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ON THE CONNECTIVITY OF MANIPULATOR FREE WORKSPACE

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ABSTRACT :

This paper presents a new topological characterization of the free workspace of manipulators moving among obstacles. The free workspace is the set of positions and orientations that the end-effector of the manipulator can reach, according to the joint limits of each of its links, and taking into account the obstacles of the environment. A classification of new connectivity properties of the free workspace is proposed, corresponding to different types of motions (point to point motions, following of continuous trajectories) that the manipulator can perform in the cartesian space. For each property, a necessary and sufficient condition is given, which permits to verify the connectivity of the whole free workspace and leads to all the connected subspaces in it. These properties have been implemented in a Robotics C.A.D. system using octree representation of spaces. Some applications are presented, which show that this work is of primary interest for off-line tasks preparing, and is a new contribution to the problem of robotic cells layout design.

1- INTRODUCTION

When designing a robotic cell, one has to face difficult problems which require complex geometric reasoning in the 3-D cartesian space. Very powerful aids must be at the disposal of the designer. Thus, several robotics C.A.D. (Computer Aided Design) systems such as ROBCAD (from Tecknomatix) and CATIA (from Dassault System) have emerged. They involve powerful graphic functions and can perform realistic motions simulations. However, all these systems are characterized by a trial and error approach as they do not provide *a priori* information about the *global* geometric and kinematic performances of the manipulator in the cell^{1,2}. The operator has to simulate the movement of the robot all along the trajectories. Then, the system verifies if the joints of the manipulator remain within their respective limits during the motions, and collision with environment obstacles are often automatically detected. However, the operator himself must modify the layout of the cell when the task is unfeasible, due to obstacle obstructions or kinematic impossibilities.

The goal of our study is to give a global geometric and kinematic description of the manipulator capabilities in the cell. This is achieved through a new topological analysis of the *free workspace* of the manipulator. The *free workspace* is the set of positions and orientations that the manipulator's end-effector can reach, according to the joint limits and the obstacles. The *free workspace* is a generalization to cluttered environments of the *workspace*, which has already drawn the interest of many researchers. Typical characterizations of manipulator workspace are :

- the two well known projections in the cartesian space³ "*reachable workspace*" (projection of the whole workspace) and "*dextrous workspace*" (projection of the part of the workspace where the end-effector can reach any orientation)

- determination of plane sections of the workspace with fixed hand orientation (*work-areas* analysis⁴)

- shape analysis : characterization of *holes and voids* in the workspace⁵ and evaluation of criteria such as volume, connectedness, compactness of the workspace^{6,7}

- the *approach angles and lengths*⁸ which give local information in the workspace

- characterization of the *aspects* of a manipulator⁹. The N_a aspects A_j , $j \in J = \{1, \dots, N_a\}$ of a given manipulator are theoretically defined as follows :

- 1/ each A_j is open and path-connected,

- 2/ Let m be the dimension of the workspace : for any joint vector in A_j , the determinant of any $m \times m$ sub-matrix extracted from the jacobian matrix is different from zero, except if this minor is equal to zero everywhere in the joint space.

Aspects are defined for both redundant and non-redundant manipulators. For non-redundant manipulators, the aspects are the singularity-free domains of the configuration space. Given two adjacent aspects A_j and $A_{j'}$ of a non-redundant manipulator, the hyper-surface separating them is a subset of a "branch of singularity" : it is the set of configurations which nullify a factor of the determinant D of the Jacobian matrix, while the other factors are of a given sign.

No actual industrial manipulator can change posture without passing through a singularity. Thus, the image under the geometrical operator of an aspect describes the set of positions and orientations that the manipulator can reach in configurations corresponding to a given posture. In other words, the geometrical operator of a non-redundant (industrial) manipulator is one-to-one from each aspect onto its image. For example, an anthropomorphic manipulator moving in a vertical plane can reach a given point in aspect "elbow up" or in aspect "elbow down" (see fig. 1.a). Generally, due to the joint limits, the images of the aspects are

not coincident in the workspace, so that the manipulator cannot reach all points in each aspect. When changing aspect, the end-effector has to cross an hyper-surface in the workspace, which is the image of a branch of singularity (see fig. 1.b). Thus, the determination of the images of the aspects are of great interest when studying the feasibility of continuous trajectories, for which a full control of the end-effector is required .

- the *well-connected-workspace* property characterizes the ability of a manipulator to move between any two points in its workspace without changing aspect¹⁰ : this property means that the images of the aspects are all coincident, which occurs for manipulators with special geometric properties and without joint limits.

However, none of these studies takes into account the effects of obstacles. The obstacles lying within the scope of a manipulator do modify not only the shape but the topology of its workspace¹¹.

In this paper, a classification of new connectivity properties of manipulator free workspace is proposed, corresponding to different types of motions that the manipulator can perform. For each property, a necessary and sufficient condition is given, which permits to verify the connectivity of the whole workspace and leads to all the connected subspaces in it. These properties have been implemented in a Robotics C.A.D. system, using octree representation of spaces. Some applications are presented in the last section.

2- OBSTACLES ANALYSIS AND MANIPULATOR FREE WORKSPACE

The problem is to find a convenient way of representing the effects of obstacles on the manipulator's ability to perform motions. There exists two possibilities :

- obstacles analysis in the joint space of the manipulator and characterization of the configuration free space^{12,13},
- obstacles analysis in the operational space¹⁴ and characterization of the manipulator free workspace.

2-1 Configuration free space of a manipulator

Let Q be the set of joint angles within their respective limits. The configuration free space Qf of a manipulator is defined as the set of joint angles in Q for which no part of the manipulator intersects the obstacles. Let $\mathcal{M}(\mathbf{q})$ be defined as the volume occupied by the manipulator at configuration \mathbf{q} , and let \mathcal{O} be the set of obstacles :

$$Qf = \{ \mathbf{q} \in Q / \mathcal{M}(\mathbf{q}) \cap \mathcal{O} = \emptyset \}$$

The dimension of Qf is n , which is the number of independent joints of the manipulator (usually $n=6$). When the manipulator has joint limits, the configuration free space can be considered as a subset of \mathbb{R}^n .

In the configuration free space, the coordinated movements of the manipulator's links among the obstacles are translated into the motion of one abstract n -component point. Therefore, the configuration free space is very useful when finding a continuous path between two locations in the environment (which is the well known path planning problem^{15,16}, and is not the subject of our paper).

On the other hand, some authors have studied some topological properties of the configuration free space, according to the manipulator and the obstacles. The aim was to evaluate the geometric and kinematic performances of the manipulator^{17,18}. However, the configuration free space analysis is not a convenient way of describing the manipulator's *global* ability to perform motions in

the cartesian space. Indeed, the configuration free space is abstract and does not give straightforward informations on the manipulator's ability to achieve given tasks (they are usually defined in the space of positions and orientations of the end-effector). In addition, the configuration free space topology is not necessarily related to the manipulator's global ability to perform motions : the configuration free space may be non path-connected, whereas the manipulator can always move between any two points in its free workspace (see section 2.3)

2-2 Free workspace of a manipulator

The free workspace Wf of a manipulator describes the set of positions and orientations that the end-effector can reach without collision between any part of the robot and the obstacles of the environment. It can be defined as the image under the geometrical operator \mathbf{f} of the configuration free space :

$$Wf = \mathbf{f}(Qf)$$

The free workspace is defined in the operational space OS_m ¹⁹. A full description of the end-effector situation requires three position coordinates and three orientation coordinates, in which case $OS_m = \mathbb{R}^3 \times SO(3)$ (where $SO(3)$ is the group of proper rotations in \mathbb{R}^3). When only the position coordinates are specified, the operational space coincides with the cartesian space: $OS_m = \mathbb{R}^3$. The free workspace permits to describe the coordinated movements of the manipulator's links among the obstacles by representing one point (attached to the end-effector) in the "physical" space where the tasks are defined. This representation is of great interest when designing the layout of a robotic cell.

While a lot of authors have studied the configuration free space, very little work addresses the free workspace of a manipulator. Most of the time, the free workspace is determined by removing from the workspace W the regions occupied by the obstacles lying in W ^{20,21}. This approximation consists in ignoring the collisions involving the links of the manipulator which are not the end-effector. This leads to important errors when the first links are obstructed. Some authors as Shiller¹⁴ characterize the "shadows" of obstacles. They describe, in the operational space, the forbidden regions due to obstacles obstructions. Then, the free workspace can be geometrically defined as $W_f = W - S$, where S is the set of all the shadows lying in W . However, this definition is still insufficient, because it does not describe the *connectivity* of the free workspace, as shown in following section 2.3.

2-3 Qualitative analysis of the free workspace connectivity

We present in this section examples illustrating some connectivity properties of the free workspace. All presented figures have been obtained using tools that will be described in section 4.

2-3-1 Ability to move between any two points in W_f

In figure 2a, the task to be achieved is the assembling in P_1 , P_2 , and P_3 of pieces to be grasped from the table (which is supposed to be under the robot's workarea and consequently has no effect on the workspace). The free workspace of the robot is path-connected, and the table has been placed in such a way that a part of it (A) be reachable (figure 2b). However, this layout does not allow the feasibility of the task, as no path exists between A and point P_1 (although the first link of the robot never collides the obstacle). In fact, the configuration free space is not path-connected : the free workspace is composed of two independant "sub-

workspaces" which overlap, and between which the manipulator cannot move (more details concerning this example will be given in section 4.4).

Figure 3.a depicts a 3-DOF planar robot working in an enclosed space. It can be proved that the corresponding configuration free space is composed of three path-connected components. However, thanks to the disposition of their images in the cartesian space (the sub-workspaces), any two points in the free workspace can be linked together by the robot's end-effector (see figure 3.b).

These two examples show that the ability of the manipulator to move its end-effector between any two points in its free workspace cannot be characterized by the classical path-connectivity property, neither in the free workspace, nor in the configuration free space.

2-3-2 Ability to move between any n points in Wf

The ability to move between any two points in Wf and the ability to move between any n points (with $n > 2$) in Wf are not equivalent properties. Indeed, the manipulator from previous example is unable to move between points P_1 , P_2 , and P_3 (see figure 4.a), because these three points cannot be reached in a same sub-workspace (see figure 4.b, none of them recover the three points)

2.3.3- Ability of the robot to move along given continuous trajectories

An other family of tasks (such as continuous welding or cutting operations) requires tracking continuous trajectories, Then, it is necessary to verify that the manipulator does not change aspect during motion. This problem has already been studied by Borrel in the case of environments without obstacles⁹ (see section 1).

When tracking a trajectory, the joint limits as well as an obstacle may oblige the robot to change aspect. In figure 5, trajectory T_c is unfeasible : point X_1 can be reached only in aspect $q_2 > 0$ (elbow down) because of the joints limits, while point X_2 can be reached only in aspect $q_2 < 0$ (elbow up) because of the obstacle.

Thus, it is necessary to generalize the aspects to the case of cluttered environments.

3- THEORITICAL STUDY OF THE FREE WORKSPACE CONNECTIVITY

The previous examples show that the ability of a manipulator to travel through its free workspace Wf is not straightforward. The topology of the configuration free space Qf does not provide sufficient information (if Qf is path-connected, then the manipulator will be able to move between any n points in Wf , but this is not a necessary condition). On the other hand, the classical path-connectivity property is insufficient to characterize the topology of Wf (if Wf is not path-connected, then the manipulator cannot link any n points in Wf ; but the converse is false). The free workspace can be regarded as a global reorganization, via the geometrical operator, of the configuration free space. Consequently, the free workspace topology, and thus the ability of the manipulator to travel through its free workspace, will naturally depend upon the spatial disposition of the images under the geometrical operator of some basic subspaces of Qf , which are the path-connected components of Qf and those of the free aspects (see thereafter). Nevertheless, our purpose is to show that the global manipulator's ability to perform motions in a cluttered environment can be completely characterized by a stringent analysis of its free workspace, independently of the configuration free space topology. Further, it will be shown in section 4 that the images of the basic subspaces can be characterized quite independently of Qf .

3.1- Notations and preliminary definitions

- Let $\mathbf{f} : \mathbb{R}^n \rightarrow \text{OS}_m$ be the geometrical operator of the robot.

- We assume that there are p path-connected components of the configuration free space Qf . We shall denote them as $\{Qf_i : i \in I\}$ where I is the set $\{1, \dots, p\}$:

$$Qf = \cup_{i \in I} (Qf_i) \quad \text{and} \quad \forall i \neq j, i \in I, j \in I, \quad Qf_i \cap Qf_j = \emptyset$$

- Then $\forall i \in I : Wf_i = \mathbf{f}(Qf_i)$ denotes the image of Qf_i under the geometrical operator. Each Wf_i is necessarily connected, because Qf_i is connected and \mathbf{f} is continuous. However, the Wf_i 's are not necessarily the connected components of Wf because they may overlap.

Preliminary definition 1 : The Wf_i 's will be called the *sub-workspaces*.

- In order to take into account the obstacles in the characterization of the aspects, we state the following definition :

Preliminary definition 2 : We shall define the *free aspects* Af_j as the intersection of the aspects A_j with the configuration free space : $\forall j \in J, Af_j \triangleq A_j \cap Qf$

Then, the free aspects are not necessarily path-connected. Given a free aspect Af_j , we shall denote its p_j path-connected components as $\{Af_{jk}, k \in I(j)\}$ where $I(j)$ is the set $\{1, \dots, p_j\}$:

$$\forall j \in J, Af_j \triangleq \cup_{k \in I(j)} Af_{jk}, \quad Af_{jk} \cap Af_{jk'} = \emptyset \text{ if } k \neq k'$$

We shall denote by Wf_j the image under f of the free aspect A_j , and by Wf_{jk} the image under f of the path-connected component A_{jk} :

$$\begin{aligned}\forall j \in J, Wf_j &\triangleq f(A_j) \\ \forall j \in J, \forall k \in I(j), Wf_{jk} &\triangleq f(A_{jk})\end{aligned}$$

Preliminary definition 3 : The Wf_{jk} 's will be called the *w-aspects*.

3.2- Definition of the 2-connectivity property (P₂)

Definition 1 : The free workspace will be *2-connected* if, by definition, the manipulator can move its end-effector between any two points in Wf .

Theorem 1 : Wf is 2-connected if and only if :

$$Wf^2 = \cup_{i \in I} Wf_i^2 \quad (1)$$

(A^2 means the cartesian product of A by itself)

Proof 1 : Wf is 2-connected means that for any two points in Wf , there exists a continuous path in the configuration free space which permits to link the two points. In other words, there must exist a sub-workspace Wf_i which contains both of these points :

$$\forall \mathbf{X}_1, \mathbf{X}_2 \in Wf, \exists i \in I, \mathbf{X}_1 \in Wf_i \text{ and } \mathbf{X}_2 \in Wf_i$$

which can be rewritten as in (1).

The free workspace Wf depicted in figure 3 can be proved to be 2-connected.

3.3- Definition of the n-connectivity property (P_n)

As shown in section 2.3.2, the 2-connectivity property does not ensure that the manipulator can move between any n points in its free workspace, when $n > 2$. The following definition gives a stronger property.

Definition 2 : The free workspace will be *n-connected* if, by definition, the manipulator can move its end-effector between an arbitrary number n of points in Wf . This property can be written as follows :

$$\forall n > 1, \forall (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \in Wf^n, \exists i \in I / (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_n) \in Wf_i^n \quad (2)$$

Theorem 2 : Wf is n -connected if and only if Wf is coincident with one sub-workspace :

$$\exists i \in I / Wf = Wf_i \quad (3)$$

Proof 2 : *Sufficient condition* :

(3) \Rightarrow (2) is obvious.

Necessary condition :

Suppose that (3) is false : $\forall i \in I, Wf_i \neq Wf$. Thus, for any i in I , $Df_i = Wf_i - Wf$ is nonempty. Let p be the number of sub-workspaces. Let p points $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p$ be chosen in Wf such that for any $j \leq p$, \mathbf{X}_j belongs to Df_j . Then, there exists a p -uple $(\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p)$ in Wf^p such that : $\forall i \in I, (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p) \cap Df_i^p \neq \emptyset$, which means :

$$\exists (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p) \in Wf^p / \forall i \in I, (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_p) \notin Wf_i^p. \text{ Thus, (2) is false.}$$

Figure 6 shows a case where the free workspace is n -connected.

3.4- n^+ -connectivity (P_{n^+}) and n^{++} -connectivity ($P_{n^{++}}$) properties

A manipulator whose free workspace is n -connected can move between any n points in it, but not necessarily in each of its reachable configurations at these points, as shown in figure 7. In the three cases, points P_1 and P_2 can be *linked together*. In addition, these points can always be *reached* in both configurations "elbow up" and "elbow down". However :

- In case 1, due to the spherical obstacle and the joint limits, the manipulator cannot move from point P_1 to P_2 in aspect "elbow down". Moreover, it cannot change aspect between points P_1 and P_2 , because of the rectangular obstacles. Thus, trajectory $\{P_1, P_2\}$ can be achieved only in aspect "elbow up".

- In case 2, trajectory $\{P_1, P_2\}$ can be achieved with any choice of configuration at point P_1 or P_2 , but not at both points in the same time (the manipulator cannot change aspect between the two points).

- In case 3, trajectory $\{P_1, P_2\}$ can be achieved, whatever the configuration at point P_1 and P_2 .

The following n^+ -connectivity and n^{++} -connectivity properties will respectively deal with case 2 and case 3.

3-4-1 n^+ -connectivity

Definition 3 : The free workspace Wf will be n^+ -connected if, by definition, the manipulator can move between an arbitrary number n of points in Wf , whatever the reachable configuration at one of these n points. This can be written as :

$$\forall n > 1, \forall (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n-1}, \mathbf{q}) \in Wf^{n-1} \times Qf, \exists i \in I / (\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n-1}, \mathbf{q}) \in Wf_i^{n-1} \times Qf_i \quad (4)$$

Theorem 3 : Wf is n^+ -connected if and only if each sub-workspace is coincident with the whole free workspace :

$$\forall i \in I / Wf = Wf_i \quad (5)$$

Proof 3 : Sufficient condition

Let $\mathbf{q} \in Qf$ and let $\mathbf{X}_0 \in Wf$ such that : $\mathbf{X}_0 = \mathbf{f}(\mathbf{q})$. As (5) is true : $\forall i \in I, \mathbf{X}_0 \in Wf_i$.

Let i_0 in I such that $\mathbf{X}_0 \in Wf_{i_0}$ and $\mathbf{q} \in Qf_{i_0}$.

Similarly, let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n-1} \in Wf$, then $\forall i \in I, \mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n-1} \in Wf_i$, which is also true for $i = i_0$. In conclusion : $\mathbf{q} \in Qf_{i_0}$ and $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n-1} \in Wf_{i_0}$, thus (4) is true.

Necessary condition

Suppose that (5) is false : $\exists i \in I$ such that $Wf_i \neq Wf$.

Let $\mathbf{q} \in Qf_i$ and let $\mathbf{X}_1, \mathbf{X}_2, \dots, \mathbf{X}_{n-1} \in Wf - Wf_i$, then (4) is false.

An example is presented in figure 8 where the free workspace is n^+ -connected.

3-4-2 n^{++} -connectivity

Definition 4 : The free workspace Wf is n^{++} -connected if, by definition, the manipulator can move between an arbitrary number n of points in Wf , whatever the reachable configuration at each of these n points.

Theorem 4 : Wf is n^{++} -connected if and only if Wf is composed of one single sub-workspace.

Proof 4 : The n^{++} -connectivity property means that between any n configurations in the configuration free space, there exists a continuous path which links them. In other words, the configuration free space is path-connected, which is equivalent to theorem 4.

Remark : The free workspace of a manipulator moving in an environment without obstacle is always n^{++} -connected.

3.5- Definition of the t -connectivity property (P_t)

The previous properties describe four increasing levels of connectivity in the free workspace. The last one (the n^{++} -connectivity) allows the manipulator having a n^{++} -connected free workspace Wf to perform any discrete trajectory in Wf with any reachable configuration at each point of the trajectory. However, as shown in previous section 2.3.3, the manipulator cannot necessarily follow any continuous trajectory in Wf , as it may need to change aspect during motion. When the manipulator is non-redundant, the change of aspect forces the manipulator to move its end-effector through an hyper-surface in the workspace, which is the image of a singularity. This is a serious drawback when the manipulator must carry out tasks that require a control of the end-effector all along its trajectory. In order to ensure the feasibility of tasks such as continuous welding, cutting, spraying, or assembling operations, we will propose the t -connectivity property. A manipulator whose free workspace Wf is t -connected will be able to track any continuous trajectory in Wf without changing aspect.

Before stating such a property, it is necessary to study the feasibility of a given continuous trajectory in the free workspace. This needs to generalize the results stated by P. Borrel⁹ to the case of cluttered environments.

Borrel have shown that a continuous trajectory T_C can be tracked without changing aspect if and only if T_C is enclosed in the image of one aspect. It can be easily proved that this result generalizes to cluttered environments as follows :

if a continuous trajectory T_C can be tracked without changing aspect, T_C is necessarily enclosed in the image of a connected component of a free aspect. In other words, T_C must entirely lie in a w-aspect Wf_{jk} .

This condition is also sufficient when the manipulator is non-redundant. Indeed, the geometrical operator f of a non-redundant manipulator is a point-to-point mapping of each aspect on to its image. Thus, if T_C entirely lies in a w-aspect Wf_{jk} , there exists a continuous path in Af_{jk} whose image under f is exactly T_C .

Remark : The previous condition is false when the manipulator is redundant, because the geometrical operator of such robots is not a point-to-point mapping of each aspect on to its image^{9,22}. As the following t- and t⁺-connectivity properties will be based upon this condition, they will concern only non-redundant manipulators. Note that a redundant manipulator can change aspect without moving its end-effector⁹, so that it can perform continuous trajectories more easily than a non-redundant one.

Definition 5 : The free workspace Wf of a non-redundant manipulator will be *t-connected* if, by definition, every continuous trajectory in Wf can be tracked by the end-effector, without changing aspect.

Theorem 5 : Wf is t-connected if and only if Wf is coincident with a w-aspect :

$$\exists j \in J \text{ and } \exists k \in I(j) / Wf = Wf_{jk} \quad (6)$$

Proof 5 : Let T_C be a continuous trajectory in Wf . It has been stated that T_C could be tracked without changing aspect if and only if :

$\exists j \in J$ and $\exists k \in I(j) / T_C \subset Wf_{jk}$. Then the proof is quite similar to proof 2.

Figure 9 shows a case where the free workspace is t-connected.

3.6- Definition of the t⁺-connectivity property (P_{t⁺})

When the free workspace Wf of a non-redundant manipulator is t-connected, every continuous trajectory in Wf can be tracked without changing aspect. However, the configuration cannot be freely chosen, because all points in Wf cannot necessarily be reached in any aspect. That is why we state the following property :

Definition 6 : The free workspace Wf of a non-redundant manipulator will be *t⁺-connected* if, by definition, every continuous trajectory in Wf can be tracked without changing aspect, and whatever the configuration at any one point along the trajectory. This means that all points in Wf can be reached in any aspect. Thus, the t⁺-connectivity property is the generalization to cluttered environments of the "well-connected workspace property" stated by Brad Paden¹⁰.

Theorem 7 : Wf is t⁺-connected if and only if the w-aspects are all coincident with Wf :

$$\forall j \in J, \forall k \in I_j \quad Wf = Wf_{jk} \quad (7)$$

Proof 5 : The proof is quite similar to proof 3.

The free workspace depicted in figure 10 is t^+ -connected.

3.7- Generalization to connected subspaces

When the free workspace Wf of a manipulator is not connected, it is useful to characterize the connected subspaces of Wf . Each connectivity property can be easily generalized to any subspace of Wf . Then, the maximal connected subspaces of Wf are defined as follows :

Definition 7 : The maximal 2-connected subspaces SWf_2 are the larger regions in Wf where any two points can be linked together by the manipulator's end-effector. They are defined as the larger subspaces such that :

$$SWf_2 \subset \cup_{i \in I} Wf_i^2 \quad (8)$$

Definition 8 : The maximal n -connected subspaces SWf_n are the larger regions in Wf where an arbitrary number n of points can be linked together by the manipulator's end-effector. Thus, the maximal n -connected subspaces are the manipulator's sub-workspaces :

$$SWf_n = Wf_i \quad (9)$$

for any i in I .

Definition 9 : The maximal n^+ -connected subspaces SWf_{n^+} are the larger regions in Wf where an arbitrary number n of points can be linked together by the manipulator's end-effector, whatever the reachable configuration at one of these n points. We prove easily¹¹ that the maximal n^+ -connected subspaces are given by :

$$SWfn^+ = (\cap_{j \in I'} Wf_j) - (\cup_{k \notin I'} Wf_k) \quad (10)$$

for any set I' in I such that $SWfn^+$ is non-empty.

Definition 10 : The maximal n^{++} -connected subspaces $SWfn^{++}$ are the larger regions in Wf where an arbitrary number n of points can be linked together by the manipulator's end-effector, whatever the reachable configuration at each of these n points. It has been proved¹¹ that the maximal n^{++} -connected subspaces are given by :

$$SWfn^{++} = Wf_i - (\cup_{k \neq i} Wf_k) \quad (11)$$

for any i in I such that $SWfn^{++}$ is non-empty.

Definition 11 : The maximal t -connected subspaces $SWft$ are the larger regions in Wf where any continuous trajectory can be tracked by the manipulator without changing aspect. Thus, the maximal t -connected subspaces are the w -aspects :

$$SWft = WAf_{jk} \quad (12)$$

for any j in J and for any k in $I(j)$.

Definition 12: The maximal t^+ -connected subspaces $SWft^+$ are the larger regions in Wf where any continuous trajectory can be tracked by the manipulator without changing aspect, and whatever the configuration at any one point along the trajectory. The maximal t^+ -connected subspaces have been proved²² to be the path-connected components of :

$$(\cap_{j \in J', k \in I(j)} W A f_{jk}) - (\cup_{j \in J', l \in I(j)} W A f_{jl}) - (\cup_{m \in J} W A f_m) \quad (13)$$

for any subset J' of indices in J which leads to non-empty subspaces.

3.8- Comments

These six connectivity properties can be divided into two families :

- the ability to perform point to point motions (2-, n-, n+-, and n++-connectivity properties) :

$$(P_{n++}) \Rightarrow (P_{n+}) \Rightarrow (P_n) \Rightarrow (P_2)$$

- the ability to track any continuous trajectory without changing aspect (t- and t+-connectivity properties):

$$(P_{t+}) \Rightarrow (P_t)$$

As the $(P_t) / (P_{t+})$ properties do not apply for redundant manipulators, a "t++-connectivity" property allowing the choice of configurations all along the continuous trajectories, would not have any sense. Indeed, a non-redundant manipulator can reach a given point in an aspect with only one configuration.

The 2-connectivity property is intended for tasks involving motions between only two points, such as handling or simple assembling operations. Thus, this property is poor. It is also difficult to verify, as cartesian products calculations are required, which lead to high-dimensional spaces construction. For this reason, equivalent formulations of (1) and (8) have been proposed¹¹ :

(1) $\Leftrightarrow Wf = (\cap_{k \in K} (\cup_{i \in I_k} Wf_i)) \cap (\cup_{k \in K} (\cap_{i \in I_k} Wf_i))$ for some family $\{I_k\}_{k \in K}$ of subsets of I .

Similarly : (8) $\Leftrightarrow SWf2 = (\cap_{k \in K} (\cup_{i \in I_k} Wf_i)) \cap (\cup_{k \in K} (\cap_{i \in I_k} Wf_i))$ for any family $\{I_k\}_{k \in K}$ of subsets of I such that $SWf2$ is non-empty.

Simple rules have also been given¹¹, which considerably reduce the number of families to analyse.

The n-connectivity properties are well suited for most of robotic tasks, provided that no continuous trajectory following be involved.

Finally, the t-connectivity properties concern all tasks that require tracking continuous trajectories, such as continuous cutting, welding, or spraying operations.

The n⁺- and n⁺⁺-connectivity properties (those allowing the choice of the configurations) are particularly usefull when minimizing criteria (such as cycle time) for trajectories to be achieved by redundant manipulators.

4- ALGORITHMIC ANALYSIS

In this section, we present a method for implementing the previous theoretical results in a robotic C.A.D. system. Some classical tools have already been developed, which permit to perform an optimized scanning of the joint space of a manipulator moving in a cluttered environment. These tools will not be described here, they use a C.S.G description of the robot and of the environment, and involve efficient intersection tests based on the hierarchical data structures of the scene²³.

4.1- Basic subspaces construction

Now, all connectivity properties can be verified if some "basic subspaces" are known. These are the sub-workspaces Wf_i , and the w-aspects WAf_{jk} . Then, the use of classical boolean operations (intersection, union, difference) will lead to the necessary and sufficient conditions.

A natural way of getting the sub-workspaces is to extract the path-connected components of the configuration-free space Qf , and then to compute their images in the operational space, using the geometrical operator of the manipulator. However, this method is not at all optimized. Indeed, the geometrical operator must be twice computed (the configuration free space must be completely constructed before computing the images of the path-connected components). In addition, a special modelization of Qf would be required, which has no other uses for our analysis.

That is why we will propose a method for directly mapping the basic subspaces in the operational space. Here are two remarks which are intended for only non-redundant manipulators :

Remark 1 : The images under f of the path-connected components of the free aspects are the path-connected components of the images under f of the free aspects.

Indeed, the geometric operator of a non-redundant manipulator is a point-to-point mapping of each aspect on to its image. Thus, the path-connectedness of an aspect is equivalent to the path-connectedness of its image.

This remark permits to map the w-aspects WAf_{jk} by a path-connectivity analysis *in the operational space*.

Remark 2 : By definition, the free aspects are open sets. The union of their closure is the configuration free space. Thus, each path-connected component of the configuration free space is the union of the closure of adjacent path-connected

components of free aspects. Consequently, each sub-workspace is the union of the closure of c-adjacent w-aspects (c-adjacent w-aspects means that their preimages are adjacent). Now, if we can recognise the c-adjacent w-aspects, the sub-workspaces can be computed without considering a model of the configuration free space. Let Af_{j_1} , Af_{j_2} be two distinct free-aspects. Let $S_{j_1j_2}$ be the border between Ak_{j_1} and Ak_{j_2} ($S_{j_1j_2}$ is a subset of a branch of singularity and is fully determined by the factors of the determinant of the Jacobian matrix as recalled in §1). Let $Af_{j_1k_1}$, $Af_{j_2k_2}$ be respective path-connected components of Af_{j_1} and Af_{j_2} . $Af_{j_1k_1}$, $Af_{j_2k_2}$ are adjacent if and only if the intersection of the closure of their images is non-empty and intersects the set $\mathbf{f}(S_{j_1j_2})$. Let \mathbf{q} belonging to the border between $Af_{j_1k_1}$ and $Af_{j_2k_2}$. Then, \mathbf{q} also belongs to $S_{j_1j_2}$. Thus, $\mathbf{f}(\mathbf{q})$ belongs to the closure of $WAf_{j_1k_1}$ and of $WAf_{j_2k_2}$, and also belong to $\mathbf{f}(S_{j_1j_2})$. Conversely, let \mathbf{X} belong to $\mathbf{f}(S_{j_1j_2})$, and to the closure of $WAf_{j_1k_1}$ and of $WAf_{j_2k_2}$. Then, there exists $\mathbf{q}_1 \in Af_{j_1k_1}$, $\mathbf{q}_2 \in Af_{j_2k_2}$ and $\mathbf{q}_3 \in S_{j_1j_2}$. Since a non-redundant manipulator has only one configuration per aspect, \mathbf{q}_3 is necessarily equal to \mathbf{q}_1 and \mathbf{q}_2 . Thus, $\mathbf{q} = \mathbf{q}_1 = \mathbf{q}_2 = \mathbf{q}_3$ belongs to the border between $Af_{j_1k_1}$ and $Af_{j_2k_2}$, which means that $Af_{j_1k_1}$ and $Af_{j_2k_2}$ are adjacent.

In summary, the sub-workspaces are obtained by making the union of some w-aspects, and of some sets $\mathbf{f}(S_{j_1j_2})$. Since the w-aspects are computed by first looking for the sets $S_{j_1j_2}$, the sets $\mathbf{f}(S_{j_1j_2})$ can be easily determined. Using remark 2, the computation of the sub-workspaces is much more efficient than a calculation of the images of the path-connected components of the configuration free space : it has been experimentally shown²² that the computational time decreases by 5 to 10, according to the type of robot and the complexity of the environment. Figure 11 shows the application of remark 2 for the case of a planar 2-DOF manipulator.

These first remarks do not concern redundant manipulators. For such robots, the w-aspects are useless since they only occur in the t- and t⁺-connectivity properties, which do not apply for redundant manipulators. However, the sub-

workspaces are useful for all manipulators. The following remark will permit to get the sub-workspaces of redundant manipulators as those of non-redundant ones.

Remark 3 : A n -DOF redundant manipulator working in the m -dimensional operational space can be considered as a non-redundant one by adding to the operational space, either $n-m$ independent degrees of freedom, or, when not possible, $n-m$ independent manipulator joint variables combinations, such that the corresponding jacobian matrix remains full-rank. Then, the redundant manipulator's sub-workspaces will be obtained as follows :

- get the n -dimensional sub-workspaces in the extended operational space, using previous remarks,
- project them in the m -dimensional operational space of the original redundant manipulator.

Examples : - Let a *3-revolute-joint* planar manipulator be used with only two positioning coordinates (x,y) of the end-effector (see figure 12a). Thus, this manipulator is redundant. In this case, it is possible to add a degree of freedom, which is the orientation of the end-effector.

- Let a *3-prismatic-joint* planar manipulator whose third joint axis is parallel to the first one (see figure 12b). If the two positioning coordinates (x,y) of the end-effector are specified, one dimension must be added to the operational space, which cannot be the orientation of the end-effector as it is constant. However, the joint variable $u = q_1$, (which is independent of $x = q_2$ and of $y = q_1 + q_3$) can be chosen.

4.2- Spaces modelling

The basic subspaces have to be represented using a model which be consistent with the following specifications :

- the spaces to be modelled are complex shapes,
- their calculations involve a lot of boolean operations,
- path-connectivity analyses are required.

Thus, the octree model has been chosen. The octree is a hierarchical data structure based on a recursive subdivision of space²⁴. It is particularly useful when representing complex shapes, and lends itself very well to boolean operations. In addition, path-connectivity analyses can be naturally achieved, thanks to the implicit adjacency graph of the octree structure²⁵. The octree modelling of a space S leads to a representation of S with cubes of various sizes. The smallest cubes generally lie near the border of the shape, their size determines the accuracy of the octree representation (which is called the octree resolution). The octree structures permit to modelize *3-dimensional* spaces, but they can be easily generalized to *k-dimensional* spaces²⁶ (using kd-tree structures). Figure 13a shows the quadtree ($k=2$) graphic representation of a planar shape, figure 13b depicts its tree representation.

4.3- Results

In this section, we present the connectivity analyses of the free workspace of some manipulators. All free workspaces have been defined for a point lying on the extremity of the last link of the corresponding manipulators. All connectivity analyses have been achieved with an accuracy of 1%.

4-3-1 Maximal 2-connected subspaces

Figure 14a shows the whole free workspace of a 3-DOF planar manipulator, which is path-connected, but not 2-connected. Figure 14b depicts the maximal 2-connected subspaces (note that the first three subspaces are also n-connected,

whereas the last one is only 2-connected). In each of these subspaces (and only in them), the manipulator is able to move between any two points. This connectivity analysis has been completed in 10 minutes on a 8700-VAX mini-computer.

4-3-2 Maximal n -connected subspaces

All following figures will depict the case of an anthropomorphic manipulator, whose first joint is supposed to be locked (in order to get the graphic representation more readable). Figure 15a shows the whole free workspace : it is path-connected, but not 2-connected (and thus it is not n - nor t -connected).

Figure 15b depicts the three maximal n -connected subspaces. In each of these subspaces (and only in these), the manipulator is able to link together any n points.

4-3-3 Maximal n^+ -connected subspaces

Figure 16 shows the five maximal n^+ -connected subspaces : in each of these subspaces (and only in these), the manipulator is able to move between any n points, whatever the reachable configuration at one of these n points.

4-3-4 Maximal n^{++} -connected subspaces

Figure 17 depicts the three maximal n^{++} -connected subspaces, in which (and only in which) the manipulator can move between any n points, whatever the reachable configurations at each of these n points.

4-3-5 Maximal t -connected subspaces

Figure 18 shows the six maximal t -connected subspaces, in which (and only in which) the manipulator is able to follow any continuous trajectory without changing aspect.

4-3-6 Maximal t^+ -connected subspaces

Finally, figure 19 shows the ten maximal t^+ -connected subspaces of the free workspace. In each of these subspaces (and only in these), the manipulator can track any continuous trajectory without changing aspect, and whatever the reachable configuration at any one point along the trajectory.

The full connectivity analysis of the anthropomorphic manipulator's free workspace (Fig 15 to 19) has been completed in 6 minutes on a 8700-Vax mini-computer.

5- ROBOTICS CAD APPLICATIONS

The connectivity analyses of manipulator workspace are mainly intended for robotic C.A.D. applications such as off-line tasks preparing. As shown in the following examples, the knowledge of the various connected subspaces is very useful when designing the spatial layout of a robotic plant.

5.1- Using the n -connected subspaces

Let consider the example of figure 3. In section 2.3.1, it has been stated that the manipulator could not reach point P_1 from the table. Now, the following two questions are :

1- Why is the task unfeasible?

2- How can one make it feasible?

Figure 20a clearly answer these questions : the free workspace is not n -connected, and in addition, none of the two maximal n -connected subspaces recover at the same time the three points and a part of the table. However, the task could be carried out by simply positioning the table in the left side of the scene (see figure 20b). On the other hand, figure 20c shows that the configurations cannot be freely chosen at the different workpoints, as no n^+ -connected subspaces recovers all the three points. A solution would be to change the position of the manipulator (or of the obstacle), in order to modify the topology of the free workspace.

5.2- Using the t-connected subspaces

Suppose that the task to be achieved is the continuous cutting of an L-shaped piece. The problem is to find a positioning of the piece that ensures the feasibility of this task (figure 21a). The outline must be enclosed in a t -connected subspace, as the end-effector must not leave the continuous trajectory. Figure 21b shows how to place the piece in the free workspace of the robot.

5.3- Generalization to robotized cells design

The complete design a robotized cell involves difficult geometric problems such as choosing the robot whose morphology and dimensions are best suited to the tasks to be achieved, and positioning the various components of the cell in the space. The workspace connectivity analyses are currently implemented in a robotic cells design package (SITROB) which performs the automatic positioning of manipulators in cluttered environments²⁷.

6- CONCLUDING REMARKS

The main contribution of this work is the analysis of obstacles in the operational space of the manipulator, which is the space where the tasks are naturally defined. This study is of great interest for the designer of a robotic cell.

A classification of new topological properties of manipulator free workspace has been presented : the 2-connectivity property (ability to link together any 2 points in the workspace), the n -, n^+ - and n^{++} -connectivity properties (ability to link together any n points in the workspace) and finally the t - and t^+ -connectivity properties (ability to follow any continuous trajectory in the workspace).

Based on necessary and sufficient conditions, and using octree model, a general method has been proposed to analyse the free workspace connectivity of manipulators working in cluttered environments. When the free workspace is not connected, all the connected subspaces are determined. In order to avoid important memory size requirement, only 3-dimensional spaces are considered (which can involve any combination of position and orientation coordinates).

The connectivity properties are currently implemented in an automatic robotic cell design package²⁷.

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Figure 1 : Aspects "elbow up" and "elbow down" and their images in the plane of an anthropomorphic manipulator.

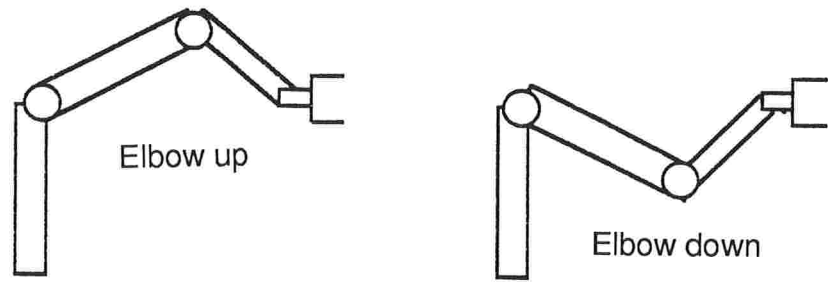


Fig 1.a

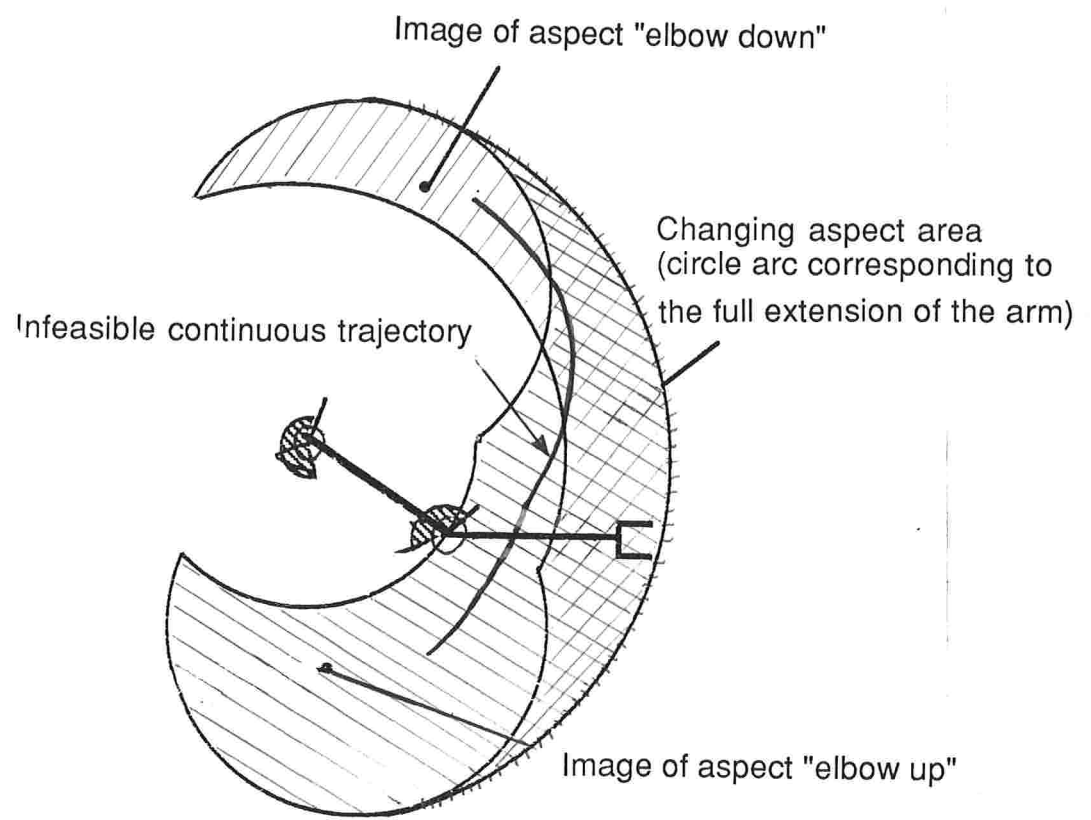


Fig. 1.b

Figure 2 : The free workspace is path connected, but no path exists between the table and point P_1 .

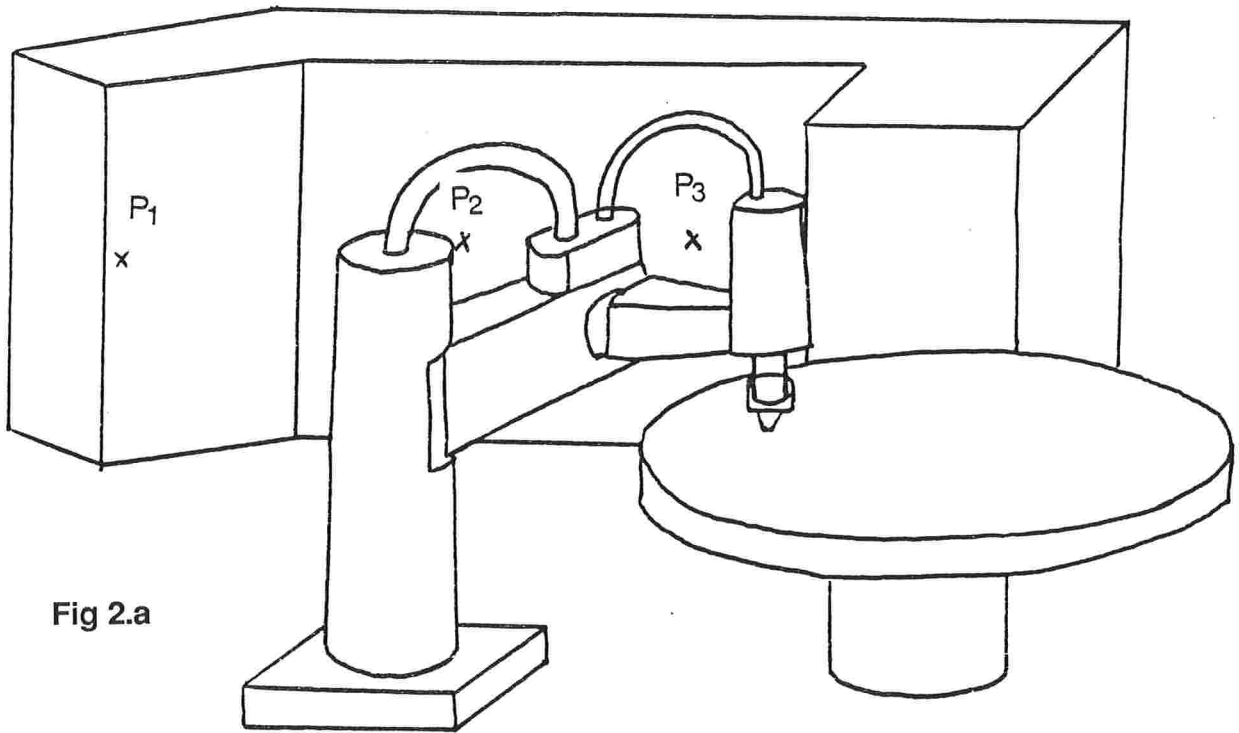


Fig 2.a

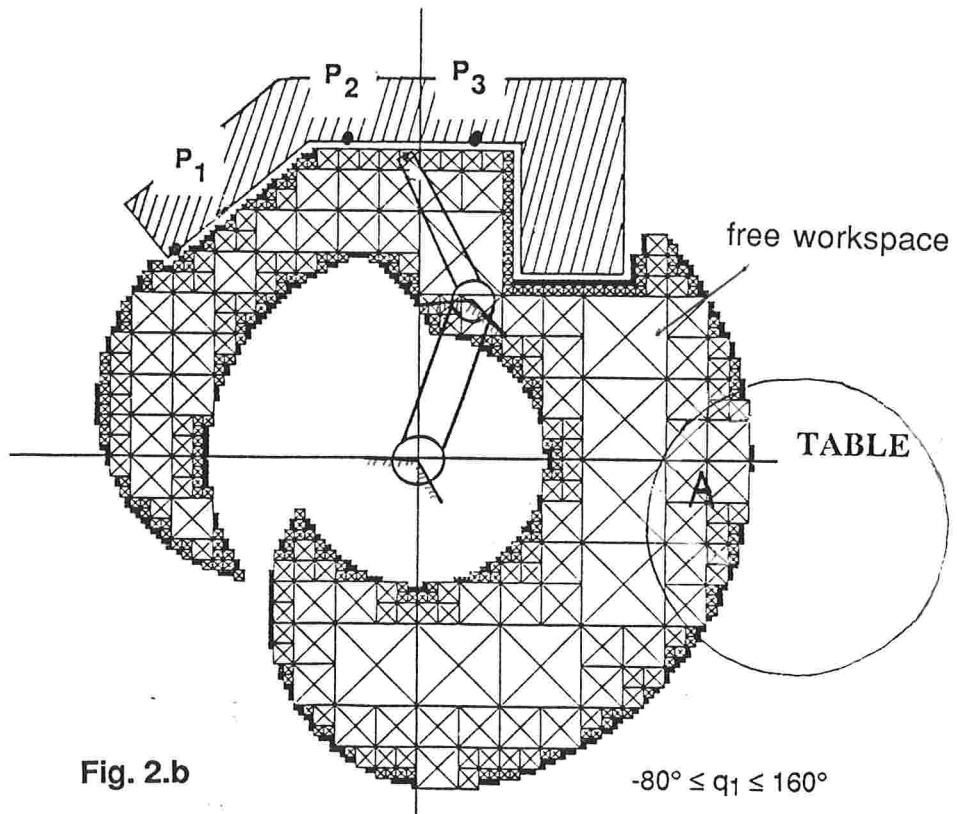


Fig. 2.b

$$-80^\circ \leq q_1 \leq 160^\circ$$

$$-135^\circ \leq q_2 \leq 100^\circ$$

Figure 3 : The free workspace is path-connected, but the configuration free space is not path-connected. However, any two points can be linked together by the manipulator.

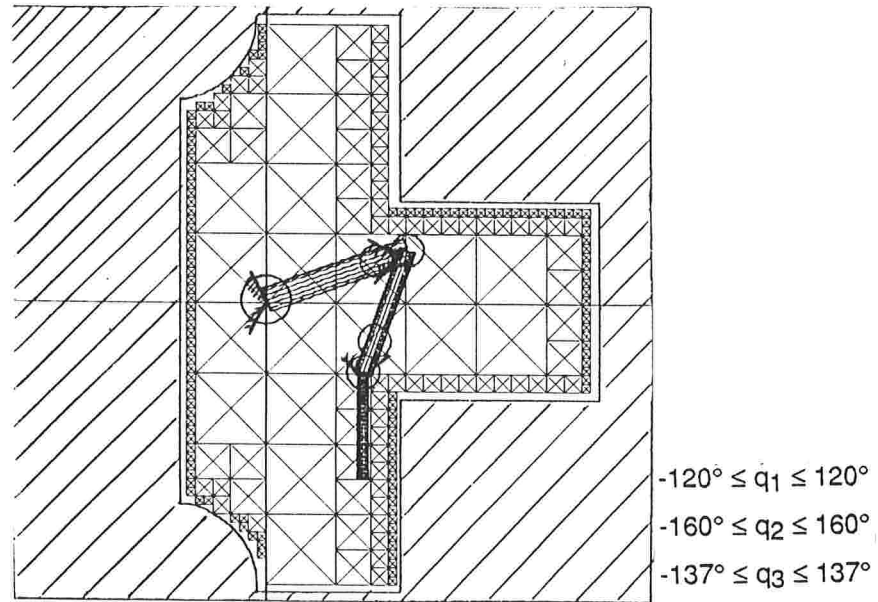


Fig 3.a : Free Workspace

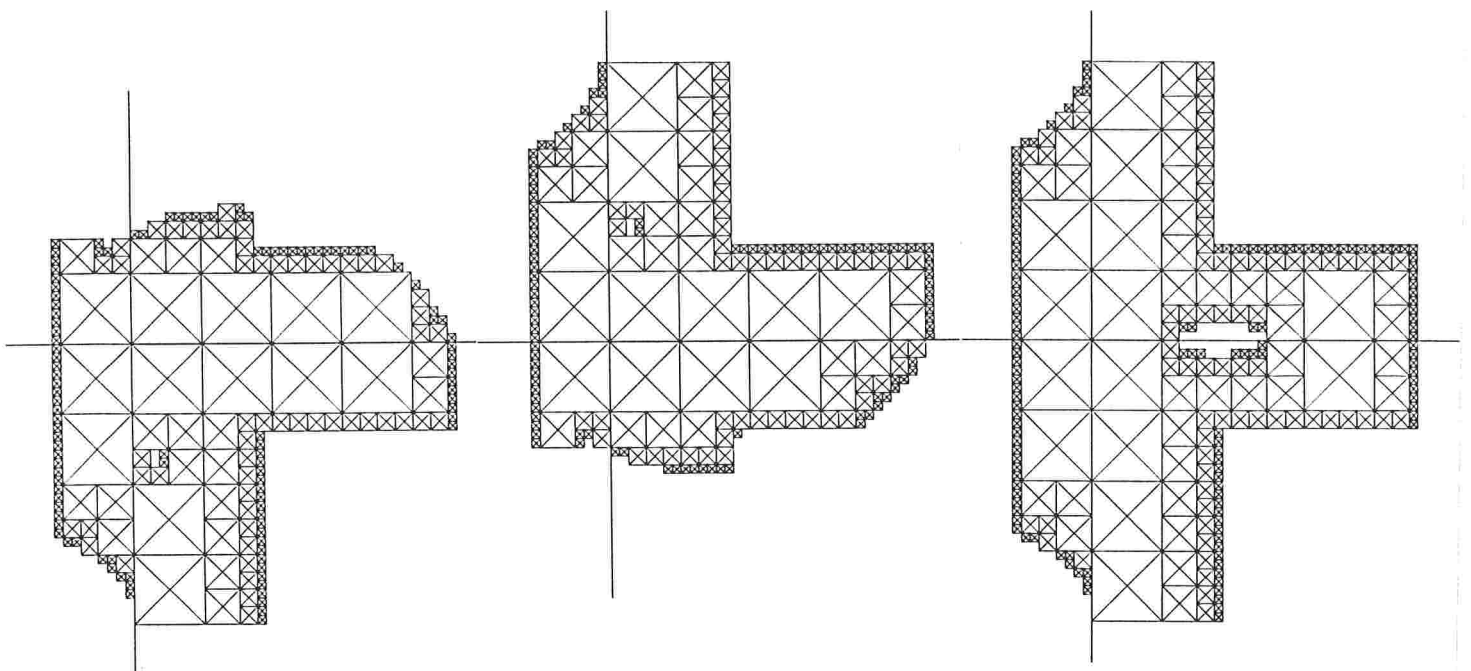


Fig. 3.b : Sub-workspaces

Figure 4 : The three points P_1 , P_2 and P_3 cannot be linked together

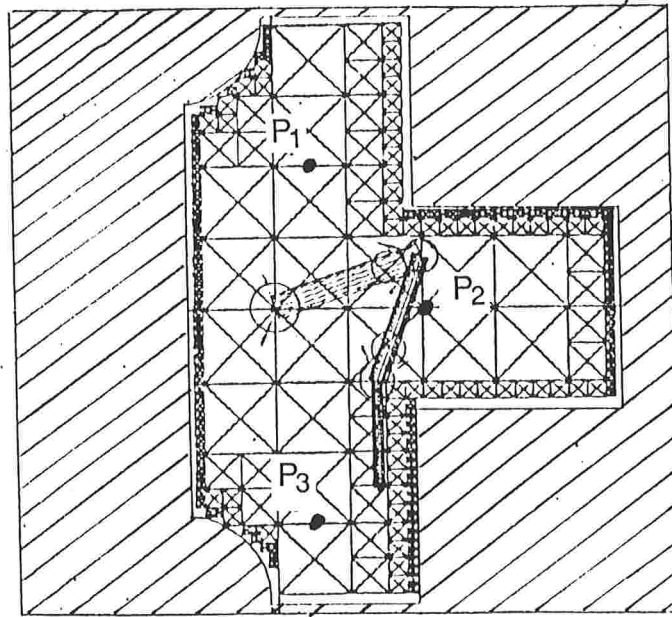


Fig 4.a

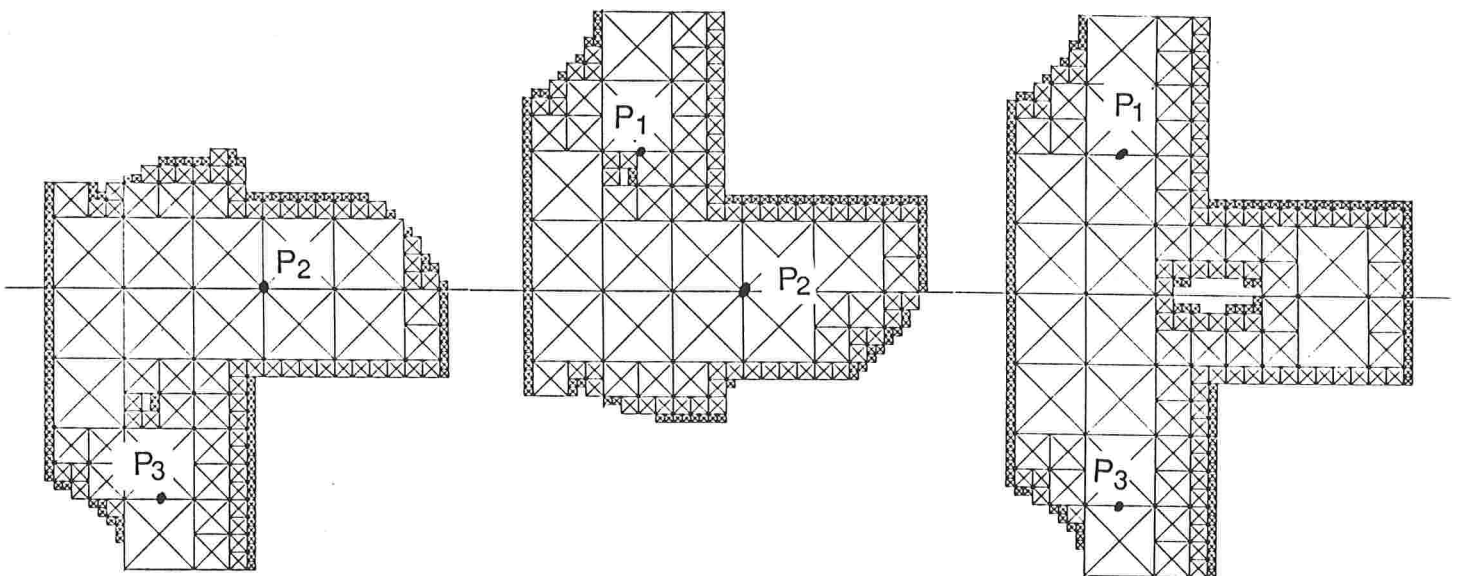


Fig. 4.b

Figure 5 : Unfeasible trajectory

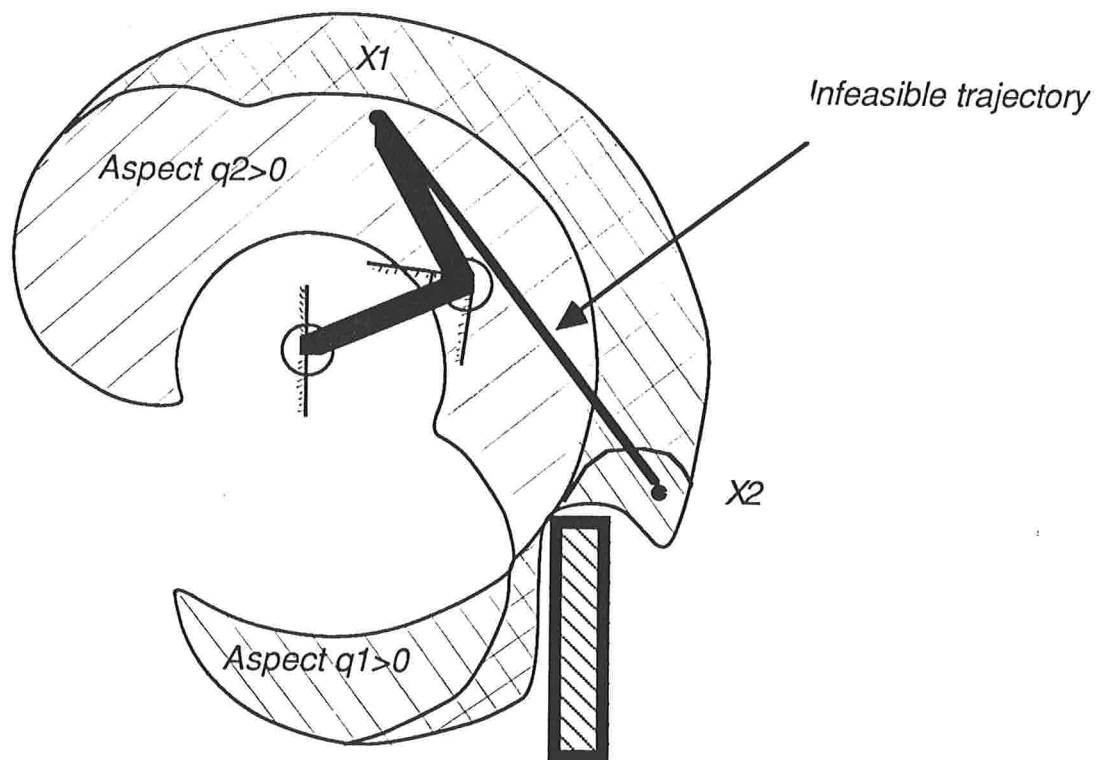


Figure 6 : A case where the free-workspace is n-connected

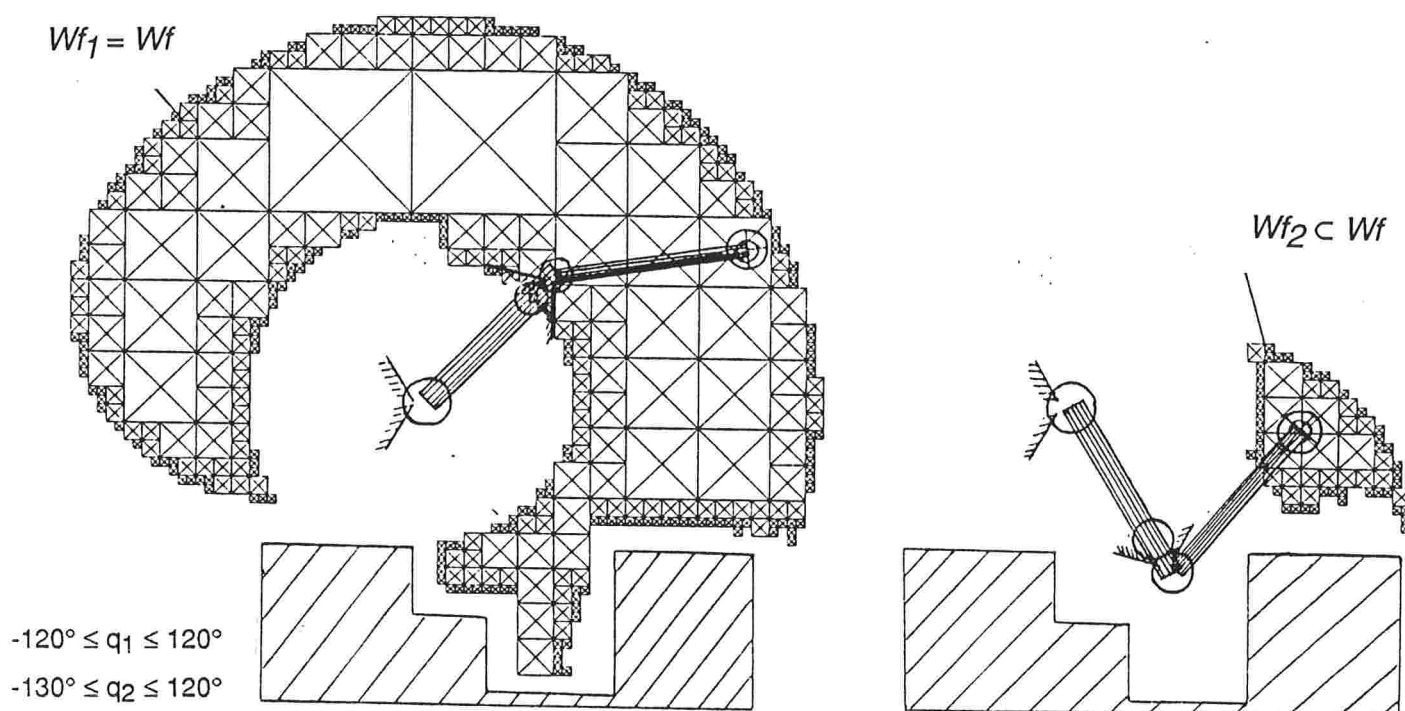


Figure 7 : Choice of the configurations for point to point motions

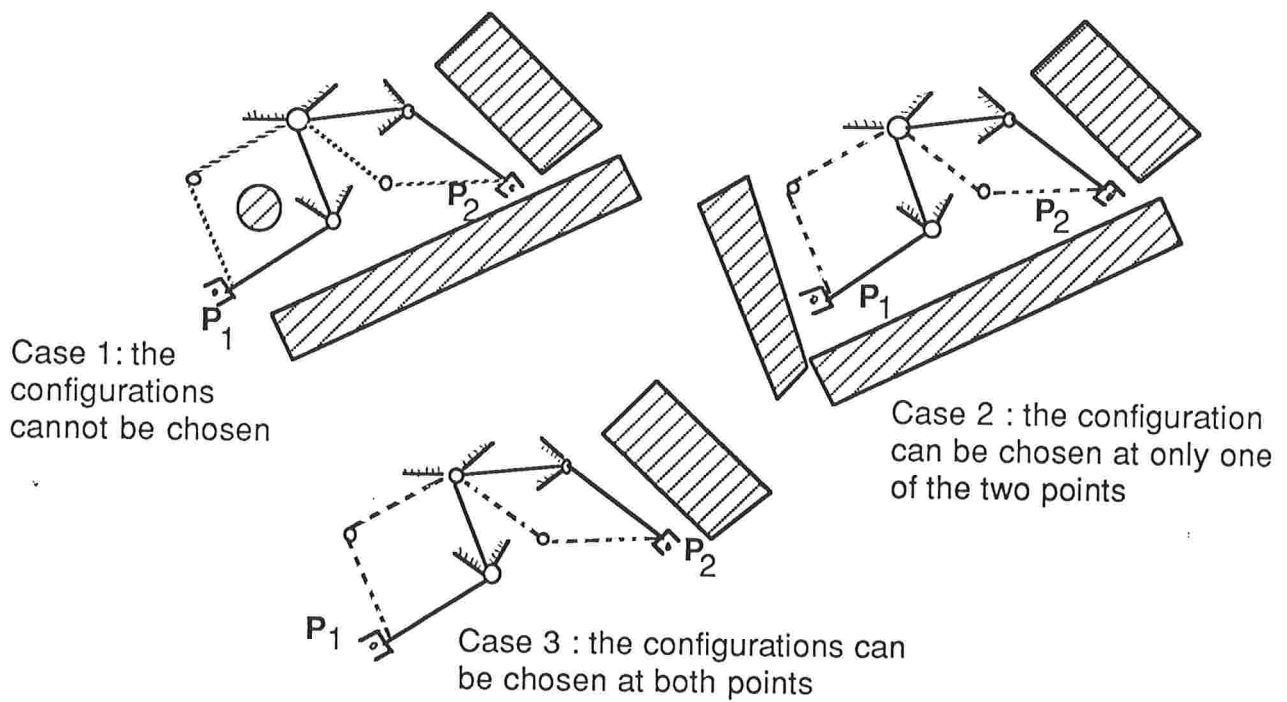
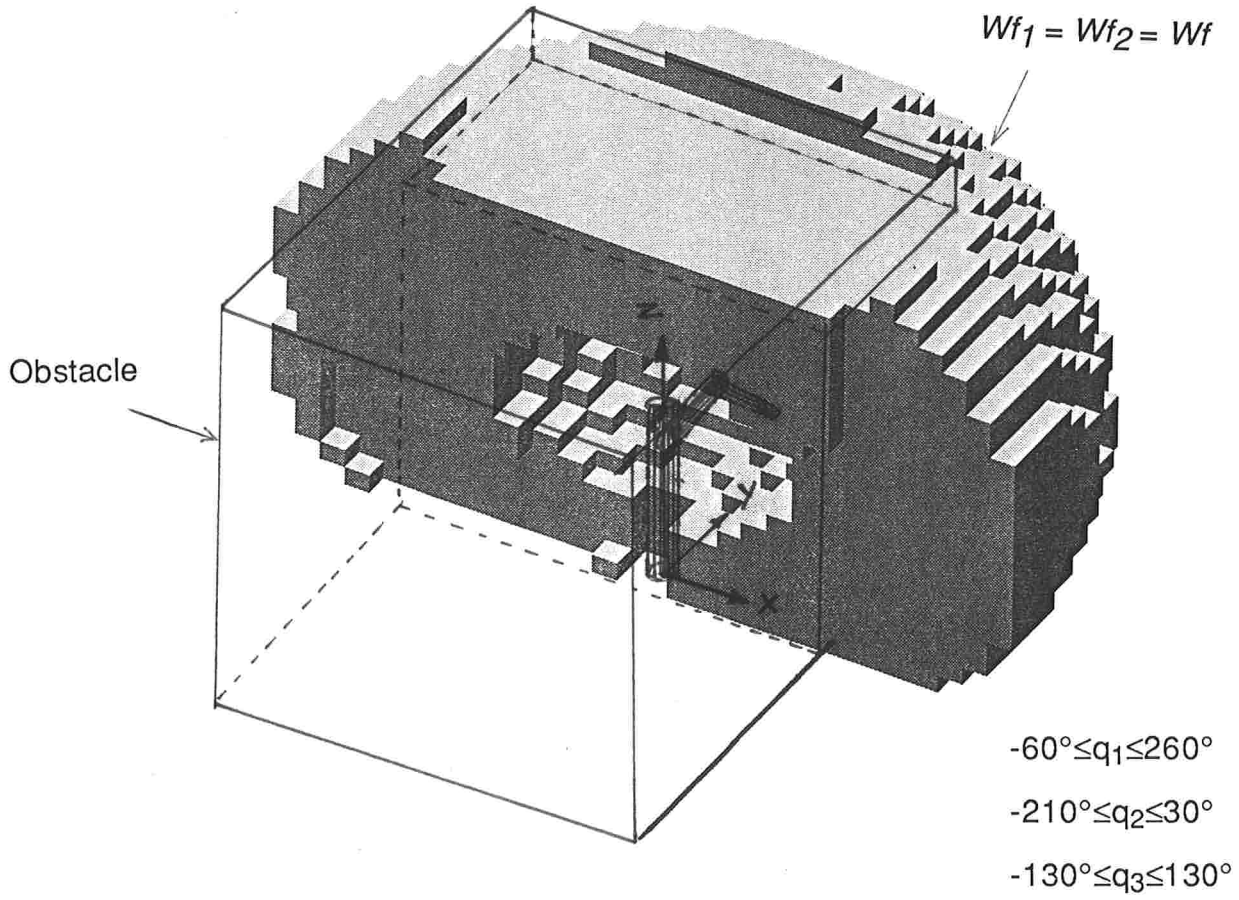
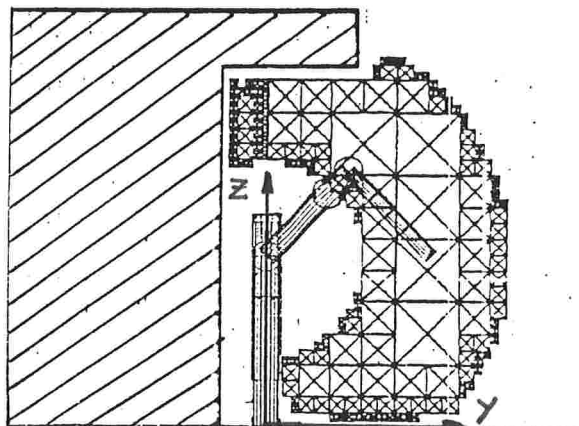


Figure 8 : A case where the free-workspace is n⁺-connected



The manipulator is represented in configuration $q_1=90^\circ, q_2=-45^\circ, q_3=90^\circ$



section of Wf in the manipulator plane

Figure 9 : A case where the free-workspace is t-connected

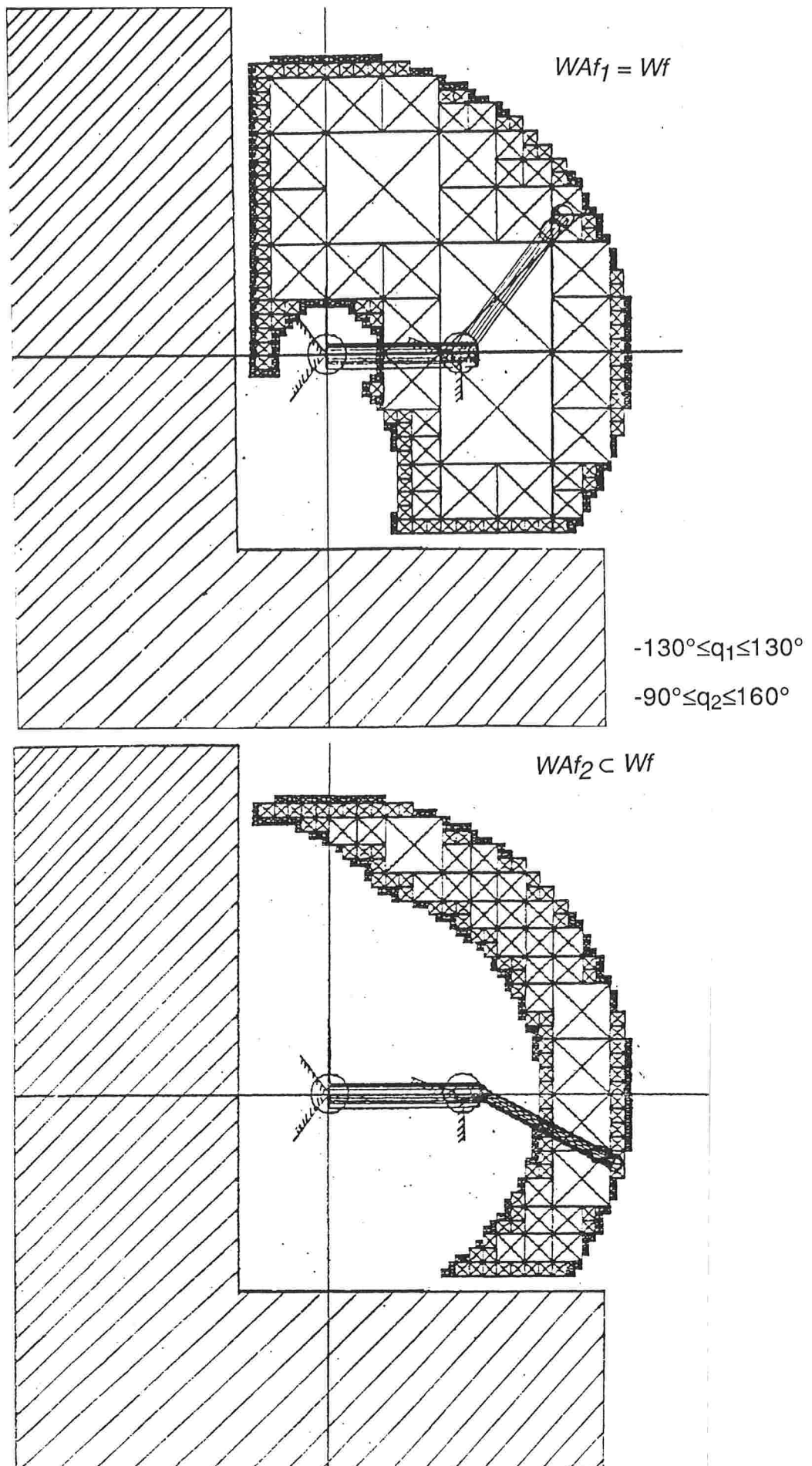


Figure 10 : A case where the free-workspace is t⁺-connected

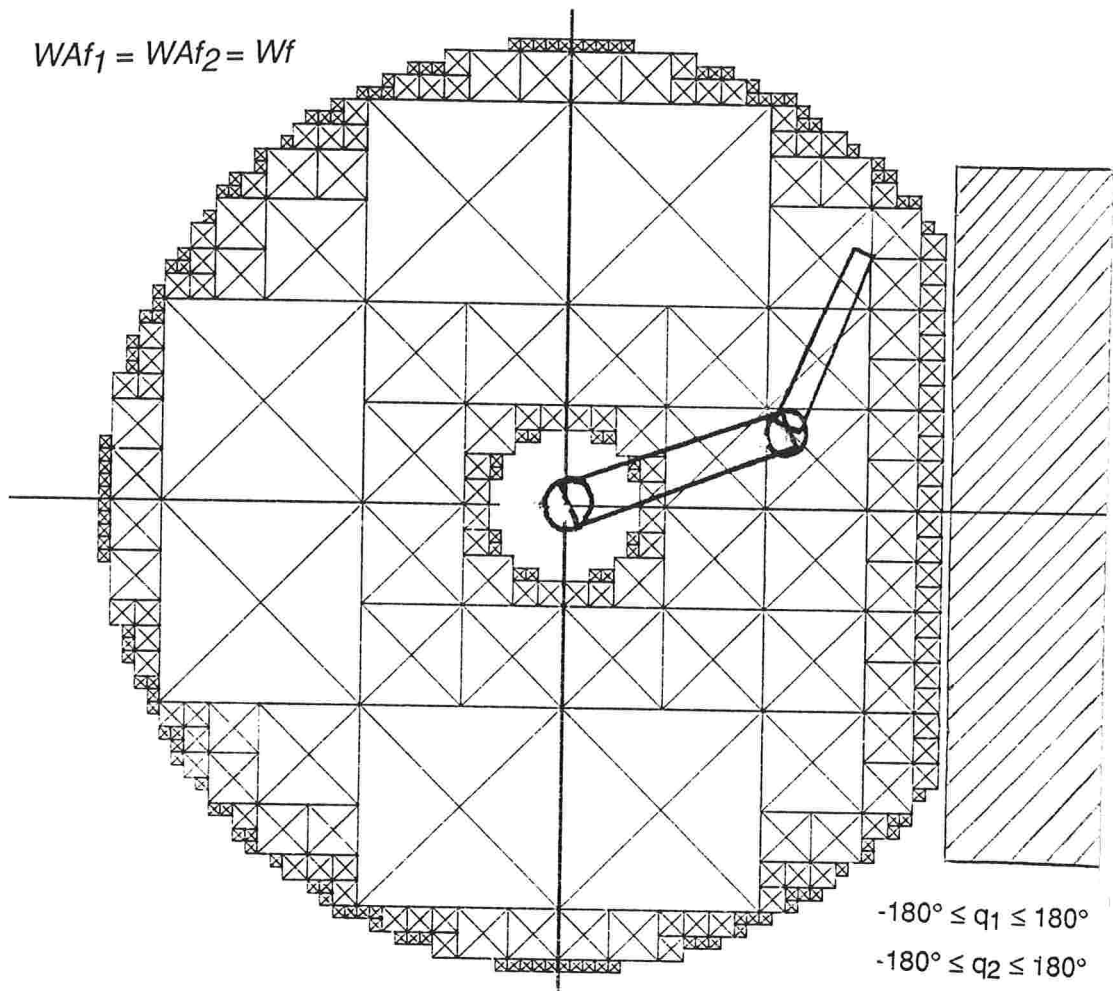


Figure 11 : Getting the sub-workspaces as union of w-aspects

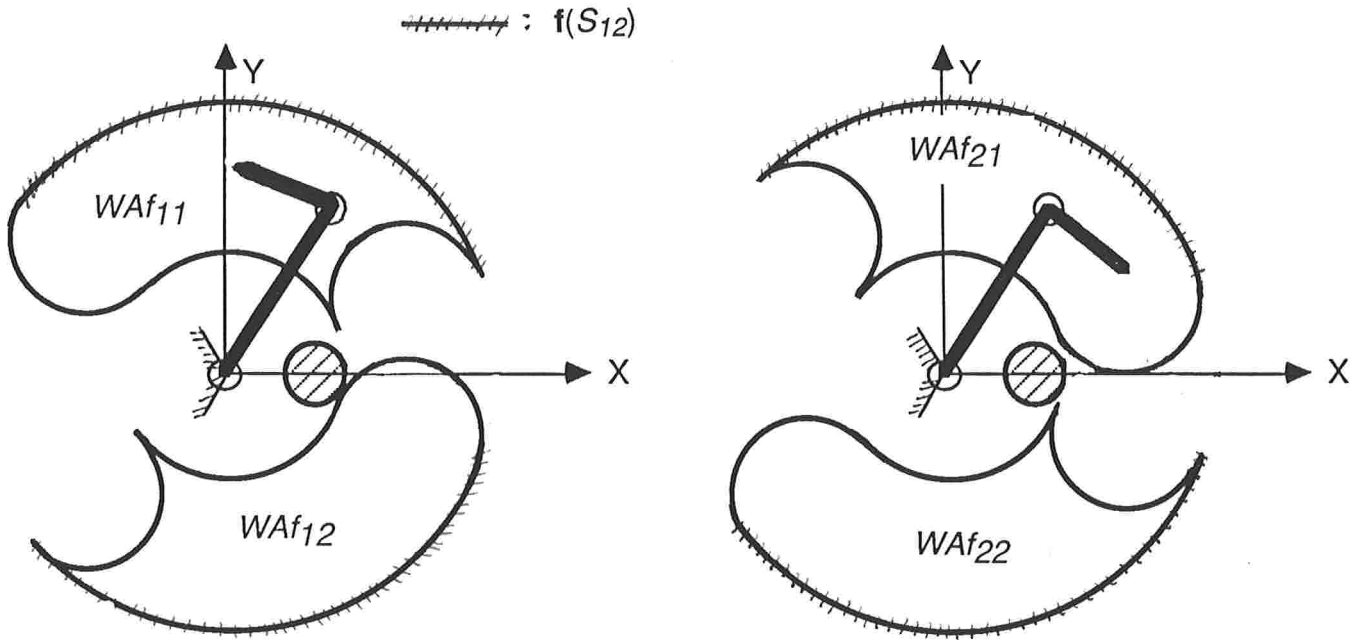


Fig 11.a

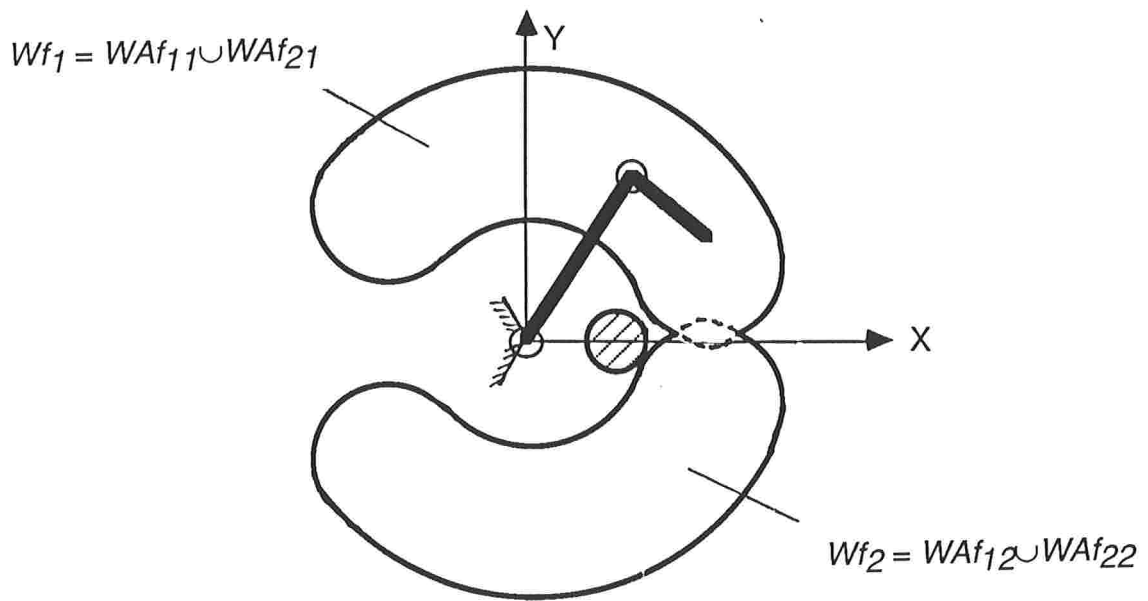


Fig 11.b

Figure 12 : Two redundant planar manipulators working in the (x, y) -cartesian space

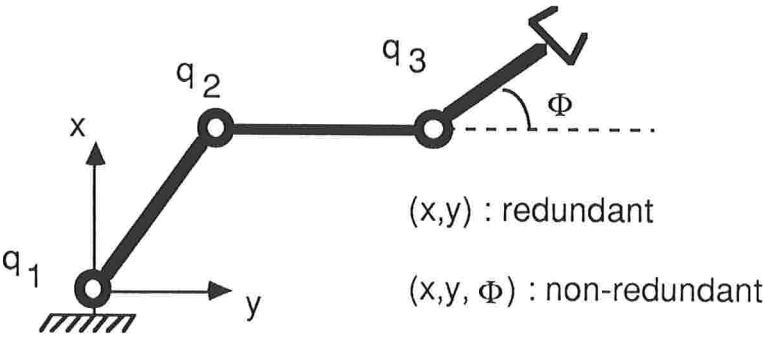


Fig. 12.a

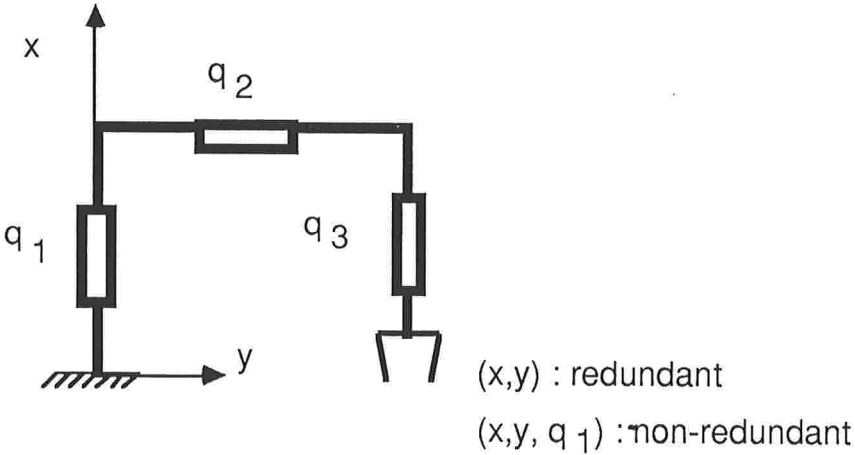


Fig. 12.b

Figure 13 : Quadtree representation of a planar shape

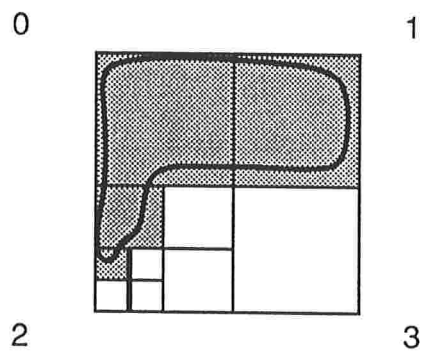


Fig. 13.a : Graphic representation

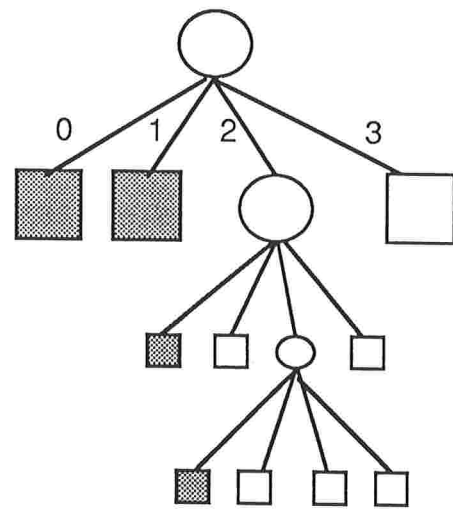


Fig. 13.b : Tree representation

Figure 14 : Free workspace and maximal 2-connected subspaces

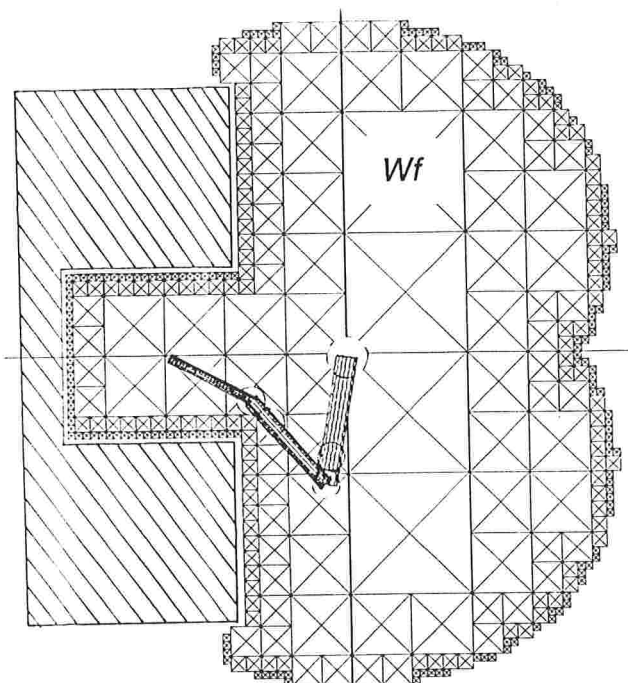


Fig. 14.a : Free workspace

$$\begin{aligned} 60^\circ &\leq q_1 \leq 300^\circ \\ -160^\circ &\leq q_2 \leq 160^\circ \\ -137^\circ &\leq q_3 \leq 137^\circ \end{aligned}$$

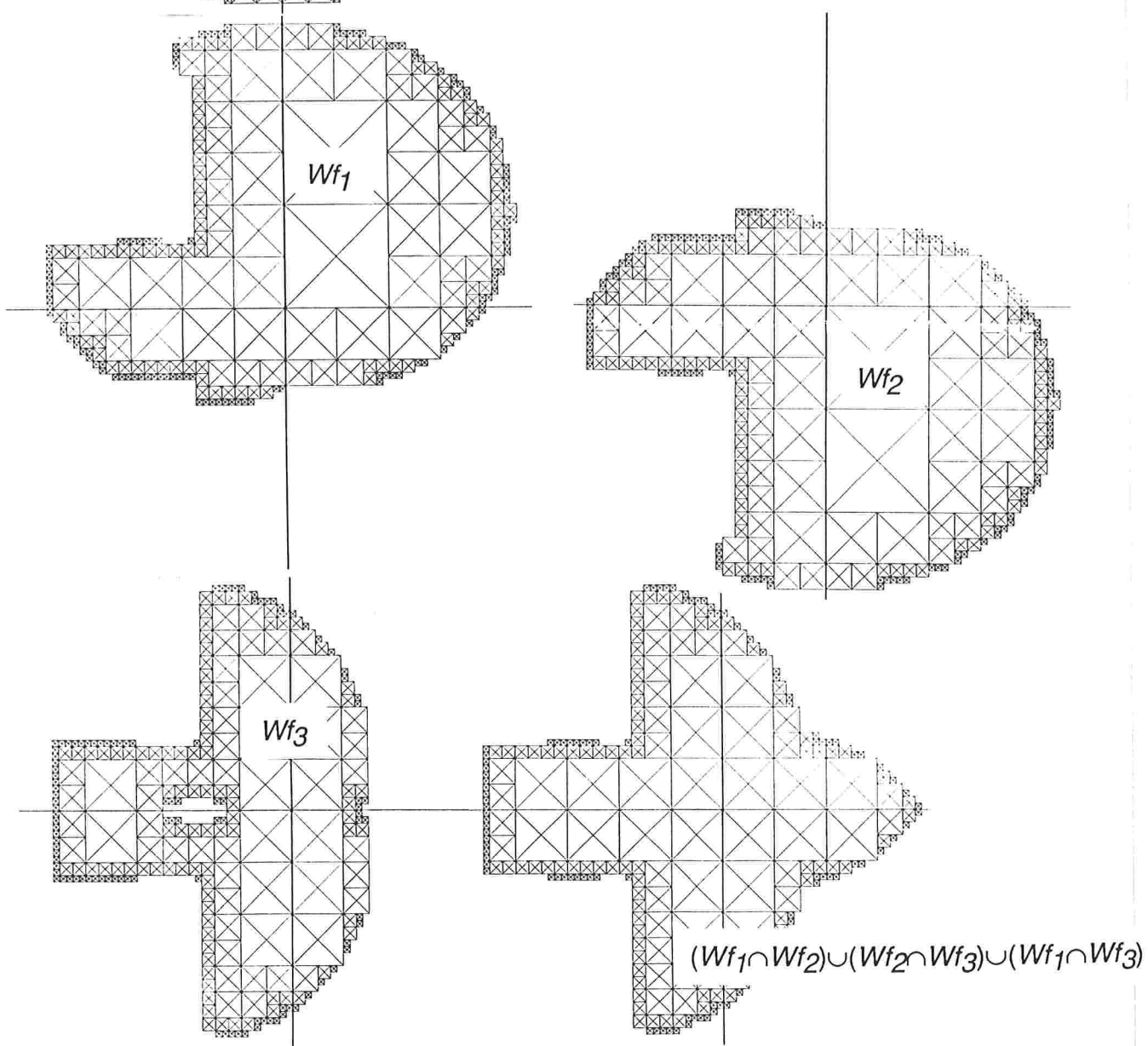


Fig. 14.b : Maximal 2-connected subspaces

Figure 15 : Free workspace and maximal n-connected subspaces

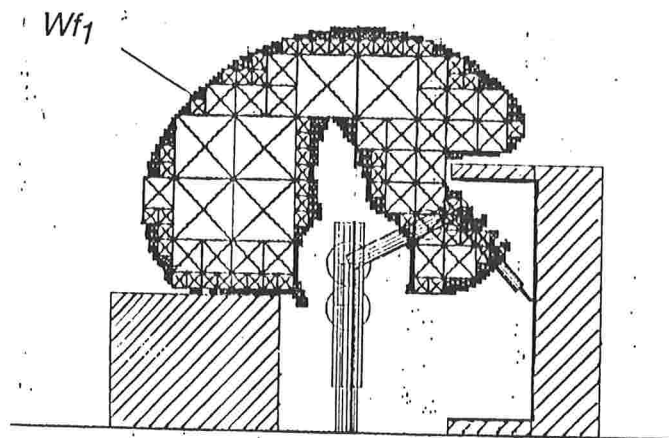
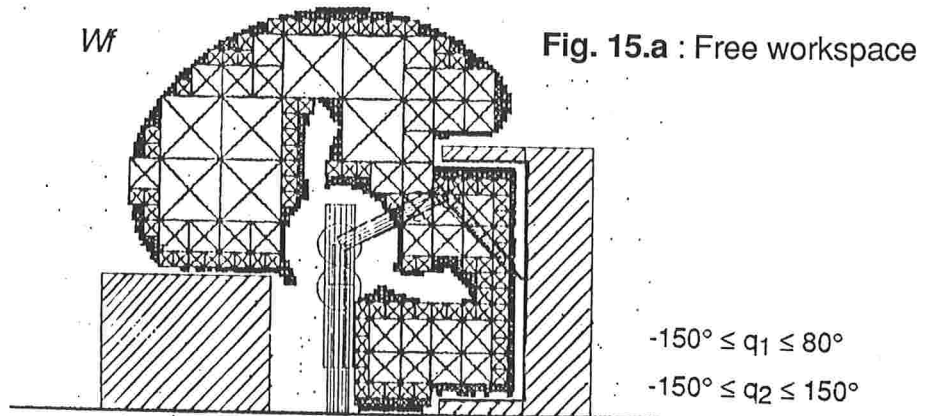


Fig. 15.b : Maximal n-connected subspaces

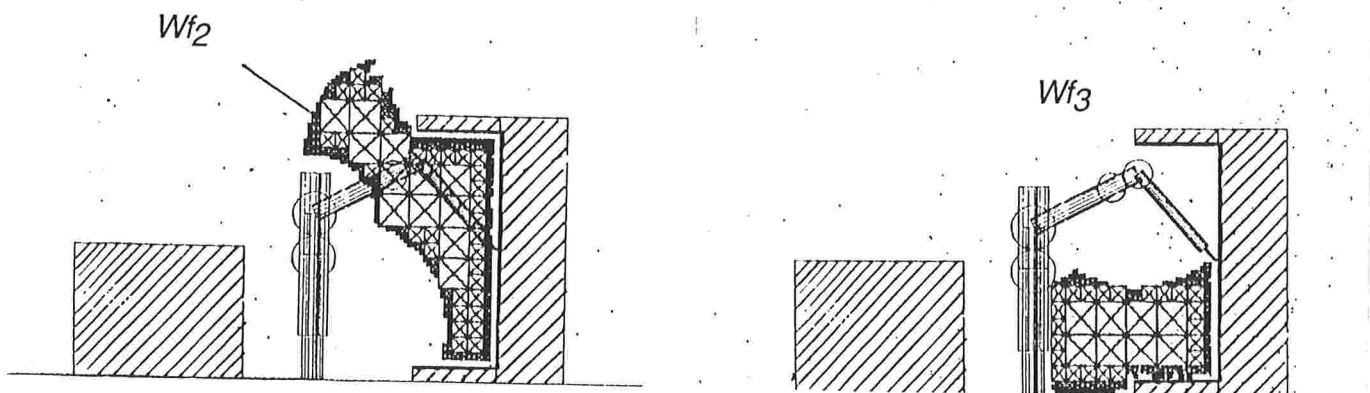


Figure 16 : Maximal n+-connected subspaces

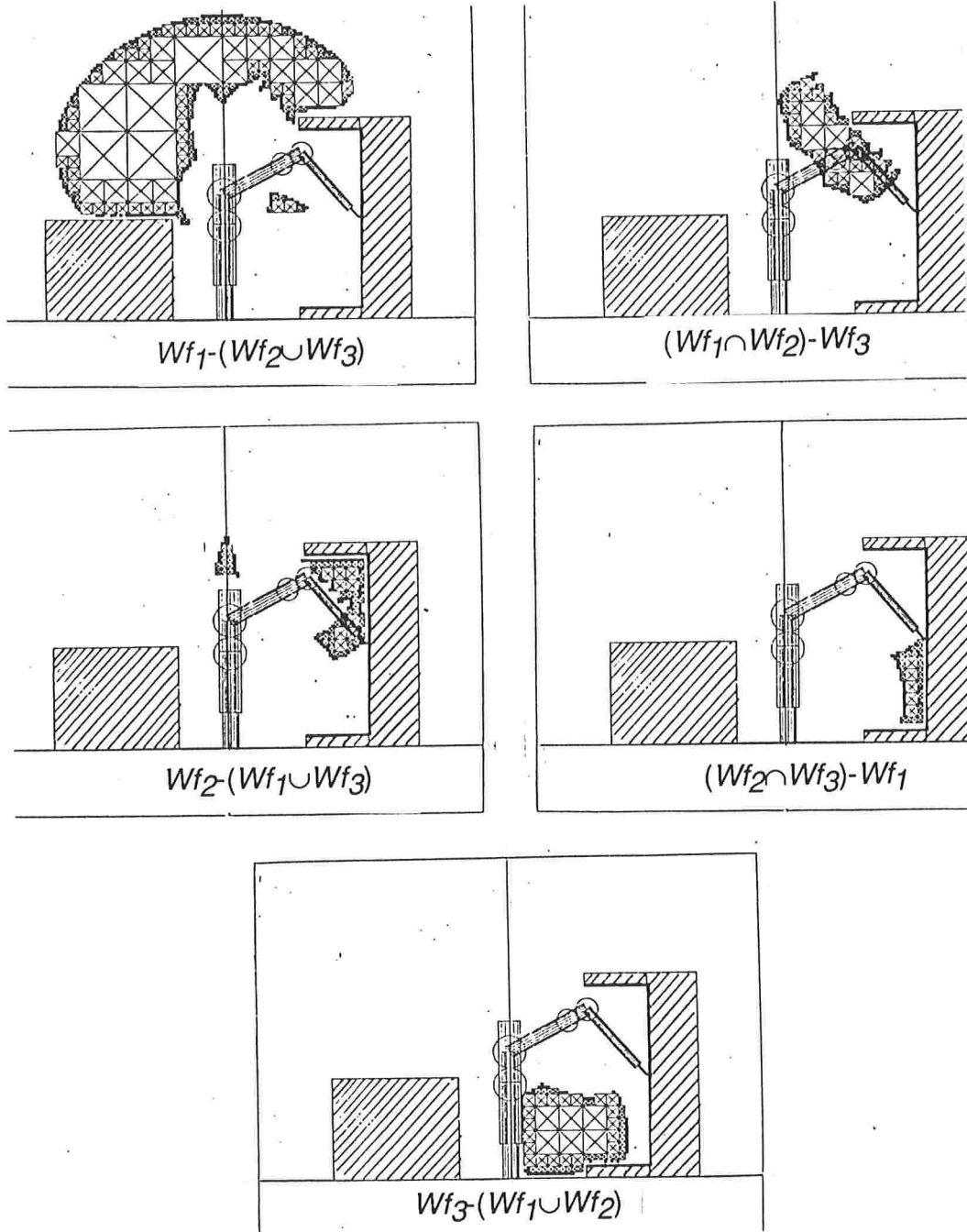


Figure 17 : Maximal n^{++} -connected subspaces

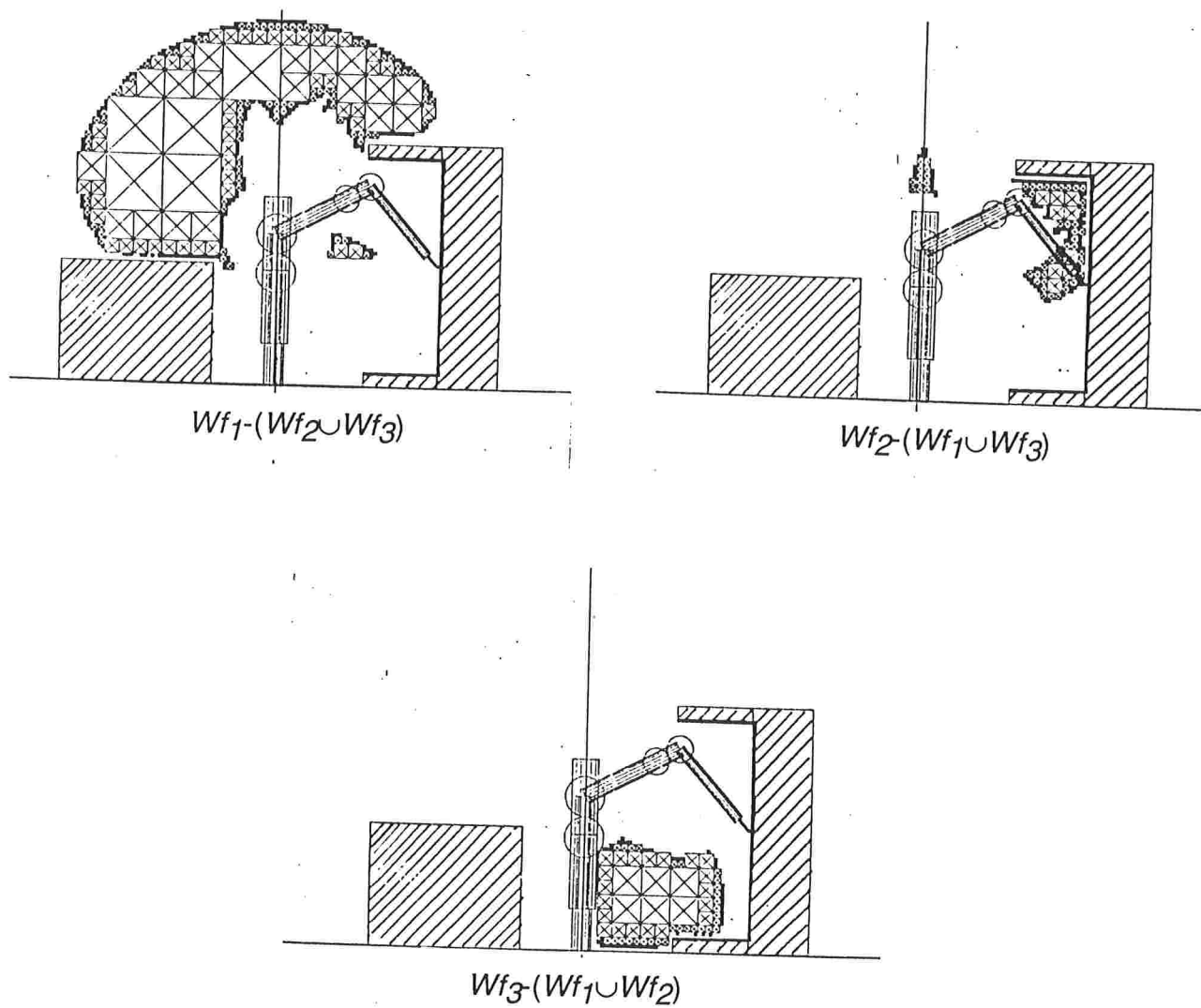


Figure 18 : Maximal t-connected subspaces

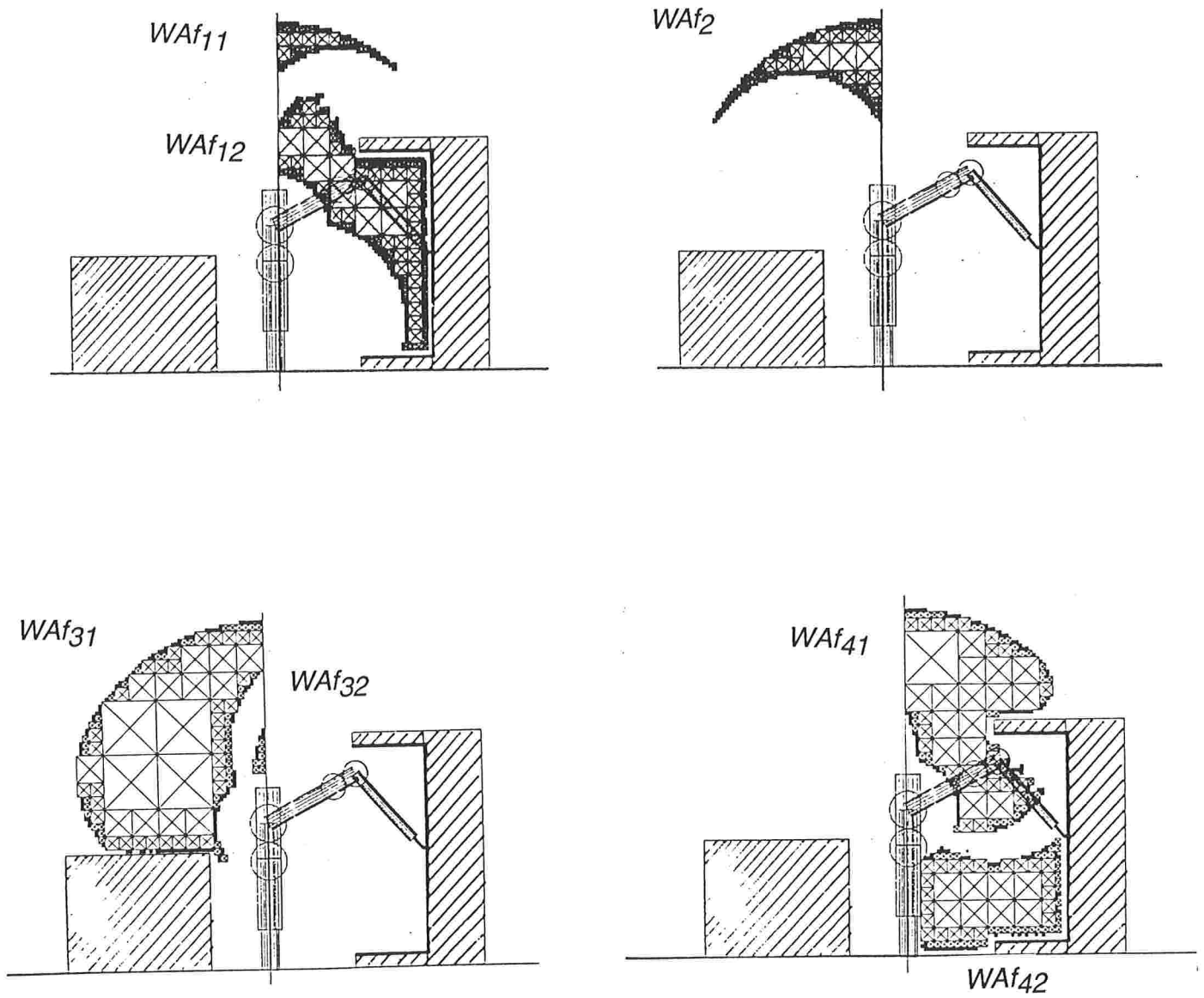


Figure 19 : Maximal t^+ -connected subspaces

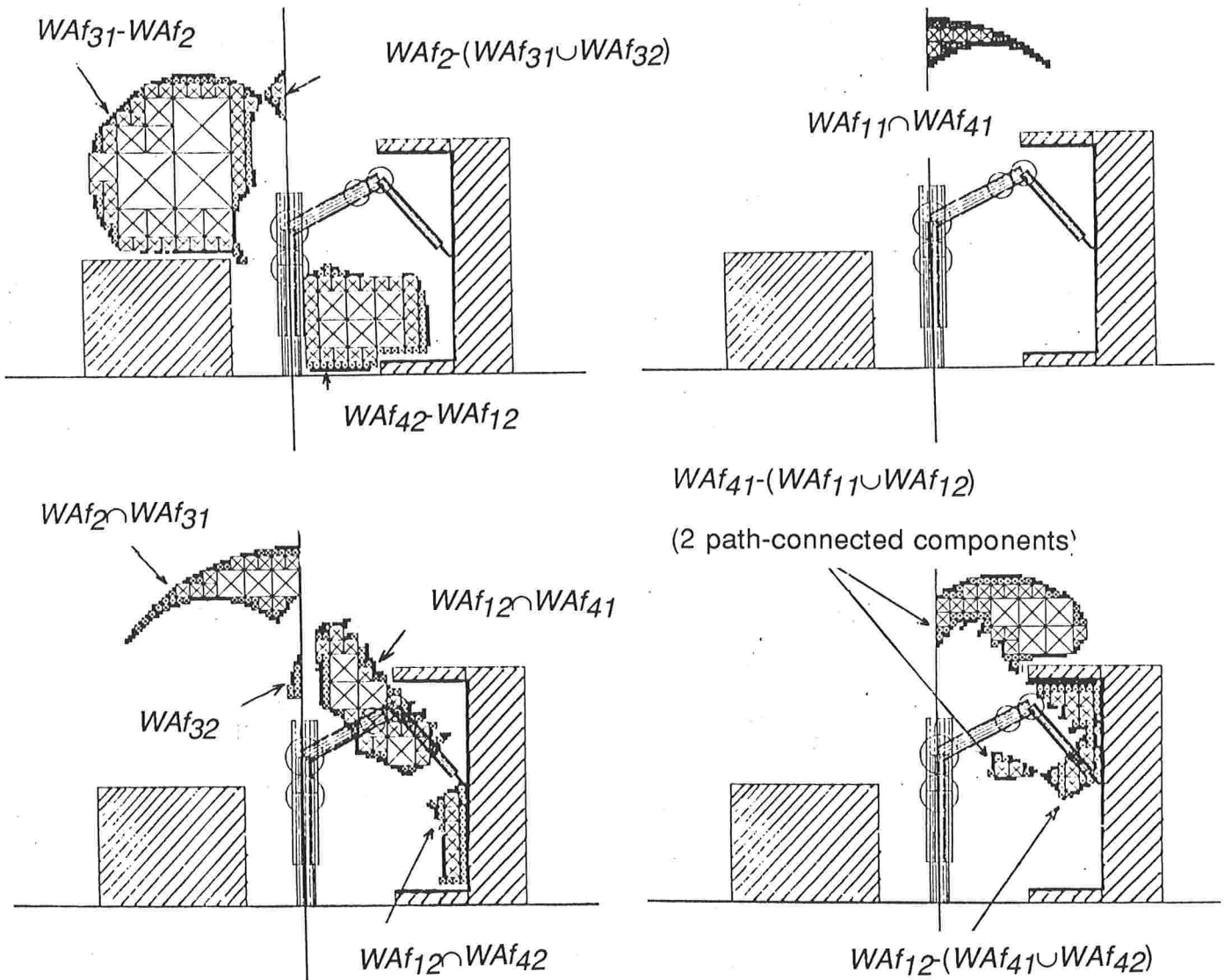


Figure 20 : Using the maximal n-connected subspaces for off-line tasks preparing

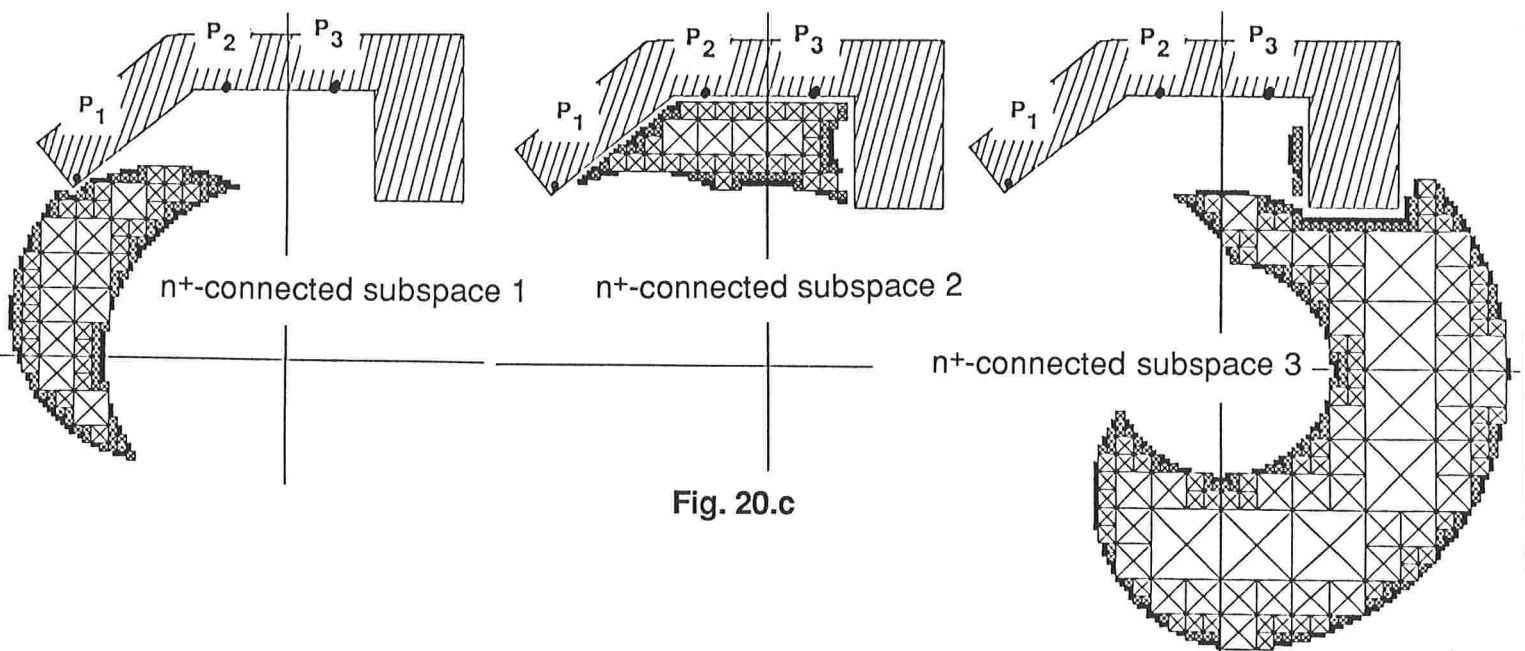
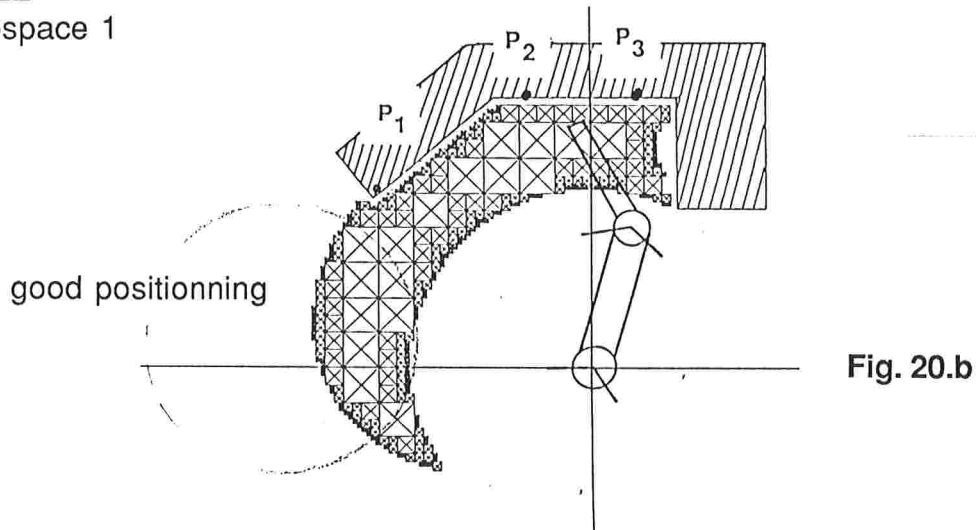
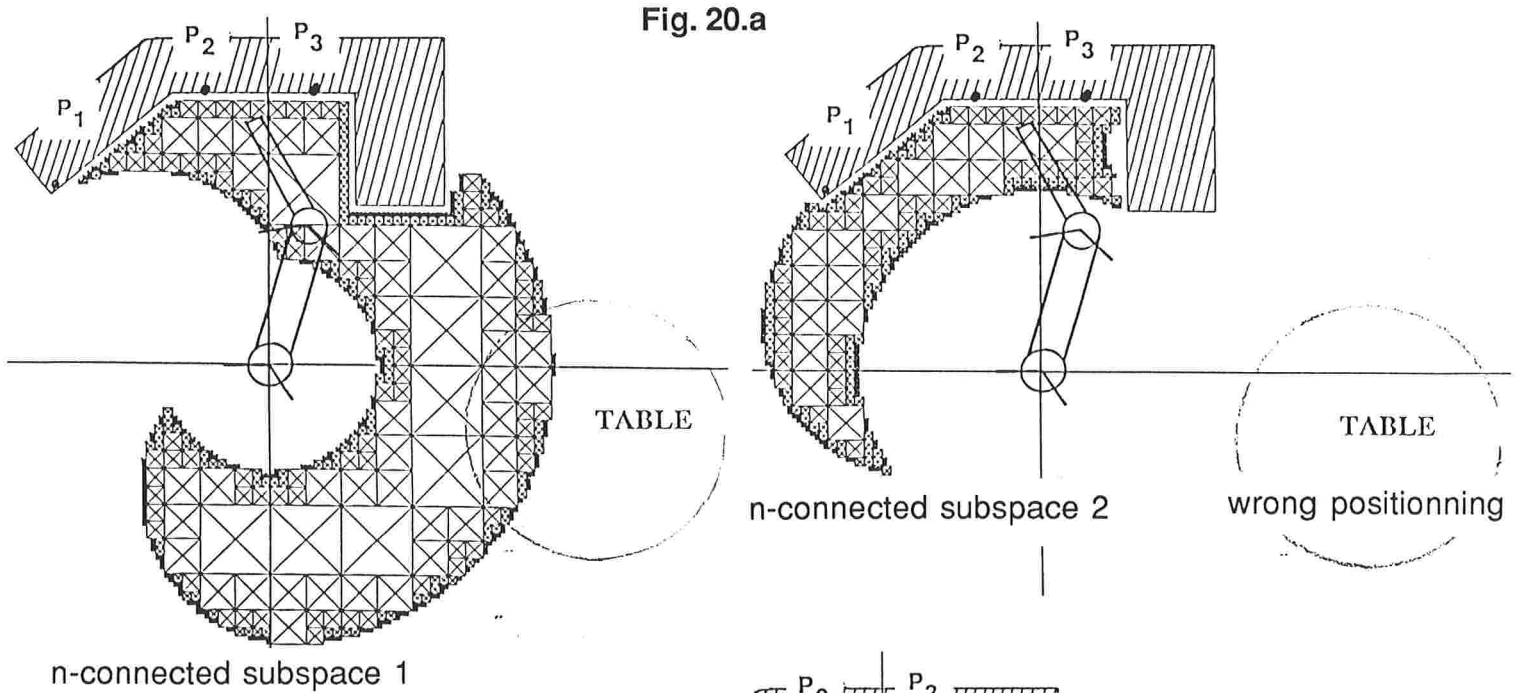


Figure 21 : Using the maximal t-connected subspaces for off-line tasks preparing

