Topological rewriting systems applied to standard bases and syntactic algebras

Cyrille Chenavier

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Abstract

We introduce topological rewriting systems as a generalisation of abstract rewriting systems, where we replace the set of terms by a topological space. Abstract rewriting systems correspond to topological rewriting systems for the discrete topology. We introduce the topological confluence property as an approximation of the confluence property. Using a representation of linear topological rewriting systems with continuous reduction operators, we show that the topological confluence property is characterised by lattice operations. Using this characterisation, we show that standard bases induce topologically confluent rewriting systems on formal power series. Finally, we investigate duality for reduction operators that we relate to series representations and syntactic algebras. In particular, we use duality for proving that an algebra is syntactic or not.

Keywords: topological confluence, standard bases, series representations and syntactic algebras.

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Contents

1. INTRODUCTION

Algebraic rewriting systems are computational models used to deduce algebraic properties through rewriting reasoning. The approach consists in orienting the generating relations in a presentation by generators and relations, *e.g.*, of a monoid, a category, a (commutative, Lie, noncommutative) algebra or an operad, into rewriting rules, and extend them into rewriting steps. The way we extend these rules

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takes into account the underlying algebraic context, but some rewriting properties have a universal formulation, independent of the context. Such properties are termination, that is, there is no infinite sequence of rewriting steps, or confluence, that is, every two rewriting sequences starting at the same term t may be continued until a common target term t', as represented on the following diagram:

$$\begin{tikzcd}
& \cdot \arrow[dr, dashed, bend left, "*"] & \\
t \arrow[ur, bend left, "*"] \arrow[dr, bend right, "*"'] & & t' \\
& \cdot \arrow[ur, dashed, bend right, "*"'] &
\end{tikzcd}$$

The binary relation $\xrightarrow{*}$ denotes rewriting sequences. Under hypotheses of termination and confluence, computing irreducible terms, also called normal forms, has algorithmic applications, for instance to the decision of the word or the ideal membership problems. It also provides effective methods for computing linear bases, Hilbert series, homotopy bases or free resolutions [1, 18, 23], and obtain constructive proofs of coherence theorems, from which we deduce an explicit description of the action of a monoid on a category [15], or of homological properties, such as finite derivation type, finite homological type [19, 30], or Koszulness [27].

When one rewrite terms of linear structures, a relation is usually oriented by rewriting one monomial into the linear combination of other monomials, and there exist three main approaches for selecting the rewritten monomial. The most classical one uses monomial orders and induces a rewriting characterisation of Gröbner bases: they are generating sets of polynomials ideals which induce confluent rewriting systems. As a consequence, effective confluence-based criteria were introduced for checking if a given set is a Gröbner basis or one of its numerous adaptations to different types of algebras or operads [7, 8, 13, 22, 25, 29]. Another approach consists in selecting the reducible monomials with more flexible orders than monomial ones, which may be used for proving Koszulness of algebras for which Gröbner bases give no result [17]. Finally, rewriting steps may be described in a functional manner [6, 16, 21], so that linear rewriting systems are represented by reduction operators. From this approach, the confluence property is characterised by means of lattice operations [10], which provides various applications to computer algebra and homological algebra: construction of Gröbner bases [11], computation of syzygies [12] or proofs of Koszulness [4, 5, 9, 24].

Rewriting methods based on monomial orders were also developed for formal power series, where the leading monomial of a series is the smallest monomial in its decomposition. Standard bases were introduced by Hironaka [20], and are analogous to Gröbner bases: they are generating sets of power series ideals such that their leading monomials generate leading monomials of the ideal. A notable difference is that standard bases are not characterised in terms of the confluence property, which is an obstruction for allowing more flexible orders than monomial ones. For instance, for the deglex order induced by $x > y > z$, the polynomials $z-y,\ z-x,\ y-y^2$ and $x-x^2$ form a standard basis of the ideal they generate in the power series ring $\mathbb{K}[[x, y, z]]$, but they do not induce a confluent rewriting system, as illustrated by the following diagram

$$\begin{tikzcd}
& x \arrow[r] & x^2 \arrow[r] & \cdots \arrow[r] & x^{2n} \arrow[r] & \cdots \\
z \arrow[ur, bend left] \arrow[dr, bend right] & & & & & \\
& y \arrow[r] & y^2 \arrow[r] & \cdots \arrow[r] & y^{2n} \arrow[r] & \cdots
\end{tikzcd}$$

However, this diagram becomes confluent when passing to the limit since the two sequences $(x^{2n})_n$ and $(y^{2n})_n$ both converge to zero for the I-adic topology, where I is the power series ideal generated by $x,\ y$ and z. Note that this asymptotical behaviour of rewriting sequences is also investigated in computer science, for instance in the probabilistic λ-calculus [14].

In the present paper, we introduce a new paradigm of rewriting by considering rewriting systems on topological spaces, from which we take into account the topological properties of rewriting sequences. We also develop the functional approach to rewriting on formal power series. We get the following two applications: we characterise standard bases in terms of a topological confluence property and we introduce a criterion for an algebra to be syntactic.

Topological rewriting systems. We introduce *topological rewriting systems*, which, by definition, are triples $(A, \tau, \to)$, where τ and $\to$ are a topology and a binary relation on A, respectively. We also introduce the *topological confluence* property, meaning that two rewriting sequences starting at the same term t may be continued to reach target terms in arbitrary neighbourhoods of a term t'. Denoting by $\Rightarrow$ the topological closure of $\overset{*}{\to}$ for the product of the discrete topology and τ, the topological confluence property is represented by the following diagram:

$$\begin{tikzcd} & \cdot \arrow[dr, Rightarrow, dashed] & \\ t \arrow[ur, bend left, "*"] \arrow[dr, bend right, "*"'] & & t' \\ & \cdot \arrow[ur, Rightarrow, dashed] & \end{tikzcd}$$

We recover abstract rewriting systems and the usual confluence property when τ is the discrete topology. Notice that in this topological framework, we are not interested in the termination property since we allow confluence "at the limit". Guided by the aforementioned applications of reduction operators to computer algebra and homological algebra, we introduce a topological adaptation of these operators. For topological vector spaces, monomials form a total family, that is, a free family generating a dense subspace, and a reduction operator maps such a monomial into a possibly infinite linear combination of smaller monomials, that is, a formal series. In Theorem 2.2.4, we extend the lattice structure introduced in [10] for the discrete topology to topological vector spaces. From this, we deduce a lattice characterisation of the topological confluence property in Theorem 3.1.7.

Topological confluence for standard bases. We show that standard bases are characterised in terms of the topological confluence property. For that, we first notice that the I-adic topology on formal power series comes from a metric δ, that we recall in Section 4, and we simply say δ-confluence for the topological confluence property associated with this metric. Morever, we show in Proposition 4.1.2 that standard bases are represented by reduction operators which satisfy the lattice criterion of topological confluence proven in Theorem 3.1.7. Thus, denoting by $\to_R$ the rewriting relation eliminating leading monomials of a set R of formal power series, our first main result is stated as follows:

> **Theorem 4.1.3.** *A subset R of $\mathbb{K}[[X]]$ is a standard basis of the ideal it generates if and only if the rewriting relation $\to_R$ is δ-confluent.*

Duality and syntactic algebras. A formal power series uniquely defines a linear form on polynomials. A representation of a series is a quotient of a polynomial algebra which factorises the linear form associated with this series, and there always exists a minimal representation, called the syntactic algebra. An algebra is said to be syntactic if it is the syntactic algebra of a formal power series. When they are noncommutative, these algebras may be thought as a generalisationof automata in the theory of formal languages through the followinggeneralisation of Kleene's Theorem: a formal power series is rational if and only if its syntactic algebra is finite-dimensional [28]. We characterise syntactic algebras in terms of duality for reduction operators. We expect that this characterisation may be used for proving that a series is rational or not. Hence, denoting by $\mathbb{K}\langle X\rangle$ the algebra of noncommutative polynomials over X, $\mathrm{nf}(T)$ the set of normal form monomials for the reduction operator T and by $\widehat{\mathbb{K}\mathrm{nf}(T)}$ the set of formal power series whose nonzero coefficients only involve elements of $\mathrm{nf}(T)$, our second main result is the following:

Theorem 4.2.2. *Let $I \subseteq \mathbb{K}\langle X\rangle$ be an ideal and let T be the reduction operator with kernel I. Then, the algebra $\mathbb{K}\langle X\rangle/I$ is syntactic if and only if there exists $S' \in \widehat{\mathbb{K}\mathrm{nf}(T)}$ such that I is the greatest ideal included in $I \oplus \ker(S')$.*

This is a duality condition since $\widehat{\mathbb{K}\mathrm{nf}(T)}$ is the kernel of the adjoint operator of T. Finally, we illustrate this criterion with examples of syntactic and non-syntactic algebras coming from [26, 28].

Organisation. In Section 2, we introduce topological reduction operators and show that they admit a lattice structure defined in terms of kernels. In Section 3.1, we introduce topological confluence and show that for topological vector spaces, it is characterised in terms of lattice operations. In Section 3.2, we relate representations of formal series to duality of reduction operators. In Section 4, we present two applications of our methods to formal power series. First, we characterise standard bases in terms of topological confluence. Then, we formulate a duality criterion for an algebra to be syntactic, and illustrate it with examples.

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2. TOPOLOGICAL REDUCTION OPERATORS

In this Section, we introduce reduction operators on topological vector spaces and show that they admit a lattice structure.

2.1. Order relation on topological reduction operators

We fix an ordered set $(G, <_{\mathrm{red}})$, a map $d : G \to \mathbb{R}_{>0}$ and a commutative field $\mathbb{K}$, equipped with the discrete topology. Let $\mathbb{K}G$ be the vector space spanned by G. For every $g \in G$, let

$$\pi_g : \mathbb{K}G \to \mathbb{K},$$

be the linear morphism mapping $v \in \mathbb{K}G$ to the coefficient of g in v. We equip $\mathbb{K}G$ with the metric δ defined as follows:

$$\delta(u, v) := \max\{d(g) \mid \pi_g(u - v) \neq 0\}. \tag{1}$$

In particular, for every $g \in G$, we have $d(g) = \delta(0, g)$. Let us denote by $(\widehat{\mathbb{K}G}, \delta)$ the completion of $(\mathbb{K}G, \delta)$. The open ball in $\widehat{\mathbb{K}G}$ of center $v \in \widehat{\mathbb{K}G}$ and radius $\epsilon > 0$ is written $\mathrm{B}(v, \epsilon)$, and the topological closure of a subset V in $\widehat{\mathbb{K}G}$ is written $\overline{V}$.

Before introducing reduction operators in Definition 2.1.6, we establish some topological properties of $\widehat{\mathbb{K}G}$, which do not depend on $<_{\mathrm{red}}$.

Proposition 2.1.1. *The metric spaces $(\mathbb{K}G, \delta)$ and $(\widehat{\mathbb{K}G}, \delta)$ are topological vector spaces. For every $g \in G$, the morphism π_g is continuous.*

Proof. For the first part of the Proposition, it is sufficient to show that $(\mathbb{K}G, \delta)$ is a topological vector space. Let $\lambda \in \mathbb{K}$, $v \in \mathbb{K}G$ and U a neighbourhood of λv, so that there exists $\epsilon > 0$ such that $\delta(\lambda v, u) < \epsilon$ implies $u \in U$. Using $\delta(\mu v_1, \mu v_2) \leq \delta(v_1, v_2)$ for every $v_1, v_2 \in \mathbb{K}G$ and $\mu \in \mathbb{K}$, $\{\lambda\} \times \mathrm{B}(v, \epsilon)$ is an open neighbourhood of (λ, v) in the inverse image of U through the scalar multiplication $(\lambda, v) \mapsto \lambda v$. Hence, the latter is continuous. Let $(v_1, v_2) \in \mathbb{K}G$ and U be a neighbourhood of $v_1 + v_2$, so that there exists $\epsilon > 0$ such that $\delta(v_1 + v_2, v) < \epsilon$ implies $v \in U$. Using $\delta(u_1 + u_2, u_1' + u_2') \geq \max(\delta(u_1, u_1'), \delta(u_2, u_2'))$, $\mathrm{B}(v_1, \epsilon) \times \mathrm{B}(v_2, \epsilon)$ is an open neighbourhood of (v_1, v_2) in the inverse image of U through the addition $(v_1 + v_2) \mapsto v_1 + v_2$. Hence, the latter is continuous.

For the second assertion, it is sufficient to show that π_g is continuous at 0, that is, $\pi_g^{-1}(0)$ is open. This is true since for $v \in \pi_g^{-1}(0)$, $\mathrm{B}(v, d(g))$ is included in $\pi_g^{-1}(0)$.

□

By density of $\mathbb{K}G$ in $\widehat{\mathbb{K}G}$, the continuous morphism π_g induces a continuous morphism, still written π_g, from $\widehat{\mathbb{K}G}$ to $\mathbb{K}$.

Definition 2.1.2. The *support* of $v \in \widehat{\mathbb{K}G}$ is the set $\mathrm{supp}(v) := \{g \in G \mid \pi_g(v) \neq 0\}$.

Notice that π_g being continuous and $\mathbb{K}$ being equipped with the discrete topology, we have $\pi_g(v) = \pi_g(u)$ for every $u \in \mathbb{K}G$ such that $\delta(u, v)$ is small enough.

In Formula (4), we describe elements of $\widehat{\mathbb{K}G}$ in terms of formal series. For that, we need preliminary results on supports that we present in Lemma 2.1.3. In the latter, we use the following notation: for $v \in \widehat{\mathbb{K}G}$ and $\epsilon > 0$, we let

$$\mathrm{supp}(v)_{\geq \epsilon} := \{g \in \mathrm{supp}(v) \mid d(g) \geq \epsilon\}. \tag{2}$$

Lemma 2.1.3. *For every $u, v \in \widehat{\mathbb{K}G}$, $\mathrm{supp}(u+v)$ is included in $\mathrm{supp}(u) \cup \mathrm{supp}(v)$. Moreover, the sets $\mathrm{supp}(v)_{\geq \epsilon}$ are finite and $\mathrm{supp}(v)$ is countable.*

Proof. Let us show the first assertion. Let g in the complement of $\mathrm{supp}(v_1) \cup \mathrm{supp}(v_2)$ in G and let $(u_n)_n$ and $(v_n)_n$ be sequences in $\mathbb{K}G$ converging to u and v, respectively. For n large enough, $\pi_g(u_n)$ and $\pi_g(v_n)$ are equal to 0, so that $\pi_g(u + v) = 0$.

Let us show the second assertion. For every $\epsilon > 0$, there exists $u \in \mathbb{K}G$ such that $\delta(u, v) < \epsilon$, that is, $d(g) < \epsilon$ for every $g \in \mathrm{supp}(v - u)$. From the first part of the Lemma, $\mathrm{supp}(v)_{\geq \epsilon}$ is included in the finite set $\mathrm{supp}(u)$. Moreover, $\mathrm{supp}(v)$ is the countable union of the finite sets $\mathrm{supp}(v)_{\geq 1/n}$, where $n \in \mathbb{N}$, so that it is countable.

□

For every strictly positive integer n, let $G^{(n)} := \{g \in G \mid 1/n \leq d(g) < 1/(n-1)\}$, and for every $v \in \widehat{\mathbb{K}G}$, let

$$v_n := \sum_{g \in V_n} \pi_g(v) g \in \mathbb{K}G, \tag{3}$$

where $V_n := \mathrm{supp}(v) \cap G^{(n)}$, that is, $V_n = \{g \in \mathrm{supp}(v) \mid 1/n \leq d(g) < 1/(n-1)\}$. The sequence of partial sums $(v_1 + \cdots + v_n)_n$ converges to v and the sets V_n form a partition of $\mathrm{supp}(v)$. Hence, we may identify v with the following formal series:

$$v = \sum_{n \in \mathbb{N}} v_n = \sum_{g \in \mathrm{supp}(v)} \pi_g(v) g. \tag{4}$$

In the following Proposition, we present a necessary and sufficient condition such that $\mathbb{K}G$ is a complete metric space.

Proposition 2.1.4. *We have $\widehat{\mathbb{K}G} = \mathbb{K}G$ if and only if $0 \in \mathbb{R}$ does not belong to the topological closure of $\mathrm{im}(d)$ in $\mathbb{R}$.*

Proof. We have $0 \notin \overline{\mathrm{im}(d)}$ if and only if there exists $n > 0$ such that $d(g) \geq 1/n$, for every $g \in G$. With the notations of (4), that means that each $v \in \widehat{\mathbb{K}G}$ is equal to $v_1 + \cdots + v_n$, that is, $v \in \mathbb{K}G$.

□

Example 2.1.5. Let X be a finite set of indeterminates and let G be the set of (non)commutative monomials over X, so that $\mathbb{K}G$ is isomorphic to the free (non)commutative polynomial algebra over X. If d is constant equal to 1, then the metric δ of (1) satisfies $\delta(f, g) = 1$ whenever $f \neq g$, so that $\widehat{\mathbb{K}G}$ is the (non)commutative polynomial algebra equipped with the discrete topology. If $d(m) = 1/2^n$ for every (non)commutative monomial of degree n, then $\widehat{\mathbb{K}G}$ is the set of (non)commutative formal power series over X and δ is the metric of the I-adic topology, where I is the two-sided power series ideal generated by X. In Section 4, we investigate this topological vector space and rewriting systems on it in more details.

Now, we introduce reduction operators and present some of their basic properties.

Definition 2.1.6. A *reduction operator* relative to $(G, <_{\text{red}}, d)$ is a continuous linear projector of $\widehat{\mathbb{K}G}$ such that for every $g \in G$, $T(g) \neq g$ implies $g' <_{\text{red}} g$, for every $g' \in \text{supp}(T(g))$.

The set of reduction operators is written $\mathbf{RO}\,(G, <_{\text{red}}, d)$ and for $T \in \mathbf{RO}\,(G, <_{\text{red}}, d)$, we let

$$\text{nf}(T) := \{g \in G \mid T(g) = g\} \quad \text{and} \quad \text{red}(T) := \{g \in G \mid T(g) \neq g\}.$$

The element $g \in G$ is called a *T-normal form* if $g \in \text{nf}(T)$ or *T-reducible* if $g \in \text{red}(T)$.

Proposition 2.1.7. *Let $T \in \mathbf{RO}\,(G, <_{\text{red}}, d)$. The subspaces $\text{im}(T)$ and $\ker(T)$ are the closed subspaces spanned by $\text{nf}(T)$ and $\{g - T(g) \mid g \in \text{red}(T)\}$, respectively.*

Proof. The set $\text{nf}(T)$ is included in $\text{im}(T)$ and the latter is closed since it is the inverse image of $\{0\}$ by the continuous map $\text{id}_V - T$. Hence, the closure $\overline{\mathbb{K}\text{nf}(T)}$ of $\mathbb{K}\text{nf}(T)$ is included in $\text{im}(T)$. In the same manner, we show that $\overline{\mathbb{K}\{g - T(g) \mid g \in \text{red}(T)\}} \subseteq \ker(T)$.

Let us show the converse inclusions. Let $v \in \widehat{\mathbb{K}G}$ and $(v_n)_n$ be a sequence in $\mathbb{K}G$ converging to v. For every $n \in \mathbb{N}$, there exist $u_n \in \mathbb{K}\text{nf}(T)$ and $w_n \in \mathbb{K}\text{red}(T)$ such that $v_n = u_n + w_n$. By continuity and linearity of T, $T(v)$ is the limit of the sequence $(z_n)_n$, where $z_n := u_n + T(w_n)$. The latter belongs to $\mathbb{K}\text{nf}(T)$. If $v \in \text{im}(T)$, we have $v = T(v)$, so that $v = \lim(z_n)$ belongs to $\overline{\mathbb{K}\text{nf}(T)}$. If $v \in \ker(T)$, we have $v = v - T(v)$, so that $v = \lim(w_n - T(w_n))$ belongs to $\overline{\mathbb{K}\{g - T(g) \mid g \in \text{red}(T)\}}$. □

The following Lemma is used to prove Proposition 2.1.9, where we prove that reduction operators form a poset.

Lemma 2.1.8. *Let $T,\ T' \in \mathbf{RO}\,(G, <_{\text{red}}, d)$, $g \in G$ and $v \in \widehat{\mathbb{K}G}$.*

1. *If for every $g' \in \text{supp}(v)$, we have $g' < g$, then g does not belong to $\text{supp}(T(v))$.*
2. *If $\ker(T) \subseteq \ker(T')$, then $\text{nf}(T') \subseteq \text{nf}(T)$.*

Proof. First, we show Point 1. If $v \in \mathbb{K}G$, the result is a consequence of Lemma 2.1.3. If v does no belong to $\mathbb{K}G$, let v_n as in (3), so that v is the limit of $(w_n)_n$, where $w_n := v_1 + \cdots + v_n$. Each $g' \in \text{supp}(w_n)$ is strictly smaller than g, so that $g \notin \text{supp}(T(w_n))$. By continuity, $(T(w_n))_n$ converges to $T(v)$, so that $g \notin \text{supp}(T(v))$.

Let us show Point 2 by contrapositive. Let us assume that there exists $g \in \text{nf}(T')$ such that $g \notin \text{nf}(T)$. The vector $v := g - T(g)$ belongs to $\ker(T)$ but does not belong to $\ker(T')$: otherwise, we would have $g = T'(T(g))$, which is not possible from Point 1. □

Proposition 2.1.9. *The binary relation $\preceq$ on $\mathbf{RO}\,(G, <_{\text{red}}, d)$ defined by $T \preceq T'$ if $\ker(T') \subseteq \ker(T)$ is an order relation.*

Proof. The relation $\preceq$ is reflexive and transitive. Moreover, $\ker(T) = \ker(T')$ implies $\text{nf}(T) = \text{nf}(T')$ from Point 2 of Lemma 2.1.8. From Proposition 2.1.7, $\text{nf}(T) = \text{nf}(T')$ implies $\text{im}(T) = \text{im}(T')$. Hence, T and T' are two projectors with same kernels and images, so that they are equal and $\preceq$ is antisymmetric. □

2.2. Elimination maps

From Propositions 2.1.7 and 2.1.9, the kernel map induces an injection of $\mathbf{RO}\,(G, <_{\text{red}}, d)$ into closed subspaces of $\widehat{\mathbb{K}G}$. In this Section, we introduce a sufficient condition such that this injection is surjective. Moreover, we deduce a lattice structure on reduction operators in Theorem 2.2.4.

Throughout the Section, we assume that $<_{\mathrm{red}}$ is a total order and d is an elimination map, where this notion is introduced in the following definition. Before, recall that $G^{(n)}$ denotes elements $g \in G$ such that $1/n \leq d(g) < 1/(n-1)$.

Definition 2.2.1. We say that d is an *elimination map* with respect to $(G, <_{\mathrm{red}})$ if it is non-decreasing and if the sets $G^{(n)}$ equipped with the order induced by $<_{\mathrm{red}}$ are well-ordered sets.

Under these hypotheses, we may introduce the notion of leading monomial. The order $<_{\mathrm{red}}$ being total, for every $v \in \mathbb{K}G$, there exists a greatest element in the support of v, written $\max(\mathrm{supp}(v))$. For $v \in \widehat{\mathbb{K}G}$, we let $v = \sum v_n$ as in (4) and we denote by n_0 the smallest n such that $v_n \neq 0$. The map d being non-decreasing, $\max(\mathrm{supp}(v_{n_0}))$ is the greatest element of $\mathrm{supp}(v)$. We let

$$\mathrm{lm}(v) := \max(\mathrm{supp}(v_{n_0})) \quad \text{and} \quad \mathrm{lc}(v) = \pi_{\mathrm{lm}(v)}(v).$$

The elements $\mathrm{lm}(v) \in G$ and $\mathrm{lc}(v) \in \mathbb{K}$ are respectively called the *leading monomial* and the *leading coefficient* of v.

The construction of the inverse of ker is based on a property of leading monomials of closed subspaces presented in Proposition 2.2.3. In order to show the latter, we need the following intermediate lemma.

Lemma 2.2.2. *Let V be a subspace of $\widehat{\mathbb{K}G}$ and let n be an integer. There exists a family $(v_g)_{g \in G^{(n)}} \subset V$ such that the following hold:*

- $v_g = 0$ *if and only if there is no $v \in V$ such that* $\mathrm{lm}(v) = g$*;*
- *if $v_g \neq 0$, then* $\mathrm{lm}(v_g) = g$, $\mathrm{lc}(v_g) = 1$ *and for $g' \neq g$ in $G^{(n)}$, $g \notin \mathrm{supp}(v_{g'})$.*

Proof. We proceed by induction along the well-founded order $<_{\mathrm{red}}$: assume that for every $g' <_{\mathrm{red}} g$ in $G^{(n)}$, $v_{g'}$ is constructed. If there is no $v \in V$ such that $\mathrm{lm}(v) = g$, we let $v_g = 0$. Else, we choose such a v with $\mathrm{lc}(v) = 1$ and we let $v_g = v - \sum \pi_{g'}(v) v_{g'}$, where the sum is taken over $g' \in G^{(n)}$ such that $g' <_{\mathrm{red}} g$.

□

In the statement of Proposition 2.2.3, we use the notion of a *total basis* of a topological vector space V, that is, a free family which spans a dense subspace of V.

Proposition 2.2.3. *Let V be a subspace of $\widehat{\mathbb{K}G}$. There exists a family $(v_g)_{g \in G}$ of $\overline{V}$ such that the following hold:*

- $v_g = 0$ *if and only if there is no $v \in V$ such that* $\mathrm{lm}(v) = g$*;*
- *if $v_g \neq 0$, then* $\mathrm{lm}(v_g) = g$, $\mathrm{lc}(v_g) = 1$ *and $g \notin \mathrm{supp}(v_{g'})$ for every $g' \neq g$.*

In particular, nonzero elements of this family form a total basis of $\overline{V}$.

Proof. For every $g \in G$, let us define by induction the sequence $(v_g^n)_n$ as follows: $v_g^n = 0$ if $d(g) < 1/n$, v_g^n is chosen as in Lemma 2.2.2 if $g \in G^{(n)}$, that is, if $1/n \leq d(g) < 1/(n-1)$, and

$$v_g^n = v_g^{n-1} - \sum_{g' \in G^{(n)}} \pi_{g'}(v_g^{n-1}) v_{g'}^n,$$

if $d(g) \geq 1/(n-1)$. For every n, m and g, $\mathrm{lm}(v_g^n - v_g^m)$ belongs to $G^{(k)}$, for $k \geq \min(n, m)$. Hence, we have $\delta(v_g^n, v_g^m) < 1/\min(n, m)$, so that $(v_g^n)_n$ is Cauchy and converges to $v_g \in \overline{V}$. By construction, $(v_g)_g$ satisfies the first two properties stated in the Proposition. Let us show that its nonzero elements form a total basis. They form a free family since the leading monomials of its elements are pairwise distinct. Moreover, this family is dense since for every $v \in \overline{V}$ and every $\epsilon > 0$, the element

$$v_\epsilon := \sum_{g \in \mathrm{supp}(v)_{\geq \epsilon}} \pi_g(v) v_g,$$

$\mathrm{supp}(v)_{\geq \epsilon}$ being defined as in (2), belongs to $\mathrm{B}(v, \epsilon)$. Indeed, assume by contradiction that there exists $g' \in \mathrm{supp}(v - v_\epsilon)$ such that $d(g') \geq \epsilon$. There exists $g \in \mathrm{supp}(v)_{\geq \epsilon}$ such that $g' \in \mathrm{supp}(v_g)$ but $g' \neq g$. The maximal g' for $<_{\mathrm{red}}$ is the leading monomial of $v - v_\epsilon \in V$, so that $v_{g'} \neq 0$. This is a contradiction since this condition implies $g' \notin \mathrm{supp}(v_g)$.

□

With the notations of the previous Proposition, the elements v_g are limits of sequences $(v_g^n)_n$ of elements of V such that $\mathrm{lm}(v_g^n) = g$. The family of nonzero v_g's being total, for every $v \in \overline{V}$, $\mathrm{lm}(v) = g$ for some v_g, so that it belongs to $\mathrm{lm}(V) := \{\mathrm{lm}(u) \mid u \in V\}$. Hence, the following formula holds:

$$\mathrm{lm}(V) = \mathrm{lm}(\overline{V}). \tag{5}$$

We can now introduce the main result of this Section.

Theorem 2.2.4. *Assume that $<_{\mathrm{red}}$ is a total order and that d is an elimination map with respect to $(G, <_{\mathrm{red}})$. The kernel map induces a bijection between $\mathbf{RO}\,(G, <_{\mathrm{red}}, d)$ and closed subspaces of $\widehat{\mathbb{K}G}$. In particular, $\mathbf{RO}\,(G, <_{\mathrm{red}}, d)$ admits lattice operations $(\preceq, \wedge, \vee)$, defined by*

- $T_1 \preceq T_2$ *if* $\ker(T_2) \subseteq \ker(T_1)$*;*
- $T_1 \wedge T_2 := \ \ker^{-1}\left(\overline{\ker(T_1) + \ker(T_2)}\right)$*;*
- $T_1 \vee T_2 := \ker^{-1}\left(\ker(T_1) \cap \ker(T_2)\right)$.

Proof. It is sufficient to show that ker is surjective. Consider a closed subspace V of $\widehat{\mathbb{K}G}$ and let $(v_g)_g$ as in Proposition 2.2.3. Consider the linear map $T : \widehat{\mathbb{K}G} \to \widehat{\mathbb{K}G}$ defined by $T(g) = v_g - g$ if $v_g \neq 0$ and $T(g) = g$, otherwise. This map defines a linear projector, compatible with $<_{\mathrm{red}}$ and is continuous at 0 since for every $v \in \widehat{\mathbb{K}G}$, we have $\delta(T(v), 0) \leq \delta(v, 0)$. Hence, T is continuous, so that it is a reduction operator. Moreover, the set of nonzero v_g's is a total basis of $\ker(T)$ and V from Propositions 2.1.7 and 2.2.3, so that we have $\ker(T) = V$.

□

Example 2.2.5. We assume that 0 does not belong to the closure of $\mathrm{im}(d)$, for instance, d is constant equal to 1. We fix a well-order $<$ on G and we choose $<_{\mathrm{red}}$ equal to $<$, so that d is an elimination map. Moreover, the topology induced by δ is the discrete topology, so that every subspace is closed. Hence, we recover a result from [10]: ker induces a bijection between subspaces of $\mathbb{K}G$ and reduction operators. The interpretations of the lattice operations on $\mathbf{RO}\,(G, <_{\mathrm{red}}, d)$ were given in [10, Proposition 2.3.6] and [12, Proposition 2.1.4]: $T_1 \wedge T_2$ characterises the equivalence relation on $\mathbb{K}G$ induced by the binary relation $v \to T_i(v)$, $v \in \mathbb{K}G$ and $i = 1, 2$, and $\ker(T_1 \vee T_2)$ is isomorphic to the space of syzygies for T_1 and T_2.

Example 2.2.6. Assume that G is infinite and equipped with a well-order $<$ such that d is strictly decreasing. In particular, 0 belongs to the closure of $\mathrm{im}(d)$. For $<_{\mathrm{red}}$, we choose the opposite order of $<$, that is, $g <_{\mathrm{red}} g'$, whenever $g' < g$. Hence, d is an elimination map, $\widehat{\mathbb{K}G}$ is the set of formal series as in (4), where the sum may be infinite, and ker induces a bijection between closed subspaces of formal series and reduction operators. We point out that there exist subspaces which are not closed, as illustrated in the following Proposition.

Proposition 2.2.7. *Let $G := \{g_1 < g_2 < \cdots\}$ be a countable well-ordered set and $d : G \to \mathbb{R}_{>0}$, defined by $d(g_n) = 1/n$. The subspace $V := \mathbb{K}\{g_n - g_{n+1} \mid n \geq 1\} \subset \widehat{\mathbb{K}G}$ is dense and different from $\widehat{\mathbb{K}G}$.*

Proof. For every $n \geq 1$, g_n does not belong to V but is equal to $\lim(g_n - g_k)_k$.

□

3. CONFLUENCE AND DUALITY

In this Section, we investigate the rewriting and duality properties of reduction operators.

3.1. Topological confluence

Throughout this Section, we fix a set $F \subseteq \mathbf{RO}\,(G, <_{\mathrm{red}}, d)$, where d is an elimination map with respect to $(G, <_{\mathrm{red}})$. Our objective is to introduce a confluence-like property for F. First, we recall from [2] classical notions of rewriting theory and introduce a topological adaptation of rewriting systems.

An *abstract rewriting system* is a pair $(A, \to)$, where A is a set and $\to$ is a binary relation on A. We write $a \to b$ instead of $(a, b) \in\to$. We denote by $\leftrightarrow$ and $\overset{*}{\to}$ the symmetric and the reflexive transitive closures of $\to$, respectively. Hence, $\overset{*}{\leftrightarrow}$ is the reflexive transitive symmetric closure of $\to$. If $a \overset{*}{\to} b$, we say that a *rewrites* into b. We say that a and b are *joinable* if there exists c such that both a and b rewrite into c. We say that $\to$ is *confluent* if whenever a rewrites into b and c, then b and c are joinable.

A *topological rewriting system* is a triple $(A, \tau, \to)$, where (A, τ) is a topological space and $\to$ is a rewriting relation on A. We denote by $\Rightarrow$ the topological closure of $\to$ for the product topology $\tau_A^{\mathrm{d}} \times \tau$, where τ_A^{d} is the discrete topology on A. In other words, we have $a \Rightarrow b$ if and only if every neighbourhood V of b contains b' such that a rewrites into b'. By this approach, we formulate the following topological notion of confluence.

Definition 3.1.1. The rewriting relation $\to$ is τ*-confluent* if whenever a rewrites into b and c, then there exists d such that $b \Rightarrow d$ and $c \Rightarrow d$.

Notice that if $\tau = \tau_A^{\mathrm{d}}$, then τ-confluence is equivalent to confluence.

We equip $\widehat{\mathbb{K}G}$ with the topology induced by δ and we associate to F the topological rewriting system $(\widehat{\mathbb{K}G}, \delta, \to_F)$ defined by

$$v \to_F T(v), \ \forall v \in \widehat{\mathbb{K}G}, \ T \in F.$$

Moreover, we consider the operator $\wedge F \in \mathbf{RO}\,(G, <_{\mathrm{red}}, d)$ and the set $\mathrm{nf}(F) \subseteq G$ defined as follows:

$$\wedge F := \ker^{-1}\left(\overline{\sum_{T \in F} \ker(T)}\right), \ \mathrm{nf}(F) := \bigcap_{T \in F} \mathrm{nf}(T).$$

From Point 2 of Lemma 2.1.8, $\mathrm{nf}(\wedge F)$ is included in $\mathrm{nf}(F)$, and we let

$$\mathrm{obs}(F) := \mathrm{nf}(F) \setminus \mathrm{nf}(\wedge F).$$

The elements of $\mathrm{obs}(F)$ are called the *obstructions* of F.

Definition 3.1.2. A set $F \subseteq \mathbf{RO}(G,\ <_{\mathrm{red}},\ d)$ is said to be *confluent* if $\mathrm{obs}(F) = \emptyset$.

In Theorem 3.1.7, we show that F is confluent if and only if it induces a δ-confluent rewriting relation. For that, we need the following intermediate definition and results.

Definition 3.1.3. Let $v \in \ker(\wedge F)$. A decomposition $v = \sum_{i=0}^{n} \lambda_i(g_i - T_i(g_i)) + r$, where $\lambda_i \neq 0$, $T_i \in F$ and $r \in \ker(\wedge F)$, of v is said to be *admissible* if $g_i \leq \mathrm{lm}(v)$, for every $1 \leq i \leq n$, and $\mathrm{lm}(r) < \mathrm{lm}(v)$.

The following lemma establishes the link between the confluence property and reduction of S-polynomials into zero in our framework.

Lemma 3.1.4. *If $\to_F$ is δ-confluent, then for $g \in G$ and $T, T' \in F$, $(T - T')(g) \Rightarrow_F 0$ and $(T - T')(g)$ admits an admissible decomposition.*

Proof. Let us show the first part of the lemma. Assume by induction that for every $n \geq 0$, a finite composition R_n of elements of F has been constructed such that $\delta(R_n(T - T')(g), 0) < 1/n$. By induction on the well-ordered set $G^{(n+1)}$, there exists a finite composition R of elements of F such that

$$\mathrm{supp}(R \circ R_n(T(g))_{\geq 1/(n+1)} \cap \mathrm{supp}(R \circ R_n(T'(g))_{\geq 1/(n+1)} \subseteq \mathrm{nf}(F). \tag{6}$$

Letting $R_{n+1} := R \circ R_n$, g rewrites into $v := R_{n+1}(T(g))$ and $v' := R_{n+1}(T'(g))$. By δ-confluence, there exists w such that $v \Rightarrow w$ and $v' \Rightarrow w$, and from (6), each $g' \in G$ with $d(g') \geq n+1$ of $\mathrm{supp}(v)$ and $\mathrm{supp}(v')$ are normal forms. Hence, $\pi_{g'}(v) = \pi_{g'}(v')$, so that $\delta(R_n(T - T')(g), 0) < 1/(n+1)$. Hence, $(T - T')(g) \Rightarrow_F 0$.

The second part of the lemma is a consequence of the the first one and the following fact: if $v \Rightarrow_F 0$, then v admits an admissible decomposition. Indeed, for every n, we let $v = (v - R_n(v)) + R_n(v)$, where R_n is a finite composition of elements of F such that $\delta(R_n(v), 0) < 1/n$. Then, $v - R_n(v)$ is a linear combination of elements of the form $g - T(g)$, with $g \leq \mathrm{lm}(v)$. Moreover, for n large enough, we have $\mathrm{lm}(R_n(v)) < \mathrm{lm}(v)$, so that v admits an admissible decomposition. ☐

Proposition 3.1.5. *If $\to_F$ is δ-confluent, then every $v \in \ker(\wedge F)$ admits an admissible decomposition.*

Proof. We adapt the proof of [3, Lemma 4.2] to our situation.

By density of $\sum_{T \in F} \ker(T)$ in $\ker(\wedge F)$, v admits a decomposition $v = \sum_{i \geq 1}^{n} \lambda_i(g_i - T_i(g_i)) + r$, such that $\mathrm{lm}(r) < \mathrm{lm}(v)$. Without lose of generality, assume that $(g_i)_i$ is not increasing. If $g_1 \leq \mathrm{lm}(v)$, then the chosen decomposition is admissible. Otherwise, we show by induction on the greatest $k \geq 2$ such that $g_k = g_1$, that v admits another decomposition

$$\sum_{i=1}^{n'} \lambda'_i(g'_i - T'_i(g'_i)) + r', \ g'_i \geq g'_{i+1}, \ g'_1 < g_1, \ \mathrm{lm}(r') < \mathrm{lm}(v). \tag{7}$$

If $k = 2$, we have $\lambda_2 = -\lambda_1$, so that $v = \lambda_1(T_2 - T_1)(g_1) + \sum_{i=3}^{n} \lambda_i(g_i - T_i(g_i)) + r$. From Lemma 3.1.4, $(T_1 - T_2)(g_1)$ admits an admissible decomposition $\sum_{i=1}^{n'} \lambda'_i(g'_i - T'_i(g'_i)) + r'$. In particular, we have $g'_1 < g_1$, so that $\sum_{i=1}^{n'} \lambda'_i(g'_i - T'_i(g'_i)) + \left(\sum_{i=3}^{n} \lambda_i(g_i - T_i(g_i)) + r + r'\right)$ is a decomposition of v such as in (7). If $k \geq 3$, we write

$$v = \lambda_1(T_2 - T_1)(g_1) + \left((\lambda_1 + \lambda_2)(g_1 - T_2(g_1)) + \sum_{i=3}^{k} \lambda_i(g_1 - T_i(g_1)) + \sum_{i=k+1}^{n} \lambda_i(g_i - T_i(g_i)) + r\right).$$

The first term of the sum admits an admissible decomposition from Lemma 3.1.4. Moreover, the element $v' := v - \lambda_1(T_2 - T_1)(g_1)$ admits a decomposition such as in (7) by induction hypothesis. The sum of these two decomposition gives a decomposition of v such as in (7). By applying inductively the decomposition (7) to v, we deduce that it admits an admissible decomposition. ☐

Proposition 3.1.6. *Let $v, v' \in \widehat{\mathbb{K}G}$.*

1. *$v - v' \in \ker(\wedge F)$ if and only if for every $\epsilon > 0$, there exists $v_\epsilon \in \mathrm{B}(0, \epsilon)$ such that $v \overset{*}{\leftrightarrow}_F v' + v_\epsilon$.*
2. *If F is confluent, then $v \Rightarrow_F (\wedge F)(v)$.*

Proof. Let us show Point 1. First, assume that $v_\epsilon \in \mathrm{B}(0,\epsilon)$ exists such that $v \overset{*}{\leftrightarrow}_F v' + v_\epsilon$. By definition and by continuity of $\wedge F$, $v - v' - v_\epsilon$ belongs to its kernel and $(\wedge F)(v_\epsilon)$ goes to zero when ϵ does. Hence, $v - v' \in \ker(\wedge F)$. Conversely, $v - v' \in \ker(\wedge F)$ implies that $v = \sum_{i\geq 1} \lambda_i(g_i - T_i(g_i)) + v'$, where the sum maybe infinite, $T_i \in F$ and the sequence $(d(g_i))_i$ goes to 0. For every $k \geq 1$, letting $v_k := \sum_{i\geq k} \lambda_i(g_i - T_i(g_i))$, we have $T_k(v_k + v') = T_k(v_{k+1} + v')$, so that $v \overset{*}{\leftrightarrow}_F v_k + v'$. The sequence $(v_k)_k$ goes to zero, which shows the direct implication of Point 1.

Let us show Point 2. By induction, assume that a finite composition R_n of elements of F has been constructed such that $\mathrm{supp}(R_n(v)) \cap G^{(k)}$ is included in $\mathrm{nf}(\wedge F)$, for every $k < n$. By confluence of F, $\mathrm{red}(\wedge F)$ is the union of the sets $\mathrm{red}(T)$, where $T \in F$. By induction on the well-ordered set $G^{(n)}$, we construct a finite composition R of elements of F such that $\mathrm{supp}(R(R_n(v)))\cap G^{(n)}$ is included in $\mathrm{nf}(\wedge F)$. We let $R_{n+1} := R \circ R_n$. By this iterative construction, we get a sequence $(v_n)_n$, where $v_n := R_n(v)$, such that $v \overset{*}{\to}_F v_n$ and $\delta((\wedge F)(v), v_n) \leq 1/n$. Passing to the limit, we get $v \Rightarrow_F (\wedge F)(v)$. □

Theorem 3.1.7. *Let $F \subseteq \mathbf{RO}(G,\ <_{\mathrm{red}},\ d)$. Then, F is confluent if and only if $\to_F$ is δ-confluent.*

Proof. Assume that F is confluent and let $v \in \widehat{\mathbb{K}G}$ which rewrites both into v_1 and v_2. From 2 of Proposition 3.1.6, $v_i \Rightarrow_F (\wedge F)(v_i)$, for $i = 1, 2$. We also have $v_1 \overset{*}{\leftrightarrow}_F v_2$, so that $(\wedge F)(v_1) = (\wedge F)(v_2)$ from 1 of Proposition 3.1.6. Hence, $\to_F$ is δ-confluent.

If $\to_F$ is δ-confluent, from Proposition 3.1.5, every $v \in \ker(\wedge F)$ admits an admissible decomposition $\sum \lambda_i(g_i - T_i(g_i))$ such that $\mathrm{lm}(v)$ is the greatest of the g_i's. Moreover, the g_i's may be chosen in such a way that they belong to $\mathrm{red}(T_i)$, so that $\mathrm{lm}(v)$ is T-reducible for $T \in F$. Hence, $\mathrm{red}(\wedge F)$ being equal to $\bigcup\{\mathrm{lm}(v) \mid v \in \ker(\wedge F)\}$, it is the union of the sets $\mathrm{red}(T)$, $T \in F$, so that F is confluent. □

3.2. Duality and series representations

In this Section, we fix a countable set G, equipped with a well-order $<$. We fix a strictly decreasing map $d : G \to \mathbb{R} > 0$. Considering $<^{\mathrm{op}}$, the opposite order of $<$, $\widehat{\mathbb{K}G}$ is the set of formal series over G and d is an elimination map, as pointed out in Example 2.2.6. We consider the following two sets of reduction operators: $\mathbf{RO}(G, <, 1)$ and $\mathbf{RO}(G, <^{\mathrm{op}}, d)$, where 1 is the function $g \mapsto 1$. For simplicity, we say reduction operators on $\mathbb{K}G$ and $\widehat{\mathbb{K}G}$, and we denote these sets by $\mathbf{RO}(G, <)$ and $\mathbf{RO}(\widehat{G}, <^{\mathrm{op}})$, respectively.

We denote by $\mathbb{K}G^*$ the algebraic dual of $\mathbb{K}G$. For $\varphi \in \mathbb{K}G^*$ and $v \in \mathbb{K}G$, let $\langle \varphi \mid v\rangle \in \mathbb{K}$ by the result obtained by applying φ to v. In the sequel, we identify $\mathbb{K}G^*$ to $\widehat{\mathbb{K}G}$ through the isomorphism $\mathbb{K}G^* \to \widehat{\mathbb{K}G}$, $\varphi \mapsto \sum\langle \varphi \mid g\rangle g$. For T an endomorphism of $\mathbb{K}G$, we denote by $T^* : \widehat{\mathbb{K}G} \to \widehat{\mathbb{K}G}$ the adjoint operator defined by $T^*(\varphi) = \varphi \circ T$. For a subspace $V \subseteq \mathbb{K}G$, we denote by $V^\perp \subseteq \widehat{\mathbb{K}G}$ the orthogonal space of V, that is, the set of $\varphi \in \widehat{\mathbb{K}G}$ such that $V \subseteq \ker(\varphi)$.

Proposition 3.2.1. *For $T \in \mathbf{RO}(G, <)$, the operator $T^! := \mathrm{id}_{\widehat{\mathbb{K}G}} - T^*$ is a reduction operator on $\widehat{\mathbb{K}G}$. Moreover, we have $\mathrm{nf}(T^!) = \mathrm{red}(T)$, $\mathrm{red}(T^!) = \mathrm{nf}(T)$ and $\ker(T^!) = \ker(T)^\perp$.*

Proof. The operator T being a projector, T^* and $T^!$ are projectors. For every $g \in G$, $T^*(g)$ is equal to $\sum\langle g \mid T(g')\rangle g'$, so that $T^!(g) = g - \sum\langle g \mid T(g')\rangle g'$. If $g <^{\mathrm{op}} g'$, that is, $g' < g$, g does not belong to $\mathrm{supp}(T(g'))$, that is, $\langle g \mid T(g')\rangle = 0$. Hence, $T^!(g) = g - \sum\langle g \mid T(g')\rangle g'$, where the sum is taken over all g''s such that $g' \leq^{\mathrm{op}} g$. If $g \in \mathrm{red}(T)$, then $\langle g \mid T(g')\rangle = 0$, for every $g' \leq^{\mathrm{op}} g$, so that $T^!(g) = g$. If $g \in \mathrm{nf}(T)$, then $\langle g \mid T(g)\rangle = 1$, so that $T^!(g) = -\sum\langle g \mid T(g')\rangle g'$, the sum is taken over g''s such that $g' <^{\mathrm{op}} g$. Hence, $\mathrm{red}(T)$ and $\mathrm{nf}(T)$ are included in $\mathrm{nf}(T^!)$ and $\mathrm{red}(T^!)$, respectively, and by using that $\mathrm{red}(T) \cup \mathrm{nf}(T) = G$, these inclusions are equalities. Moreover, denoting by δ the metric defined such as in (1), for every $v \in \mathbb{K}G$, we have $\delta(T^!(v), 0) \leq \delta(v, 0)$, so that $T^!$ is continuous at 0, hence continuous. Hence, $T^!$ is a reduction operator on $\widehat{\mathbb{K}G}$. It remains to show the relation on kernels. For $\varphi \in \ker(T^!)$ and $g \in \mathrm{red}(T)$, we have $\langle \varphi \mid g - T(g)\rangle = \langle T^!(\varphi) \mid g\rangle = 0$, so that φ vanishes over the set $\{g - T(g) \mid g \in \mathrm{red}(T)\}$. This set forms a basis of $\ker(T)$, so that $\varphi \in \ker(T)^\perp$. Conversely let $\varphi \in \ker(T)^\perp$ and $v \in \mathbb{K}G$. We have $\langle T^!(\varphi) \mid v\rangle = \langle \varphi \mid v - T(v)\rangle = 0$, so that $\varphi \in \ker(T^!)$. □

From Proposition 3.2.1, we have a map

$$\mathbf{RO}(G,<) \to \mathbf{RO}(\widehat{G},<^{\mathrm{op}}),\ T \mapsto T^{!}. \tag{8}$$

Moreover, if V and W are subspaces of $\mathbb{K}G$, then $V \subseteq W$ implies $W^{\perp} \subseteq V^{\perp}$. Hence, the equality $\ker(T^{!}) = \ker(T)^{\perp}$, for $T \in \mathbf{RO}(G,<)$, implies that the map (8) is strictly decreasing.

We finish this Section by relating the duality for reduction operators to representations of series.

Definition 3.2.2. Let $S \in \widehat{\mathbb{K}G}$ be a formal series.

- The *representations category* of S is the category defined as follows:
  - objects are triples (V,α,φ), where V is a vector space, $\alpha : V \to \mathbb{K}G$ is a linear map, and φ a linear form on V such that $S = \varphi \circ \alpha$;
  - a morphism between two representations (V,α,φ) and (V',α',φ') is a linear map $\phi : V \to V'$ such that $\phi \circ \alpha = \alpha'$ and $\varphi' \circ \phi = \varphi$.
- A representation is said to be *surjective* if α is surjective.
- A *representation by operator* is a representation $(\mathbb{K}\mathrm{nf}(T), T, S_{|\mathbb{K}\mathrm{nf}(T)})$, where $S_{|\mathbb{K}\mathrm{nf}(T)}$ is the restriction of S to $\mathbb{K}\mathrm{nf}(T)$. In this case, we say that S is *represented* by T.

We point out that two representations are isomorphic if and only if there exists a morphism of representations between them which is an isomorphism as a linear map and that a representation by operator is surjective.

The following proposition means that being a surjective representation is a duality condition.

Proposition 3.2.3. *A surjective representation of $S \in \widehat{\mathbb{K}G}$ is isomorphic to a representation by operator of S. Moreover, S is represented by a reduction operator T if and only if $S \in \ker(T^{!})$.*

Proof. Assume that (V,α,φ) is a surjective representation of S, so that V is the quotient of $\mathbb{K}G$ by $\ker(\alpha)$. Let T be the reduction operator such that $\ker(T) = \ker(\alpha)$, so that there is an isomorphism $\phi : V \to \mathbb{K}\mathrm{nf}(T),\ \alpha(u) \mapsto T(u)$, with inverse $\phi^{-1}(u) = \alpha(u)$. In order to show that ϕ is a morphism of representations, we only have to show that $\varphi(v) = S(\phi(v))$, for every $v \in V$. Given $u \in \mathbb{K}G$ such that $\alpha(u) = v$, we have $\varphi(v) = S(u)$ and $S(\phi(v)) = S(\phi(\alpha(u))) = S(T(u)) = S(u)$. Hence, ϕ is an isomorphism of representations. The second assertion of the proposition is due to the fact that the relation $S = S \circ T$, means $S = T^{*}(S)$, that is, $S \in \ker(T^{!})$. □

Finally, we classify series represented by a single reduction operator.

Proposition 3.2.4. *Let T be a reduction operator on $\mathbb{K}G$. Then, T^{*} induces an isomorphism between $\widehat{\mathbb{K}\mathrm{nf}(T)}$ and series represented by T.*

Proof. From Proposition 3.2.3, $S \in \widehat{\mathbb{K}G}$ is represented by T if and only if $S \in \ker(T^{!})$. Moreover the set of $g - T^{!}(g)$, $g \in \mathrm{red}(T^{!}) = \mathrm{nf}(T)$, forms a total basis of $\ker(T^{!})$. Hence, there exists a unique $S' \in \widehat{\mathbb{K}\mathrm{nf}(T)}$ such that $S = S' - T^{!}(S') = T^{*}(S')$. The map $S \mapsto S'$ is the inverse of T^{*}. □

4. APPLICATIONS TO FORMAL POWER SERIES

In this Section, we apply the theory of topological reduction operators presented in the previous Sections to formal power series. For that, we recall some notions introduced in Example 2.1.5, and we fix some conventions and notations.

In the following two Section, we fix a set $X := \{x_1, \cdots, x_n\}$ of indeterminates. We denote by $\mathbb{K}[X]$ and $\mathbb{K}\langle X\rangle$ the commutative and noncommutative polynomial algebras over X, respectively. As vector spaces, these algebras have a basis composed of commutative and noncommutative monomials, respectively, the noncommutative being identified to words. A *monomial order* is a well-order on monomials, compatible with multiplication. A (non)commutative *Gröbner basis* of a (two-sided) ideal I of $\mathbb{K}[X]$ or $\mathbb{K}\langle X\rangle$, is a generating subset R of I such that $\mathrm{lm}(R)$ is a generating subset of the monomial ideal $\mathrm{lm}(I)$. In other words, R is a (non)commutative Gröbner basis if and only if for every $f \in I$, there exists $g \in R$ such that $\mathrm{lm}(g)$ divides $\mathrm{lm}(f)$. We recall that this is equivalent to the fact that the polynomial reduction induced by R is a confluent rewriting relation. In this case, the irreducible monomials for the polynomial reduction form a linear basis of the quotient algebra A/I, where $A = \mathbb{K}[X]$ or $\mathbb{K}\langle X\rangle$.

Denote by A the algebra $\mathbb{K}[X]$ or $\mathbb{K}\langle X\rangle$. For a subset S of A, we denote by $I(S)$ the two-sided ideal generated by S. We equip A with the $I(X)$-*adic* topology, that is, the topology induced by the metric $\delta(f, g) = 1/2^n$, where n is the smallest degree of a monomial occurring in the decomposition of $f - g$. The sets $\mathbb{K}[[X]]$ and $\mathbb{K}\langle\langle X\rangle\rangle$ of commutative and noncommutative formal power series, respectively, are the completions of the corresponding algebras. Note that the support of a formal power series $\sum \alpha_m m$, where the sum is taken over monomials, is the set of monomials m such that α_m is different from 0.

4.1. Topological confluence and standard bases

Throughout this Section, we only deal with commutative formal power series. Let $<$ be a monomial order on commutative monomials, which is assumed to be compatible with degrees: $\deg(m) < \deg(m')$ implies $m < m'$. Let $<_{\mathrm{red}} := <^{\mathrm{op}}$ be the opposite order of $<$, so that the leading monomial of a formal power series is the smallest element of its support with respect to $<$.

Definition 4.1.1. Let I be an ideal of $\mathbb{K}[[X]]$. A *standard basis* of I is a generating set R of I such that $\mathrm{lm}(R)$ generates the monomial ideal $\mathrm{lm}(I)$.

Let us consider the map d which maps every monomial m to $1/2^{\deg(m)}$, so that d is an elimination map and the metric induced by d is precisely the metric δ of the $I(X)$-adic topology. Our purpose is to relate standard bases for power series ring ideals to the confluence property of reduction operators. For that, for any $f \in \mathbb{K}[[X]]$, we denote by $T(f)$ the reduction operator whose kernel is the closed ideal generated by f: $T(f) := \ker^{-1}(\overline{I(f)})$. Explicitly, for every monomial m, $T(f)$ is defined by the following recursive formulas:

- if m is not divisible by $\mathrm{lm}(f)$, then $T(f)(m) = m$,
- if $m = \mathrm{lm}(f)m'$, then $T(f)(m) = 1/\mathrm{lc}(f)\Big(T(f)(m'(\mathrm{lc}(f)\mathrm{lm}(f) - f))\Big)$.

In particular, $\mathrm{red}(T(f))$ is the monomial ideal spanned by $\mathrm{lm}(f)$. Finally, for a subset $R \subseteq \mathbb{K}[[X]]$, we denote by $F(R) := \{T(f) \mid f \in R\}$.

Proposition 4.1.2. *A subset R of $\mathbb{K}[[X]]$ is a standard basis of the ideal it generates if and only if $F(R)$ is a confluent set of reduction operators.*

Proof. We denote by $I(R)$ the ideal of $\mathbb{K}[[X]]$ generated by R. By definition of $\wedge F(R)$, its kernel is equal to $\overline{I(R)}$. Hence, $\mathrm{red}(\wedge F(R))$ is equal to $\mathrm{lm}(\overline{I(R)})$, and from (5), the latter is equal to $\mathrm{lm}(I(R))$. Moreover, $\mathrm{red}(F(R))$ is the union of the sets $\mathrm{lm}(T(f))$, $f \in R$, that is, it is the monomial ideal spanned by $\mathrm{lm}(R)$. Hence the statement of the proposition is due to the following sequence of equivalences: $F(R)$ is confluent if and only if $\mathrm{red}(\wedge F(R))$ is equal to $\mathrm{red}(F(R))$, that is, if and only if $\mathrm{lm}(I(R))$ is the monomial ideal spanned by $\mathrm{lm}(R)$, that is, if and only if R is a standard basis of $I(R)$. □

As for Gröbner bases and S-polynomials, there is a criterion in terms of *S-series* for a generating set of an ideal of a power series ring to be a standard basis [3, Theorem 4.1]. From Theorem 3.1.7 and Proposition 4.1.2, we obtain the following formulation of this criterion in terms of the δ-confluence property.

Theorem 4.1.3. *A subset R of $\mathbb{K}[[X]]$ is a standard basis of the ideal it generates if and only if the rewriting relation $\to_{F(R)}$ is δ-confluent.*

Example 4.1.4. Consider the example of the introduction: $X := \{x, y, z\}$, $<$ is the deglex order induced by $x > y > z$ and $R := \{z - y, z - x, y - y^2, x - x^2\}$. Then, $\mathrm{lm}(R)$ is equal to $\{x, y, z\}$, and constant coefficients of elements of the power series ideal generated by R are equal to 0, so that R is a standard basis of this ideal. Hence, the rewriting relation $\to_{F(R)}$ is δ-confluent. However, it is not confluent as illustrated by the following diagram, where, for simplicity, we remove the subscript $F(R)$:

$$\begin{tikzcd}
 & x \arrow[r] & x^2 \arrow[r] & \cdots \arrow[r] & x^{2n} \arrow[dr, Rightarrow, bend left] & \\
z \arrow[ur, bend left] \arrow[dr, bend right] & & & & & 0 \\
 & y \arrow[r] & y^2 \arrow[r] & \cdots \arrow[r] & y^{2n} \arrow[ur, Rightarrow, bend right] &
\end{tikzcd}$$

4.2. Duality and syntactic algebras

Throughout out this Section, we only deal with noncommutative objects: formal power series, algebras, Gröbner bases ... Hence, we omit the adjective noncommutative. Moreover, we only deal with two-sided ideals, so that we also omit the adjective two-sided.

Our purpose is to relate duality for reduction operators to syntactic algebras. Let $S \in \mathbb{K}\langle\langle X \rangle\rangle$ be a formal power series. The *syntactic ideal* of S, written I_S, is the greatest ideal included in $\ker(S)$. The *syntactic algebra* of S is the quotient algebra $A_S := \mathbb{K}\langle X \rangle / I_S$. The series S is said to be *rational* if A_S is finite-dimensional as a vector space. Moreover, an algebra is said to be *syntactic* if it is the syntactic algebra of a formal power series. Let us illustrate this notion with an example coming from [26]: $X = \{x_0, x_1\}$, and

$$S := \sum_{w \in X^*} \mathrm{val}(w) w, \tag{9}$$

where $\mathrm{val}(w) \in \mathbb{N}$ is the integer whose binary expression is equal to w. From [26], this series is rational. We propose another proof of this result by providing a Gröbner basis of the syntactic ideal of S.

Proposition 4.2.1. *The syntactical ideal of* (9) *admits the following Gröbner basis:*

$$R := \left\{ \begin{array}{ll} f_1 := x_0x_0 - 3x_0 + 2 & f_2 := x_0x_1 - x_1 - 2x_0 + 2 \\ f_3 := x_1x_0 - 2x_1 - x_0 + 2 & f_4 := x_1x_1 - 3x_1 + 2 \end{array} \right\} \subset \mathbb{K}\langle X \rangle.$$

In particular, (9) *is rational.*

Proof. The ideal generated by f_1 is included in $\ker(S)$, since for every words w and w', we have

$$\begin{aligned}
\langle S \mid wf_1w' \rangle &= \langle S \mid wx_0x_0w' \rangle - 3\langle S \mid wx_0w' \rangle + 2\langle S \mid ww' \rangle \\
&= 2^{|x_0x_0w'|}\mathrm{val}(w) + \langle S \mid x_0x_0w' \rangle - 3\Big(2^{|x_0w'|}\mathrm{val}(w) + \langle S \mid x_0w' \rangle\Big) + 2\Big((2^{|w'|}\mathrm{val}(w) + \langle S \mid w' \rangle\Big) \\
&= 2^{|w'|}\Big(4 - 6 + 2).\mathrm{val}(w) + 2^{|w'|}\langle S \mid f_1 \rangle + \langle S \mid w' \rangle\Big(1 - 3 + 2\Big) \\
&= 0.
\end{aligned}$$

By using analogous arguments, we show that ideals generated by other f_i's are also included in $\ker(S)$. We easily show that for the deglex order induced by $x_0 < x_1$, all the S-polynomials of f_i's reduce into zero, so that R forms a Gröbner basis of the ideal I it generates. Hence, the algebra $A := \mathbb{K}\langle X\rangle/I$ is 3-dimensional, with a basis composed of $1, x_0$ and x_1. Moreover, by computing codimensions, we check that $\ker(S)$ is equal to $I \oplus \mathbb{K}\{1, x_0\}$, so that I is the syntactic ideal of S, and A is its syntactic algebra. $\square$

We associate to an algebra $A = \mathbb{K}\langle X\rangle/I$, the reduction operator $T := \ker^{-1}(I)$ on $\mathbb{K}\langle X\rangle$ with kernel I. This operator is computed as follows: if R is a Gröbner basis of I, then $T(f)$ is the unique normal form of $f \in \mathbb{K}\langle X\rangle$ through polynomial reduction. In particular, a series is represented by A if and only if it is represented by T.

Theorem 4.2.2. *Let $I \subseteq \mathbb{K}\langle X\rangle$ be an ideal and let T be the reduction operator with kernel I. Then, the algebra $\mathbb{K}\langle X\rangle/I$ is syntactic if and only if there exists $S' \in \widehat{\mathbb{K}\mathrm{nf}(T)}$ such that I is the greatest ideal included in $I \oplus \ker(S')$.*

Proof. First, observe that the sum of the statement of the Theorem is direct since for every subspace $V \subseteq \mathbb{K}\mathrm{nf}(T)$, $I + V$ is direct, indeed, the leading monomial of a nonzero $f \in I$ does not belong to $\mathrm{nf}(T)$. Moreover, from Proposition 3.2.4, S is represented by A if and only if there exists $S' \in \widehat{\mathbb{K}\mathrm{nf}(T)}$ such that $S = T^*(S')$. Hence, for every $g \in \mathbb{K}\langle X\rangle$, we have $\langle S \mid g\rangle = \langle S' \mid T(g)\rangle$, so that

$$\langle S \mid g\rangle = \langle S \mid (g - T(g)) + T(g)\rangle = \langle S' \mid T(g)\rangle. \tag{10}$$

The element $g - T(g)$ belongs to I, which is included in $\ker(S)$. Thus, from (10), $\ker(S) = I \oplus \ker(S')$. The statement of the theorem follows since S is represented by its syntactic algebra [28]. $\square$

Example 4.2.3.

1. Consider the series S as in (9) and let T be the reduction operator with kernel the syntactic ideal I_S. From Proposition 4.2.1, $\mathrm{nf}(T)$ is equal to $\{1, x_0, x_1\}$, so that $S = T^*(S')$, $S' \in \mathbb{K}\mathrm{nf}(T)$. By evaluating S at 1, x_0 and x_1, we get $S' = x_1$. We easily check that I_S is the greatest ideal included in $I_S \oplus \ker(S') = I_S \oplus \mathbb{K}\{1, x_0\}$.

2. Consider the algebra $A := \mathbb{K}\langle X\rangle/I$, where $X := \{x, y\}$ and I is the ideal generated by 2-letter words. In [28], it is proven that this algebra is not syntactic. Here, we propose another proof, based on duality. The reduction operator T with kernel I maps every word of length at least 2 to 0. Let $S' = \alpha + \beta x + \gamma y \in \mathbb{K}\mathrm{nf}(T)$. If $\beta = \gamma = 0$, then $I \oplus \ker(S')$ is equal to $I \oplus \mathbb{K}\{x, y\}$, and if not, it contains $I \oplus \mathbb{K}\{\gamma x - \beta y\}$. Since both $I \oplus \mathbb{K}\{x, y\}$ and $I \oplus \mathbb{K}\{\gamma x - \beta y\}$ are ideals, $I \oplus \ker(S')$ contains an ideal strictly greater than I. Hence, the criterion of Theorem 4.2.2 does not hold, that is, A is not syntactic.

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