An efficient domain decomposition method with cross-point treatment for Helmholtz problems

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Abstract

The parallel finite-element solution of large-scale time-harmonic scattering problems is addressed with a non-overlapping domain decomposition method (DDM). It is well known that the efficiency of this method strongly depends on the transmission condition enforced on the interfaces between the subdomains. Local conditions based on high-order absorbing boundary conditions (HABCs) are well suited for configurations without cross points *(where more than two subdomains meet)*. In this work, we extend this approach to efficiently deal with cross points. Two-dimensional finite-element results are presented.

Keywords: Helmholtz solvers, finite elements, domain decomposition, fast comput. technique

1 Introduction

Optimized Schwarz DDMs are currently a very promising approach for the parallel solution of high-frequency time-harmonic problems. With these methods, subproblems of smaller sizes are solved in parallel using direct solvers, and are combined in an iterative procedure [1, 2, 4].

The convergence rate of the DDM procedure depends on the transmission condition enforced on the interfaces between the subdomains. Local conditions based on HABCs represent a good compromise between basic impedance conditions *(which lead to suboptimal convergence)* and the exact Dirichlet-to-Neumann (DtN) map related to the complementary of the subdomain *(which is expensive to compute)*. They are well suited for configurations without cross points [1], but a direct application of this approach with cross points does not provide satisfactory results.

Noting that cross points actually are corners for the subdomains, we propose a novel strategy which consists in incorporating a corner treatment developed for HABCs [3] into the DDM procedure for configurations with cross points and right angles.

2 DDM method

We consider a 2D Helmholtz problem defined on a rectangular computational domain Ω:

$$\begin{cases} \Delta u + k^2 u = s, & \text{in } \Omega, \\ \partial_{n_f} u - \imath k u = 0, & \text{on each } \Gamma_f, \end{cases}$$

where k is the wavenumber, s is a source term, Γ_f is an edge of the domain, and ∂_{n_f} is the exterior normal derivative, with $f = 1 \dots 4$.

The domain Ω is partitioned into a structured grid of non-overlapping rectangular subdomains Ω_I, with $I = 1 \dots N^{\text{dom}}$. Each edge $\Gamma_{I,f}$ can be either a *boundary edge* (if $\subset \partial\Omega$) or an *interface edge* (if $\not\subset \partial\Omega$). In the standard DDM procedure, the solution u_I of each subdomain Ω_I is obtained by solving the subproblem

$$\begin{cases} \Delta u_I + k^2 u_I = s, & \text{in } \Omega_I, \\ \partial_{n_{I,f}} u_I - \imath k u_I = 0, & \text{on each } \Gamma_{I,f} \subset \partial\Omega, \\ \partial_{n_{I,f}} u_I + \mathcal{B} u_I = g_{I,f}, & \text{on each } \Gamma_{I,f} \not\subset \partial\Omega, \end{cases}$$

where $\mathcal{B}$ is an impedance operator and $g_{I,f}$ is a transmission variable. For any interface edge, the transmission variable is computed using

$$g_{I,f} = \partial_{n_{I,f}} u_J + \mathcal{B} u_J = -g_{J,g} + 2\mathcal{B} u_J, \quad (1)$$

where $g_{J,g}$ and u_J belong to the neighboring subdomain Ω_J with the shared edge $\Gamma_{I,f} = \Gamma_{J,g}$. Each iteration of the DDM procedure then consists in solving concurrently the subproblems and updating the transmission variables using equation (1). See [1] for further details.

With our approach, the transmission operator $\mathcal{B}$ is an approximation of an exact half-space DtN operator, where a rational approximation of the square root is used in the symbol. The application of $\mathcal{B}$ on u_I is written as

$$\mathcal{B} u_I = -\imath k \alpha \left[u_I + \frac{2}{M} \sum_{i=1}^{N} c_i \left(u_I + \varphi_{I,f,i} \right) \right],$$

where $\alpha = e^{\imath\phi/2}$, $c_i = \tan^2(i\pi/M)$, $M = 2N+1$, N and ϕ are parameters, and $\{\varphi_{I,f,i}\}_{i=1\dots N}$ are 1D auxiliary fields living on the edge $\Gamma_{I,f}$.

$$\begin{cases} \partial_{\tau_{I,f}\tau_{I,f}}\varphi_{I,f,i} + k^2\big((\alpha^2 c_i + 1)\varphi_{I,f,i} + \alpha^2(c_i+1)u_I\big) = 0, & \text{on } \Gamma_{I,f}, \\ \partial_{n_{I,f'}}\varphi_{I,f,i} - ik\varphi_{I,f,i} = 0, & \text{on each } P_{I,ff'} \subset \partial\Omega, \\ \partial_{n_{I,f'}}\varphi_{I,f,i} + \mathcal{C}\big(\varphi_{I,f,i},\, \varphi_{I,f',1},\, \ldots,\, \varphi_{I,f',N}\big) = g_{I,f,i}, & \text{on each } P_{I,ff'} \not\subset \partial\Omega. \end{cases} \tag{2}$$

For each interface edge $\Gamma_{I,f}$, each auxiliary field $\varphi_{I,f,i}$ is governed by a 1D Helmholtz equation (first equation in system (2), where ∂_{τ_i} is the tangent derivative). Because of the second-order partial derivative, a boundary condition is required at each extremity of the edge [3], which are corners of the subdomain. In the DDM procedure, that condition becomes a transmission condition if the adjacent edge is an interface.

The corner shared by $\Gamma_{I,f}$ and any adjacent edge $\Gamma_{I,f'}$ is denoted $P_{I,ff'} = \Gamma_{I,f} \cap \Gamma_{I,f'}$. Depending on the type of $\Gamma_{I,f'}$, the auxiliary field $\varphi_{I,f,i}$ verifies one of the two last equations of system (2). In the last one, $\mathcal{C}$ is a linear function taking $\varphi_{I,f,i}$ and all the auxiliary fields living on $\Gamma_{I,f'}$ *(the expression is easily obtained from [3])*. The transmission variable $g_{I,f,i}$ verifies

$$g_{I,f,i} = -g_{J,f,i} + 2\mathcal{C}\big(\varphi_{J,f,i}, \varphi_{J,g',1}, \ldots, \varphi_{J,g',N}\big),$$

where $\Gamma_{I,f}$ and $\Gamma_{J,f}$ have the same position in Ω_I and Ω_J, and $\Gamma_{I,f'} = \Gamma_{J,g'}$ is the shared edge.

3 Preliminary finite element results

To analyse the efficiency of the method, we consider the scattering of a plane wave by the unit disk in a squared domain partitioned into 6 subdomains (figure 1). A Neumann BC is used on the boundary of the disk, and the basic ABC is prescribed on the exterior border. Simulations are performed with P1 elements and a Galerkin method adapted from [1] using `GetDDM` [4]. The GMRES is used on the top of the procedure.

The convergence is faster when the HABC transmission condition is used with the cross-point treatment (figure 2). If the number of auxiliary fields N is large enough, the number of iteration does not vary when increasing the frequency or the mesh density. The procedure always converges towards the correct solution, even without the treatment *(results not shown)*.

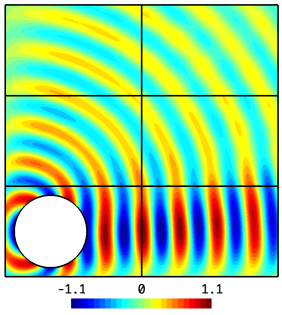

Figure 1: Configuration and reference solution.

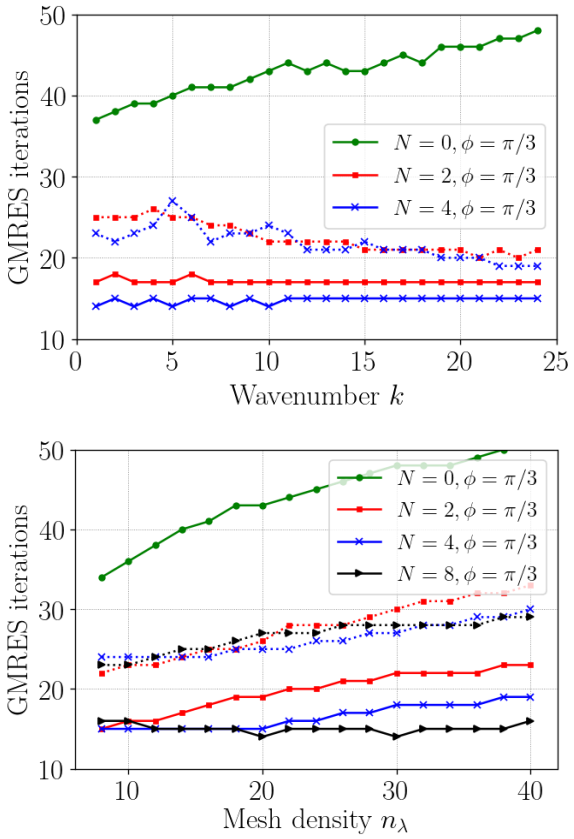

Figure 2: Number of iterations to reach relative residual 10^{-6} *vs* wavenumber k (with $n_\lambda = 15$) and mesh density n_λ (with $k = 2\pi$), without *(dot. lines)* or with *(cont. lines)* c.-p. treatment.

Our approach can be used with other exterior boundary conditions. In future works, we will consider other physical waves and combination with preconditioning techniques.

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