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# On the computation of Nash equilibria in games on graphs

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## Abstract

In this talk, I will show how one can characterize and compute Nash equilibria in multiplayer games played on graphs. I will present in particular a construction, called the *suspect game construction*, which allows to reduce the computation of Nash equilibria to the computation of winning strategies in a two-player zero-sum game.

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## 1 Introduction

Multiplayer concurrent games over graphs allow to model rich interactions between players. Those games are played as follows. In a state, each player chooses privately and independently an action, defining globally a move (one action per player); the next state of the game is then defined as the successor (on the graph) of the current state using that move; players continue playing from that new state, and form a(n infinite) play. Each player then gets a reward given by a payoff function (one function per player). In particular, objectives of the players may not be contradictory: those games are non-zero-sum games, contrary to two-player games used for controller or reactive synthesis [10, 7].

Using solution concepts borrowed from game theory, one can describe the interactions between the players, and in particular describe their rational behaviours. One of the most basic solution concepts is that of Nash equilibria [8]. A Nash equilibrium is a strategy profile where no player can improve her payoff by unilaterally changing her strategy. The outcome of a Nash equilibrium can therefore be seen as a rational behaviour of the system. While very much studied by game theoretists (e.g. over matrix games), such a concept (and variants thereof) has been only rather recently studied over games on graphs. Probably the first works in that direction are [5, 4, 11, 12].

Computing Nash equilibria requires to (i) find a good behaviour of the system; (ii) detect deviations from that behaviour, and identify deviating players (called deviators); (iii) punish them. Variants of Nash equilibria (like subgame-perfect equilibria, robust equilibria, *etc*) require slightly different ingredients, but they are mostly of a similar vein.

In this talk, we will first recall some basics of game theory over matrix games. Those games are not sufficient in a verification context: indeed, explicit states are very useful when



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modelling systems or programs, but are missing in matrix games. However stability notions like Nash equilibria or other solution concepts borrowed from game theory, are very relevant. We will thus present the model of concurrent multiplayer games (played on graphs), which extends in a natural way standard models used in verification with multiplayer interactions. We will explain how Nash equilibria can be characterized and computed in such general games. We will also discuss some existence results for Nash equilibria.

A reference note for this talk is [1]. Related notes are [10], which discussed the use of two-player zero-sum games in verification, and [6], which discussed solution concepts in multiplayer turn-based games on graphs.

To give a taste of the approach, we informally discuss below a simple scenario. The general case with concurrent games and more general payoff functions will be handled by the suspect game abstraction, which somehow generalizes the simple scenario below. For readers not familiar at all with games, you can skip the discussion and wait until the talk.

## 2 Discussion on a simple scenario

We fix a turn-based and deterministic game  $\mathcal{G}$  with set of players  $\mathcal{P}$ , and we assume that the payoff function for each player  $A \in \mathcal{P}$  is given by a Boolean prefix-independent objective  $\phi_A$  (that is, the player gets +1 if the play satisfies  $\phi_A$ , and 0 otherwise). Each player  $A$  plays using a (deterministic) strategy  $\sigma_A$ , and once a strategy is fixed for every player, we have a strategy profile  $\sigma = (\sigma_A)_{A \in \mathcal{P}}$ . In such a game, the player objective is to make the generated play satisfy her formula. Hence, if the unique outcome of  $\sigma$  satisfies  $\phi_A$ , then player  $A$  is satisfied. Otherwise player  $A$  will try to be more satisfied by changing her strategy; the new strategy is then called a (single-player) *deviation*. The strategy profile  $\sigma$  will then be a (pure) *Nash equilibrium* if it is resistant to single-player deviations.

In our simple setting, it will be rather easy to characterize deviations that are *profitable* to player  $A$ : once a deviation by player  $A$  has occurred, we assume that all other players (which we note as a coalition  $(\neg A)$ ) play optimally in  $\sigma$  for the objective  $\neg\phi_A$ .<sup>1</sup> That way, if the outcome of the strategy does only visit winning states for  $(\neg A)$  (which we denote  $W_{(\neg A)}$ ), then no deviation for player  $A$  can be profitable; conversely if the outcome visits a winning state for  $A$  (by determinacy, this is the negation of the previous case), then there will be a profitable deviation for player  $A$ . One can then characterize Nash equilibria as follows:

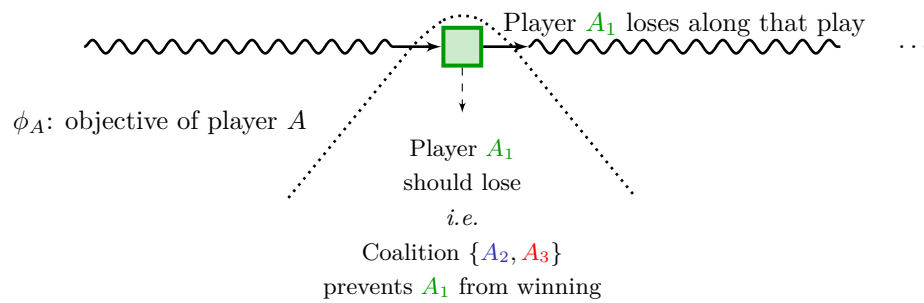
► **Proposition 1.** *Let  $\rho$  be an infinite path in  $\mathcal{G}$  from the initial vertex  $v_{\text{init}}$ . Then,  $\rho \models \Phi_{NE}$  if and only if there is a Nash equilibrium  $\sigma$  from  $v_{\text{init}}$  such that the outcome of  $\sigma$  is  $\rho$ , where*

$$\Phi_{NE} = \bigwedge_{A \in \mathcal{P}} \left( \neg\phi_A \Rightarrow \mathbf{G}W_{(\neg A)} \right)$$

The situation is illustrated on Figure 1. Note that by determinacy, “Player  $A_1$  should lose” can be replaced by “Coalition  $\{A_2, A_3\}$  prevents  $A_1$  from winning”.

A solution to compute Nash equilibria is then to compute for every  $A \in \mathcal{P}$  the set  $W_{(\neg A)}$  (or equivalently  $W_A$ ), and to compute an infinite path in the game which satisfies formula  $\Phi_{NE}$  (which can be done for instance by enumerating the possible set of losing players, and then finding an adequate ultimately periodic play). Obviously, for specific winning conditions, more efficient algorithms can be designed, see for instance [13, 2].

<sup>1</sup> Such a strategy is sometimes called a *threat* or a *trigger* strategy.



■ **Figure 1** General shape of a Nash equilibrium in the simple setting (example with three players  $A_1$ ,  $A_2$  and  $A_3$ ).

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