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Sebastien Courtin, Annick Laruelle

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Sebastien Courtin [*] and Annick Laruelle [†‡]

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Abstract

This paper deals with rules that specify the collective acceptance or rejection of a proposal with several dimensions. We introduce the notions of separability and weightedness in this context. We provide a partial characterization of separable rules and show the independence between separability and weightedness.

KEYWORDS: Multi-dimensional rules, Weightedness, Separability relation, Decision making process.

JEL Classification: C71, D71

1 Introduction

Many collective decision processes at a local, national or international level use a simple voting scheme: the question is unidimensional, meaning that individuals answer "yes" or "no" to a single question. We refer to Andjiga et al. [2003] and Laruelle and Valenciano [2008] for a detailed description of these processes. Some extensions include the consideration of non-binary choices: Felsenthal and Machover [1997] or Laruelle and Valenciano [2012] introduce decision systems in which "abstention" or "non-participation" are permitted; Freixas [2005] and Freixas et al. [2014] study (j,k)– games where multiple dimensions concern both the input and the output.

[*]Normandie Univ, Unicaen, CREM, UMR CNRS 6211, France. Email: sebastien.courtin@unicaen.fr;

[†]Departamento de Fundamentos del Análisis Económico I, Universidad del País Vasco, Avenida Lehendakari Aguirre, 83, E-48015 Bilbao, Spain; a.laruelle@ikerbasque.org;

[‡]IKERBASQUE, Basque Foundation of Science, 48011, Bilbao, Spain.

Here, we are interested in another type of extension. Consider a situation in which a group of experts have to decide whether a candidate qualifies for a promotion. Several criteria can be taken into account: performance, ability to work in a team, and leadership. Each expert gives a positive or negative evaluation of the candidate for each criterion. The question is to choose the decision-making rule that will aggregate all evaluations into a promotion or non-promotion decision. A similar problem occurs when the success of a student is based on tests taken at different periods or on different subjects. This paper models these situations through multi-dimensional decision-making rules, that is, rules with several voters casting a binary choice (yes/no) on several dimensions resulting in a dichotomous (yes/no) decision. More precisely, we model decision processes where: (i) There are several voters. (ii) There are several dimensions. (iii) Each voter expresses a binary choice (”yes” or ”no”) on each dimension. (iv) A decision process maps each choice to a final binary decision (”yes” or ”no”). We also study some properties of these rules, such as separability and weightedness, and show that these notions are not related. It is worth noting that multi-dimentional rules can be seen as voting rules (simple games) whose set of voters has a product structure.

A multi-dimensional decision process is separable if it can be decomposed voter by voter or dimension by dimension. In other words, we can represent some multi-dimensional systems by combining single-dimensional systems. We show that not all rules are separable, by means of examples. We then provide some necessary and sufficient conditions which guarantee separability (Propositions 1, 2, 3 and 4). Remarks 1 to 6 give the specific structure of some separable multi-dimensional rules. The properties of separability are related to well-known paradoxes in the theory of voting: the referendum paradox (see Feix et al. [2004]) and the Ostrogorski paradox (Ostrogorski [1970]).

A multi-dimensional decision process is weighted if it can be represented by a mean of weights and a threshold for the system. When the system has only one dimension, Taylor and Zwicker [1992] produce sufficient and necessary conditions to ensure weightedness. This result is generalized to systems with multiple dimensions (Theorem 1 and 2) and the independence between separability and weightedness is provided.

This paper is structured as follows: Section 2 introduces the general framework of multi-dimensional rules. Section 3 presents separability conditions, while Section 4 defines and characterizes weighted multi-dimensional rules. Finally, Section 5

shows that separability and weightedness are different notions.

2 Definition of multi-dimensional rules

Let $N = \{1,...,n\}$ be a set of n voters, and $M = \{1,...,m\}$ be a set of m dimensions. Generic voters will be denoted by i, j, k, while dimensions will be denoted by a, b, c. Each voter faces a binary choice and casts a positive or negative vote on each dimension. A multi-dimensional rule aggregates the votes cast by the n voters on the m dimensions in a dichotomous outcome, positive or negative.

A possible result of a vote, or a multi-dimensional configuration (simply configuration), specifies, for each voter, the vote cast on each dimension. Let $s_i^a \in \{0,1\}$ be the vote cast by voter i on dimension a. The interpretation of $s_i^a = 0$ is that i casts a negative vote, while $s_i^a = 1$ means that i casts a positive vote. If $\mathbb{Z}_n^m$ denotes the set of $n \times m$ matrices of 0 and 1, a configuration can be represented by a matrix $\mathbf{S} \in \mathbb{Z}_n^m$, where the element of row i and column a is s_i^a:

$$\mathbf{S} = \begin{pmatrix} s_1^1 & \cdots & s_1^m \\ \vdots & & \vdots \\ \vdots & & \vdots \\ s_n^1 & \cdots & s_n^m \end{pmatrix}.$$

The i^{th} row, that we denote by $S_i = (s_i^1,...,s_i^m)$, represents voter i's opinion on the m dimensions, while a^{th} column, denoted by $S^a = (s_1^a,...,s_n^a)^T$, represents the n voters' votes on dimension a.[1] Let s_i denote the number of positive votes cast by voter i, that is, $s_i = \sum_{a \in M} s_i^a$ while s^a denotes the number of positive votes cast on dimension a: we have $s^a = \sum_{i \in N} s_i^a$. The total number of positive votes is denoted by $s = \sum_{i \in N} \sum_{a \in M} s_i^a$. Generic configurations will be denoted by $\mathbf{S}$, $\mathbf{T}$. We will write $\mathbf{S} \leq \mathbf{T}$ if $s_i^a \leq t_i^a$ for any $i \in N$, $a \in M$. Matrix $\mathbf{I}_n^m$ is the $n \times m$ matrix of 1, and $\mathbf{O}_n^m$ is the $n \times m$ matrix of 0.

A multi-dimensional rule specifies the configurations of $\mathbb{Z}_n^m$ that lead to a positive final outcome and those that lead to a negative outcome. We refer to the configurations that lead to a positive outcome as winning configurations (the other ones are referred to as losing configurations). A voting rule can be represented by a function

[1] Note that, in general, voters have no preference as to the outcome. There is no strategic consideration.

w that associates a value 0 to losing configurations and 1 to winning configurations. Some properties are required. Basically, we extend the properties displayed by a single dimension binary voting rule.

Definition 1. *A multi-dimensional rule with n voters and m dimensions is a function*

$$\begin{aligned} \mathbf{w}: \quad & \mathbb{Z}_n^m \to \{0,1\} \\ & \mathbf{S} \longmapsto \mathbf{w}(\mathbf{S}) \end{aligned}$$

that satisfies the following properties:

1. *Positive unanimity:* $\mathbf{w}(\mathbf{I}_n^m) = 1$.
2. *Negative unanimity:* $\mathbf{w}(\mathbf{O}_n^m) = 0$.
3. *Monotonicity:* $\mathbf{w}(\mathbf{S}) = 1 \Rightarrow \mathbf{w}(\mathbf{T}) = 1$ *for any* $\mathbf{S} \leq \mathbf{T}$.

Positive unanimity requires that if all voters vote 'yes' on all dimensions, the outcome should be positive. Negative unanimity requires that if all voters vote 'no' on all dimensions, the outcome should be negative. Monotonicity requires that if a vote configuration is winning then, any other configuration with more support by all voters is also winning.

The monotonicity allows us to define a minimal winning configuration: **S** is minimal winning if $\mathbf{w}(\mathbf{S}) = 1$ and for any configuration $\mathbf{T} \neq \mathbf{S}$ with $\mathbf{T} \leq \mathbf{S}$ we have $\mathbf{w}(\mathbf{T}) = 0$. Let $\mathcal{M}(\mathbf{w})$ be the set of minimal winning configurations of rule **w**. It is another possible representation of the voting rule.

When there is a single voter ($n = 1$), we refer to the rule as a decision rule and it is denoted by d. We have

$$\begin{aligned} d: \quad & \mathbb{Z}_1^m \to \{0,1\} \\ & S_i \to d(S_i), \end{aligned} \quad .$$

Similarly, when there is a single dimension, we refer to the rule as a voting rule and it is v. Again we have

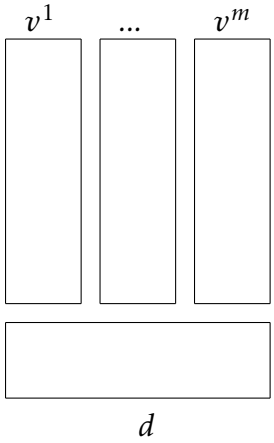
$$\begin{aligned} v: \quad & \mathbb{Z}_n^1 \to \{0,1\} \\ & S^a \to v(S^a). \end{aligned}$$

The set of minimal winning configurations of a decision (voting) rule is denoted by $\mathcal{M}(d)$ ($\mathcal{M}(v)$). Both d and v satisfy the properties of unanimity and monotonicity.

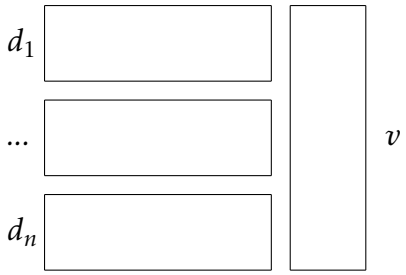
3 Separable Multi-dimensional rules

Some multi-dimensional rules can be represented as a composition of voting and decision rules. They are separable in the sense that the vote can be decomposed dimension by dimension or voter by voter. More precisely, if $\mathbf{w}$ is dimension separable, the rule can be decomposed into two stages: in the first stage, a vote on each dimension is held and then, a decision is made among the different dimensions. In the introductory example, the assessments of the experts would first be aggregated criterion by criterion to obtain a global evaluation in terms of performance, ability to work in a team, and leadership. Then, the three criteria would be aggregated. If $\mathbf{w}$ is voter-separable then, the rule can be decomposed into two stages: in the first stage, the decision per voter is determined, and, in the second stage, the final vote is cast among voters. In the example, each expert would first form their own evaluation of the candidate. Next, the experts' opinions would be aggregated.

Definition 2. *Let* $\mathbf{w} : \mathbb{Z}_n^m \to \{0,1\}$ *be a multi-dimensional rule. If there exist* m *voting rules* v^1, ..., $v^m : \mathbb{Z}_n^1 \to \{0,1\}$ *and one decision rule* $d : \mathbb{Z}_1^m \to \{0,1\}$, *such that for any* $\mathbf{S} \in \mathbb{Z}_n^m$ *we have* $\mathbf{w}(\mathbf{S}) = d(v^1(S^1),...,v^m(S^m))$ *then, we say that* $\mathbf{w}$ *is dimension-separable, and write* $\mathbf{w} = d\left[v^1,...,v^m\right]$.



Definition 3. *Let* $\mathbf{w} : \mathbb{Z}_n^m \to \{0,1\}$ *be a voting rule. If there exist* n *decision rules* d_1, ..., $d_n : \mathbb{Z}_1^m \to \{0,1\}$ *and one voting rule* $v : \mathbb{Z}_n^1 \to \{0,1\}$, *such that for any* $\mathbf{S} \in \mathbb{Z}_n^m$ *we have* $\mathbf{w}(\mathbf{S}) = v((d_1(S_1),...,d_n(S_n))^T)$ *then, we say that* $\mathbf{w}$ *is voter-separable, and write* $\mathbf{w} = v\,[d_1,...,d_n]$.



Several questions arise with separability. The first question is whether all rules are separable. The answer is negative: all possible cases can occur as will be illustrated in Section 5; we will provide examples of rules that are dimension-separable but not voter-separable or voter-separable but not dimension-separable (rules $\mathbf{w_1}$ and $\mathbf{w_4}$), and examples of rules that are both dimension- and voter-separable (rules $\mathbf{w_3}$ and $\mathbf{w_6}$) or neither dimension- nor voter-separable (rules $\mathbf{w_2}$ and $\mathbf{w_5}$).

Nevertheless, we have some necessary conditions for a multi-dimensional rule to be dimension- or voter-separable.

Definition 4. *Consider a multi-dimensional rule* $\mathbf{w}$ *represented by its minimal winning configurations* $\mathcal{M}(\mathbf{w})$*. Let* $\mathcal{S}^a(\mathbf{w})$ *be the set of the* a^{th} *columns of its minimal winning configurations that are non-null. That is, for* $a = 1,...m$*, let* $\mathcal{S}^a(\mathbf{w}) = \{S^a \neq \mathbf{O}_n^1 | \mathbf{S} \in \mathcal{M}(\mathbf{w})\}$*. A multi-dimensional rule satisfies condition* α *when* $\mathbf{w}(\mathbf{T}) = 1$ *for any* $\mathbf{T}$ *such that for any* $a = 1,...m$ *we have* $T^a \in \mathcal{S}^a(\mathbf{w})$ *.*

We first give a necessary condition to ensure that a multi-dimensional rule is dimension-separable.

Proposition 1. *If a multi-dimensional rule is dimension-separable then condition* α *is satisfied.*

Proof. Consider a configuration $\mathbf{T}$ such that $T^a \in \mathcal{S}^a(\mathbf{w})$ for any $a = 1,...,m$. There exists $\mathbf{S} \in \mathcal{M}(\mathbf{w})$ such that $S^a = T^a$ with $\mathbf{w}(\mathbf{S}) = 1$ and $\mathbf{w}(\tilde{\mathbf{S}}) = 0$ where $\tilde{S}^a = \mathbf{O}_n^1$ and $\tilde{S}^b = S^b$ for $b \neq a$. Moreover, $\mathbf{w}$ is dimension separable, *i.e.* $\mathbf{w}(\mathbf{S}) = d(v^1(S^1),...,v^m(S^m)) = 1$, and $\mathbf{w}(\tilde{\mathbf{S}}) = d(v^1(\tilde{S}^1),...,v^m(\tilde{S}^m)) = 0$. The only difference is $\tilde{S}^a \neq S^a$ and $\tilde{S}^a = \mathbf{O}_n^1$. Thus, $v^a(S^a) = 1$. Since $T^a = S^a$, it follows that $\mathbf{w}(\mathbf{T}) = d(\mathbf{I}_1^m) = 1$.

□

According to this proposition, any configuration built from columns of minimal winning configurations has to be winning when the multi-dimensional rule is

dimension-separable. For some special rules, condition α is a necessary and sufficient condition.

Proposition 2. *Let a multi-dimensional rule* $\mathbf{w}$ *such that for all minimal winning configurations* $\mathbf{S} \in \mathcal{M}(\mathbf{w})$ $S^a \neq \mathbf{O}_n^1$, *for any* $a \in M$. *We have:* $\mathbf{w}$ *is dimension-separable if and only if condition* α *is verified.*

Proof. Necessary condition:
Same proof as in Proposition 1.

Sufficient condition:
Let us assume that for any configuration $\mathbf{T}$ such that $T^a \in \mathcal{S}^a(\mathbf{w})$, $\mathbf{w}(\mathbf{T}) = 1$.
Let us construct m voting rules $v^1, ..., v^m : \mathbb{Z}_n^1 \to \{0,1\}$ and one decision rule $d : \mathbb{Z}_1^m \to \{0,1\}$, such that for any $\mathbf{S} \in \mathbb{Z}_n^m$ we have $\mathbf{w}(\mathbf{S}) = d(v^1(S^1),...,v^m(S^m))$.
For any $S^a \in \mathbb{Z}_n^1$ choose $v^a(S^a) = 1$ if and only if there exists $\tilde{S}^a \in \mathcal{S}^a(\mathbf{w})$ with $S^a \geq \tilde{S}^a$, and for any $R \in \mathbb{Z}_1^M$ let $d(R) = 1$ if and only if $R = \mathbf{I}_1^m$.

First, we have to show that $v^1, ..., v^m$ and d satisfy the properties of voting rules and a decision rule respectively. For any $a \in M$: $v^a(\mathbf{0}_n^1) = 0$ since $\mathbf{0}_n^1 \notin \mathcal{S}^a(\mathbf{w})$; $v^a(\mathbf{I}_n^1) = 1$ since there exist some $S^a \subseteq \mathbf{I}_n^1$ with $S^a \in \mathcal{S}^a(\mathbf{w})$; and v^a is monotonic by construction. For d it is obvious that all properties are satisfied since d is the unanimity rule.

Second, we have to show that $\mathbf{w}(\mathbf{S}) = d(v^1(S^1),...,v^m(S^m))$. This is the case given that d is the unanimity rule: we have $\mathbf{w}(\mathbf{S}) = 1$ if and only if $v^a(S^a) = 1$ for all $a \in M$.

□

From the proof it can be seen that whenever there is a non-null column among the minimal winning configurations, whenever w is dimension-separable, the decision rule is the unanimity rule. Then, we obtain as a corollary the following result:

Corollary 1. *Let a multi-dimensional rule* $\mathbf{w}$ *such that for all minimal winning configurations* $\mathbf{S} \in \mathcal{M}(\mathbf{w})$, $S^a \neq \mathbf{O}_n^1$ *for any* $a \in M$. *If* $\mathbf{w}$ *is dimension-separable then,* $\mathbf{w}$ *can be written as* $\mathbf{w} = d\left[v^1,...,v^m\right]$ *with* $\mathcal{M}(v^a) = \{S^a \in \mathbb{Z}_n^1 | S^a \in \mathcal{S}^a(\mathbf{w})\}$ *and* $\mathcal{M}(d) = \left\{\mathbf{I}_1^m\right\}$.

A similar approach can be adopted for voter-separability.

Definition 5. *Consider a multi-dimensional rule* $\mathbf{w}$ *represented by its minimal winning configurations* $\mathcal{M}(\mathbf{w})$. *Let* $\mathcal{S}_i(\mathbf{w})$ *be the set of the* i^{th} *rows of its minimal winning configurations. That is, for* $i = 1,...n$, *let* $\mathcal{S}_i(\mathbf{w}) = \{S_i \neq \mathbf{O}_1^m | \mathbf{S} \in \mathcal{M}(\mathbf{w})\}$. *A multi-dimensional rule satisfies condition* β *when* $\mathbf{w}(\mathbf{T}) = 1$ *for any* $\mathbf{T}$ *such that for any* $i = 1,...n$ *we have* $T_i \in \mathcal{S}_i(\mathbf{w})$.

Proposition 3. *If a multi-dimensional rule is voter-separable then condition β is satisfied.*

Proof. The proof is similar to the proof of the dimension rule. □

Proposition 4. *Let a multi-dimensional rule* $\mathbf{w}$ *such that for all minimal winning configurations* $\mathbf{S} \in \mathcal{M}(\mathbf{w})$, $S_i \neq \mathbf{O}_1^m$ *for any* $i \in N$. $\mathbf{w}$ *is voter-separable if and only if condition* β *is satisfied.*

Proof. The proof is similar to the proof of the dimension rule. □

Again, the same observation can be made: whenever there are non-null rows among the minimal winning configurations and the rule is voter-separable then, the voting rule is the unanimity rule.

Corollary 2. *Let a multi-dimensional rule* $\mathbf{w}$ *such that for all minimal winning configurations* $\mathbf{S} \in \mathcal{M}(\mathbf{w})$, $S_i \neq \mathbf{O}_1^m$ *for any* $i \in N$. *If* $\mathbf{w}$ *is voter-separable then,* $\mathbf{w}$ *can be written as* $\mathbf{w} = v[d_1, ..., d_n]$ *with* $\mathcal{M}(d_i) = \{S_i \in \mathbb{Z}_1^m | S_i \in \mathcal{S}_i(\mathbf{w})\}$ *and* $\mathcal{M}(v) = \left\{\mathbf{I}_n^1\right\}$.

Let us now make some remarks about the separability of some multi-dimensional rules.

Remark 1. *If* $|\mathcal{M}(\mathbf{w})| = 1$ *then* $\mathbf{w}$ *is dimension-separable and voter-separable. This proof follows from the definition of voter and dimension separability.*

Remark 2. *If* $\mathcal{M}(\mathbf{w}) = \{\mathbf{S} \mid s = 1\}$ *then* $\mathbf{w}$ *is dimension- and voter-separable. Rule* $\mathbf{w}$ *can be written as* $\mathbf{w} = d_1[v_1, ..., v_1]$ *and* $\mathbf{w} = v_1[d_1, ..., d_1]$ *with* $\mathcal{M}(d_1) = \{S_i \in \mathbb{Z}_1^m | s_i = 1\}$ *and* $\mathcal{M}(v_1) = \{S^a \in \mathbb{Z}_n^1 | s^a = 1\}$.

A rule, in which the minimum requirement for a minimal configuration is to have one positive vote regardless of the number of dimensions and voters, is dimension- and voter-separable. Note that this is not true when more that one positive vote is the minimum requirement (see $\mathbf{w}_2$*).*

Remark 3. *If* $\mathcal{M}(\mathbf{w}) = \{\mathbf{S} \mid s^1 =, ... = s^m = k\}$ *then* $\mathbf{w}$ *is dimension-separable. Rule* $\mathbf{w}$ *can be written as* $\mathbf{w} = d[v_1, ..., v_1]$ *with* $\mathcal{M}(d) = \left\{\mathbf{I}_1^m\right\}$ *and* $\mathcal{M}(v_1) = \{S^a \in \mathbb{Z}_n^1 | s^a = k\}$.

Remark 4. *If* $\mathcal{M}(\mathbf{w}) = \{\mathbf{S} \mid s_1 = ... = s_n = t\}$ *then* $\mathbf{w}$ *is voter-separable. Rule* $\mathbf{w}$ *can be written as* $\mathbf{w} = v[d_1, ..., d_1]$ *with* $\mathcal{M}(d_1) = \{S_i \in \mathbb{Z}_1^m | s_i = t\}$ *and* $\mathcal{M}(v) = \left\{\mathbf{I}_n^1\right\}$.

Remark 3 (respectively 4) concerns rules where for all configurations we want at least k (respectively t) positive votes by dimension (resp. voter) for each dimension

(resp. voter). Note that it is easy to show, with Proposition 1 (resp. 3), that these rules are not voter- (resp. dimension) separable.

The following remark concerns rules where for each winning configuration we want a positive vote for all voters for at least t dimensions.

Remark 5. *If* $\mathcal{M}(\mathbf{w}) = \{\mathbf{S} \mid s^a = n \text{ for } t \text{ columns}\}$ *then* $\mathbf{w}$ *is dimension-separable and can be written as* $\mathbf{w} = d[v_1, ..., v_1]$ *with* $\mathcal{M}(d) = \{S_i \in \mathbb{Z}_1^m | s_i = t\}$ *and* $\mathcal{M}(v_1) = \left\{\mathbf{I}_n^1\right\}$.

The converse is also true, *i.e* rules where a positive vote is necessary for all dimensions for at least i voters, are voter-separable.

Remark 6. *Let* $\mathcal{M}(\mathbf{w}) = \{\mathbf{S} \mid s_i = m \text{ for } p \text{ rows}\}$ *then* $\mathbf{w}$ *is voter-separable with* $\mathbf{w} = v[d_1, ..., d_1]$*with* $\mathcal{M}(d_1) = \{(1, ..., 1)\}$ *and* $\mathcal{M}(v) = \{S^a | s^a = p\}$.

Links to the social choice paradox

Note that we may have $v[d, .., d] \neq d[v, ..., v]$, even with $n = m$ and $S \in \mathcal{M}(d) \Leftrightarrow S^T \in \mathcal{M}(v)$. In the example, the result may be different if, as a first step, experts make their global opinion on the candidate (by aggregating the three criteria by simple majority) and, in the second step, the global opinions of the experts are aggregated by simple majority; or if, in the first step, each criterion is globally evaluated and, in the second step, the criteria are aggregated.

Consider $n = m = 3$ and d, and v being simple majorities.

Example 1. *Let* $\mathbf{w_{sm}}, \bar{\mathbf{w}}_{\mathbf{sm}} : \mathbb{Z}_3^3 \to \{0, 1\}$ *with* $\mathbf{w_{sm}} = d_{sm}[v_{sm}, v_{sm}, v_{sm}]$ *and* $\bar{\mathbf{w}}_{\mathbf{sm}} = v_{sm}[d_{sm}, d_{sm}, d_{sm}]$ *with* $\mathcal{M}(d) = \left\{(1,1,0), (1,0,1), (0,1,1)\right\}$ *and* $\mathcal{M}(v) = \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$. *These two rules do not coincide as we can find a configuration such that* $\mathbf{w_{sm}}(\mathbf{S}) \neq \bar{\mathbf{w}}_{\mathbf{sm}}(\mathbf{S})$. *Choose* $\mathbf{S} = \begin{pmatrix}1&1&0\\1&0&0\\0&1&0\end{pmatrix}$. *We have* $v_{sm}\begin{pmatrix}1\\1\\0\end{pmatrix} = 1$, $v_{sm}\begin{pmatrix}1\\0\\1\end{pmatrix} = 1$, $v_{sm}\begin{pmatrix}0\\0\\0\end{pmatrix} = 0$, $d_{sm}(1,1,0) = 1$, *and thus* $\mathbf{w_{sm}}(\mathbf{S}) = 1$. *Likewise, we have* $d_{sm}(1,1,0) = 1$, $d_{sm}(1,0,0) = 0$, $d_{sm}(0,1,0) = 0$, $v_{sm}\begin{pmatrix}1\\0\\0\end{pmatrix} = 0$ *and thus,* $\bar{\mathbf{w}}_{\mathbf{sm}}(\mathbf{S}) = 0$.

This result is related to the discursive dilemma or the doctrinal paradox (see List and Puppe [2009]) in judgment aggregation. It is referred to as the Ostrogorski

paradox (Ostrogorski [1970]) in social choice theory: the paradox holds whenever $v[d,..,d] \neq d[v,...,v]$ where d and v are simple majorities. A majority of voters who vote positively on a majority of issues is not equivalent to a majority of issues on which a majority of voters vote positively.

How can we go further in the analysis of the Ostrogorski paradox? In Social Choice literature, many authors propose an answer to these questions. Deb and Kesley [1987] provide a necessary and sufficient condition for the appearance of the paradox, which is based on the number of voters and dimensions. The Ostrogorski paradox may prevail if and only if $n.m - 6.m.x - 2.n.y - 4xy \geq 0$, where x (resp. y)= 1 if n (resp. m) is even, and x (resp. y)= 0.5 if n (resp. m) is odd. Such a condition implies that the paradox is almost unavoidable unless restricted to low-dimensional voting problems. For example, it is easy to show that for $m = 2$, the paradox does not hold. Mbih and Valeu [2016] evaluate the likelihood of the paradox under classical probabilistic models. One of their main conclusions is that the occurrence of the paradox increases when both the number of voters and issues rise, conjecturing a limit value around 30%. Laffond and Lainé [2006] choose to introduce a restriction in the domain of admissible preferences of the voters. They prove that, for any number of candidates and voters, there is no paradox if the vote in the configuration is "single-switch". This means that for each voter there is an ordering of the votes on all dimensions such that all votes exhibit a unique switch between a positive one and a negative one. Another way to avoid the paradox is to go beyond the simple majority rule. Deb and Kesley [1987] also show that the avoidance of the Ostrogorski paradox normally requires very large qualified majorities in determining the winners on each dimension [2].

Another related paradox is the referendum paradox (see Laffond and Lainé [2000]) in which two-stage majorities are not equivalent to a global majority. Again,we have an example with $n = m = 3$.

Example 2. *Let* $\dot{\mathbf{w}}_{\mathbf{sm}} \in \mathbb{Z}_3^3 \to \{0,1\}$ *with* $\dot{\mathbf{w}}_{\mathbf{sm}}(\mathbf{S}) = \begin{cases} 1 \text{ if } s > 4 \\ 0 \text{ otherwise} \end{cases}$ *and* $\mathbf{w}_{\mathbf{sm}} \in \mathbb{Z}_3^3 \to \{0,1\}$*:* $\mathbf{w}_{\mathbf{sm}} = d_{sm}[v_{sm}, v_{sm}, v_{sm}]$. *We have* $\dot{\mathbf{w}}_{\mathbf{sm}} \neq \mathbf{w}_{\mathbf{sm}}$ *as we can find a configuration such that* $\mathbf{w}_{\mathbf{sm}}(\mathbf{S}) \neq \dot{\mathbf{w}}_{\mathbf{sm}}(\mathbf{S})$. *Choosing again* $\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ *we have* $\dot{\mathbf{w}}_{\mathbf{sm}}(\mathbf{S}) = 0$ *and* $\mathbf{w}_{\mathbf{sm}}(\mathbf{S}) = 1$.

[2] Deb and Kesley [1987] introduce a generalized Ostrogorski paradox, in which the simple majorities are replaced by qualified majorities.

It is evident that one condition to avoid the paradox is to have a global rule which is dimension-separable. Now consider the rules given by $\mathcal{M}(\mathbf{w}) = \{\mathbf{S}, ..., \bar{\mathbf{S}}\}$ with $s \geq \frac{n.m}{2}, ..., \bar{s} \geq \frac{n.m}{2}$. According to Remark 2, such rules are not dimension-separable, meaning that the paradox cannot be avoided with qualified majorities for the global rule. However, as suggested by Laffond and Lainé [2000], a global rule where we have at least 75% of positive votes by dimension is immune to the paradox regardless of the decision rule.

Proposition 5. *Let* $\mathcal{M}(\mathbf{w}) = \{\mathbf{S} \mid s^a = \frac{3}{4}.n \text{ for all } a\}$ *then* $\mathbf{w}$ *can be written as* $\mathbf{w} = d\left[v_{\frac{3}{4}}, v_{\frac{3}{4}}, v_{\frac{3}{4}}\right]$ *where* $v_{\frac{3}{4}}$ *the* $\frac{3}{4}$*–majority with* $\mathcal{M}(v_{\frac{3}{4}}) = \{S^a \in \mathbb{Z}_n^1 \left| s^a = \frac{3}{4}.n\right.\}$.

Proof. See Laffond and Lainé [2000] □

4 Weighted Multi-dimensional rule

In students evaluations, different periods and different subjects may be weighted differently: some tests may be more important than others. Therefore, we introduce a multi-dimensional weighted rule.

A multi-dimensional rule with n voters and m dimensions

$$\begin{aligned} \mathbf{w}: \quad & \mathbb{Z}_n^m \to \{0,1\} \\ & \mathbf{S} \longmapsto \mathbf{w}(\mathbf{S}) \end{aligned}$$

is weighted if there exist a quota q and a $m * n$ matrix $U = (u_i^a)_{i \in N, a \in M}$ such that $\mathbf{w}(\mathbf{S}) = 1$ if and only if $u(\mathbf{S}) \geq q$ where $u(\mathbf{S}) = \sum_{i \in N} \sum_{a \in M} u_i^a s_i^a$. We denote a weighted rule by $\mathbf{w} \equiv [q; U]$.

Particular cases include weighted voting rules ($n = 1$) and weighted decision rules ($m = 1$). In a weighted voting rule, each vote has a specific weight and in a weighted decision rule, each dimension is weighted.

Taylor and Zwicker [1992, 1999] prove that a voting rule is weighted if and only if the voting rule is trade robust.[3] The meaning of this theorem is very simple: consider a voting rule with two winning coalitions. Assume that there is an exchange of simply two voters between the two winning coalitions. According to the trade

[3] Theorem A in Taylor and Zwicker [1992] concerns all voting rule whereas Theorem 2.4.2 in Taylor and Zwicker [1999] is given for monotonic voting rules.

robust property, at least one of the coalitions should still be a winning configuration after the trade, which is consistent with the fact that the simple game is weighted. It is indeed the case since one of the coalitions must have lost weight to become a loser as the result of an exchange, which led the other to gain weight, therefore reinforcing its winning position.

We first generalize the notion of trade robustness.

Definition 6. *A sequence of voting configurations* $\mathcal{C} = (\mathbf{S}, ..., \bar{\mathbf{S}})$ *is a trading transform if it can be converted into a sequence* $\mathcal{C}' = (\mathbf{T}, ..., \bar{\mathbf{T}})$ *by trading votes, with* $|\mathcal{C}| = |\mathcal{C}'|$*. Formally,*

$\sum_{\mathbf{S}\in\mathcal{C}} s_i^a = \sum_{\mathbf{T}\in\mathcal{C}'} t_i^a$ *for all* $i \in N$ *and all* $a \in M$

Example 3. *Three subjects* ($n = 3$) *are taught over three semesters* ($m = 3$)*. Let two voting configurations* $\mathbf{S} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *and* $\bar{\mathbf{S}} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ *and two other configurations* $\mathbf{T} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ *and* $\bar{\mathbf{T}} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ *obtained from* $\mathbf{S}$ *and* $\bar{\mathbf{S}}$ *by the trade of* s_1^1 *and* $\bar{s}_1^1$ *and the trade of* s_3^1 *and* $\bar{s}_3^1$ *.*

The notion of trade robustness for multi-dimensional rules is given by the following definition.

Definition 7. *A multi-dimensional rule is trade robust if, for a sequence* $\mathcal{C}$ *of winning configurations, and a sequence* $\mathcal{C}'$ *obtained from* $\mathcal{C}$ *by trades, at least one configuration in* $\mathcal{C}'$ *is winning.*

Once again, consider a rule with only two winning configurations. Let a trade in which a voter exchanges a positive vote in the first winning configuration for a negative vote in the second. This trade can transform the two winning configurations into two losing configurations, though not if the rule is weighted. Intuitively, the loss of weight (negative vote instead of a positive vote) in the first configuration implies a gain of weight in the second winning configuration, which cannot be a losing configuration.

We can now generalize Theorem 2.4.2 of Taylor and Zwicker [1999] results in the context of multi-dimensional rules.

Theorem 1. *A multi-dimensional rule* $\mathbf{w}$ *is weighted if and only if* $\mathbf{w}$ *is trade robust.*

Proof. The proof is in the appendix. □

Note that our result is true only for monotonic multi-dimensional rules. However, one can easily extend our result to rules without the monotonicity condition. Note also that the proof makes no reference to the unanimity conditions. So one can rule out these two conditions in a more generalized definition of a multi-dimensional rule, and then directly extend Theorem A of Taylor and Zwicker [1992].

Definition 8. *Generalized rules*

- *A generalized multi-dimensional rule denoted by* $\hat{\mathbf{w}}$ *will be any triple* $(N, M, \mathcal{W}(\hat{\mathbf{w}}))$ *with* $\mathcal{W}(\hat{\mathbf{w}})$ *being an arbitrary collection of configurations.*
- *A generalized voting rule denoted by* $\hat{v}$ *will be any pair* $(N, \mathcal{W}(\hat{v}))$ *with* $\mathcal{W}(\hat{v})$ *being an arbitrary collection of configurations with only one dimension.*

Theorem 2. *A generalized multi-dimensional rule* $\hat{\mathbf{w}}$ *is weighted if and only if* $\hat{\mathbf{w}}$ *is trade robust.* [4]

We omit the the proof as it is similar to the one of Theorem 1 (it only uses Theorem A of Taylor and Zwicker [1992] in the ($\Leftarrow$) part).

5 Weightedness versus separability

The following examples illustrate that the notions of separability and weightedness are different. Each of them is defined by its set of minimal winning configurations.

Consider that two subjects ($n = 2$) are taught in m periods. Rules $\mathbf{w}_1$ to $\mathbf{w}_4$ involve 2 periods. Rule $\mathbf{w}_5$ is concerned with three periods, whereas rule $\mathbf{w}_6$ deals with five periods. Every semester, there is a test on each subject, with $s_i^a = 1$ if the student passes test i in semester a (and $s_i^a = 0$ if the student fails). Rule $\mathbf{w}$ ultimately determines the student's success.

For each rule, we examine the separability, or not, and the weightedness, or not.

1. ($\mathbf{w}_1$): the student has to be successful in both subjects in at least one semester in order to pass;

$$\mathcal{M}(\mathbf{w}_1) = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right\}.$$

[4] The definition of trade robustness is independent of the class of games defined.

Rule $\mathbf{w_1}$ is dimension-separable (see Remark 5); but $\mathbf{w_1}$ is not voter-separable since we have $\mathbf{w}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 0$, a contradiction with Proposition 3. $\mathbf{w}_1$ is not weighted either, as can be shown by choosing $\tilde{\mathbf{S}} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, $\bar{\mathbf{S}} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $\tilde{\mathbf{T}} = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $\bar{\mathbf{T}} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}$.

By symmetry of separability properties, one can easily find a rule which is not dimension-separable, not weighted but voter-separable.

2. ($\mathbf{w}_2$) at least three tests (no matter the period or the subject)

$$\mathcal{M}(\mathbf{w}_2) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Rule $\mathbf{w_2}$ is not dimension-separable since we have $\mathbf{w}\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} = 0$ a contradiction with Proposition 1. $\mathbf{w_2}$ is not voter-separable either since we have $\mathbf{w}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$, a contradiction with Proposition 3. Finally, $\mathbf{w}_2$ is weighted as illustrated by a quota $q = 6$ and $U = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$.

3. ($\mathbf{w}_3$) all tests

$$\mathcal{M}(\mathbf{w}_3) = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}.$$

Based on Remark 1, $\mathbf{w}_3$ is both dimension- and voter-separable. Taking $q = 8$ and $u = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ show that $\mathbf{w}_3$ is weighted.

4. ($\mathbf{w_4}$) in the first subject, the test in the first period is compulsory, as is at least one test in the second period;

$$\mathcal{M}(\mathbf{w_4}) = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

$\mathbf{w_4}$ can be written $\mathbf{w_4} = d_4\left[v_4^a, v_4^b\right]$ with $\mathcal{M}(d_4) = \left\{\mathbf{I}_1^m\right\}$, $\mathcal{M}(v_4^a) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$ and $\mathcal{M}(v_4^b) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$. $\mathbf{w_4}$ is not voter-separable since we have $\mathbf{w}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = 0$, a contradiction with Proposition 3. Rule $\mathbf{w_4}$ is weighted, as illustrated with $q = 6$ and $U = \begin{pmatrix} 4 & 2 \\ 1 & 2 \end{pmatrix}$.

Once again, by symmetry, one can easily find a weighted rule which is voter-separable but not dimension-separable.

5. ($\mathbf{w_5}$) at least three tests for the first two periods and at least one test for the third period (regardless of the subjects).

$$\mathcal{M}(\mathbf{w_5}) = \left\{\begin{array}{l} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \\ \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \end{array}\right\}.$$

$\mathbf{w}_5$ is not dimension-separable since $\mathbf{w}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 0$, a contradiction with 1. $\mathbf{w}_5$ is not voter-separable since $\mathbf{w}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix} = 0$, a contradiction with Proposition 3. Moreover, $\mathbf{w}_5$ is not weighted as it can be illustrated by taking $\tilde{\mathbf{S}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\bar{\mathbf{S}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, $\tilde{\mathbf{T}} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ and $\bar{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$.

6. ($\mathbf{w_6}$) for subject one, at least 2 tests during the first three semesters and at least one test in the last two semesters; it is of no importance for subject two

$$\mathcal{M}(\mathbf{w}_6) = \left\{\begin{array}{l} \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{array}\right\}.$$

Rule $\mathbf{w}_6$ can be written as $\mathbf{w}_6 = d_6[v_6, v_6, v_6, v_6, v_6] = v_6[d_6, d_6]$ with $\mathcal{M}(d_6) =$

$\left\{(1,1,0,1,0),(1,1,0,0,1),(1,0,1,1,0),(1,0,1,0,1),(0,1,1,1,0),(0,1,1,0,1)\right\}$ and $\mathcal{M}(v_6) = \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$. Rule $\mathbf{w}_6$ is not weighted as illustrated by choosing $\tilde{\mathbf{S}} = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\bar{\mathbf{S}} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$, $\tilde{\mathbf{T}} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and $\bar{\mathbf{T}} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$.

The following table summarizes the previous results.

Table 1: *Rules Comparison*

	Weighted	*Dimension separable*	*Voter separable*
w_1	no (no)	yes (no)	no (yes)
w_2	yes	no	no
w_3	yes	yes	yes
w_4	yes (yes)	yes (no)	no (yes)
w_5	no	no	no
w_6	no	yes	yes

Appendix

Proof of Theorem 1

Proof. ($\Rightarrow$) Assume $\mathbf{w}$ is a weighted multi-dimensional rule, let $\mathcal{C} = (\mathbf{S},...,\bar{\mathbf{S}})$ be a sequence of winning configurations, and $\mathcal{C}' = (\mathbf{T},...,\bar{\mathbf{T}})$ be a second sequence of configurations obtained from $\mathcal{C}$ by trades. Note that the sum $u(\mathbf{S}) + ... + u(\bar{\mathbf{S}})$ can be rewritten as $\sum_{\mathbf{S}\in\mathcal{C}} \sum_{i\in N} \sum_{a\in M} u_i^a s_i^a$, that is $\sum_{a\in M} \sum_{i\in N} u_i^a (\sum_{\mathbf{S}\in\mathcal{C}} s_i^a)$. By definition, $\sum_{\mathbf{S}\in\mathcal{C}} s_i^a = \sum_{\mathbf{T}\in\mathcal{C}'} t_i^a$ for all $i \in N$ and all $a \in M$ it follows that $u(\mathbf{S}) + ... + u(\bar{\mathbf{S}}) = u(\mathbf{T}) + ... + u(\bar{\mathbf{T}})$.

Now consider $u(\hat{\mathbf{T}}) = max\{u(\mathbf{T})|\mathbf{T} \in \mathcal{C}'\}$. We have $u(\hat{\mathbf{T}}) \geq \frac{u(\mathbf{S})+...+u(\bar{\mathbf{S}})}{|\mathcal{C}|} \geq \frac{q.|\mathcal{C}|}{|\mathcal{S}|} = q$.

Then, $\hat{\mathbf{T}}$ is winning and $\mathbf{w}$ is trade robust.

($\Leftarrow$) Assume $\mathbf{w}$ is trade robust. We define a voting rule $v^*(\mathbf{w})$ with $n*m$ voters by replicating m times the set of voters (one per dimension). Let $N^* = N^1 \cup ... \cup N^m$ be the set of $n*m$ voters. Let $S^* = ((S^1)^T,...,(S^m)^T))$. Consider now the voting rule with $n*m$ voters, defined on N^* by

$$\mathcal{M}(v^*(\mathbf{w})) = \{S^* \in \mathcal{M}(v^*(\mathbf{w})) \Leftrightarrow \mathbf{S} \in \mathcal{M}(\mathbf{w})\}$$

If $\mathbf{w}$ is trade robust then by definition, $v^*(\mathbf{w})$ is trade robust. It follows by Theorem 2.4.2 of Taylor and Zwicker [1999] that $v^*(\mathbf{w})$ is weighted: $v^*(\mathbf{w}) \equiv [q; U^*]$, with $U^* = (u_1^1, ..., u_n^m)$ where u_i^a is the weight assigned to the member in N^* for each $1 \leq a \leq m$ and $1 \leq i \leq n$. From U^* we define $U = \begin{pmatrix} u_1^1 & \cdots & u_1^m \\ : & & : \\ u_n^1 & \cdots & u_n^m \end{pmatrix}$. It follows that $\mathbf{w}$ is weighted.

This concludes the proof. □

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References

N. Andjiga, F. Chantreuil, and D. Lepelley. La mesure du pouvoir de vote. *Mathematiques et Sciences Humaines*, 163:111–145, 2003.

R. Deb and D. Kesley. On constructing a generalized Ostrogorski paradox: necessary and sufficient conditions. *Mathematical Social Sciences*, 14:161–174, 1987.

M. Feix, D. Lepelley, V. Merlin, and J.L. Rouet. The probability of US presidential type election paradoxes. *Economic Theory*, 23:227–258, 2004.

D.S. Felsenthal and M. Machover. Ternary voting games. *International Journal of Game Theory*, 26:335–351, 1997.

J. Freixas. The Shapley Shubik power index for games with several levels of approval in the input and output. *Decision Support Systems*, 39:185–195, 2005.

J. Freixas, B. Tchantcho, and N. Tedjeugang. Voting games with abstention: Linking completeness and weightedness. *Decision support systems*, 57:172–177, 2014.

G. Laffond and J. Lainé. Representation in majority tournaments. *Mathematical Social Sciences*, 39:35–53, 2000.

G. Laffond and J. Lainé. Single-switch preferences and the ostrogorski paradox. *Mathematical Social Sciences*, 52:49–66, 2006.

A. Laruelle and F. Valenciano. *Voting and collective decision-making*. Cambridge University Press, 1st edition, 2008.

A. Laruelle and F. Valenciano. Quaternary dichotomous voting rules. *Social Choice and Welfare*, 38:431–454, 2012.

C. List and C. Puppe. Judgement aggregation a survey. In C. List and C. Puppe, editors, *Handbook of Rational and Social Choice*. Oxford University Press, 2009.

B. Mbih and A. Valeu. La vulnérabilité de la règle de la majorité aux paradoxes d'anscombe et d' Ostrogorski. *Revue internationales des economistes de langue française*, 1:171–191, 2016.

M. Ostrogorski. *Democracy and the Organization of Political Parties, vol. I and II*. Haskell House Publishers, New York, 1970.

A.D. Taylor and W. Zwicker. A characterization of weighted voting. *Proc Am Math Soc*, 115:1089–1094, 1992.

A.D. Taylor and W. Zwicker. *Simple games: desirability relations, trading, and pseudoweightings.* Princeton University Press, New Jersey, 1999.