



Explicit Rieffel induction module for quantum groups

Damien Rivet

► To cite this version:

| Damien Rivet. Explicit Rieffel induction module for quantum groups. 2019. hal-02347447

HAL Id: hal-02347447

<https://hal.science/hal-02347447>

Preprint submitted on 5 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Explicit Rieffel induction module for quantum groups

Damien Rivet

Abstract

For \mathbb{G} an algebraic (or more generally, a bornological) quantum group and \mathbb{B} a closed quantum subgroup of \mathbb{G} , we build in this paper an induction module by explicitly defining an inner product which takes its value in the convolution algebra of \mathbb{B} , as in the original approach of Rieffel [Rie74]. In this context, we study the link with the induction functor defined by Vaes. In the last part we illustrate our result with parabolic induction of complex semi-simple quantum groups with the approach suggested by Clare [Cla13][CCH16].

1 Introduction

Let G be a locally compact group and B a closed subgroup of G . One can build unitary representations of G from those of B with the unitary induction procedure due to Mackey [Mac52], who also developed the concept of imprimitivity. Rieffel [Rie74] gave an alternative and more general formulation in the C^* -algebraic setting by using C^* -Hilbert modules. In short, there exists a Hilbert $C^*(B)$ -module $\mathcal{E}(G)$, with a left unitary action of $C^*(G)$, such that for a unitary representation of \mathbb{B} on any Hilbert module V , $\mathcal{E}(\mathbb{G}) \otimes_{C^*(B)} V$ is the induced unitary representation of G .

In the case where \mathbb{G} is a locally compact quantum group and \mathbb{B} a closed quantum subgroup, induction procedures have been developed by Kustermans [Kus02] and Vaes [Vae05]. Vaes was able to formulate this in a wide framework and to state imprimitivity theorems. In this paper we develop an approach closer to the original one of Rieffel, by directly defining the induction module $\mathcal{E}(G)$.

The main difficulty is that, unless \mathbb{B} is also an open subgroup of \mathbb{G} (the case treated in [KKSS]), we don't have an inclusion of $C^*(\mathbb{B})$ into $C^*(\mathbb{G})$ and so it is not possible to define a conditional expectation from $C^*(\mathbb{G})$ to $C^*(\mathbb{B})$. In his original paper Rieffel avoided this issue by considering the convolution algebra $C_c(G)$ of compactly supported functions on G , instead of the full space $C^*(G)$. Then he defined a weak conditional expectation $C_c(G) \rightarrow C_c(B)$.

The bornological setting for quantum groups developed by Voigt [Voi08] allows us to consider algebras with comparable properties and then to define an analogue of the weak conditional expectation. One of the main goals of this

paper is to show that, in this particular case, the induction functor we obtain is the same as the one defined by Vaes. We thus get a more direct way to compute induced representations for regular bornological quantum groups and to apply the powerful imprimitivity theorems. We remark that the class of bornological quantum groups is an extremely large subclass of locally compact quantum groups. The only known obstruction to being bornological is non-regularity and regularity is already a necessary condition in Vaes imprimitivity theorem. The class of bornological quantum groups includes compact quantum groups, complex semi-simple quantum groups and classical locally compact groups.

In the last part, we illustrate the general construction with the example of principal series representations of a semi-simple complex quantum group G_q [VY19a]. Specifically, in analogy to the classical case in [CCH16], we build a module using a G_q -space G_q/N_q , which implements the parabolic induction. The notation G_q/N_q is meant to suggest a homogeneous space with respect to a quantum analogue of the classical unipotent subgroup although we do not actually use any such subgroup in its definition. As well as giving a noncommutative geometry perspective on the parabolic induction functor for G_q we can thus provide a new description of the structure of the reduced C^* -algebra $C_r^*(G_q)$ following the result of [VY19b].

2 Bornological quantum groups

Bornological quantum groups, defined by Voigt [Voi08], are a generalization of algebraic quantum groups introduced by Van Daele [Dae98] where most of the interesting properties stay valid. In the section we recall some technical points on bornological quantum groups. It can be ignored by just replacing bornological by algebraic quantum groups.

Definition 2.1. *A bornological quantum group is an essential bornological $*$ -algebra $\mathcal{A}(\mathbb{G})$ satisfying the approximation property, together with a comultiplication $\Delta : \mathcal{A}(\mathbb{G}) \rightarrow M(\mathcal{A}(\mathbb{G}) \otimes \mathcal{A}(\mathbb{G}))$, such that all Galois maps associated to Δ are isomorphisms, and a faithful left invariant positive functional $\phi_{\mathbb{G}}$.*

We recall that the Galois maps associated to Δ refer to the four following maps

$$\begin{aligned} \gamma_l : f \otimes g &\mapsto \Delta(f)(g \otimes 1), & \gamma_r : f \otimes g &\mapsto \Delta(f)(1 \otimes g), \\ \rho_l : f \otimes g &\mapsto (f \otimes 1)\Delta(g), & \rho_r : f \otimes g &\mapsto (1 \otimes f)\Delta(g), \end{aligned}$$

which by assumption are isomorphisms of $\mathcal{A}(\mathbb{G}) \otimes \mathcal{A}(\mathbb{G})$.

Remark 1. We modify the original definition of [Voi08] by adding a $*$ -structure. We do not treat technical details in this paper, which will be treated in [Riv20].

Let $\mathcal{A}(\mathbb{G})$ a bornological quantum group endowed with a compatible $*$ -structure $f \mapsto \bar{f}$. (This notation, inspired by the commutative case, is unconventional but motivated by the fact that we will use another $*$ -structure denoted by $f \mapsto f^*$ for the convolution algebra).

Remark 2. In this paper the tensor notation \otimes designates, either the natural tensor product in the category of bornological vector spaces (Voigt uses $\hat{\otimes}$) or the tensor product of Hilbert modules. The meaning will be clear from the context.

The hypothesis about Galois maps is the crucial point which allows us to define what we need for our Rieffel-like approach. In particular we define the associated convolution algebra $\mathcal{D}(\mathbb{G})$ where $\mathcal{D}(\mathbb{G}) = \mathcal{A}(\mathbb{G})$ as a bornological vector space, endowed with a $*$ -algebra structure defined by

$$\begin{aligned} f * g &= (\text{id} \otimes \phi_{\mathbb{G}})[(1 \otimes S^{-1}(g))(\Delta(f))], \quad \forall f, g \in \mathcal{D}(\mathbb{G}), \\ f^* &= \overline{S(f)}\delta_{\mathbb{G}}. \end{aligned}$$

where $\delta_{\mathbb{G}} \in M(\mathcal{A}(\mathbb{G}))$ designates the modular element associated with the Haar state $\phi_{\mathbb{G}}$, defined by the property

$$(\phi_{\mathbb{G}} \otimes \text{id})(\Delta(f)) = \phi_{\mathbb{G}}(f)\delta_{\mathbb{G}} \in M(\mathcal{A}(\mathbb{G})), \quad \forall f \in \mathcal{A}(\mathbb{G}).$$

We also mention the notable property

$$\phi_{\mathbb{G}}(S(f)) = \phi_{\mathbb{G}}(f\delta_{\mathbb{G}}).$$

Remark 3. • $\mathcal{D}(\mathbb{G})$ has in fact a structure of Hopf algebra, as the dual quantum group of $\mathcal{A}(\mathbb{G})$ (with comultiplication denoted by $\hat{\Delta}$) but we use it rarely.

- In [Voi08], $\mathcal{D}(\mathbb{G})$ is defined as a subspace of $\mathcal{A}(\mathbb{G})'$ instead of being the same space as $\mathcal{A}(\mathbb{G})$. The way to pass from one picture to another is to consider the map $f \mapsto \phi_{\mathbb{G}}(\cdot f)$, which is a Hopf $*$ -isomorphism. In our picture, the natural pairing between $\mathcal{A}(\mathbb{G})$ and $\mathcal{D}(\mathbb{G})$ is expressed as

$$(f, g) = \phi_{\mathbb{G}}(gf), \quad f \in \mathcal{A}(\mathbb{G}), \quad g \in \mathcal{D}(\mathbb{G}).$$

See Theorem 1.11 in [VY19a] for consequences of this convention.

- Associated to a left corepresentation α on V of $\mathcal{A}(\mathbb{G})$ one can obtain a representation of $\mathcal{D}(\mathbb{G})$ via

$$f * v = (\phi_{\mathbb{G}} \otimes \text{id})((\rho(f) \circ S^{-1} \otimes \text{id})(\alpha(v))), \quad f \in \mathcal{D}(\mathbb{G}), \quad v \in V.$$

Where $\rho(f)$ designates the right multiplication by f .

We denote by $L^2(\mathbb{G})$ the GNS construction associated with $(\mathcal{A}(\mathbb{G}), \phi_{\mathbb{G}})$. Note we have the remarkable identity

$$\langle f, g \rangle_{L^2(\mathbb{G})} = \phi_{\mathbb{G}}(\bar{f}g) = \epsilon(f^* * g), \quad \forall f, g \in \mathcal{A}(\mathbb{G}).$$

We suppose that $\mathcal{A}(\mathbb{G})$ has a universal C^* -algebra $C_0^u(\mathbb{G})$ which, endowed with the extension of $\Delta : C_0^u(\mathbb{G}) \mapsto M(C_0^u(\mathbb{G}) \hat{\otimes} C_0^u(\mathbb{G}))$, is a locally compact group. For technical reasons we always suppose that the unbounded element

$\delta_{\mathbb{G}}^{\frac{1}{2}}$ affiliated to $C_0^u(\mathbb{G})$ belongs to $M(\mathcal{A}(\mathbb{G}))$ (which seems to be valid for all examples of our interest). The generality of these points will also be treated in [Riv20]. The dual locally compact quantum group of $C_0^u(\mathbb{G})$ will be denoted by $C^*(\mathbb{G})$.

Remark 4. The reader must be careful, in whole paper we juggle with $\mathcal{A}(\mathbb{G})$ and $\mathcal{D}(\mathbb{G})$ using everywhere both structures of algebras, which can be confusing.

3 Closed quantum subgroups

Definition 3.1. A bornological quantum group $\mathcal{A}(\mathbb{B})$, equipped with a bounded surjective $*$ -morphism of bornological quantum groups $\pi : \mathcal{A}(\mathbb{G}) \rightarrow \mathcal{A}(\mathbb{B})$ is called a closed quantum subgroup of $\mathcal{A}(\mathbb{G})$.

Let $\mathcal{A}(\mathbb{B})$ such a subgroup with a left Haar state $\phi_{\mathbb{B}}$ (with modular discriminant $\delta_{\mathbb{B}}$). The map π induces right coaction of $\mathcal{A}(\mathbb{B})$ on $\mathcal{A}(\mathbb{G})$

$$(id \otimes \pi) \circ \Delta : \mathcal{A}(\mathbb{G}) \rightarrow M(\mathcal{A}(\mathbb{G}) \otimes \mathcal{A}(\mathbb{B}))$$

and left coaction defined in the same way, but those two coactions do not induce a unitary map $\mathcal{D}(\mathbb{B}) \rightarrow M(\mathcal{D}(\mathbb{G}))$. We fix this problem by defining

$$\gamma = \pi(\delta_{\mathbb{G}}^{-\frac{1}{2}})\delta_{\mathbb{B}}^{\frac{1}{2}} \in M(\mathcal{D}(\mathbb{B}))$$

which is a group-like element. Now we modify the map π into

$$E : \mathcal{D}(\mathbb{G}) \rightarrow \mathcal{D}(\mathbb{B}), \quad E(f) = \pi(f)\gamma.$$

We then set for all f in $\mathcal{D}(\mathbb{G})$ and for all $h \in \mathcal{D}(\mathbb{B})$

$$\begin{aligned} f \cdot h &= (id \otimes \phi_{\mathbb{B}})[(1 \otimes S^{-1}(h))((id \otimes E)(\Delta(f)))], \\ h \cdot f &= (\phi_{\mathbb{B}} \otimes id)[((S^{-1} \circ E \otimes id)\Delta(f))(h \otimes 1)]. \end{aligned}$$

Proposition 3.2. The map $E : \mathcal{D}(\mathbb{G}) \rightarrow \mathcal{D}(\mathbb{B})$, $E(f) = \pi(f)\gamma$, has the two following properties :

1. $E(f^*) = E(f)^*$,
2. $E(f \cdot h) = E(f) * h$ and $E(h \cdot f) = h * E(f)$.

Proof.

$$\begin{aligned} E(f^*) &= E(\overline{S(f)}\delta_{\mathbb{G}}) \\ &= \overline{S(\pi(f))}\pi(\delta_{\mathbb{G}})\gamma \\ &= \overline{S(\pi(f))}\pi(\delta_{\mathbb{G}}^{\frac{1}{2}})\delta_{\mathbb{B}}^{\frac{1}{2}} \\ &= \overline{S(\pi(f)\gamma)}\delta_{\mathbb{B}} = E(f)^*. \\ E(f \cdot h) &= E((id \otimes \phi_{\mathbb{B}})[(1 \otimes S^{-1}(h))(id \otimes \pi)(\Delta(f))(1 \otimes \gamma)]) \\ &= (id \otimes \phi_{\mathbb{B}})[(1 \otimes S^{-1}(h))(\pi \otimes \pi)(\Delta(f))(1 \otimes \gamma)(\gamma \otimes 1))] \\ &= (id \otimes \phi_{\mathbb{B}})[(1 \otimes S^{-1}(h))(\Delta(\pi(f)\gamma))] \\ &= E(f) * h. \end{aligned}$$

The last equality can be shown in the same way. \square

Now we set $\hat{\pi} : \mathcal{D}(\mathbb{B}) \rightarrow M(\mathcal{D}(\mathbb{G}))$, where for $h \in \mathcal{D}(\mathbb{B})$, $\hat{\pi}(h)$ is defined as a multiplier by $\hat{\pi}(h) * f = h \cdot f$ and $f * \hat{\pi}(h) = f \cdot h$. To make sure that for $h \in \mathcal{D}(\mathbb{B})$, $\hat{\pi}(h)$ indeed defined a multiplier we check that

$$(g * f) \cdot h = g * (f \cdot h) \text{ and } h \cdot (f * g) = (h \cdot f) * h \quad \forall f, g \in \mathcal{D}(\mathbb{G}).$$

The compatibility between the right and left action follows directly from the second point of 3.2.

Proposition 3.3. *The map $\hat{\pi}$ is a $*$ -morphism.*

Proof. The morphism property follows from the associativity

$$(k * h) \cdot f = k \cdot (h \cdot f) \text{ and } f \cdot (k * h) = f \cdot k \cdot h,$$

where $f \in \mathcal{D}(\mathbb{G})$ and $h, k \in \mathcal{D}(\mathbb{B})$. This can be checked directly. The compatibility with the $*$ -structure can be expressed as the equalities

$$\hat{\pi}(h^*) * f = (f^* * \hat{\pi}(h))^* \text{ and } f * \hat{\pi}(h^*) = (\hat{\pi}(h) * f)^*.$$

which are clearly true. \square

Proposition 3.4. *The extension $\pi : C_0^u(\mathbb{G}) \rightarrow C_0^u(\mathbb{B})$ identify $L^\infty(\mathbb{B})$ as a closed quantum subgroup of $L^\infty(\mathbb{G})$ (in the sens of Definition 2.5 of [Vae05]).*

Proof. The map $\hat{\pi} : \mathcal{D}(\mathbb{B}) \rightarrow M(\mathcal{D}(\mathbb{G}))$ naturally extends to the following two morphisms

$$\begin{array}{ccc} C^*(\mathbb{B}) & \xrightarrow{\hat{\pi}} & M(C^*(\mathbb{G})) \\ \downarrow & & \downarrow \\ \mathcal{L}(\mathbb{B}) & \xrightarrow{\hat{\pi}} & \mathcal{L}(\mathbb{G}) \end{array}$$

and one can check that the density of $\mathcal{D}(\mathbb{B})$ in $C^*(\mathbb{B})$ and $\mathcal{L}(\mathbb{B})$ ensure that the diagram commutes. \square

4 The induction module $\mathcal{E}(\mathbb{G})$

We equip the space $\mathcal{D}(\mathbb{G})$ with the right action of $\mathcal{D}(\mathbb{B})$ defined above (induced by $\hat{\pi}$).

Proposition 4.1. *The sesquilinear map $\langle \cdot, \cdot \rangle_{\mathcal{D}(\mathbb{B})}$ defined for $f, g \in \mathcal{D}(\mathbb{G})$ by*

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{B})} = E(f^* * g),$$

defines a $\mathcal{D}(\mathbb{B})$ -valued inner product.

The $\mathcal{D}(\mathbb{B})$ -linearity and compatibility with the involution of the above sesquilinear map follow immediately from proposition 3.2. It only remains to check the strict positivity, which we obtain by first proving the following result.

Proposition 4.2. *The map $\rho_\bullet : \mathcal{D}(\mathbb{G}) \rightarrow B(L^2(\mathbb{B}), L^2(\mathbb{G}))$ defined by*

$$\rho_f(\eta) = f \cdot \eta, \quad \forall f \in \mathcal{D}(\mathbb{G}), \forall \eta \in \mathcal{D}(\mathbb{B}),$$

verifies

$$\langle f, g \rangle_{\mathcal{D}(\mathbb{B})} = \rho_f^* \rho_g,$$

where $\langle f, g \rangle_{\mathcal{D}(\mathbb{B})}$ is seen as an element of $B(L^2(\mathbb{B}))$.

Proof. First we claim that as an operator, $(\rho_f)^*$ is equal to $\langle f, \cdot \rangle_{\mathcal{D}(\mathbb{B})}$. For this, note that using $\epsilon_{\mathbb{B}}(E(x)) = \epsilon_{\mathbb{G}}(x)$, for any $x \in \mathcal{D}(\mathbb{G})$, we obtain $\epsilon_{\mathbb{B}}(E(x * y)) = \epsilon_{\mathbb{G}}(x^* * y) = \langle x, y \rangle_{L^2(\mathbb{G})}$ for any $y \in \mathcal{D}(\mathbb{G})$. Therefore, for all $\eta \in \mathcal{A}(\mathbb{B})$ and $\xi \in \mathcal{A}(\mathbb{G})$ we have

$$\begin{aligned} \langle \rho_f \eta, \xi \rangle &= \langle f \cdot \eta, \xi \rangle_{L^2(\mathbb{G})} \\ &= \epsilon_{\mathbb{B}}(\langle f \cdot \eta, \xi \rangle_{\mathcal{D}(\mathbb{B})}) \\ &= \epsilon_{\mathbb{B}}(\eta^* * \langle f, \xi \rangle_{\mathcal{D}(\mathbb{B})}) \\ &= \langle \eta, \langle f, \xi \rangle \rangle_{L^2(\mathbb{B})}. \end{aligned}$$

We also have :

$$\begin{aligned} \langle f, g \rangle_{\mathcal{D}(\mathbb{B})} * \eta &= \langle f, \rho_g \eta \rangle_{\mathcal{D}(\mathbb{B})} \\ &= \rho_f^* \rho_g \eta. \end{aligned}$$

□

This concludes the proof of Proposition 4.1.

Definition 4.3. *The Hilbert $C^*(\mathbb{B})$ -module completed from $\mathcal{D}(\mathbb{G})$ with respect to the inner product is denoted $\mathcal{E}(\mathbb{G})$ and we call it the induction module (associated to \mathbb{B}).*

See [Lan95] for details about the completion. The space $\mathcal{E}(\mathbb{G})$ is innately equipped with a left $C^*(\mathbb{G})$ -action, which commutes with the right $C^*(\mathbb{B})$ -action. We then get our induction bi-module

$$C^*(\mathbb{G})\mathcal{E}(\mathbb{G})_{C^*(\mathbb{B})}.$$

Now, for α a representation of $C^*(\mathbb{B})$ on a A -Hilbert module K (where A is any C^* -algebra) we consider, following Rieffel's definition for induced representations in [Rie74], the A -Hilbert module

$$\text{Ind}_{\mathbb{B}}^{\mathbb{G}} V = \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V,$$

where the tensor product is completed with respect to the interior inner product ([Lan95]).

5 Link with Vaes' approach to induction

Still considering our quantum groups \mathbb{G} and \mathbb{B} , we illustrate in what follows the induction procedure of [Vae05]. We start with $L^\infty(\mathbb{G})$ seen as a Von Neumann algebraic group and its closed subgroup $L^\infty(\mathbb{B})$ and we consider a representation of \mathbb{B} on an A -Hilbert module V , seen as a $*$ -morphism $\alpha : C_0^u(\mathbb{B}) \rightarrow \mathfrak{B}(V)$. From this we take in consideration the Hilbert module

$$L^2(\mathbb{G}) \otimes V,$$

equipped with a left action $\beta_l : \mathcal{L}(\mathbb{B}) \rightarrow \mathfrak{B}(L^2(\mathbb{G}) \otimes V)$, a right action $\beta_r : \mathcal{L}(\mathbb{G}) \rightarrow \mathfrak{B}(L^2(\mathbb{G}) \otimes V)$ and also a right action $L^\infty(\mathbb{G}) \rightarrow \mathfrak{B}(L^2(\mathbb{G}) \otimes V)$, determined on dense subspaces by

- $\beta_l(h)(\xi \otimes v) = (\hat{\pi} \otimes \alpha)(\hat{\Delta}(h))(\xi \otimes v)$, $h \in \mathcal{D}(\mathbb{B})$, $\xi \otimes v \in L^2(\mathbb{G}) \otimes V$,
- $\beta_r(f)(\xi \otimes v) = (\xi * f^*) \otimes v$, $f \in \mathcal{D}(\mathbb{G})$,
- $(\xi \otimes v).f = \xi f \otimes v$, $f \in \mathcal{A}(\mathbb{G})$.

One then considers the $\mathcal{L}(\mathbb{G})$ - $\mathcal{L}(\mathbb{G})$ -bimodule

$$\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V),$$

where \mathcal{I} is defined by

$$\mathcal{I} = \{u \in B(L^2(\mathbb{B}), L^2(\mathbb{G})), ux = \hat{\pi}'(x)u \ \forall x \in \mathcal{L}(\mathbb{B})'\}$$

and naturally equipped with a $\mathcal{L}(\mathbb{G})$ - $\mathcal{L}(\mathbb{B})$ -bimodule structure. The Hilbert module $\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V)$ has a structure of equivariant $\mathcal{L}(\mathbb{G})$ - $\mathcal{L}(\mathbb{G})$ -correspondence so it can be written ([Vae05, Proposition 3.7]) as

$$\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V) = L^2(\mathbb{G}) \otimes \text{Ind } V$$

where $\text{Ind } V$ is a canonically defined A -Hilbert module, and defines the induced representation in the sens of Vaes. The aim of this section is to establish an equivalence of representations

$$\text{Ind } V \cong \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V$$

For that purpose we equip $L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V$ with its natural $\mathcal{L}(\mathbb{G})$ - $\mathcal{L}(\mathbb{G})$ -correspondence structure (see [Vae05, section 3]) with actions given for $\xi \otimes f \otimes v \in L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V$ by

- $\gamma_l(g)(\xi \otimes f \otimes v) = [(\lambda \otimes \lambda)(\hat{\Delta}(g))(\xi \otimes f)] \otimes v$, $\forall g \in \mathcal{D}(\mathbb{G})$,
- $\gamma_r(g)(\xi \otimes f \otimes v) = \xi * g^* \otimes v$, $\forall g \in \mathcal{D}(\mathbb{G})$,
- $(\xi \otimes f \otimes v).h = \xi g \otimes f \otimes v$, $\forall h \in \mathcal{A}(\mathbb{G})$.

Here λ designates the convolution action, that is $\lambda(g)f = g * f$.

Proposition 5.1. *We have an equivalence of equivariant W^* - $\mathcal{L}(\mathbb{G})$ - $\mathcal{L}(\mathbb{G})$ -bimodules*

$$\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V) \cong L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V.$$

Lemma 5.2. *The morphism ρ_\bullet (4.2) defines a injection with dense image $\mathcal{E}(\mathbb{G}) \rightarrow \mathcal{I}$ (with respect to the weak topology of $B(L^2(\mathbb{B}), L^2(\mathbb{G}))$). Its image is denoted \mathcal{I}_0 .*

Proof. $\mathcal{D}(\mathbb{G})^{op}$ can be embedded in $\mathcal{L}(\mathbb{G})'$ via

$$\iota(f)\xi = \xi * f, \quad f \in \mathcal{D}(\mathbb{G}), \quad \xi \in \mathcal{D}(\mathbb{G}).$$

Likewise for $\mathcal{D}(\mathbb{B})$ in $\mathcal{L}(\mathbb{B})'$. Moreover, the map $\hat{\pi}'$ defined in section 4 of [Vae05] restricted as a morphism $\mathcal{D}(\mathbb{B}) \rightarrow \mathcal{L}(\mathbb{G})'$ is given by

$$\hat{\pi}'(h)\xi = \xi \cdot h, \quad h \in \mathcal{D}(\mathbb{B}), \quad \xi \in \mathcal{D}(\mathbb{G}).$$

Thus, for $x = \rho_f$ and $u = \iota(h)$ ($f \in \mathcal{D}(\mathbb{G}), h \in \mathcal{D}(\mathbb{B})$) and $\eta \in \mathcal{D}(\mathbb{B})$, we have

$$\begin{aligned} x(u(\eta)) &= x(\eta * h) \\ &= f \cdot (\eta * h) \\ &= (f \cdot \eta) \cdot h \\ &= \hat{\pi}'(x)(u(\eta)) \end{aligned}$$

So it is clear that $\mathcal{E}(\mathbb{G})$ embeds in \mathcal{I} . It remains to show that the image is dense. Let $\xi \in L^2(\mathbb{G})$ and $\eta \in L^2(\mathbb{B})$. Suppose we have

$$\langle \xi, f \cdot \eta \rangle = 0$$

for all $f \in \mathcal{E}(\mathbb{G})$. Let then $u \in \mathcal{I}$. We want to obtain that $\langle \xi, u(\eta) \rangle = 0$. Let $\varepsilon > 0$, there exist

1. $a \in \mathcal{A}(\mathbb{B})$ s.t. $\|\eta - a\|_{L^2(\mathbb{B})} \leq \varepsilon$ (density of $\mathcal{A}(\mathbb{B})$),
2. $b \in \mathcal{A}(\mathbb{B})$ s.t. $\|b * a - a\|_{L^2(\mathbb{B})} \leq \varepsilon$ (approximation property),
3. $c \in \mathcal{A}(\mathbb{G})$ s.t. $\|u(b) - c\|_{L^2(\mathbb{G})} \leq \varepsilon$ (density of $\mathcal{A}(\mathbb{G})$).

Now, there exist $k_1, k_2, k_3 > 0$ (depending only on the norms of ξ, η and u) s.t.

1. $|\langle \xi, u(\eta) \rangle - \langle \xi, u(a) \rangle| \leq k_1 \varepsilon,$
2. $|\langle \xi, u(a) \rangle - \langle \xi, u(b * a) \rangle| \leq k_2 \varepsilon,$ and we note that $u(b * a) = u(b) \cdot a,$
3. $|\langle \xi, u(b) \cdot a \rangle - \langle \xi, c \cdot \eta \rangle| \leq k_3 \varepsilon,$

Finally, since $\langle \xi, c \cdot \eta \rangle = 0$ we have

$$|\langle \xi, u(\eta) \rangle| \leq (k_1 + k_2 + k_3) \varepsilon,$$

So $\langle \xi, u(\eta) \rangle = 0$ and we are done. \square

Lemma 5.2 allows us to consider the linear map

$$\begin{aligned}\Psi : \mathcal{D}(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V &\rightarrow \mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V) \\ \xi \otimes f \otimes v &\mapsto \Delta(\xi)(f \otimes 1) \otimes v\end{aligned}$$

where on the right hand side we identify $\mathcal{E}(\mathbb{G})$ with its image in \mathcal{I} .

We also need the following lemma to deal with regular vectors in a representation

Lemma 5.3. *Let K be a representation of $C^*(\mathbb{G})$ on any Hilbert module, the space*

$$\mathcal{D}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{G})} K$$

equipped with the left convolution action of $\mathcal{D}(\mathbb{G})$ is a dense sub- $\mathcal{D}(\mathbb{G})$ -representation of K .

Proof. The inclusion is given by the map determined by $f \otimes v \in \mathcal{D}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{G})} K \mapsto f * v$. The associativity of the convolution ensures that it is an intertwiner and the density comes from the fact that $\mathcal{D}(\mathbb{G})$ is dense in $C^*(\mathbb{G})$ and that K is a non-degenerate $C^*(\mathbb{G})$ -module and thus essential. \square

This module is denoted by K^∞ and one can see it as a left $\mathcal{A}(\mathbb{G})$ -corepresentation where the coaction is given by $f \otimes v \mapsto \Delta(f) \otimes v$.

Proof of proposition 5.1. In this proof we will use Sweedler notation to clarify the calculations, even though it is not well defined in the bornological framework. One has to keep in mind that we only use the coproduct when we use convolution products or actions and so each expression in calculus can be written formally in terms of Galois maps and then do make sense. For example $f_{(1)}g \otimes f_{(2)}$ formally means $\gamma_l(f \otimes g)$.

First, we show that the map Ψ is compatible with inner-products. Before beginning the calculation, we observe that for $h \in \mathcal{D}(\mathbb{B})$, $\eta \in \mathcal{A}(\mathbb{G})$ and $w \in V^\infty$, one can re-express the diagonal convolution action via

$$\begin{aligned}h * (\eta \otimes w) &= [(\hat{\pi} \otimes \alpha)(\hat{\Delta}(h))](\eta \otimes w) \\ &= \phi_{\mathbb{B}}(S^{-1}(\pi(\eta_{(1)})w_{(1)})h)\eta_{(2)} \otimes w_{(2)} \\ &= \eta_{(2)} \otimes (\pi(S^{-1}(\eta_{(1)}))h) * w.\end{aligned}$$

Let $\xi \otimes f \otimes v$ and $\eta \otimes g \otimes w$ be in $\mathcal{A}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{G}) \otimes_{\mathcal{D}(\mathbb{B})} V^\infty$. Using the above

calculation in the equality (*) below, we obtain

$$\begin{aligned}
& \langle \xi_{(1)} f \otimes \xi_{(2)} \otimes v, \eta_{(1)} g \otimes \eta_{(2)} \otimes w \rangle_{\mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V)} \\
&= \left\langle \xi_{(2)} \otimes v, \langle \xi_{(1)} f, \eta_{(1)} g \rangle_{C^*(\mathbb{B})} * (\eta_{(2)} \otimes w) \right\rangle_{L^2(\mathbb{G}) \otimes V} \\
&= \langle \xi_{(2)} \otimes v, \phi_{\mathbb{G}}(\overline{\xi_{(1)}} f \eta_{(1)} g_{(1)}) (\pi(\eta_{(2)} g_{(2)}) \gamma) * (\eta_{(3)} \otimes w) \rangle_{L^2(\mathbb{G}) \otimes V} \\
&\stackrel{(*)}{=} \langle \xi_{(2)} \otimes v, \phi_{\mathbb{G}}(\overline{\xi_{(1)}} f \eta_{(1)} g_{(1)}) \eta_{(4)} \otimes (\pi(S^{-1}(\eta_{(3)}) \pi(\eta_{(2)} g_{(2)}) \gamma) * w) \rangle_{L^2(\mathbb{G}) \otimes V} \\
&= \langle \xi_{(2)} \otimes v, \phi_{\mathbb{G}}(\overline{f} \overline{\xi_{(1)}} \eta_{(1)} g_{(1)}) \eta_{(2)} \otimes (\pi(g_{(2)} \gamma) * w) \rangle_{L^2(\mathbb{G}) \otimes V} \\
&= \phi_{\mathbb{G}}(\overline{\xi_{(2)}} \eta_{(2)}) \langle v, \phi_{\mathbb{G}}(\overline{f} \overline{\xi_{(1)}} \eta_{(1)} g_{(1)}) (\pi(g_{(2)} \gamma) * w) \rangle_V \\
&= \left\langle \xi \otimes v, \eta \otimes (\langle f, g \rangle_{C^*(\mathbb{B})} * w) \right\rangle_{L^2(\mathbb{G}) \otimes V} \\
&= \langle \xi, \eta \rangle_{L^2(\mathbb{G})} \left\langle v, \langle f, g \rangle_{C^*(\mathbb{B})} * w \right\rangle_V \\
&= \langle \xi \otimes f \otimes v, \eta \otimes g \otimes w \rangle_{L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V}.
\end{aligned}$$

Now we can extend $\Psi : L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_{C^*(\mathbb{B})} V \rightarrow \mathcal{I} \otimes_{\mathcal{L}(\mathbb{B})} (L^2(\mathbb{G}) \otimes V)$. It is clear that this map Ψ intertwines the right action of $L^\infty(\mathbb{G})'$ and $\mathcal{L}(\mathbb{G})$. we check for left actions. We have

$$[(\lambda \otimes \lambda)(\hat{\Delta}(g))](\xi \otimes f) \otimes v = \phi_{\mathbb{G}}(S^{-1}(\xi_{(1)} f_{(1)}) g) \xi_{(2)} \otimes f_{(2)} \otimes v$$

and its image under Ψ is

$$\phi_{\mathbb{G}}(S^{-1}(\xi_{(1)} f_{(1)}) g) \xi_{(2)} f_{(2)} \otimes \xi_{(3)} \otimes v = g * \xi_{(1)} f \otimes \xi_{(2)} \otimes v$$

so we are done.

To finish we check that $\Phi : \mathcal{I} \otimes_{\pi_l} (L^2(\mathbb{G}) \otimes V) \rightarrow L^2(\mathbb{G}) \otimes \mathcal{E}(\mathbb{G}) \otimes_\alpha V$ defined for $f \in \mathcal{D}(\mathbb{G})$, $\xi \in \mathcal{A}(\mathbb{G})$ and $v \in V$ by $\Phi(f \otimes \xi \otimes v) = \tau((S^{-1} \otimes \text{id})(\Delta(\xi))(f \otimes 1)) \otimes v$ (where τ is the flip map), is the inverse bijection of Ψ . \square

Theorem 5.4. *The representations $\text{Ind } V$ and $\mathcal{E}(G_q) \otimes_{C^*(B_q)} V$ are equivalent.*

Proof. We still follow proposition 3.7 of [Vae05], which establishes the link between representations and correspondences and apply it to proposition 5.1. \square

6 Parabolic Induction

In this section we give an explicit Rieffel induction module associated to the functor of parabolic induction for complex semi-simple quantum groups. This is an analogue of the construction due to Clare in [Cla13] for classical semi-simple Lie groups.

6.1 Preliminaries

We follow the notations and conventions of [VY19a]. Let \mathfrak{g} be a complex semi simple Lie algebra and let G_q be the associated simply connected complex semi-simple quantum group and K_q its maximal compact quantum subgroup (with its multiplicative unitary $W \in M(\mathcal{A}(\hat{K}_q) \otimes \mathcal{A}(K_q))$). We write $\mathcal{U}_q(\mathfrak{g})$ for the associated quantized enveloping algebra and $\mathcal{U}_q^{\mathbb{R}}(\mathfrak{k})$ for the same algebra equipped with the involution $*$, seen as the complexification of the quantized enveloping algebra of the compact form K_q . We recall that the algebra of representative functions on the Drinfeld double $G_q = K_q \bowtie \hat{K}_q$ is defined by (see [VY19a, Definition 3.18])

$$\mathcal{A}(G_q) = \mathcal{A}(K_q) \otimes \mathcal{A}(\hat{K}_q),$$

with coproduct

$$\Delta_{G_q}(a \otimes f) = W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes f_{(2)})W_{32}.$$

It is equipped with a left Haar state $\phi_{K_q} \otimes \psi_{\hat{K}_q}$, where ϕ_{K_q} is the Haar state on $\mathcal{A}(K_q)$ and $\psi_{\hat{K}_q}$ the right Haar state on $\mathcal{A}(\hat{K}_q)$ (dual to the right Haar state ψ_{K_q}).

Let \mathbf{P} is the weight lattice associated to \mathfrak{g} and $\mathcal{U}_q^{\mathbb{R}}(\mathfrak{t}) = \text{span}\{K_\lambda, \lambda \in \mathbf{P}\}$. For each $\mu \in \mathbf{P}$ we define $e^\mu \in \mathcal{U}_q^{\mathbb{R}}(\mathfrak{t})'$ by

$$e^\mu(K_\lambda) = q^{(\lambda, \mu)}.$$

In this way we can identify the algebra of functions on the torus subgroup T of K_q as

$$\mathcal{A}(T) = \text{span}\{e^\mu, \mu \in \mathbf{P}\} \subset \mathcal{U}_q^{\mathbb{R}}(\mathfrak{t})',$$

where \mathbf{P} is the weight lattice associated to \mathfrak{g} and where e^μ is defined by

T is naturally identified with $\text{spec}(\mathcal{A}(T))$ and we note that for any $\lambda \in \mathfrak{t}^*$ we obtain a character of the $*$ -algebra $\mathcal{A}(T)$ by

$$(K_{i\lambda}, e^\mu) = q^{i(\lambda, \mu)}$$

and this yields an identification $T \cong i(\mathfrak{t}^* / \frac{2\pi}{\log(q)} Q^\vee)$ where $Q^\vee = \text{Hom}(\mathbf{P}, \mathbb{Z})$ is the coroot lattice, see [VY19a, Section 5.11]. We will not use this identification in what follows.

We define the restriction map $\pi : \mathcal{A}(K_q) \rightarrow \mathcal{A}(T)$ via

$$\pi(a) = a|_{\mathcal{U}_q^{\mathbb{R}}(\mathfrak{t})}.$$

The Borel subgroup $B_q = T \bowtie \hat{K}_q$ is defined via $\mathcal{A}(B_q) = \mathcal{A}(T) \otimes \mathcal{A}(\hat{K}_q)$ (see [VY19a, section 3.7] with coproduct

$$\Delta(a \otimes f) = \widetilde{W}_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes f_{(2)})\widetilde{W}_{32},$$

twisted by the bicharacter $\widetilde{W} = (\pi \otimes \text{id})(W)$. It is a closed subgroup of G_q with restriction map

$$\pi \otimes \text{id} : \mathcal{A}(G_q) \rightarrow \mathcal{A}(B_q).$$

We denote by ϕ_T the Haar functional on $\mathcal{A}(T)$. The functional $\phi_T \otimes \psi_{\hat{K}_q}$ is left invariant on $\mathcal{A}(B_q)$ (it will be denoted ϕ_{B_q}) and we have

$$\delta_{B_q} = \mathbf{1} \otimes K_{-4\rho}$$

for the associated modular element (see the proof of [VY19a, Proposition 3.19]). We thus obtain our map $E : \mathcal{D}(G_q) \rightarrow \mathcal{D}(B_q)$, $E(a \otimes f) = \pi(a) \otimes f K_{-2\rho}$, for all $a \otimes f$ in $\mathcal{D}(K_q) \otimes \mathcal{D}(\hat{K}_q)$. We also note that the convolution algebra can be explicitly defined in as $\mathcal{D}(B_q) = \mathcal{D}(T) \bowtie \mathcal{D}(\hat{K}_q)$ with twisted product given for $x \bowtie u$ and $y \bowtie v \in \mathcal{D}(T) \bowtie \mathcal{D}(\hat{K}_q)$ by

$$(x \bowtie u)(y \bowtie v) = x(y_{(1)}, u_{(1)})y_{(2)} \bowtie u_{(1)}(y_{(3)}, S^1(u_{(3)}))v,$$

where we consider here the coalgebra structures of $\mathcal{D}(T)$ and $\mathcal{D}(\hat{K}_q)$ and the natural pairing between those two Hopf algebras. The coproduct on $\mathcal{D}(B_q)$ is simply the untwisted coproduct of $\mathcal{D}(T) \otimes \mathcal{D}(\hat{K}_q)$.

For the discrete quantum group $\mathcal{A}(\hat{K}_q)$ we denote by $\hat{\mathbf{1}}_{K_q}$ the element corresponding to the Haar state $\phi_{K_q} \in \mathcal{D}(K_q)$ (where here $\mathcal{D}(K_q)$ is seen as a subspace of $\mathcal{A}(K_q)'$). It is the identity element for the convolution in $\mathcal{D}(\hat{K}_q)$.

The quotient map

In the classical case, with $G = KAN$, principal series representation are induced from characters of the the Borel subgroup $B = MAN$. Explicitly, we choose first a character μ of M and λ of A and the identification $MA = B/N$ allows us to extend $\mu \otimes \lambda$ to a character of B . In this way we obtain the principal series representation

$$\text{Ind}_B^G \mu \otimes \lambda.$$

In the quantum case we don't have an analog for the subgroup N . But we do have a quotient map

$$\hat{K}_q \twoheadrightarrow A_q,$$

where A_q is simply a notation for the weight lattice \mathbf{P} , which serves as a quantum analog for A .

There are now two versions of the map π . First with the canonical identification of $*$ -algebras $\mathcal{A}(K_q) = \mathcal{D}(\hat{K}_q)$ and $\mathcal{A}(T) = \mathcal{D}(A_q)$, one can consider

$$\pi : \mathcal{D}(\hat{K}_q) \rightarrow \mathcal{D}(A_q),$$

which is a $*$ -morphism and induces a structure of left $\mathcal{D}(\hat{K}_q)$ -module on $\mathcal{D}(A_q)$. Secondly, using the identifications of vector spaces $\mathcal{A}(\hat{K}_q) \cong \mathcal{D}(\hat{K}_q)$ and $\mathcal{A}(A_q) \cong \mathcal{D}(A_q)$ one can also look at

$$\pi : \mathcal{A}(\hat{K}_q) \rightarrow \mathcal{A}(A_q)$$

which is a conditional expectation in the sense of Proposition 3.2 (observing that K_q and T are unimodular) and in particular $\pi(fK_\lambda) = \pi(f)K_\lambda$. This map has also the notable property

$$\phi_{\hat{K}_q}(f) = \phi_{A_q}(\pi(f)),$$

where $\phi_{\hat{K}_q}$ (resp. φ_{A_q}) is the left Haar measure on \hat{K}_q (resp. A_q) s.t. $\phi_{\hat{K}_q}(\hat{1}_{K_q}) = 1$ (resp. $\varphi_{A_q}(\hat{1}_T) = 1$). Indeed observe that writing $\mathcal{F}_{K_q}(a) : \mathcal{A}(K_q) \rightarrow \mathcal{A}(\hat{K}_q)$ the Fourier map, we have $\phi_{\hat{K}_q}(\mathcal{F}_{K_q}(a)) = \epsilon(a)$ for all $a \in \mathcal{A}(K_q)$. This last version of π is what we call the quotient map.

Remark 5. We claim that the $*$ -morphism $\pi : \mathcal{D}(\hat{K}_q) \rightarrow \mathcal{D}(A_q)$ makes $\mathcal{D}(A_q)$ into a left $\mathcal{D}(\hat{K}_q)$ -module. Later, we will need to convert (see remark 3) this structure on $\mathcal{D}(A_q)$ into the corresponding left $\mathcal{A}(\hat{K}_q)$ -comodule structure on $\mathcal{A}(A_q)$ and it can be expressed simply as

$$\alpha_{\mathcal{A}(A_q)} : \pi(f) \mapsto f_{(1)} \otimes \pi(f_{(2)})$$

for any $f \in \mathcal{A}(\hat{K}_q)$. It is not clear a priori that this map is well defined but it follows from the fact that for $g \in \mathcal{D}(\hat{K}_q)$ and $\pi(f) \in \mathcal{A}(A_q)$, the action of g on $\pi(f)$ is given by

$$\begin{aligned} g * \pi(f) &= \pi(f * g) \\ &= \hat{\phi}(S^{-1}(f_{(1)})g)\pi(f_{(2)}) \end{aligned}$$

Therefore we identify the expression of the associated coaction. With the same kind of argument we can also express $\alpha_{\mathcal{A}(A_q)}(h) = \hat{\pi}(h_{(1)}) \otimes h_{(2)}$, $h \in \mathcal{A}(A_q)$.

6.2 The parabolic induction module : two constructions

In this section we build a Hilbert module which implements the parabolic induction functor. We first define this module as a balanced tensor product $\mathcal{E}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$ (details below). Then we build a geometric picture of it by introducing a quantum homogeneous space that we denote in by G_q/N_q .

We set $L_q = A_q \times T$ (which is a classical group), its convolution algebra is $\mathcal{D}(L_q) = \mathcal{D}(T) \otimes \mathcal{D}(A_q)$ (with the ordinary tensor product hopf algebra structure). The $*$ -morphism $(\text{id} \otimes \pi) : \mathcal{D}(B_q) \rightarrow \mathcal{D}(L_q)$ provides the right $\mathcal{D}(L_q)$ -module $\mathcal{D}(L_q)$ with a left $\mathcal{D}(B_q)$ -action. For $(\mu, \lambda) \in \mathbf{P} \times \mathfrak{t}^*$, one can build the one dimensional representation of B_q

$$\mathcal{D}(L_q) \otimes_{\mathcal{D}(L_q)} \mathbb{C}_{\mu, \lambda},$$

where $\mathbb{C}_{\mu, \lambda}$ corresponds to the character $\mathbb{C}_\mu \otimes \mathbb{C}_\lambda$ of $\mathcal{D}(L_q) = \mathcal{D}(T) \otimes \mathcal{D}(A_q)$ given by $a \mapsto (a, e^\mu)$, $a \in \mathcal{D}(T)$ and $f \mapsto \lambda(f)$, $f \in \mathcal{D}(A_q)$. Because of the essentialness of $\mathcal{D}(L_q)$, it is clear that $\mathcal{D}(L_q) \otimes_{\mathcal{D}(L_q)} \mathbb{C}_{\mu, \lambda}$ is one dimensional and we again denote it by $\mathbb{C}_{\mu, \lambda}$, as a module of B_q . We recall the definition

of parabolically induced representation associated to this character ([VY19a, chapter 5])

$$\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} = \{ \xi \in M(\mathcal{A}(G_q)) \mid (\text{id} \otimes \pi_{B_q}) \Delta_{G_q}(\xi) = \xi \otimes (e^\mu \otimes K_{2\rho+\lambda}) \}.$$

with a inner product given by

$$\langle a \otimes f, b \otimes g \rangle_{\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}} = \phi_{K_q}(\bar{a}b)$$

Proposition 6.1. *The representations $\mathcal{E}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda}$ and $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ of $C^*(G_q)$ are isomorphic.*

Proof. We define the map

$$\begin{aligned} \mathcal{E}(G_q) \otimes_{\mathcal{D}(B_q)} \mathbb{C}_{\mu, \lambda} &\longrightarrow \text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda} \\ (a \otimes f) \otimes 1 &\longmapsto (a \otimes f) \cdot (e^\mu \otimes K_\lambda), \end{aligned}$$

where for $\xi \in \mathcal{A}(G_q)$ we set

$$\xi \cdot (e^\mu \otimes K_\lambda) = \phi_{B_q}(S^{-1}(e^\mu \otimes K_\lambda) \pi_{B_q}(\xi_{(2)})(1 \otimes K_{-2\rho})) \xi_{(1)},$$

this is analogous to the definition of right action in section 3 except that K_λ belongs to $M(\mathcal{A}(\hat{K}_q))$. We first note that this map is well defined, it follows from the group-like property of $e^\mu \otimes K_\lambda$ and one obtains that for $X \in \mathcal{D}(B_q)$ we have $X * (e^\mu \otimes K_\lambda) = (\mu \otimes \lambda)(X)(e^\mu \otimes K_\lambda)$ therefore $(a \otimes f) \cdot X \otimes 1$ and $(a \otimes f) \otimes (\mu \otimes \lambda)(X)$ have the same image.

The fact that $(a \otimes f) \cdot (e^\mu \otimes K_{2\rho+\lambda})$ belongs to $\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda}$ is again due to the group-like property of $e^\mu \otimes K_{2\rho+\lambda}$. Indeed for $\xi \in \mathcal{A}(G_q)$ we have

$$\begin{aligned} (\text{id} \otimes \pi_{B_q}) \Delta_{G_q}(\xi \cdot (e^\mu \otimes K_\lambda)) &= \xi_{(1)} \otimes \pi_{B_q}(\xi_{(2)}) \phi_{B_q}(S^{-1}(e^\mu \otimes K_\lambda) \pi_{B_q}(\xi_{(3)})(1 \otimes K_{-2\rho})) \\ &\stackrel{(1)}{=} \xi_{(1)} \otimes \pi_{B_q}(\xi_{(2)}) \phi_{B_q}(S^{-1}(e^\mu \otimes K_{2\rho+\lambda}) \pi_{B_q}(\xi_{(3)})) \\ &\stackrel{(2)}{=} (\xi \cdot (e^\mu \otimes K_\lambda)) \otimes (e^\mu \otimes K_{2\rho+\lambda}). \end{aligned}$$

Here we have used the following properties: (1) KMS property : $\hat{\psi}(fg) = \hat{\psi}(K_{2\rho} \rightharpoonup g \leftarrow K_{-2\rho} f)$ cf 3.2.3 in [VY19a].

(2) We write $\pi_{B_q}(\xi_{(2)}) = \pi_{B_q}(\xi_{(2)})(e^\mu \otimes K_{2\rho+\lambda}) S^{-1}(e^\mu \otimes K_{2\rho+\lambda})$ and we use the left invariance of ϕ_{B_q} .

It is clear that the map intertwines the $\mathcal{D}(G_q)$ -actions (because the left $\mathcal{D}(G_q)$ -action on $\mathcal{D}(G_q)$ commutes with the right $\mathcal{D}(B_q)$ -action).

Finally we show that the map conserves the scalar product. It is enough to work with elements of the form $(a \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1$ with $a \in \mathcal{A}(K_q)$ (its image is $a \cdot e^\mu \otimes K_{2\rho+\lambda}$). So let $(a \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1$ and $(b \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1$ in $\mathcal{E}(G_q) \otimes_{\mathcal{D}(B_q)} (\mathcal{D}(L_q) \otimes_{\mathcal{D}(L_q)} \mathbb{C}_{\mu, \lambda})$. Before starting our calculation we first remark that $e^\mu * 1 = \varphi(S^{-1}(e^\mu) e^\mu) 1$ so

e^μ acts as the identity on $\mathbb{C}_{\mu,\lambda}$ and we thus have $(a \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1 = (a \cdot e^\mu \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1$. Therefore,

$$\begin{aligned} \langle (a \cdot e^\mu \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1, (b \otimes \hat{\mathbf{1}}_{K_q}) \otimes 1 \rangle &= \langle a \cdot e^\mu \otimes \hat{\mathbf{1}}_{K_q}, b \otimes \hat{\mathbf{1}}_{K_q} \rangle * 1 \\ &= (((a \cdot e^\mu)^* * b) \otimes \hat{\mathbf{1}}_{K_q}) * 1 \\ &= \phi_{K_q}(\overline{a \cdot e^\mu} b_{(1)}) \varphi_T(\pi(b_{(2)}) e^{-\mu}) \\ &= \phi_{K_q}(\overline{a \cdot e^\mu} b \cdot e^\mu) \\ &= \langle a \cdot e^\mu \otimes K_{2\rho+\lambda}, b \cdot e^\mu \otimes K_{2\rho+\lambda} \rangle_{\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu,\lambda}} \end{aligned}$$

□

As announced, we now set $G_q/N_q = K_q \times A_q$, with function algebra

$$\mathcal{A}(G_q/N_q) := \mathcal{A}(K_q) \otimes \mathcal{A}(A_q),$$

equipped with its natural structure of untwisted $*$ -algebra and with the positive functional $\phi_{G_q/N_q} = \phi_{K_q} \otimes \phi_{A_q}(\cdot K_{-4\rho})$.

We endow it with a left $\mathcal{A}(G_q)$ -coaction given, for $a \otimes \pi(f) \in \mathcal{A}(G_q/N_q)$, by

$$\Delta_{G_q/N_q}'(a \otimes \pi(f)) = W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes \pi(f_{(2)})) W_{32} \in \mathcal{M}(\mathcal{A}(G_q) \otimes \mathcal{A}(G_q/N_q)),$$

(which is well defined, see remark 5) and a right $\mathcal{A}(L_q)$ -coaction

$$\Delta_{G_q/N_q}'(a \otimes h) = a_{(1)} \otimes h_{(1)} \otimes \pi(a_{(2)}) \otimes h_{(2)} \in \mathcal{A}(G_q/N_q) \otimes \mathcal{A}(L_q).$$

Proposition 6.2. *Those two coactions commute.*

Proof. We claim that we have

$$(\text{id} \otimes \hat{\pi} \otimes \text{id} \otimes \hat{\pi})(\Delta_{G_q/N_q}'(a \otimes h)) = (\text{id} \otimes \text{id} \otimes \pi \otimes \text{id})[\Delta_{G_q}(a \otimes \hat{\pi}(h))]$$

For this we recall the expression for W and W^{-1} . Let $(u_{ij}^\sigma) \in \mathcal{A}(K_q)$ denote the matrix coefficient associated to a weight basis of an irreducible representation σ of K_q and let $(\omega_{ij}^\sigma) \in \mathcal{A}(\hat{K}_q)$ denote the elements of the dual basis. Then we have

$$W = \sum_{i,j,\sigma} u_{ij}^\sigma \otimes \omega_{ij}^\sigma, \quad W^{-1} = \sum_{i,j,\sigma} S(u_{ij}^\sigma) \otimes \omega_{ij}^\sigma,$$

where the sums run over all equivalent classes of irreducible representations. We have

$$\begin{aligned} &(\text{id} \otimes \text{id} \otimes \pi \otimes \text{id})[\Delta_{G_q}(a \otimes \hat{\pi}(h))] \\ &= \sum_{i,j,\sigma,r,s,\nu} a_{(1)} \otimes \omega_{ij}^\sigma \hat{\pi}(h_{(1)}) \omega_{rs}^\nu \otimes \pi(S(u_{ij}^\sigma) a_{(2)} u_{rs}^\nu) \otimes \hat{\pi}(h_{(2)}) \\ &\stackrel{(*)}{=} \sum_{i,\sigma,r,\nu} a_{(1)} \otimes \omega_{ii}^\sigma \omega_{rr}^\nu \hat{\pi}(h_{(1)}) \otimes \pi(S(u_{ii}^\sigma) \pi(a_{(2)}) \pi(u_{rr}^\nu)) \otimes \hat{\pi}(h_{(2)}) \\ &= \sum_{i,\sigma,r,\nu} a_{(1)} \otimes \omega_{ii}^\sigma \hat{\pi}(h_{(1)}) \otimes \pi(a_{(2)}) \otimes \hat{\pi}(h_{(2)}) \\ &= a_{(1)} \otimes \hat{\pi}(h_{(1)}) \otimes \pi(a_{(2)}) \otimes \hat{\pi}(h_{(2)}) \\ &= (\text{id} \otimes \hat{\pi} \otimes \text{id} \otimes \hat{\pi})(\Delta_{G_q/N_q}'(a \otimes h)), \end{aligned}$$

(see the proof of [VY19a, Lemma 5.18] for a similar calculation). For the equality $(*)$ we use the fact that the elements of $\hat{\pi}(\mathcal{A}(A_q)) \subset M(\mathcal{A}(\hat{K}_q))$ are diagonal (keeping in mind that $M(\mathcal{A}(\hat{K}_q)) = \sum_{\sigma} \text{End}(V_{\sigma})$) and then commute with diagonal elements ω_{ii}^{σ} . We have also that (following remark 5)

$$(\text{id} \otimes \text{id} \otimes \text{id} \otimes \hat{\pi})(\Delta_{G_q/N_q}(a \otimes h)) = \Delta_{G_q}(a \otimes \hat{\pi}(h)).$$

Now we can prove the proposition and first we rewrite above equalities using the leg notation

$$\begin{aligned} (\hat{\pi} \otimes \hat{\pi})_{24} \circ \Delta'_{G_q/N_q} &= \pi_3 \circ \Delta_{G_q} \circ \hat{\pi}_2 \\ \hat{\pi}_4 \circ \Delta_{G_q/N_q} &= \Delta_{G_q} \circ \hat{\pi}_2 \end{aligned}$$

Now observe that we have from the one hand

$$\begin{aligned} (\hat{\pi} \otimes \hat{\pi})_{46} \circ (\Delta'_{G_q/N_q})_{34} \circ \Delta_{G_q/N_q} &= ((\hat{\pi} \otimes \hat{\pi})_{24} \circ \Delta'_{G_q/N_q})_{34} \circ \Delta_{G_q/N_q} \\ &= (\pi_3 \circ \Delta_{G_q} \circ \hat{\pi}_2)_{34} \circ \Delta_{G_q/N_q} \\ &= \pi_5 \circ (\Delta_{G_q})_{34} \circ \Delta_{G_q} \circ \hat{\pi}_2 \end{aligned}$$

and on the other hand

$$\begin{aligned} (\hat{\pi} \otimes \hat{\pi})_{46} \circ (\Delta_{G_q/N_q})_{12} \circ \Delta'_{G_q/N_q} &= (\Delta_{G_q})_{12} \circ (\hat{\pi} \otimes \hat{\pi})_{24} \circ \Delta'_{G_q/N_q} \\ &= (\Delta_{G_q})_{12} \circ \pi_3 \circ \Delta_{G_q} \circ \hat{\pi}_2 \\ &= \pi_5 \circ (\Delta_{G_q})_{12} \circ \Delta_{G_q} \circ \hat{\pi}_2 \end{aligned}$$

and we conclude the proof using the coassociativity of Δ_{G_q} and injectivity of $\hat{\pi}$. \square

Proposition 6.3. *The functional $\phi_{G_q/N_q} = \phi_{K_q} \otimes \phi_{A_q}(\cdot K_{-4\rho})$ is invariant with respect to the left coaction Δ_{G_q/N_q} of $\mathcal{A}(G_q)$.*

Proof. Before beginning our calculation we remark that, knowing that $\phi_T(\pi(f)) = \phi_{\hat{K}_q}(f)$ and $\psi_{\hat{K}_q}(f) = \phi_{\hat{K}_q}(fK_{-4\rho})$, we have $\phi_{A_q}(\pi(f)K_{-4\rho}) = \phi_{A_q}(\pi(fK_{-4\rho})) = \psi_{\hat{K}_q}(f)$.

Let $a \otimes f \in \mathcal{A}(K_q) \otimes \mathcal{A}(\hat{K}_q)$. Now, using the invariance of the Haar state $\phi_{K_q} \otimes \psi_{\hat{K}_q}$ on $\mathcal{A}(G_q)$, we have

$$\begin{aligned} &(\text{id} \otimes \text{id} \otimes \phi_{G_q/N_q})(\Delta_{G_q/N_q}(a \otimes \pi(f))) \\ &= (\text{id} \otimes \text{id} \otimes \phi_{K_q})(W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)})W_{32})\phi_{A_q}(\pi(f_{(2)})K_{-4\rho}) \\ &= (\text{id} \otimes \text{id} \otimes \phi_{K_q})(W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)})W_{32})\psi_{\hat{K}_q}(f_{(2)}) \\ &= (\text{id} \otimes \text{id} \otimes \phi_{K_q} \otimes \psi_{\hat{K}_q})(W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes f_{(2)})W_{32}) \\ &= \phi_{K_q}(a)\psi_{\hat{K}_q}(f)\mathbf{1}_{G_q} \\ &= \phi_{K_q}(a)\phi_{A_q}(\pi(f)K_{-4\rho})\mathbf{1}_{G_q}. \end{aligned}$$

□

Proposition 6.4. *The map $\text{id} \otimes \pi : \mathcal{A}(G_q) \rightarrow \mathcal{A}(G_q/N_q)$ intertwines the left- $\mathcal{A}(G_q)$ -coactions (where $\mathcal{A}(G_q)$ is simply considered with its natural comodule structure).*

Proof. This is immediate from the fact that for $a \otimes f \in \mathcal{A}(G_q)$ we have

$$(\text{id} \otimes \text{id} \otimes \text{id} \otimes \pi)(W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes f_{(2)})W_{32}) = W_{32}^{-1}(a_{(1)} \otimes f_{(1)} \otimes a_{(2)} \otimes \pi(f_{(2)}))W_{32}.$$

□

One can see $\mathcal{A}(G_q/N_q)$ as a $\mathcal{D}(L_q)$ -module with the convolution action induced by the right coaction Δ'_{G_q/N_q} of $\mathcal{A}(L_q)$:

$$(a \otimes f) \cdot (\tau \otimes h) = a \cdot \tau \otimes f \cdot h, \quad \tau \otimes h \in \mathcal{D}(L_q)$$

where we set

$$f \cdot h = \phi_{A_q}(S(h)f_{(2)}K_{2\rho})f_{(1)}.$$

Remark 6. We use the notation \cdot even though it has not exactly the same meaning as in Section 3. The deformation by $K_{2\rho}$ is motivated by the L_q -relatively invariance of the functional ϕ_{G_q/N_q} (see [Cla13, section 2] for the geometric explanation in the classical case). For us it will be made clear in calculations by the identity

$$\hat{\mathbf{1}}_T \cdot h = hK_{-2\rho}.$$

which follows from the fact that if we consider g and h in $\mathcal{A}(A_q)$ we have $g \cdot h = \hat{\varphi}(S(h)g_{(2)}K_{2\rho})g_{(1)} = \hat{\varphi}(S(h_{(1)})g)h_{(2)}K_{-2\rho}$.

From Proposition 6.4 we obtain a left $\mathcal{D}(G_q)$ convolution action on $\mathcal{A}(G_q/N_q)$ which commutes with the $\mathcal{D}(L_q)$ -action defined above. Our goal is now to turn $\mathcal{A}(G_q/N_q)$ into a pre-Hilbert $\mathcal{D}(L_q)$ -module, equivalent to $\mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$. First we check that the map

$$\begin{aligned} \Lambda : \mathcal{D}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q) &\longrightarrow \mathcal{A}(G_q/N_q) \\ a \otimes f \otimes \tau \otimes h &\longmapsto (a \otimes \pi(f)) \cdot (\tau \otimes h) \end{aligned}$$

is a surjective map which intertwines the actions of $\mathcal{D}(G_q)$. The surjectivity comes from the surjectivity of π and the fact that the $\mathcal{D}(L_q)$ -action is essential.

Finally we equip $\mathcal{A}(G_q/N_q)$ with a $\mathcal{D}(L_q)$ -inner product. We set

$$\langle a \otimes h, b \otimes k \rangle_{\mathcal{D}(L_q)} := \pi(a^* * b) \otimes (h^* * k)K_{-2\rho}.$$

In Sweedler notation, this would be expressed informally as $\langle a \otimes h, b \otimes k \rangle_{\mathcal{D}(L_q)} = \phi(\bar{a}b_{(1)})\pi(b_{(2)}) \otimes \hat{\varphi}(h^*k_{(1)})k_{(2)}K_{-2\rho}$.

Proposition 6.5. *The map Λ is compatible with this $\mathcal{D}(L_q)$ -seilinear form. That is, for $(a \otimes f) \otimes (\tau \otimes h)$ and $(b \otimes g) \otimes (\zeta \otimes k)$ in $\mathcal{E}(G_q) \otimes_{\mathcal{D}(B_q)} \mathcal{D}(L_q)$, we have*

$$\langle (a \otimes f) \otimes (\tau \otimes h), (b \otimes g) \otimes (\zeta \otimes k) \rangle_{\mathcal{D}(L_q)} = \langle (a \cdot \tau \otimes \pi(f) \cdot h), (b \cdot \zeta \otimes \pi(g) \cdot k) \rangle_{\mathcal{D}(L_q)}$$

and $\langle \cdot, \cdot \rangle_{\mathcal{D}(L_q)}$ is therefore an inner product on $\mathcal{A}(G_q/N_q)$.

Proof. It is enough to show the equality for two elements of the form $(a \otimes \hat{\mathbf{1}}_{K_q}) \otimes (\tau \otimes h)$ and $(b \otimes \hat{\mathbf{1}}_{K_q}) \otimes (\zeta \otimes k)$. We observe that we have

$$\begin{aligned} \langle a \otimes \hat{\mathbf{1}}_{K_q}, b \otimes \hat{\mathbf{1}}_{K_q} \rangle_{\mathcal{D}(B_q)} &= E((a \otimes \hat{\mathbf{1}}_{K_q})^* * (b \otimes \hat{\mathbf{1}}_{K_q})) \\ &= \pi(a^* * b) \otimes \hat{\mathbf{1}}_{K_q} K_{-2\rho} \\ &= \pi(a^* * b) \otimes \hat{\mathbf{1}}_{K_q} \end{aligned}$$

Following Remark 6 we have $\Lambda(a \otimes \hat{\mathbf{1}}_{K_q} \otimes \tau \otimes h) = a \cdot \tau \otimes h K_{-2\rho}$, so on the right it gives

$$\begin{aligned} \langle a \cdot \tau \otimes h K_{-2\rho}, b \cdot \zeta \otimes k K_{-2\rho} \rangle_{\mathcal{D}(L_q)} &= \pi(\tau^* \cdot a^* * b \cdot \zeta) \otimes ((h K_{-2\rho})^* * k K_{-2\rho}) K_{2\rho} \\ &\stackrel{(*)}{=} (\tau^* * \pi(a^* * b) * \zeta) \otimes (h^* * k), \end{aligned}$$

(*) for $x, y \in \mathcal{A}(A_q)$ we have $(x * y) K_{-2\rho} = \phi_{A_q}(S(x) y_{(2)}) y_{(1)} K_{2\rho} = \phi_{A_q}(S(x K_{2\rho}) y_{(2)} K_{2\rho}) y_{(1)} K_{2\rho} = x K_{2\rho} * y K_{2\rho}$.

On the left we have :

$$\begin{aligned} \langle a \otimes \hat{\mathbf{1}}_{K_q} \otimes \tau \otimes h, b \otimes \hat{\mathbf{1}}_{K_q} \otimes \zeta \otimes k \rangle_{\mathcal{D}(L_q)} &= \left\langle \tau \otimes h, \langle a \otimes \hat{\mathbf{1}}_{K_q}, b \otimes \hat{\mathbf{1}}_{K_q} \rangle_{C^*(B_q)} * (\zeta \otimes k) \right\rangle_{C^*(L_q)} \\ &= \langle \tau \otimes h, ((a^* * b) \otimes \hat{\mathbf{1}}_{K_q}) * (\zeta \otimes k) \rangle_{C^*(L_q)} \\ &= \langle \tau \otimes h, (\pi(a^* * b) * \zeta) \otimes k \rangle_{C^*(L_q)} \\ &= (\tau^* * \pi(a^* * b) * \zeta) \otimes (h^* * k). \end{aligned}$$

□

We can now state the following result.

Theorem 6.6. *The pre-Hilbert $\mathcal{D}(L_q)$ -module $\mathcal{A}(G_q/N_q)$ can be completed into a Hilbert $C^*(L_q)$ -module $\mathcal{E}(G_q/N_q)$ and we have*

$$\mathcal{E}(G_q/N_q) \cong \mathcal{E}(G_q) \otimes_{C^*(B_q)} C^*(L_q)$$

as G_q -representations. The tensor product $\mathcal{E}(G_q/N_q) \otimes_{C^*(L_q)} -$ defines a functor from the category of unitary $C^*(L_q)$ -representations to the category of unitary $C^*(G_q)$ -representations which coincides with parabolic induction.

By the Fourier transform, we have

$$C^*(L_q) \cong C_0(\hat{L}_q) = C_0(\mathbf{P} \times T)$$

such that the characters of $C^*(L_q)$ become the evaluation maps

$$\text{ev}_{(\mu, \lambda)} : C_0(\mathbf{P} \times T) \rightarrow \mathbb{C}_{\mu, \lambda}.$$

Theorem 6.6 shows that the Hilbert space bundle of principal series representations $\mathcal{H} = (\text{Ind}_{B_q}^{G_q} \mathbb{C}_{\mu, \lambda})_{\mu, \lambda}$ over $\mathbf{P} \times T$ is isomorphic to the $C_0(\mathbf{P} \times T)$ -Hilbert module

$$\mathcal{E}(G_q/N_q) \otimes_{C_0(\mathbf{P} \times T)} C_0(\mathbf{P} \times T) \cong \mathcal{E}(G_q/N_q).$$

According to [VY19a], the action of the Weyl group W on $\mathbf{P} \times T$ extends to a action by $C^*(G_q)$ -linear maps on $\mathcal{H} \cong \mathcal{E}(G_q/N_q)$ and then Theorem 7.1 of [VY19b] can be interpreted as follows.

Corollary 6.7. *Let G_q be a complex semi-simple quantum group. Then*

$$C_r^*(G_q) \cong \mathfrak{K}(\mathcal{E}(G_q/N_q))^W$$

where \mathfrak{K} indicates the algebra of compact operators in the sense of Hilbert modules.

In the classical case, this result has been first obtained in [Was87] and reformulated in [CCH16] with the Rieffel induction framework that we used in this paper.

References

- [CCH16] Pierre Clare, Tyrone Crisp, and Nigel Higson. Parabolic induction and restriction via C^* -algebras and Hilbert C^* -modules. *Compos. Math.*, 152(6):1286–1318, 2016.
- [Cla13] Pierre Clare. Hilbert modules associated to parabolically induced representations. *J. Operator Theory*, 69(2):483–509, 2013.
- [Dae98] A. Van Daele. An algebraic framework for group duality. *Advances in Mathematics*, 140(2):323 – 366, 1998.
- [KKSS] Mehrdad Kalantar, Paweł Kasprzak, Adam Skalski, and Piotr M. Sołtan. Induction for locally compact quantum groups revisited. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics*, page 1–23.
- [Kus02] Johan Kustermans. Induced corepresentations of locally compact quantum groups. *J. Funct. Anal.*, 194(2):410–459, 2002.

- [Lan95] E.C. Lance. *Hilbert C^* -Modules: A Toolkit for Operator Algebraists*. Lecture note series / London mathematical society. Cambridge University Press and London Mathematical Society and Hitchin, N.J. and Cassels, J.W.S., 1995.
- [Mac52] George W. Mackey. Induced representations of locally compact groups. I. *Ann. of Math. (2)*, 55:101–139, 1952.
- [Rie74] Marc A. Rieffel. Induced representations of C^* -algebras. *Advances in Math.*, 13:176–257, 1974.
- [Riv20] Damien Rivet. PhD thesis, Upcoming, 2020.
- [Vae05] Stefaan Vaes. A new approach to induction and imprimitivity results. *J. Funct. Anal.*, 229(2):317–374, 2005.
- [Voi08] Christian Voigt. Bornological quantum groups. *Pacific J. Math.*, 235(1):93–135, 2008.
- [VY19a] Christian Voigt and Robert Yuncken. COMPLEX SEMISIMPLE QUANTUM GROUPS AND REPRESENTATION THEORY. working paper or preprint, June 2019.
- [VY19b] Christian Voigt and Robert Yuncken. The plancherel formula for complex semisimple quantum groups, 2019.
- [Was87] Antony Wassermann. Une démonstration de la conjecture de Connes-Kasparov pour les groupes de Lie linéaires connexes réductifs. *C. R. Acad. Sci. Paris Sér. I Math.*, 304(18):559–562, 1987.