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# Relativity at first order and gravitational deflection of light

## *An interplay between history and didactics*

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*Abstract:* Einstein's failure to derive in his Zurich notebook the equations of general relativity in vacuum was, as well documented, mainly due to his 1912 metric which ignored the influence of gravitation on its spatial part, even at first order. We argue that a complete application of the equivalence principle using relativity at first order in  $V/c$  would have enabled him to obtain the correct deflection of light (as well as probably Mercury's perihelion advance) as soon as 1911. The key point is that the transformations  $x'=x-Vt$ ,  $t'=t-Vx/c^2$  (Lorentz 1895) allow to discuss not only time issues (as Einstein did in 1907 and 1911) but also issues concerning space; in particular, they imply not only a "time dilation"  $T'=T(1-Vv/c^2)$  for moving clocks but also a "length contraction"  $l'=l(1+Vv/c^2)$  for moving rods, two inverse relations leading simply to the correct metric at first order. More generally, these transformations present today a great pedagogical interest for the teaching of special relativity at an elementary level.

*Keywords:* Einstein, Light deflection, Lorentz transformations, Equivalence Principle.

### 1. 1895 Lorentz transformations and relativity before 1905

In 1818, Fresnel interpreted Arago's observations of stars through a prism by: "*The motion of our globe must have no observable influence on apparent reflexion*". In order to explain it, he calculated the requisite velocity of light in the prism with respect to the ether  $v = c/n + V(1 - 1/n^2)$ . Fresnel's issue concerning geometrical optics has become Lorentz' one concerning electromagnetism after Maxwell theory of light. In 1895 Lorentz introduced his theory of corresponding states for the study of moving dielectrics [1] to which he addressed already in 1892. One of his major successes was the explanation of Fresnel's ether drag formula from Maxwell equations and the electromagnetic (e.m) force on microscopic charges. The originality of his 1895 approach lied in a change of variables

$$\vec{r}' = \vec{r} - \vec{V}t, \quad t' = t - \vec{V} \cdot \vec{r}/c^2 \quad (V \ll c) \quad (1)$$

and of fields  $\vec{E}' = \vec{E} + \vec{V} \times \vec{B}$ ,  $\vec{B}' = \vec{B} - \vec{V} \times \vec{E}/c^2$ , bringing the study of charges inside a dielectric in uniform motion at global velocity  $\vec{V}$  to that of charges in a dielectric at rest. The change of spatial variables was natural since it amounts to following the dielectric (frame  $R'$ ) in his translation in the ether frame  $R$ . The introduction of  $t'$  was new and its necessity probably was a surprise for Lorentz. This may be a reason why he spoke of  $t'$  as a “*fictitious variable*” (the “*local time*”) <sup>1</sup> although he did not hesitate to explain Fresnel’s formula as well as Doppler and aberration effects by simply rewriting the phase  $\omega t - \vec{k} \cdot \vec{r}$  of a plane wave with the prime variables. Today, we know, after Poincaré [5], that (1) is the infinitesimal version of the 1905 transformations named by him “Lorentz transformations” (L.T) which, besides rotations, leave invariant the equations of e.m and Lorentz 1904 law of dynamics  $\vec{F} = d\vec{p}/dt$  with  $\vec{p} = m\gamma(v)\vec{v}$  ( $\gamma(v) = \sqrt{1 - v^2/c^2}$ ). This allows us to speak of “relativity at first order” [6] for the physical results involving (1). Einstein himself indirectly testifies to this relativity in his June 1905 paper [7] when, after claiming that “*the same laws of electrodynamics and optics will be valid for all coordinate systems in which the equations of Mechanics holds*”, he immediately adds “*as has already been shown for quantities of first order*”.

Lorentz’ work has been largely referred to, and extended, in the e.m community before 1905. As soon as 1898, in a paper which has interested Einstein in September 1899 [8], Wien [9] recalled the expression of  $t'$  and no less than 13 experiments (with positive or negative results) agreeing with it. The same year, Liénard [10] presented Lorentz’ work with complements concerning the transformations of charge densities, of e.m volumic forces and Maxwell tensor. In his well-known 1901 treatise “Electricité et Optique” [11] (following his 1899 Sorbonne lectures), Poincaré expressed in Chap. VI as a theorem the invariance of Maxwell eqs. for dielectrics with use of  $t', \vec{r}'$ , and conjectured its validity at any order ; in Chap. VII, he derived Liénard’s results in relation to their possible influence on “*observable mechanical effects*”. Independently, in 1900, in a paper intended to Lorentz *Festschrift* [2], he used  $\vec{r}'$  and  $t'$  as coordinates of events in  $R'$  in order to obtain the change  $L' = L(1 + V/c)$  of the length of a wave packet emitted by a resonator slowly moving in the same direction. His goal was to deduce from the fields transformation the change  $E'_\gamma = E_\gamma(1 - V/c)$  of the wave packet energy  $E_\gamma$  and that of momentum  $p_\gamma = E_\gamma/c$ , and to discuss the recoils of the resonator in frames  $R$  and  $R'$ . This paper, quoted by Einstein in 1906 [12], contained implicitly the variation of the resonator mass  $\Delta m = -E_\gamma/c^2$  [6].

As we now show, relativity at first order is also the one used in fact by Einstein in his 1907 [13] and 1911 [14] applications of the “Equivalence Principle” (E.P) (section 2), and his “bumpy road” to General Relativity [15] is in large part due to his ignorance that relativity at first order allows to address length issues (section 3). In conclusion, we briefly discuss the interest of this relativity for present teaching.

## 2. Einstein’s 1907 and 1911 applications of the Equivalence Principle and light deflection; the 1912 incomplete metric

In chap. V *Principle of relativity and gravitation* of his 1907 review article on relativity [13], Einstein assumes “*the complete physical equivalence of a gravitational field and the corresponding acceleration of the reference system*”. Assimilating an accelerated frame  $\Sigma$  to a continuous succession of instantaneous inertial frames  $R_t$ , he considers two identical clocks of  $\Sigma$ , one located at origin  $O$  ( $x = 0$ ) with proper time  $t$ , and one at  $H$  ( $x = h$ ) on the axis  $x$  of the acceleration. Being equally accelerated, the walk of clocks is the same in any inertial reference frame, e.g.  $\delta t$  in  $R_t$ . However, the simultaneity being not the same in  $R_t$  and in  $R_{t+\delta t}$  which moves at velocity  $V = a\delta t$  with respect to

<sup>1</sup> A more serious reason is his attachment to the ether at that time. In 1900, Poincaré presents  $t'$  as the result of the synchronization of watches in  $R'$  [2] and in 1902 he quotes local time “Lorentz’ ingenious invention” in his letter to the Nobel committee [3]. In his 1905 treatise [4], Abraham deduces  $t'$  from  $x' = x - Vt$  and the requirement that  $x = ct$  implies  $x' = ct'$ .

$R_t$ , when the clock at  $H$  indicates  $\delta t$ , the corresponding one in  $R_{t+\delta t}$  indicates (cf. Eq. (1))  $\delta t' = \delta t - Vh/c^2 = \delta t(1 - ah/c^2) \neq \delta t$ . From this apparent contradiction, Einstein deduces the relation between the proper times of the clocks at  $H$  and  $O$ :  $\tau_H = \tau_O(1 + ah/c^2)$ . Using the E.P ( $a = g$ ) and noting that  $gh$  is the gravitational potential difference between  $H$  and  $O$ , he extends this relation at first order in the potential  $\varphi$  to  $\tau = t(1 + \varphi/c^2)$ ,  $t$  being the proper time at infinity (where  $\varphi = 0$ ). As well known, he predicts from this relation the gravitational redshift. In the following of the paper, the new writing of Maxwell eqs. in  $\Sigma$ , leads him to a spatially dependent light velocity (measured with respect to time  $t$ ):  $c(\vec{r}) = c(1 + \varphi(\vec{r})/c^2)$  from which he quotes the possibility of an (unobservable) gravitational deflection of light by the Earth.

In 1911 [14], Einstein comes back to his 1907 use of the E.P, firstly because his “former treatment of the subject does not satisfy” him, secondly because light deflection by the sun “amounts almost one second of arc” and could be measured. He thinks of a source at the height  $h$  sending a light signal of frequency  $\nu_h$  and energy  $E_h$ , towards a receptor on the ground. At reception after the travel time  $h/c$ , everything occurs as if the receptor was in motion with an upwards velocity  $V = gh/c$ . The received energy is  $E_0 = E_h(1 + V/c)$  (first order transformation of the energy of a “light complex”) and the frequency is  $\nu_0 = \nu_h(1 + V/c)$  (Doppler effect)<sup>2</sup>. Einstein then interprets the difference between  $\nu_0$  and  $\nu_h$  (a paradoxical one since the field is stationary with respect to Newtonian time  $t$ ) as being due to a difference in the proper times of (identical) clocks at  $z = 0$  and  $z = h$ . He deduces again the 1907 relation  $\tau = t(1 + \varphi/c^2)$ . Considering like in 1907 that lengths are not affected by acceleration, he recovers the light velocity  $c(\vec{r}) = c(1 + \varphi(\vec{r})/c^2)$  and applies Huygens principle to obtain the angular deviation  $\delta\theta = 2GM/rc^2$  (half of the well-known correct value) for light passing the sun at the distance  $r$ .

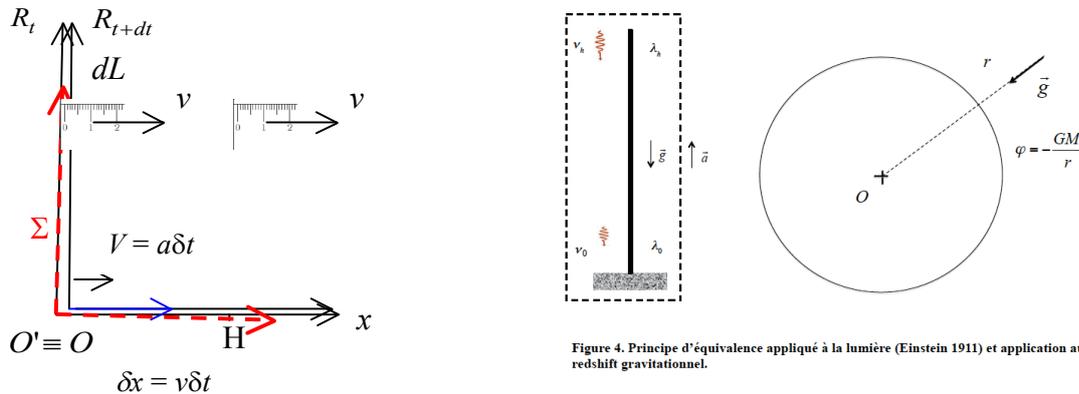


Figure 4. Principe d'équivalence appliqué à la lumière (Einstein 1911) et application au redshift gravitationnel.

In 1912, as well as in 1907 and 1911, Einstein does not consider any issue for lengths. Finally, he introduces in March

$$ds^2 = (1 + 2\varphi/c^2)dt^2 - (dx^2 + dy^2 + dz^2)/c^2$$

in the Least Action Principle describing free-fall [17]. The metric  $ds^2$  with  $\varphi = -GM/r$  for the potential due to a spherical mass  $M$  at origin, will be in large part responsible for his « bumpy road » [15, 18] to GR, before his explanation in November 1915 of Mercury perihelion advance and light deflection.

<sup>2</sup> Clearly Einstein's reasoning on energy, which endows the light complex with the potential energy  $ghE_h/c^2$ , is related to his interest in quanta. On 13<sup>th</sup> May 1911 [16, Doc. 267 p. 187] Einstein informs Michele Besso that, concerning light quanta, he “rummage[s] through the consequences as carefully as possible so as to learn about the range of applicability of this conception”. On June 9 [16, Doc. 269 p. 190], Ernest Solvay invites him to speak on “Specific heat and the theory of quanta” leaving to Sommerfeld “The application of the theory of quanta to a series of problems in physics” and telling him that he intends himself to speak of energy and gravitation. No surprise that Einstein's paper is presented on June 21 and begins with a discussion on energy!

### 3. Relativity at first order and the tests of GR

#### 3.1. The issues of lengths and wavelengths forgotten by Einstein

Let us note first that a slice of length  $\lambda$  of a plane wave (phase amplitude  $2\pi$ ) propagating along the  $x$ -axis lies in the region  $0 < x - ct < \lambda$ , which is mathematically similar to that  $0 < x - vt < l$  occupied by a rule of apparent length  $l$  moving along this axis at velocity  $v$ . A simple calculation shows that this region reads in  $R'$  [6]:  $0 < x' - v't' < l'$  with  $l' = (1 + vV/c^2)l$  and  $v' = v - V(1 - v^2/c^2)$  (of which Fresnel's formula is a particular case). For  $v = c$  and  $l = \lambda$ , one recovers Doppler effect, which allows to consider the wavelength as a standard length moving at velocity  $c$ .<sup>3</sup> As a consequence, Einstein should have written in 1911  $\lambda_0 = \lambda_h(1 - gh/c)$  for wavelengths and (with the same reasoning as for frequencies) should have obtained the proper lengths in the direction of the acceleration  $l_h = l_0(1 - \varphi/c^2)$ .

Another derivation, based on Einstein's 1907 view of an accelerated system as a succession of inertial frames, consists in considering a rule of infinitesimal length  $dL$  in uniform motion at velocity  $v$  in  $R_t$ , which at time  $t$  passes through  $O$  and which at time  $t+\delta t$  passes through the point  $H$  of abscissa  $\delta x = v\delta t$  [6]. Its length measured when it passes at  $H$  is then in  $R_{t+\delta t}$ :  $dL' = dL(1 + vV/c^2) = dL(1 + va\delta t/c^2) = dL(1 + a\delta x/c^2)$ . Once more one deduces the space-dependence of proper lengths in  $\Sigma$ :  $dL_H = dL_O(1 - a\delta x/c^2)$ .

#### 3.2. Consequences for the metric: light deflection and Mercury perihelion shift

The above reasoning either on wavelengths or on moving rules, leads to the metric, in presence of the radial gravitational field of a spherical mass  $M$ :

$$ds^2 = (1 - 2GM/rc^2)dt^2 - [(1 + 2GM/rc^2)dr^2 + r^2d\Omega^2]/c^2.$$

Remarkably, this metric is Schwarzschild's one at first order in the gravitational potential. With it, Einstein should have obtained the correct value of light deflection as well as Mercury perihelion advance (of which he has been thinking from 1907 with his friend Michele Besso).<sup>4</sup> In addition, it satisfies the equation  $R_{\mu\nu} = 0$  at first order outside matter, an equation he has probably discussed by the end of 1912 with his friend Marcel Grossmann, but which was left aside because of its incompatibility with the incomplete 1912 metric.

#### 3.3. Conclusion

In present teaching, special relativity is taught through the exact L.T, which prevents from discussing it in detail before university. As we have seen, 1895 L.T on the contrary are simple to introduce as a correction to Galileo's ones (note 1). In addition, they are deeply connected to historical relativistic issues (Fresnel (1818), Lorentz (1895), Poincaré (1900), Einstein (1907) and (1911)) concerning many domains of physics: geometrical optics, e.m., mechanics, gravitation and the E.P, and even the tests of general relativity. The interested reader can find in [6] suggestions for the derivation from (Eq. 1) of relativistic kinematics and dynamics, the key point being the search for invariants.<sup>5</sup>

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<sup>3</sup> Similarly the relation  $T' = (1 - vV/c^2)T$  between the time intervals in  $R'$  and  $R$  corresponding to some proper interval of a clock moving at velocity  $v$  in  $R$ , is obtained from  $T' = T - V\Delta x/c^2$  (Eq. (1)) with  $\Delta x = vT$ .

<sup>4</sup> In all rigor, the time-part of the metric is uniquely determined by the E.P at first order but the calculation of Mercury's perihelion involves the knowledge that second order terms are not present in it [19].

<sup>5</sup> For example (cf. section 3), the relations  $l' = (1 + vV/c^2)l$  and  $(1 - v'^2/c^2) = (1 - v^2/c^2)(1 + 2vV/c^2)$  (obtained from  $v' = v - V(1 - v^2/c^2)$  and describing the gap between  $v$  and  $c$ ), immediately lead to the invariance of  $l/\sqrt{1 - v^2/c^2}$ , equal to  $l_0$  in the proper frame of the rule where  $v = 0$  (length contraction). Similarly, the invariance of  $(x - vt)/\sqrt{1 - v^2/c^2}$  implies L.T for the  $x$ -coordinate of an event.

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