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Penalized ordinal logistic regression using cumulative logits

Clémence Karmann, Anne Gégout, Aurélie Gueudin

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Journée scientifique FCH :

"Méthodes et modèles pour comprendre les réseaux biologiques"
January 15th 2018

Introduction

- Analyze links between a variable (response) and covariates

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- Variable selection problem: identify relevant covariates

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- Variable selection problem: identify relevant covariates
- Ordinal response

Overview

- 1 Cumulative logit model
 - Generalities
 - Coefficients
- 2 Estimation, inference
 - Lasso estimation of the coefficients β
 - Penalty parameter and variable selection
- 3 Simulation studies
- 4 Application to network inference of zero-inflated data

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Multi-class regression: cumulative logit model

Idea

Generalization of the logistic regression for a response Y with $K > 2$ **ordered** categories.

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Generalization of the logistic regression for a response Y with $K > 2$ **ordered** categories.

If we have p covariates X_1, \dots, X_p , we model

$p_{\beta}^j(x) := \mathbb{P}_{\beta}(Y \leq j | X = x)$ for $j = 1, \dots, K - 1$, by:

$$\text{logit } p_{\beta}^j(x) = \alpha_j + \beta_1 x_1 + \dots + \beta_p x_p,$$

i.e.:

$$p_{\beta}^j(x) = \frac{\exp(\alpha_j + \beta_1 x_1 + \dots + \beta_p x_p)}{1 + \exp(\alpha_j + \beta_1 x_1 + \dots + \beta_p x_p)}$$

Coefficients

Pour $j = 1, 2, \dots, K - 1$:

$$\text{logit } \mathbb{P}_\beta(Y \leq j | X = x) = \alpha_j + \beta_1 x_1 + \dots + \beta_p x_p$$

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Interpretation of coefficients

The coefficient β_i measures the conditional dependence between Y and X_i given $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_p$.

\rightsquigarrow Nullity of coefficients β

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$$\underset{\substack{\alpha_1 < \dots < \alpha_{K-1} \\ \beta \in \mathbb{R}^p}}{\operatorname{argmax}} \left[\log \mathcal{L}_{\alpha, \beta} - \lambda \|\beta\|_1 \right]$$

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equivalent to (lagrangian duality):

$$\underset{\substack{\alpha_1 < \dots < \alpha_{K-1} \\ \|\beta\|_1 \leq \tau}}{\operatorname{argmax}} \log \mathcal{L}_{\alpha, \beta}$$

Frank-Wolfe algorithm

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$$\min_{x \in C} f(x); f, C \text{ convex}$$

Iteration k:

- $s^{(k)} \in \operatorname{argmin}_{s \in C} {}^t \nabla f(x^{(k-1)}) \cdot s$
- $x^{(k)} = (1 - \gamma_k)x^{(k-1)} + \gamma_k s^{(k)}, \gamma_k = \frac{2}{k+1}$

↳ linear approximation of the target function

Variable selection

$$\begin{aligned} \operatorname{argmax}_{\substack{\alpha_1 < \dots < \alpha_{K-1} \\ \|\beta\|_1 \leq \tau}} \log \mathcal{L}_{\alpha, \beta} \end{aligned}$$

☞ How to perform variable selection?

Revisited knockoffs

- Inspired from Barber and Candès (2015)
- Intuitive and suitable to any regression framework
- Provides a sorting of covariates

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Idea

The idea is to use a matrix of knockoffs of covariates whose structure is similar to X but independent from Y :

- If X_i enters the model after its KO $\rightsquigarrow X_i$ does not belong to the model
- Otherwise $\rightsquigarrow X_i$ is more likely to be relevant

Revisited knockoffs

Procedure

- 1 We construct the knockoffs matrix \tilde{X} by swapping (randomly) the rows of X

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- 3 For $i \in \{1, \dots, p\}$, $W_i := T_i \wedge T_{i+p} \times \begin{cases} +1 & \text{if } T_i < T_{i+p} \\ -1 & \text{if } T_i \geq T_{i+p} \end{cases}$

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- 4 Change detection methods applied to the sorted positive statistics W_i to select variables for which statistics W is positive and small

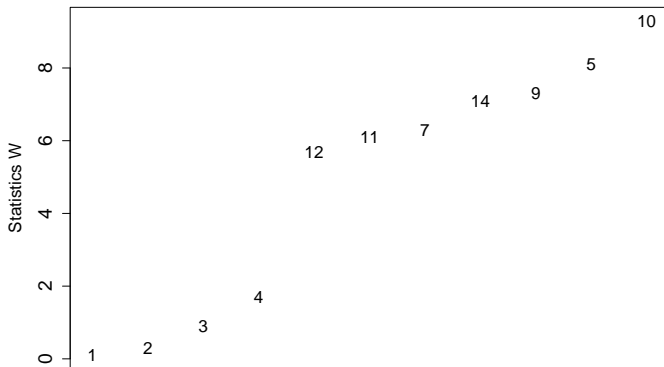


Figure: Example of sorted positive statistics W_i . Only variables X_1 , X_2 , X_3 and X_4 belong to the model (in this case, $\beta = (8, 6, 4, 2, 0, \dots, 0)$).

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- Covariates X are simulated as gaussian s.t. X_i conditionally independent to X_j with probability $1 - p$, $p = 0.6$

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- Regression coefficients $\beta = (8, 6, 4, 2, 0, \dots, 0)$

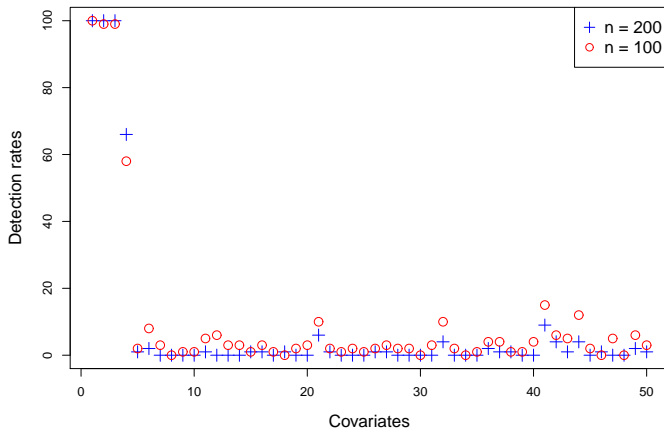


Figure: Detection rates on 100 repetitions after applying revisited knockoffs method.

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Zero-inflated data simulation

Model to simulate data which looks like our kind of data (positive, zero-inflated) and such that we know the theoretical graph structure → latent Gaussian graphical model:

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- Final data are then $Z = Ber \cdot X$.

Network inference

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Retrieve links of conditional dependence between variables X_i (known in theory thanks to the matrix Σ^{-1}) with the observations Z .

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- 2 Two networks can be built: the 'and' or the 'or' versions

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- $n = 200$ samples, $p = 200$ variables
- 'chain' structure
- zero-inflation: $\approx 12\%$, varying from 0 to 64%

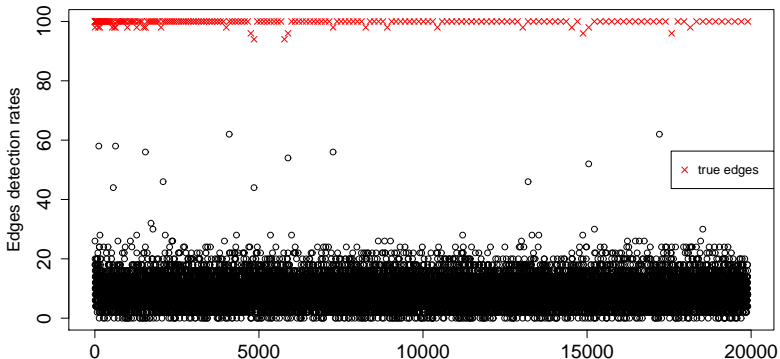


Figure: Edges detection rates (network 'and') on 50 repetitions after applying revisited KO method. Circles and crosses represent respectively false and true edges.

Thank you !