

Supplementary Material for “Localization in Spherical Shell Buckling”

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This Supplementary Materials provides developments not included in the paper. Equations given in the paper are cited here by their equation numbers preceded by the letter ‘P’.

1. FORMULATION OF THE SHALLOW SHELL MODEL.

We begin by deriving several conditions on the stress function $\tilde{f}(s)$. First, multiply the equilibrium equation (P-1.14a) by s then integrate from 0 to s_0 to obtain

$$\int_0^{s_0} (p + (N_\theta + N_\omega)) s \, ds = ((s M_\theta)' - M_\omega + s N_\theta w') \Big|_0^{s_0} = 0$$

where the vanishing of the right hand side follows from the assumed pole and boundary conditions. Because $s(N_\theta + N_\omega) = (s f)'$, the result above can be stated as $s f \Big|_0^{s_0} = -p s_0^2/2$ or $s_0 f_0(s_0) = -p s_0^2/2$ because $f(0) = 0$ if N_θ is finite at the pole. Then, by subtracting off the uniform solution $f_0 = -p s/2$, one obtains for the additional stress function defined in (P-2.2)

$$\tilde{f}(s_0) = 0. \tag{1}$$

A second condition on \tilde{f} is imposed by the boundary condition $u(s_0) = 0$. From

$$(1 + \nu) \frac{u}{s} + w = E_\omega = N_\omega - \nu N_\theta = -\frac{1-\nu}{2} p + \tilde{f}' - \nu \frac{\tilde{f}}{s},$$

together with $\tilde{f}(s_0) = 0$ and $u(s_0) = 0$, it follows that

$$\tilde{f}'(s_0) = \tilde{w}(s_0). \tag{2}$$

With (1) and (2) in hand, the compatibility equation (P-1.17) can be integrated with the result

$$\frac{1}{s} (s \tilde{f})' = \tilde{w} + G_2(\tilde{w}, s) \quad \text{and} \quad \tilde{f} = -\frac{1}{s} \int_s^{s_0} (\tilde{w}(x) + G_2(\tilde{w}, x)) x \, dx \tag{3}$$

with $G_2(\tilde{w}, s) = \int_s^{s_0} \frac{1}{2x} \tilde{w}'^2(x) \, dx$ as given in the paper below (P-2.3). The requirement that $N_\theta = \tilde{f}/s$ is bounded as $s \rightarrow 0$ implies that the integral in (3) goes to zero as s^2 as $s \rightarrow 0$. This is not an extra constraint on \tilde{w} , rather it will be met for any solution \tilde{w} satisfying the equilibrium equation and boundary conditions.

The energy of the shell and loading system for prescribed pressure p is

$$\Phi = \int_0^{s_0} \left(\frac{1}{2} (M_\theta K_\theta + M_\omega K_\omega + N_\theta E_\theta + N_\omega E_\omega) + p w \right) s \, ds \tag{4}$$

This expression can be reduced to $\Phi(\tilde{w}, p)$ given in (P-2.3) when the substitutions $w = w_0 + \tilde{w}$ and $f = f_0 + \tilde{f}$ are made, when \tilde{f} is expressed in terms of \tilde{w} using (3), and when all terms involving v explicitly are eliminated by straightforward integrations. In addition, we have made use of results such as

$$\int_0^{s_0} \left(\int_s^{s_0} \frac{1}{x} \tilde{w}'^2(x) \, dx \right) s \, ds = \frac{1}{2} \int_s^{s_0} \tilde{w}'^2(s) s \, ds$$

Terms involving only w_0 and f_0 have been dropped from $\Phi(\tilde{w}, p)$ because these terms do not participate in the variational solution process.

Write (P-2.3) as

$$\Phi(\tilde{w}, p) = \int_0^{s_0} \left\{ \frac{1}{2} (A_2(\tilde{w}) - p B_2(\tilde{w})) + \tilde{w}(s) G_2(\tilde{w}, s) + \frac{1}{2} [G_2(\tilde{w}, s)]^2 \right\} s \, ds$$

with $A_2(\tilde{w}) = (\nabla^2 \tilde{w})^2 + \tilde{w}^2$ and $B_2(\tilde{w}) = \frac{1}{2} \tilde{w}'^2$. Also, let

$$A_2(w + v) = A_2(w) + 2A_{11}(w, v) + A_2(v)$$

with $A_{11}(w, v) = \nabla^2 w \nabla^2 v + w v$ and $A_{11}(w, v) = A_{11}(v, w)$ such that $A_{11}(w, w) = A_2(w)$. Define a similar notation for B_2 and B_{11} , where $B_{11}(w, v) = \tilde{w}' \tilde{v}' / 2$.

2. KOITER’S STANDARD INITIAL POST-BUCKLING EXPANSION.

As indicated in the paper,

$$\tilde{w}(s) = \xi w_1(s) + \xi^2 w_2(s) + \xi^3 w_3(s) + \dots \quad \frac{p}{p_c} = 1 + a \xi + b \xi^2 + \dots \quad \text{and} \quad \tilde{f}(s) = \xi f_1(s) + \xi^2 f_2(s) + \xi^3 f_3(s) + \dots$$

The variational equation is

$$\delta \Phi(\tilde{w}, p) = \int_0^{s_0} \left\{ (A_{11}(\tilde{w}, \delta w) - p B_{11}(\tilde{w}, \delta w)) + G_2(\tilde{w}, s) \delta w + (\tilde{w} + G_2(\tilde{w}, s)) \int_s^{s_0} \frac{1}{x} \tilde{w}' \delta w' \, dx \right\} s \, ds = 0 \tag{5}$$

Introduce the expansions in (5) and collect like powers of ξ to obtain:

$$\begin{aligned} \delta\Phi(\tilde{w}, p) = & \xi \int_0^{s_0} \{(A_{11}(w_1, \delta w) - p_c B_{11}(w_1, \delta w))\} s \, ds \\ & + \xi^2 \int_0^{s_0} \left\{ (A_{11}(w_2, \delta w) - p_c B_{11}(w_2, \delta w)) - a p_c B_{11}(w_1, \delta w) + G_2(w_1, s) \delta w \dots \right. \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. + w_1 \int_s^{s_0} \frac{1}{x} w_1' \delta w' \, dx \right\} s \, ds \\ & + \xi^3 \int_0^{s_0} \left\{ (A_{11}(w_3, \delta w) - p_c B_{11}(w_3, \delta w)) - a p_c B_{11}(w_2, \delta w) - b p_c B_{11}(w_1, \delta w) \dots \right. \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \left. + \int_s^{s_0} \frac{1}{x} w_1' w_2' \delta w \, dx \right\} s \, ds \\ & + \xi^3 \int_0^{s_0} \left\{ w_1 \int_s^{s_0} \frac{1}{x} w_2' \delta w' \, dx + (w_2 + G_2(w_1, s)) \int_s^{s_0} \frac{1}{x} w_1' \delta w' \, dx \right\} s \, ds + \mathcal{O}(\xi^4) \end{aligned}$$

The variational statement of the eigenvalue problem is

$$\int_0^{s_0} \{(A_{11}(w_1, \delta w) - p_c B_{11}(w_1, \delta w))\} s \, ds = 0.$$

With the particular choice $\delta w = w_1$ this implies

$$\int_0^{s_0} \{(A_2(w_1) - p_c B_2(w_1))\} s \, ds = 0$$

In the standard analysis, take the orthogonality condition (this is equivalent to that in the paper)

$$\int B_{11}(w_1, w_j) s \, ds = 0 \quad j = 2, 3, \dots$$

also implying

$$\int A_{11}(w_1, w_j) s \, ds = 0 \quad j = 2, 3, \dots$$

Take $\delta w = w_1$ in $\delta\Phi = 0$ and simplify the expressions with the result

$$\begin{aligned} \delta\Phi(\tilde{w}, p) = & \xi^2 \int_0^{s_0} \{-a p_c B_2(w_1) + 3 G_2(w_1, s) w_1\} s \, ds \\ & + \xi^3 \int_0^{s_0} \left\{ -b p_c B_2(w_1) + 2 w_1 \int_s^{s_0} \frac{1}{x} w_1' w_2' \, dx + 2 (w_2 + G_2(w_1, s)) G_2(w_1, s) \right\} s \, ds \\ & + \mathcal{O}(\xi^4) \end{aligned}$$

Requiring $\delta\Phi = 0$ gives the expression for a and b

$$\begin{aligned} a &= \frac{3 \int_0^{s_0} G_2(w_1, s) w_1 s \, ds}{p_c \int_0^{s_0} B_2(w_1) s \, ds} \\ b &= \frac{2 \int_0^{s_0} \{w_1 \int_s^{s_0} x^{-1} w_1' w_2' \, dx + (w_2 + G_2(w_1, s)) G_2(w_1, s)\} s \, ds}{p_c \int_0^{s_0} B_2(w_1) s \, ds} \end{aligned} \quad (6)$$

The variational equation for w_2 is

$$\int_0^{s_0} \left\{ (A_{11}(w_2, \delta w) - p_c B_{11}(w_2, \delta w)) - a p_c B_{11}(w_1, \delta w) + G_2(w_1, s) \delta w + w_1 \int_s^{s_0} \frac{1}{x} w_1' \delta w' \, dx \right\} s \, ds = 0$$

which produces (P-2.8).

3. KOITER'S EXTENDED INITIAL POST-BUCKLING EXPANSION.

As noted in the paper, the extended expansion evaluates w_2 and higher order contributions at p not p_c , i.e.,

$$\tilde{w}(s) = \xi w_1(s) + \xi^2 w_2(s, p) + \xi^3 w_3(s, p) + \dots$$

for $p < p_c$, where w_1 and p_c are the same eigenfunctions and buckling eigenvalue as in the standard analysis. Now, the expansion of the variational equation (5) gives

$$\begin{aligned} \delta\Phi(\tilde{w}, p) = & \xi \int_0^{s_0} \{(A_{11}(w_1, \delta w) - p_c B_{11}(w_1, \delta w))\} s \, ds + \xi (p_c - p) \int_0^{s_0} B_{11}(w_1, \delta w) s \, ds \dots \\ & + \xi^2 \int_0^{s_0} \left\{ (A_{11}(w_2, \delta w) - p_c B_{11}(w_2, \delta w)) + G_2(w_1, s) \delta w + w_1 \int_s^{s_0} \frac{1}{x} w_1' \delta w' \, dx \right\} s \, ds \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \xi^2 (p_c - p) \int_0^{s_0} B_{11}(w_2, \delta w) s \, ds \\ & + \xi^3 \int_0^{s_0} \left\{ (A_{11}(w_3, \delta w) - p_c B_{11}(w_3, \delta w)) + \int_s^{s_0} \frac{1}{x} w_1' w_2' \delta w \, dx \right\} s \, ds \\ & + \xi^3 \int_0^{s_0} \left\{ w_1 \int_s^{s_0} \frac{1}{x} w_2' \delta w' \, dx + (w_2 + G_2(w_1, s)) \int_s^{s_0} \frac{1}{x} w_1' \delta w' \, dx \right\} s \, ds \\ & \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad + \xi^3 (p_c - p) \int_0^{s_0} B_{11}(w_3, \delta w) s \, ds + \mathcal{O}(\xi^4) \end{aligned}$$

Before proceeding we note that the first order change in pressure, $p/p_c = 1 + a\xi + \dots$ has already been determined with a depending only on w_1 . Thus, the lowest order contribution of the term $\xi(p_c - p) \int_0^{s_0} B_{11}(w_1, \delta w) s ds$ is $-p_c a \xi^2 \int_0^{s_0} B_{11}(w_1, \delta w) s ds$. Similarly, each term of the form $\xi^j (p_c - p) \int_0^{s_0} B_{11}(w_j, \delta w) s ds$ is no larger than order ξ^{j+1} . Let $\delta w = w_1$ in $\delta\Phi$ and make use of the various properties of the eigenvalue problem. Including terms up to order ξ^3 , one finds

$$\begin{aligned} \delta\Phi(\tilde{w}, p) = & \xi(p_c - p) \int_0^{s_0} B_2(w_1) s ds + 3\xi^2 \int_0^{s_0} G_2(w_1, s) w_1 s ds \\ & + 2\xi^3 \int_0^{s_0} \left\{ w_1 \int_s^{s_0} \frac{1}{x} w_1' w_2' dx + (w_2 + G_2(w_1, s)) G_2(w_1, s) \right\} s ds + \xi^2 (p_c - p) \int_0^{s_0} B_{11}(w_2, w_1) s ds \end{aligned}$$

Requiring $\delta\Phi = 0$ gives

$$\frac{p}{p_c} = 1 + a\xi + b(p)\xi^2 + \dots$$

where a has been given earlier (it is independent of p and depends only on w_1) and

$$b(p) = \frac{2 \int_0^{s_0} \left\{ w_1 \int_s^{s_0} x^{-1} w_1' w_2' dx + (w_2 + G_2(w_1, s)) G_2(w_1, s) \right\} s ds - p_c a \int_0^{s_0} B_{11}(w_2, w_1) s ds}{p_c \int_0^{s_0} B_2(w_1) s ds}$$

The problem for w_2 is obtained from satisfying $\delta\Phi = 0$ to order ξ^2 but including the term $\xi^2 (p_c - p) \int_0^{s_0} B_{11}(w_2, \delta w) s ds$ so that

$$\int_0^{s_0} \left\{ (A_{11}(w_2, \delta w) - p B_{11}(w_2, \delta w)) - a p_c B_{11}(w_1, \delta w) + G_2(w_1, s) \delta w + w_1 \int_s^{s_0} \frac{1}{x} w_1' \delta w' dx \right\} s ds = 0$$

One can show by direct manipulation that the above variational equation produces (P-2.15). Results such as the following arise in this direct manipulation

$$\int_0^{s_0} \left\{ w_1 \int_s^{s_0} \frac{1}{x} w_1' \delta w' dx \right\} s ds = - \int_0^{s_0} \left\{ \frac{1}{s} (f_1 w_1')' \delta w \right\} s ds.$$

4. ASYMPTOTIC SOLUTION.

4.1. Analysis of the outer region.

4.1.1. Equations in the outer region.

Inserting the outer expansion (P-4.2) into the shallow shell equations of Section P-1.3, averaging with respect to the slow variable s , and solving order by order with the help of the symbolic calculation language Wolfram Mathematica, we find that

- the tangent displacement is given in terms of the deflection as

$$U_{[0]}(x) + \xi U_{[1]}(x) = i(W_{[0]}(x) + \xi W_{[1]}(x)) - \frac{1}{x} \frac{d(x W_{[0]}(x))}{dx},$$

- the deflection satisfies the amplitude equation

$$\mathcal{L}^2 W_{[0]}(x) + \frac{b}{2} W_{[0]}(x) = 0$$

where \mathcal{L} is the differential operator

$$\mathcal{L}W(x) = \frac{1}{\sqrt{x}} \frac{d(\sqrt{x} W(x))}{dx},$$

- the parameter a vanishes, as stated in Eq. (P-4.6) in the main text.

All these equations are linear, the nonlinearity being present in the polar layer only.

4.1.2. Solution in the outer solution.

To derive the solution in the outer region, we introduce a new unknown $V_{[0]}(x)$ such that $W_{[0]}(x) = \frac{V_{[0]}(x)}{\sqrt{x}}$. The amplitude equation becomes an ordinary differential equation with constant coefficients,

$$V_{[0]}''(x) + \frac{b}{2} V_{[0]}(x) = 0.$$

As we work in an infinite domain, $s_0 = \infty$, the solution must decrease exponentially for $s \rightarrow +\infty$. This requires

$$b < 0,$$

and the solution is then $V_{[0]}(x) = (C_1 + i C_2) \exp\left(-\frac{\sqrt{-\{2b\}}}{2} x\right) = (C_1 + i C_2) \exp\left(-\frac{\sqrt{-\{2b\}} \xi}{2} s\right)$, where C_1 and C_2 are real constants, which will be determined later by matching with the inner solution. Returning to the original variable $W_{[0]}$ and eliminating $\sqrt{-b}$ using equation (P-7), we obtain

$$W_{[0]}(x) = \frac{C_1 + i C_2}{\sqrt{\xi s}} \exp\left(-\left[\frac{1}{2} \left(1 - \frac{p}{4}\right)\right]^{1/2} s\right).$$

When this is inserted into the outer expansion (P-4.2), we find the deflection $\tilde{w}(s)$ as

$$\tilde{w}_{\text{out}}(s) = \xi \left(\frac{C_1 \cos s - C_2 \sin s}{\sqrt{s}} \right) \exp \left(- \left[\frac{1}{2} \left(1 - \frac{p}{4} \right) \right]^{1/2} s \right) + \dots \quad (7)$$

4.1.3. Expansion in the outer solution in the intermediate region.

For the purpose of matching, we rewrite the argument of the exponential as $-\left[\frac{1}{2}\left(1-\frac{p}{4}\right)\right]^{1/2}s = -\sqrt{\frac{-b}{2}}\xi$, see equation (P-7). When this argument is small, *i.e.* when $\left(1-\frac{p}{4}\right)^{1/2}s \ll 1$, we can replace the exponential with its Taylor expansion at linear order, $w_{\text{out}}(s) - w_0(s) = \xi^{\alpha-\frac{1}{2}}(C_1 \cos s - C_2 \sin s) \left(\frac{1}{\sqrt{s}} - \sqrt{\frac{-b}{2}}\xi\sqrt{s} + \dots\right)$, which we rewrite as

$$\tilde{w}_{\text{out}}(s) = \xi \left(\frac{C_1 - C_2}{2} (\cos s + \sin s) + \frac{C_1 + C_2}{2} (\cos s - \sin s) \right) \left(\frac{1}{\sqrt{s}} - \sqrt{\frac{-b}{2}}\xi\sqrt{s} + \dots \right). \quad (8)$$

This approximation requires $\left(1-\frac{p}{4}\right)^{1/2}s \ll 1$: it holds in the matching region.

4.2. Analysis of the inner region.

4.2.1. Non-linear equations in the inner region.

The equations in the polar layer have been obtained with the help of the symbolic calculation language Wolfram Mathematica as well, by inserting the expansion for w_{in} into the equilibrium equations of the shallow shell model. We find four equations, corresponding to the transverse and longitudinal equilibrium on one hand, and to the dominant and subdominant orders on the other hand. They read

$$\begin{aligned} \mathcal{P}^2 \hat{W}_{[0]} &= 0 \\ \hat{W}_{[0]} + \frac{1}{s} \frac{d}{ds} (s \hat{U}_{[0]}) &= 0 \\ \mathcal{P}^2 \hat{W}_{[1]} + \hat{V}_{[1]} &= -\frac{2}{s} \hat{W}'_{[0]} \hat{W}_{[0]} \\ \hat{V}'_{[1]}(s) &= -\frac{1}{2s} \hat{W}'_{[0]}{}^2 \end{aligned} \quad (9)$$

where \mathcal{P} is the operator

$$\mathcal{P}f = \Delta f + f,$$

and Δ is the Laplacian associated with the metric $\int \dots ds$,

$$\Delta f = \frac{1}{s} \frac{d}{ds} \left(s \frac{df}{ds} \right).$$

Here, $\hat{V}_{[1]}$ is an auxiliary variable,

$$\hat{V}_{[1]}(s) = \frac{1}{1-\nu} \left((1+\nu) \left(\hat{W}_{[1]} + \frac{1}{s} \frac{d}{ds} (s \hat{U}_{[1]}) \right) + \frac{1}{2} \hat{W}'_{[0]}{}^2 \right).$$

The choice of the exponent $\beta=1$ done in Section P-4.3 is such that equations (9a–b) concerning the dominant displacement $\hat{W}_{[0]}$ and $\hat{U}_{[0]}$ are *linear*, while the last two equations, which concern the subdominant displacement $\hat{W}_{[1]}$ and $\hat{U}_{[1]}$, include *non-linear source terms* in the right-hand sides. This choice of the exponent β warrants that some nonlinearity is present in the matched asymptotic analysis, which is requirement for ultimately deriving a (non-linear) bifurcation equation.

The solution to these equations are subject to smoothness conditions for $s \rightarrow 0$ (the operator Δ is singular and a generic solution of (9) diverges for $s \rightarrow 0$), and to growth conditions for $s \rightarrow \infty$: the asymptotic equivalent of a solution of the inner equations for $s \rightarrow \infty$ must be consistent with the behavior of the outer solution in the intermediate region. Overall, we find that these conditions for $s \rightarrow 0$ and $s \rightarrow \infty$ warrant that the solution can be expressed in terms of a few constant of integration, see below.

4.2.2. Solution in the inner region.

We start by the dominant order solution in the inner region. The solution to equations (9a–b) that is smooth for $s \rightarrow 0$ and grows no faster than $\hat{W}_{[0]}(s) \sim s^{-1/2}$ for $s \rightarrow \infty$ reads

$$\begin{aligned} \hat{W}_{[0]}(s) &= C_3 J_0(s) \\ \hat{U}_{[0]}(s) &= C_3 J'_0(s), \end{aligned} \quad (10)$$

where C_3 is a constant of integration that will be determined later by matching with the outer solution.

Next, we proceed to solving the subdominant order.

From (9d), we solve the auxiliary quantity $\hat{V}_{[1]}(s)$ as

$$\hat{V}_{[1]}(s) = C_4 + \frac{C_3^2}{4} ([J_0(s)]^2 + [J_1(s)]^2 - 1) \quad (11)$$

where C_4 is another constant of integration.

As a preparatory work to solving (9c), we show how the differential operator \mathcal{P} can be inverted. First, we observe that the solution of $\mathcal{P}f = g(s)$ that is smooth near $s=0$ can be obtained using the method of the variation of the constant as

$$f(s) = J_0(s) \left(C_f - \frac{\pi}{2} \int_0^s Y_0(t) g(t) t dt \right) + Y_0(s) \frac{\pi}{2} \int_0^s J_0(t) g(t) t dt.$$

where C_f is a constant of integration.

By applying this formula twice, first to the equation $\mathcal{P}f = g(s)$ and then to the equation $\mathcal{P}h = f(s)$, one can show that the solution of $\mathcal{P}^2 h = g(s)$ that is smooth near $s=0$ reads

$$\begin{aligned} h(s) &= C_6 J_0(s) + \frac{\pi}{2} C_5 \left(Y_0(s) \int_0^s [J_0(t)]^2 t dt - J_0(s) \int_0^s Y_0(t) J_0(t) t dt \right) \dots \\ &+ J_0(s) \frac{\pi^2}{4} \left[\int_0^s Y_0(t) J_0(t) \left(\int_0^t Y_0(u) g(u) u du \right) t dt - \int_0^s [Y_0(t)]^2 \left(\int_0^t J_0(u) g(u) u du \right) t dt \right] \dots \\ &+ Y_0(s) \frac{\pi^2}{4} \left[- \int_0^s [J_0(t)]^2 \left(\int_0^t Y_0(u) g(u) u du \right) t dt + \int_0^s J_0(t) Y_0(t) \int_0^t J_0(u) g(u) u du t dt \right] \end{aligned}$$

where C_5 and C_6 are constant of integration (which are the C_f 's coming from inverting \mathcal{P} twice).

Permuting the integrals over u and t , we obtain

$$\begin{aligned} h(s) &= C_6 J_0(s) + \frac{C_5}{2} s J_1(s) \dots \\ &+ J_0(s) \frac{\pi^2}{4} \int_0^s g(u) \left[Y_0(u) \int_u^s Y_0(t) J_0(t) t dt - J_0(u) \int_u^s [Y_0(t)]^2 t dt \right] u du \dots \\ &+ Y_0(s) \frac{\pi^2}{4} \int_0^s g(u) \left[J_0(u) \int_u^s Y_0(t) J_0(t) t dt - Y_0(u) \int_u^s [J_0(t)]^2 t dt \right] u du \end{aligned}$$

The integrals over t can be calculated symbolically,

$$\begin{aligned} L_{JJ}(s) &:= \int [J_0(t)]^2 t dt = \frac{s^2}{2} ([J_0(s)]^2 + [J_1(s)]^2) \\ L_{YJ}(s) &:= \int Y_0(t) J_0(t) t dt = \frac{s^2}{2} (Y_0(s) J_0(s) + Y_1(s) J_1(s)) \\ L_{YY}(s) &:= \int [Y_0(t)]^2 t dt = \frac{s^2}{2} ([Y_0(s)]^2 + [Y_1(s)]^2) \end{aligned}$$

This yields

$$\begin{aligned} h(s) &= C_6 J_0(s) + \frac{C_5}{2} s J_1(s) \dots \\ &+ J_0(s) \frac{\pi^2}{4} \int_0^s g(u) [Y_0(u) (L_{YJ}(s) - L_{YJ}(u)) - J_0(u) (L_{YY}(s) - L_{YY}(u))] u du \dots \\ &+ Y_0(s) \frac{\pi^2}{4} \int_0^s g(u) [J_0(u) (L_{YJ}(s) - L_{YJ}(u)) - Y_0(u) (L_{JJ}(s) - L_{JJ}(u))] u du \end{aligned}$$

This can be further simplified, and we find the general solution of $\mathcal{P}^2 h = g(s)$ as

$$\begin{aligned} h(s) &= C_6 J_0(s) + \frac{C_5}{2} s J_1(s) + \frac{\pi}{4} \left(J_0(s) \int_0^s g(u) [-u^2 Y_1(u)] du + Y_0(s) \int_0^s g(u) [u^2 J_1(u)] du \dots \right. \\ &\quad \left. + s J_1(s) \int_0^s g(u) [-u Y_0(u)] du + s Y_1(s) \int_0^s g(u) [u J_0(u)] du \right) \end{aligned} \quad (12)$$

We apply this general result to equation (9c), which we rewrite as $\mathcal{P}^2 \hat{W}_{[1]}(s) = g(s)$, we set $g(s) = -\hat{V}_{[1]} - \frac{2}{s} \hat{W}'_{[0]} \hat{W}''_{[0]}$, i.e., using (11),

$$g(s) = \left(-C_4 + \frac{C_3^2}{4} \right) + C_3^2 \tilde{g}(s).$$

The result is

$$\begin{aligned} \hat{W}_{[1]}(s) &= C_6 J_0(s) + \frac{C_5}{2} s J_1(s) + \left(-C_4 + \frac{C_3^2}{4} \right) \left(1 - J_0(s) - \frac{s}{2} J_1(s) \right) \dots \\ &+ \frac{\pi}{4} C_3^2 \left(J_0(s) \int_0^s \tilde{g}(u) [-u^2 Y_1(u)] du + Y_0(s) \int_0^s \tilde{g}(u) [u^2 J_1(u)] du \dots \right. \\ &\quad \left. + s J_1(s) \int_0^s \tilde{g}(u) [-u Y_0(u)] du + s Y_1(s) \int_0^s \tilde{g}(u) [u J_0(u)] du \right). \end{aligned} \quad (13)$$

where $\tilde{g}(s)$ reflects the presence of source terms in (9c-d),

$$\tilde{g}(s) = -\frac{[J_0(s)]^2 + [J_1(s)]^2}{4} - \frac{2}{s} J_0'(s) J_0''(s). \quad (14)$$

To recapitulate, the solution for the deflection is given in Eqs. (10) and (13). The expressions of $\hat{U}_{[0]}$ and $\hat{U}_{[1]}$ are not required, but can be obtained from (10b) and by inserting (11) into the definition of $\hat{V}_{[1]}(s)$ from the previous section.

4.2.3. Expansion in the intermediate region.

To match with the inner solution, we expand both the dominant displacement $\hat{W}_{[0]}(s)$ and the subdominant one $\hat{W}_{[1]}(s)$ in the intermediate region. For $s \gg 1$, the Bessel's function in (10a) has the equivalent

$$\hat{W}_{[0]}(s) = C_3 J_0(s) \sim \frac{1}{\sqrt{s}} \frac{C_3}{\sqrt{\pi}} (\cos s + \sin s)$$

An equivalent for the subdominant contribution $\hat{W}_{[1]}(s)$ is derived in Appendix A based on the form (13). It writes, still for $s \gg 1$,

$$\hat{W}_{[1]}(s) \sim \sqrt{s} \left(-(\cos s - \sin s) \left[\frac{C_5 + C_4 - \frac{C_3^2}{4}}{2\sqrt{\pi}} + \gamma' C_3^2 \right] + (\cos s + \sin s) \gamma C_3^2 \right)$$

where γ and γ' are numerical constants. The constant γ is given by

$$\gamma = \frac{\sqrt{\pi}}{4} \int_0^\infty \left(\frac{[J_0(u)]^2 + [J_1(u)]^2}{4} + \frac{2}{u} J_0'(u) J_1''(u) \right) u J_0(u) du = 0.183226 \quad (15)$$

The other constant γ' plays no role at this order in the expansion, and its numerical value is not given.

Combining the equivalents for $\hat{W}_{[0]}(s)$ and $\hat{W}_{[1]}(s)$ just obtained, we obtain from the inner expansion (P-4.3) an expansion of the buckling deflection as

$$\tilde{w}_{\text{in}}(s) \sim \xi \left(\frac{1}{\sqrt{s}} \frac{C_3}{\sqrt{\pi}} (\cos s + \sin s) - \xi \sqrt{s} (\cos s - \sin s) \left[\frac{C_5 + C_4 - \frac{C_3^2}{4}}{2\sqrt{\pi}} + \gamma' C_3^2 \right] + \xi \sqrt{s} (\cos s + \sin s) \gamma C_3^2 + \dots \right) \quad (16)$$

4.3. Matching.

An expression of the buckling deflection \tilde{w}_{out} valid for $(1 - \frac{p}{4})^{1/2} s \ll 1$ has been obtained in (8) based on the outer solution. An expression of the same quantity $\tilde{w}_{\text{in}}(s)$ predicted by the inner solution has been obtained in (16), and it is valid for $1 \ll s$. In the matching region, $1 \ll s \ll (1 - \frac{p}{4})^{-1/2}$, the expressions must agree. This requires that the coefficients in factor of the various terms $s^{\pm 1/2} (\cos s \pm \sin s)$ and coming from either expansion match. This yields four matching conditions

$$\begin{aligned} \frac{C_1 - C_2}{2} &= \frac{C_3}{\sqrt{\pi}} && \text{(coef. in factor of } \frac{\cos s + \sin s}{\sqrt{s}}) \\ \frac{C_1 + C_2}{2} &= 0 && \text{(coef. in factor of } \frac{\cos s - \sin s}{\sqrt{s}}) \\ -\sqrt{\frac{b}{2}} \frac{C_1 - C_2}{2} &= \gamma C_3^2 && \text{(coef. in factor of } (\cos s + \sin s) \sqrt{s}) \\ -\sqrt{\frac{b}{2}} \frac{C_1 + C_2}{2} &= -\frac{C_5 + C_4 - \frac{C_3^2}{4}}{2\sqrt{\pi}} - \gamma' C_3^2 && \text{(coef. in factor of } (\cos s - \sin s) \sqrt{s}) \end{aligned} \quad (17)$$

Solving the first two equations for C_1 and C_2 , we find

$$C_1 = \frac{C_3}{\sqrt{\pi}} \quad C_2 = -\frac{C_3}{\sqrt{\pi}}. \quad (18)$$

The third equation yields an equation for the amplitude C_3 , $\gamma C_3^2 + \sqrt{\frac{b}{2}} \frac{C_3}{\sqrt{\pi}} = 0$, which was announced in Eq. (P-4.9). The fourth equation will be ignored, as it yields a relation between C_4 and C_5 , which does not allow us to determine C_4 and C_5 unless we push the expansion to the next order and derive a second relation between these quantities.

Combining the previous equations, we obtain the solution for the deflection as

$$\begin{aligned} \tilde{w}_{\text{out}}(s) &= -\frac{1}{\gamma\pi} \left(1 - \frac{p}{4}\right)^{1/2} \frac{\cos(s - \frac{\pi}{4})}{\sqrt{s}} \exp\left(-\left[\frac{1}{2}\left(1 - \frac{p}{4}\right)\right]^{1/2} s\right) + \dots \\ \tilde{w}_{\text{in}}(s) &= -\frac{1}{\gamma\pi} \left(1 - \frac{p}{4}\right)^{1/2} \sqrt{\frac{\pi}{2}} \left(J_0(s) + \left[\frac{1}{2}\left(1 - \frac{p}{4}\right)\right]^{1/2} \{s Y_1(s)\} + \dots \right) \end{aligned}$$

To this order in the approximation, one can replace the curly brace above by any function having the equivalent $\sqrt{\frac{2}{\pi}} \cos(s - \frac{\pi}{4}) \sqrt{s}$ for $s \rightarrow \infty$.

APPENDIX A. EQUIVALENT OF THE INNER SOLUTION IN THE INTERMEDIATE REGION

First, we note that

$$\tilde{g}(u) = \mathcal{O}\left(\frac{1}{u}\right)$$

for $u \rightarrow \infty$. In addition, $[-u^2 Y_1(u)] = \mathcal{O}(u^{3/2} \cos u, u^{3/2} \sin u)$ and so the first integrand in (13) can be estimated as $\tilde{g}(u)[-u^2 Y_1(u)] = \mathcal{O}(u^{1/2} \cos u, u^{1/2} \sin u)$, implying that the integral grows like $\int_0^s \tilde{g}(u)[-u^2 Y_1(u)] du = \mathcal{O}(u^{1/2} \cos u, u^{1/2} \sin u)$. After multiplication by $J_0(s) = \mathcal{O}(u^{-1/2} \cos u, u^{-1/2} \sin u)$, we find that the first term on the second line of (13) is bounded for $s \rightarrow \infty$,

$$J_0(s) \int_0^s \tilde{g}(u)[-u^2 Y_1(u)] du = \mathcal{O}(1).$$

A similar reasoning applies to the second term,

$$Y_0(s) \int_0^s \tilde{g}(u)[u^2 J_1(u)] du = \mathcal{O}(1).$$

The two integrals on the last line are treated as follows. Consider the third integral in (13). An equivalent for the quantity in factor of $\tilde{g}(u)$ is $[-u Y_0(u)] = \mathcal{O}(u^{1/2} \cos u, u^{1/2} \sin u)$ so that the integrand is $\tilde{g}(u)[-u Y_0(u)] du = \mathcal{O}(u^{-1/2} \cos u, u^{-1/2} \sin u)$ and the integral converges. The same holds for the other integral. Using $s J_1(s) \sim -\frac{\sqrt{s}}{\sqrt{\pi}}(\cos s - \sin s)$ and $s Y_1(s) \sim -\frac{\sqrt{s}}{\sqrt{\pi}}(\cos s + \sin s)$, we obtain the growth of $\hat{W}_{[1]}(s)$ in equation (13) as

$$\hat{W}_{[1]}(s) \sim \sqrt{s} \times \begin{cases} -(\cos s - \sin s) \left[\frac{C_5 + C_4 - \frac{C_3^2}{4}}{2\sqrt{\pi}} + \frac{\sqrt{\pi}}{4} C_3^2 \int_0^\infty \tilde{g}(u)[-u Y_0(u)] du \right] \dots \\ + (\cos s + \sin s) \left[-C_3^2 \frac{\sqrt{\pi}}{4} \int_0^\infty \tilde{g}(u)[u J_0(u)] du \right] \end{cases}$$

We can evaluate the integrals by inserting the definition of \tilde{g} from equation (14). A symbolic integration yields the coefficient of C_3^2 on the second line as

$$\gamma := -\frac{\sqrt{\pi}}{4} \int_0^\infty \tilde{g}(u)[u J_0(u)] du = \frac{3\sqrt{3}}{16\sqrt{\pi}} = 0.183\ 226$$