16)$ associated with the 16 combinations of the bounds of the uncertainty intervals.
Controller	A stabilizing controller C^e allowing a maximum uncertainty $\tau_{max} = 1$ $C^e(s) = \dfrac{4.5049 s^3 + 0.44035 s}{1.7524 s^4 + 0.1914 s^2 + 0.0017}$
The roots of the even part of the 16 characteristic polynomials are $a_i = \{-1.7, -0.1, -0.01\}$	

5.2.3 Example 3

A 4th order unstable system is treated with five uncertain coefficients.

System	$\widetilde{G}(s) = \dfrac{n_4 s^4 + s^3 + n_2 s^2 + n_0}{0.1 s^4 + d_3 s^3 - 2s^2 + d_1 s + 1}$ $n_0 \in [0.75 - 0.5\tau_0, 0.75 + 0.5\tau_0]$, $n_2 \in [2.25 - 0.5\tau_2, 2.25 + 0.5\tau_2]$ $n_4 \in [0.11 - 0.1\tau_4, 0.11 + 0.1\tau_4]$, $d_1 \in [-\tau_1, \tau_1]$, $d_3 \in [1.5 - \tau_3, 1.5 + \tau_3]$ with $\tau_{i,\ (i=0,\ldots,4)} \in [0,\ \tau_{max}]$
	From $\widetilde{G}$ is deduced a polytope of plants Ω_G defined by the set of its vertices $G_i, (i = 1 \ldots 32)$ associated with the 32 combinations of the bounds of the uncertainty intervals.
Controller	A stabilizing controller C^e allowing a maximum uncertainty $\tau_{max} = 0.5$ $C^e(s) = \dfrac{29.23 s^3 + 395.65 s}{10 s^4 + 138.7 s^2 + 10.5}$
The roots of the even part of the 32 characteristic polynomials are $a_i = \{-15, -7, -1, -0.1\}$	

5.2.4 Example 4

This is a 5th order unstable system with four uncertain coefficients.

System	$\widetilde{G}(s) = \dfrac{6s^3 + n_2 s^2 + 22s + n_0}{-0.2s^5 - 3s^4 + d_3 s^3 + 22s^2 + d_1 s - 0.5}$ $n_0 \in [-3 - \tau_0, -3 + \tau_0]$, $n_2 \in [3 - \tau_2, 3 + \tau_2]$ $d_1 \in [-11 - \tau_1, -11 + \tau_1]$, $d_3 \in [-16 - \tau_3, -16 + \tau_3]$ with $\tau_{i,\ (i=0,\ldots,3)} \in [0,\ \tau_{max}]$
	From $\widetilde{G}$ is deduced a polytope of plants Ω_G defined by the set of its vertices $G_i, (i = 1 \ldots 16)$ associated with the 16 combinations of the bounds of the uncertainty intervals.
Controller	A stabilizing controller C^e allowing a maximum uncertainty $\tau_{max} = 1.7$ $C^e(s) = \dfrac{50.612s^3 + 65.161s}{-4.6063s^4 - 30.132s^2 - 40.469}$
The roots of the even part of the 16 characteristic polynomials are $a_i = \{-17.275, -3.1836, -0.68588, -0.038819\}$	

5.2.5 Example 5

This is a 5th order unstable system with five uncertain coefficients.

System	$\widetilde{G}(s) = \dfrac{-2.5s^5 + n_4 s^4 - 11s^3 + n_2 s^2 + 10s + n_0}{-0.2s^5 - 31s^4 + d_3 s^3 - 81s^2 + d_1 s - 2}$ $n_0 \in [30.5 - 2.5\tau_0, 30.5 + 2.5\tau_0]$, $n_2 \in [24 - 2\tau_2, 24 + 2\tau_2]$ $n_4 \in [2.15 - 0.85\tau_4, 2.15 + 0.85\tau_4]$, $d_1 \in [-72.5 - 4.5\tau_1, -72.5 + 4.5\tau_1]$, $d_3 \in [-4.1 - 1.4\tau_3, -4.1 + 1.4\tau_3]$ with $\tau_{i,\ (i=0,\ldots,4)} \in [0,\ \tau_{max}]$
	From $\widetilde{G}$ is deduced a polytope of plants Ω_G defined by the set of its vertices $G_i, (i = 1 \ldots 32)$ associated with the 32 combinations of the bounds of the uncertainty intervals.
Controller	A stabilizing controller C^e allowing a maximum uncertainty $\tau_{max} = 1$ $C^e(s) = \dfrac{50.612s^5 + 65.161s^3 - 53.913s}{-4.6063s^6 - 30.132s^4 - 40.469s^2 - 28.229}$
The roots of the even part of the 32 characteristic polynomials are $a_i = \{-28.8, -5.32, -1.57, -0.42, -0.034\}$	

5.2.6 Comparison with the robust stabilization approach given in [32] and [33].

Well known H_∞ methods [32] and [33] are used to establish robust stabilizability conditions for a family of uncertain plants and also to synthesize controllers that would stabilize the whole family. These regulators synthesis techniques can be implemented on the examples 1 to 5, in order to evaluate robustness of our approach in a simple way. For that, an interval system is associated to a family of systems with a central model G_0 also called nominal model defined through the average values of parameters of the interval model. The plant transfer function is written with a coprime factorization $G_0 = T_0 S_0^{-1}$ where T_0, S_0 are normalized such that

$$T_0(s)T_0(-s) + S_0(s)S_0(-s) = 1.$$

Then a family of plants G is characterized by H_∞ bounded perturbations to the numerator T_0 and denominator S_0 of the normalized coprime factorization of the nomimal system G_0 as

$$\mathfrak{F}_{\mathfrak{G}} = \left\{ G : G = TS^{-1},\ T = T_0 + \Delta_T, S = S_0 + \Delta_S :\ \left\| \begin{bmatrix} \Delta_S & \Delta_T \end{bmatrix} \right\|_\infty \leq \beta,\ \beta > 0 \right\} \tag{29}$$

where Δ_T, Δ_S are stable unknown transfer functions representing the uncertainty.

Then $\mathfrak{F}_{\mathfrak{G}}$ is robustly stabilizable by a single linear time invariant controller C *iff* (see Lemma 3.1 in [32])

$$\inf_C \left\| \begin{bmatrix} C(1 - G_0C)^{-1}S_0^{-1} \\ (1 - G_0C)^{-1}S_0^{-1} \end{bmatrix} \right\|_\infty \leq \frac{1}{\beta} \tag{30}$$

where the infinimum is chosen over all compensators C that stabilizes G_0.

The problem of robust stabilization of normalized coprime factor plant descriptions with H_∞ bounded uncertainty is formulated as

$$\rho(G_0, G) < \beta_{\max} \tag{31}$$

where $\beta_{\max}$ denotes a maximum β and $\rho(G_0, G)$ is the gap-metric between G_0 and G as defined in [34] and [35] with

$$\rho(G_0, G) = \max \left(\inf_{Q \in H_\infty} \left\| \begin{bmatrix} S_0 \\ T_0 \end{bmatrix} - \begin{bmatrix} S \\ T \end{bmatrix} Q \right\|_\infty,\ \inf_{Q \in H_\infty} \left\| \begin{bmatrix} S \\ T \end{bmatrix} - \begin{bmatrix} S_0 \\ T_0 \end{bmatrix} Q \right\|_\infty \right)$$

The issue of finding $\beta_{\max}$ can be solved via standard H_∞ optimization techniques (see [32], [33] and [34]). So, as for the gap metric, any plant G at a distance less than β from G_0 ($\beta < \beta_{\max}$) will be stabilized by any compensator C stabilizing the nominal system with a stability margin of β.

In the case of our approach, an interval plant is represented by a polytope of systems Ω_G defined by its vertices G_i $(i = 1, \ldots, n)$. It is thus possible to measure for each vertex of the polytope the "distance" to the central plant G_0 by the gap metric. Denote by $\rho_m = \min_{i=1,\ldots,n} \rho(G_0, G_i)$ and $\rho_M = \max_{i=1,\ldots,n} \rho(G_0, G_i)$, the maximum and the minimum of the gap-metrics between G_0 and G_i $(i = 1, \ldots, n)$.

Consequently, for each interval plant, the radius of the maximal gap ball $\beta_{\max}$ around the central plant G_0 associated with the family $\mathfrak{F}_{\mathfrak{G}}$, may be compared to the distance measured by the gap metric between G_0 and the vertices G_i $(i = 1, \ldots, n)$ associated with the polytope Ω_G. This comparison between the H_∞ method given in [32] and [33] and our approach is presented for the examples 1 to 5 in table 1 hereafter where

- n is the number of vertices G_i $(i = 1, \ldots, n)$ associated with the polytope of plants Ω_G,
- n_ρ is the number of vertices G_i $(i = 1, \ldots, n)$ satisfying $\rho(G_0, G_i)) \leq \beta_{\max}$.

Section	$G_0(s)$	$\beta_{\max}$	n	ρ_m	ρ_M	n_ρ
5.2.1	$\frac{1.25s^2+1.25}{2s^2+1}$ with $\tau_{\max}=1$	0.60964	8	0.6312	0.96777	0
5.2.1	$\frac{1.25s^2+1.25}{2s^2+1}$ with $\tau_{\max}=1.5$	0.60964	8	0.70467	0.999	0
5.2.2	$\frac{s^3+3s^2+1}{s^3-2s^2+1}$	0.3329	16	0.25884	0.42977	2
5.2.3	$\frac{0.11s^4+s^3+2.25s^2+0.75}{0.1s^4+1.5s^3-2s^2+1}$	0.24363	32	0.17906	0.2891	14
5.2.4	$\frac{6s^3+3s^2+22s-3}{-0.2s^5-3s^4-16s^3+22s^2-11s-0.5}$	0.1728	16	0.20878	0.79022	0
5.2.5	$\frac{-2.5s^5+2.15s^4-11s^3+24s^2+10s+30.5}{-0.2s^5-31s^4-4.1s^3-81s^2-72.5s-2}$	0.16273	32	0.034146	0.17463	31

Table 1: Gap metric results applied to examples 1 to 5

In table 1, we can observe for every example that $\rho_M > \beta_{\max}$. We deduce that the robust stabilization approach developed in [32] and [33] does not allow to take into account an uncertainty level as large as the one considered in the benchmark section 5.2. These results are due to the lost of the uncertainty structure with representation (29). This design limitation of H_∞ methods in regard to the uncertainty structure provides conservatism.

Finally, we conclude our numerical benchmark of section 5.2 by pointing out that like in the literature on the stabilization of interval plants, see [18], [19], [20] and [21], our approach is based on sufficient conditions which generate some inevitable conservatism. In other words, if the linear programming problem defined by equation (20) has no solution, it does not mean that there is no robust controller that stabilizes the polytope of plants. In our examples, the limit of stability is reached for a given controller when $\tau_{i,\,(i=0,\ldots,n)} = \tau_{max}$ where n is the number of uncertainty coefficient for a given interval plant. It does not imply that there is no robust stabilizing controllers if $\tau_{i,\,(i=0,\ldots,n)} > \tau_{max}$. This is illustrated by example 1 where a 2nd order controller is get with $\tau_{max} = 1$ while a 4th order controller permits $\tau_{max} = 1.5$. The issue of τ_{max} is the one of the compromise between the controller order and the robustness.

5.3 An example of a polytope of systems not belonging to the family of interval plants

Consider now a polytope of plants Ω_G defined by the 8 vertices G_i $(i = 1, \ldots, 8)$ given in table 2. That is a system whose the polynomial coefficients are not bound by some minimum and maximum values. On the other hand these coefficients can be dependant. The set of all convex combinations

of vertices G_i $(i = 1, \ldots, 8)$ forms a convex hull. Note that the systems G_i $(i = 1, \ldots, 7)$ are unstable plants.

First of all, we show that this polytope Ω_G is not stabilizable by a first order controller. For this purpose, consider the vertex G_1 of Ω_G of the form (32) with $K = 1$, $\tilde{a} = 1$ and $\tilde{b} = 1$ and the compensator (33).

$G_1(s) = \dfrac{1}{s^3 - s^2 + s + 1}$	$G_2(s) = \dfrac{-0.24771s + 2.0307}{s^3 + 3s^2 - s + 6}$
$G_3(s) = \dfrac{-0.059633s + 1.2639}{s^3 + 5}$	$G_4(s) = \dfrac{-0.31651s + 1.9281}{s^3 + 4s^2 + 1}$
$G_5(s) = \dfrac{-0.055046s + 1.7033}{s^3 - s + 8}$	$G_6(s) = \dfrac{0.0091743s + 2.5617}{s^3 - s^2 - s + 3}$
$G_7(s) = \dfrac{-0.11927s + 2.3815}{s^3 + s^2 - s + 4}$	$G_8(s) = \dfrac{-0.44954s + 0.79637}{s^3 + 6s^2 + s + 2}$

Table 2: Vertices G_i of the polytope of plants Ω_G

$$G_{(K,\tilde{a},\tilde{b})}(s) = \frac{K}{\tilde{a}s^3 - \tilde{a}s^2 + \tilde{a}s + \tilde{b}}, \qquad \{K, \tilde{a}, \tilde{b}\} \in \mathbb{R}^3 \tag{32}$$

$$C(s) = \frac{x_1 s + x_0}{y_1 s + y_0} \tag{33}$$

The resulting closed-loop polynomial is

$$\Phi(G_{(K,\tilde{a},\tilde{b})}, C)(s) = \tilde{a}y_1 s^4 + \tilde{a}(y_0 - y_1)s^3 + \tilde{a}(y_1 - y_0)s^2 + (\tilde{b}y_1 + \tilde{a}y_0 + Kx_1)s + (\tilde{b}y_0 + Kx_0)$$

Let us remark that this characteristic equation will never be Hurwitz-stable because there are two coefficients of opposite sign, the coefficient of degree 3 and the coefficient of degree 2. Consequently, the polytope of plants Ω_G can not be stabilized with a first order compensator. Therefore, a controller of higher order must be tested. A second order controller of the following general form

$$C(s) = \frac{x_2 s^2 + x_1 s + x_0}{y_2 s^2 + y_1 s + y_0} \tag{34}$$

is already considered to stabilize this polytope of plants Ω_G.

To solve this robust stabilization problem, we desire that the closed-loop characteristic polynomials $\Phi(G_i, C)$ $(i = 1, \ldots 8)$ have a same odd part. At this end, we select the two following negative distinct real roots, $b_1 = -62.203$ and $b_2 = -2.7973$. Then we find

$$C(s) = \frac{218s^2 + 82s - 6}{s^2 + 14s + 78} \tag{35}$$

With this simultaneous compensator (35), the eight closed-loop characteristic polynomials associated with the eight vertices of Ω_G are Hurwitz-stable and have a same odd part.

5.4 Discussion

The considered numerical examples are not easy to stabilize because simple compensators like constant gain cannot stabilize these uncertain systems. Moreover unstable poles and unstable zeros of these uncertain plants may change which would constitute additional difficulties. It is also useful to observe that in literature this same problem requires the solution of bilinear matrix inequalities (BMIs) which lead to nonconvex optimization problems (see for example [18], [19]) while our approach generates a simple linear programming (LP) problem for which there are efficient solvers with low computational costs. This basic computational procedure founded on linear programming permits to find robust controllers of reasonable order in many cases when the plant order and the number of vertices of the polytope grow. These simulations also show that the order of the compensators is not higher than those used in the usual applications of automatic control: it is at most of the same order that the plant order plus one.

6 Conclusion

In this paper, a new approach for stabilizing a polytope of SISO plants has been proposed which yields to a simple controller design for these systems. The Hermite-Biehler theorem and Edge theorem have been the backbones of this framework. An alternative would be to use the Hermite-Fujiwara criterion, see [36] and [37] instead of the Hermite-Biehler criterion that we apply to stabilize the closed-loop system. So we would have a feedback that would not be based on the common interlacing property but on the properties of the Hermite-Fujiwara matrices. Thus [36] has proved that a polynomial f is Hurwitz-stable *iff* the Hermite-Fujiwara matrix associated with f is positive definite. Thereafter this property has been extended to define the Hurwitz-stability of a linear combination of several polynomials, see [37]. Assuming that we consider the Hermite-Fujiwara criterion instead of the Hermite-Biehler criterion, our approach would remain unchanged until theorem 2. Then the conditions (15) of lemma 6 would be modified in order to rewrite the closed-loop characteristic polynomials of the vertices of the polytope with the Hermite-Fujiwara matrices. In this situation, we would get bilinear matrix inequalities (BMIs) conditions which would give a non-convex optimization problem. From the literature, we know that the problem of checking the solvability of a BMI system is NP-hard, see [38]. It is the reason why, the BMIs are converted to linear matrix inequalities (LMIs) with a constraint rank. However this alteration in rank-one LMI optimization problem produces an heuristic algorithm without guaranteeing to find solutions when they exist. For example, in the case of the 2nd order system associated with a polytope of plants of 8 vertices studied in section 5.2, no solution has been found with the Yalmip and SeDuMi solvers for a fourth-order controller by using the Hermite-Fujiwara's criterion. Moreover, this approach generates a higher numerical complexity than the one developed with the Hermite-Biehler criterion. This complexity is linked to the size and to the number of Hermite-Fujiwara matrices. For example, the second order system regulated with a fourth-order controller given in the section 5.2 involves to calculate two hundred Hermite-Fujiwara's matrices.
To summarize, use of the Hermite-Biehler criterion has reduced the stabilization problem of a polytope of systems to a linear programming problem for which there exists efficacious solvers. Our approach has been illustrated by examples showing that the controllers have a reasonable order in spite of the numbers of vertices of the polytope of plants to stabilize and the uncertainty regarded.

Note that these results could be extended in several ways. Notably we could study the polytopes of plants with a similar controller design than the one given in this paper but in discrete-time or for other stability domains than the unit circle or than the left half-plane of the complex plane. Finally, we could research conditions to stabilize a polytope of systems with a compensator of any order.

7 Appendix: Stabilization of Ω_G with a same $\Phi(G_i, C)^o$

In the previous sections has been proposed a design of a controller C that guarantees the same even part for the closed-loop characteristic polynomials associated with the vertices of a polytope of systems Ω_G. In this appendix are presented similar results but for guaranteeing a same odd part $\Phi(G_i, C)^o$ for the closed-loop characteristic polynomials associated with the vertices of Ω_G.

Definition 5 (*Common odd interlacing property*) Consider a finite collection of polynomials f_i^e and a polynomial f^o with their leading coefficients of same sign. Assume that for each pair (f_i^e, f^o), $(i = 1, \ldots, n)$, the roots of f_i^e and f^o are all real, negative simple and distinct defined by the following sets

$$\begin{aligned} roots(f_i^e) &= \{a_1^i, \ldots, a_\ell^i\} \\ roots(f^o) &= \{b_1, \ldots, b_\ell\} \end{aligned}$$

The polynomials f_i^e have a common odd interlacing f^o iff the ℓ roots of each f_i^e alternate with the ℓ roots of f^o as below

$$a_1^i < b_1 < a_2^i < \ldots < a_\ell^i < b_\ell < 0$$

Note that if f_i^e, $(i = 1 \ldots, n)$ have a common interlacing f^o, this implies that for any jth root, we have

$$\max_{i \in \{1,\ldots,n\}} a_j^i < b_j < \min_{i \in \{1,\ldots,n\}} a_{j+1}^i.$$

Based on this property of common odd interlacing, a corollary is deduced for testing the Hurwitz-stability of $f_i(s) = f_i^e(s^2) + sf^o(s^2)$, $(i = 1 \ldots, n)$.

Corollary 6 (see corollary 1) The polynomials $f_i(s) = f_i^e(s^2) + sf^o(s^2)$, $(i = 1, \ldots, n)$ with $f_i \in \mathbb{P}_m$ are Hurwitz-stable iff the two following conditions hold.

1. f^o has real, simple and negative roots $(b_1, \ldots, b_\ell)$.
2. The real coefficients $c_{i,k}$ $(i = 1, \ldots, n)$ and $(k = 0, \ldots, \ell)$ given by the expansions of f_i^e $(i = 1, \ldots, n)$ as the following sums of products of factors of f^o

$$f_i^e(s^2) = c_{i,0} f^o(s^2) - \sum_{k=1}^{\ell} c_{i,k} \frac{f^o(s^2)}{s^2 - b_k}. \tag{36}$$

are strictly positive.

From corollary 6, we can deduce a matrix formulation of f_i^e and f^o.

Since $f^o(s^2) = \gamma^o \prod_{j=1}^{\ell}(s^2 - b_j)$ with $\gamma^o \in \mathbb{R}_{-\{0\}}$ then relation (36) can be simplified as it follows

$$\begin{aligned} f_i^e(s^2) &= \gamma^o c_{i,0} \prod_{j=1}^{\ell}(s^2 - b_j) - \gamma^o \sum_{k=1}^{\ell} c_{i,k} \prod_{\substack{j=1 \\ j\neq k}}^{\ell}(s^2 - b_j) \\ &= \sum_{k=0}^{\ell} c_{i,k} \sum_{j=0}^{\ell-1} \upsilon_{j,k} s^{2j}, \\ &= V A^o \Psi_i^o, \qquad i = 1, \ldots, n, \end{aligned} \tag{37a}$$

$$f^o(s^2) = V H^o \tag{37b}$$

where

$$\Psi_i^o = \begin{bmatrix} c_{i,0} & \ldots & c_{i,\ell} \end{bmatrix}^T \tag{38a}$$

$$V = \begin{bmatrix} 1 & s^2 & \ldots & s^{2\ell} \end{bmatrix} \tag{38b}$$

$$A^o = \begin{bmatrix} \upsilon_{0,0} & \upsilon_{0,1} & \ldots & \upsilon_{0,\ell} \\ \upsilon_{1,0} & \upsilon_{1,1} & \ldots & \upsilon_{1,\ell} \\ \vdots & \vdots & \vdots & \vdots \\ \upsilon_{\ell,0} & \upsilon_{\ell,1} & \ldots & \upsilon_{\ell,\ell} \end{bmatrix} \tag{38c}$$

$$H^o = \begin{bmatrix} h_0^o, & h_1^o & \ldots & h_\ell^o \end{bmatrix}^T \tag{38d}$$

A reformulation of corollary 6 is now deduced.

Corollary 7 (see corollary 2) The polynomials $f_i \in \mathbb{P}_m$ $(i = 1, \ldots, n)$ are Hurwitz-stable iff the two following conditions hold.

1. f^o has real, simple and negative roots $(b_1, \ldots, b_\ell)$.
2. There exist $\Psi_i^o \in \mathbb{R}_{++}^{\ell}$ for $i = 1, \ldots, n$, given by (38a).

The Hurwitz-stability issue of a polytope of real polynomials S_f defined by (7) may be simplified by considering corollary 3 in [28].

Lemma 8 (see lemma 3) Let f_i and f_j be in $\mathbb{P}_m$, two vertices of S_f with their leading coefficients of same sign and with a same odd part f^o then the exposed edge $f_{i,j}$ defined by (8) is Hurwitz-stable iff f_i and f_j are Hurwitz-stable.

Hence, we write.

Corollary 8 (see corollary 3) If the vertices $f_i \in \mathbb{P}_m$ $(i = 1, \ldots, n)$, of a polytope of real polynomials S_f are Hurwitz-stable with a same odd part f^o and if the leading coefficients of f_i have a same sign then the polytope of real polynomials S_f is Hurwitz-stable.

That implies theorem below.

Theorem 4 (see theorem 2) Consider a convex polytope of plants Ω_G defined by (9). If there exists a compensator C satisfying the two following relations

$$\Phi(G_i, C) \in \mathcal{H}, \tag{39a}$$
$$\Phi(G_i, C)^o = \Phi(G_j, C)^o,\ i = 1, \ldots, n-1,\ j = i+1, \ldots, n. \tag{39b}$$

with the leading coefficients of $\Phi(G_i, C)$ of same sign, then C stabilizes Ω_G.

Hence the following result by considering C_v given by (17) and Λ_i^e and Λ_i^o defined by (18).

Lemma 9 (see lemma 7) Consider a convex polytope of plants Ω_G defined by (9) and A^o, H^o given by (38c), (38d) respectively. If there exists C_v and $\Psi_i^o \in \mathbb{R}_{++}^{\ell}$, $(i = 1, \ldots, n)$ satisfying

$$\Lambda_i^o C_v = H^o, \qquad i = 1, \ldots, n, \tag{40a}$$
$$\Lambda_i^e C_v = A^o \Psi_i^o \tag{40b}$$

then C stabilizes Ω_G.

Now rewrite relations (40)

$$B^o = \Theta^o \chi^o \tag{41}$$

where

$$B^o = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ H^o \\ H^o \\ \vdots \\ H^o \end{bmatrix},\ \Theta^e = \begin{bmatrix} A^o & 0 & \ldots & 0 & -\Lambda_1^e \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & A^o & -\Lambda_n^e \\ 0 & 0 & \ldots & 0 & \Lambda_1^o \\ 0 & 0 & \ldots & 0 & \Lambda_2^o \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \ldots & 0 & \Lambda_n^o \end{bmatrix},$$

$$\chi^o = \begin{bmatrix} \psi^{oT} & {C_v}^T \end{bmatrix}^T,$$
$$\psi^o = \begin{bmatrix} \Psi_1^{oT} & \Psi_2^{oT} & \Psi_3^{oT} & \ldots & \Psi_n^{oT} \end{bmatrix}^T \in \mathbb{R}_{++}^{n(\ell+1)}.$$

There is solutions ψ^o and C_v to (41) iff [29].

$$\operatorname{rank}\left(\begin{bmatrix} B^o & \Theta^o \end{bmatrix}\right) = \operatorname{rank}(\Theta^o). \tag{42}$$

By considering equation (40), the robust stabilization of Ω_G given by lemma 9 is now set as the issue of feasibility of a linear programming problem. Thus, we can deduce the next theorem.

Theorem 5 (see theorem 3) Let Ω_G be a convex polytope of plants defined by (9). Assume that (42) holds then the controller C stabilizes Ω_G if there exists a solution χ^o to the linear programming (LP) problem given by (41).

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