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Estimate for P_tD for the stochastic Burgers equation.

Giuseppe Da Prato *and Arnaud Debussche †

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Abstract

We consider the Burgers equation on $H = L^2(0,1)$ perturbed by white noise and the corresponding transition semigroup P_t. We prove a new formula for $P_tD\varphi$ (where $\varphi : H \to \mathbb{R}$ is bounded and Borel) which depends on φ but not on its derivative. Then we deduce some consequences for the invariant measure ν of P_t as its Fomin differentiability and an integration by parts formula which generalises the classical one for gaussian measures.

1 Introduction

We consider the following stochastic Burgers equation in the interval $[0,1]$ with Dirichlet boundary conditions,

$$\begin{cases} dX(t,\xi) &= (\partial_\xi^2 X(t,\xi) + \partial_\xi(X^2(t,\xi)))dt + dW(t,\xi),\ t>0,\ \xi\in(0,1), \\ X(t,0) &= X(t,1) = 0,\ t>0, \\ X(0,\xi) &= x(\xi),\ \xi\in(0,1). \end{cases} \tag{1}$$

The unknown X is a real valued process depending on $\xi \in [0,1]$ and $t \geq 0$ and dW/dt is a space-time white noise on $[0,1]\times[0,\infty)$. This equation has been studied by several authors (see [BeCaJL94], [DaDeTe94], [DaGa95], [Gy98]) and it is known that there exists a unique solution with paths in $C([0,T];L^p(0,1))$ if the initial data $x \in L^p(0,1)$, $p \geq 2$. In this article, we want to prove new properties on the transition semigroup associated to (1).

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We rewrite (1) as an abstract differential equation in the Hilbert space $H = L^2(0,1)$,

$$\begin{cases} dX = (AX + b(X))dt + dW_t, \\ \\ X(0) = x. \end{cases} \tag{2}$$

As usual $A = \partial_{\xi\xi}$ with Dirichlet boundary conditions, on the domain $D(A) = H^2(0,1) \cap H_0^1(0,1)$, $b(x) = \partial_\xi(x^2)$. Here and below, for $s \geq 0$, $H^s(0,1)$ is the standard $L^2(0,1)$ based Sobolev space. Also, W is a cylindrical Wiener process on H. We denote by $X(t,x)$ the solution.

We denote by $(P_t)_{t\geq 0}$ the transition semigroup associated to equation (2) on $\mathscr{B}_b(H)$, the space of all real bounded and Borel functions on H endowed with the norm

$$\|\varphi\|_0 = \sup_{x\in H} |\varphi(x)|, \quad \forall\, \varphi \in \mathscr{B}_b(H).$$

We know that P_t possesses a unique invariant measure ν so that P_t is uniquely extendible to a strongly continuous semigroup of contractions on $L^2(H,\nu)$ (still denoted by P_t) whose infinitesimal generator we shall denote by $\mathscr{L}$. Let $\mathscr{E}_A(H)$ be the linear span of real parts of all φ of the form

$$\varphi_h(x) := e^{i\langle h,x\rangle},\ x \in D(A).$$

We have proved in [DaDe07] that $\mathscr{E}_A(H)$ is a core for $\mathscr{L}$ and that

$$\mathscr{L}\varphi = \frac{1}{2}\mathrm{Tr}\,[QD_x^2\varphi] + \langle Ax + b(x), D_x\varphi\rangle, \quad \forall\, \varphi \in \mathscr{E}_A(H). \tag{3}$$

Here and below, D_x denotes the differential with respect to $x \in H$. When φ is a real valued function, we often identify $D_x\varphi$ with its gradient. Similarly, D_x^2 is the second differential and for a real valued function $D_x^2\varphi$ can be identify with the Hessian.

In this paper, we use a formula for $P_tD_x\varphi$ which depends on φ but not on its derivative. To our knowledge, this formula is new.

For a finite dimensional stochastic equation a formula for P_tD_x can be obtained, under suitable assumptions, using the Malliavin calculus and it is the key tool for proving the existence of a density of the law of $X(t,x)$ with respect to the Lebesgue measure, see [Ma97]. Concerning SPDEs, several results are available for densities of finite dimensional projections of the law of the solutions, see [Sa05] and the references therein. For these results, Malliavin calculus is used on a finite dimensional random variable. Malliavin calculus is difficult to generalize to a true infinite dimensional setting and it does not seem useful to give estimate on $P_tD_x\varphi$ in terms of φ. The formula we use allows a completely different approach. It relates P_tD_x to D_xP_t. In

the recent years several formulae for $D_xP_t\varphi$ independent on $D_x\varphi$ have been proved thanks to suitable generalizations of the Bismut–Elworthy–Li formula (BEL). Thus, combining our formula to estimates obtained on D_xP_t implies useful information on P_tD_x. As we shall show, these can be used to extend to the measure ν a basic integration by parts identity well known for Gaussian measures.

Let us explain the main ideas. Let $u(t,x) = P_t\varphi(x)$, under suitable conditions, it is a solution of the Kolmogorov equation

$$\begin{cases} D_tu(t,x) = \dfrac{1}{2}\mathrm{Tr}\,[QD_x^2u(t,x)] + \langle Ax + b(x), D_xu(t,x)\rangle, \\ \\ u(0,x) = \varphi(x). \end{cases} \tag{4}$$

Set $v_h(t,x) = D_xu(t,x)\cdot h$, the differential of u with respect to x in the direction h. Then formally v_h is a solution to the equation

$$\begin{cases} D_tv_h(t,x) = \mathscr{L}v_h(t,x) + \langle Ah + b'(x)h, D_xu(t,x)\rangle, \\ \\ v_h(0,x) = D_x\varphi(x)\cdot h. \end{cases} \tag{5}$$

We notice that formal computations may be made rigorous by approximating φ with elements from $\mathscr{E}_A(H)$. By (5) and variation of constants, it follows that

$$v_h(t,x) = P_t(D_x\varphi(x)\cdot h) + \int_0^t P_{t-s}(\langle Ah + b'(x)h, D_xu(s,x)\rangle)ds, \tag{6}$$

which implies

$$P_t(D_x\varphi(x)\cdot h) = D_xP_t\varphi(x)\cdot h - \int_0^t P_{t-s}(\langle Ah + b'(x)h, D_xu(s,x)\rangle)ds. \tag{7}$$

This formula allows to obtain the following estimate: For all $\varphi\in\mathscr{B}_b(H)$, $\delta>0$ and all $h\in H^{1+\delta}(0,1)$, we have

$$|P_t(D\varphi\cdot h)(x)| \le ce^{ct}(1+t^{-1/2})(1+|x|_{L^4})^8\,\|\varphi\|_0\,|h|_{1+\delta}, \tag{8}$$

where $|\cdot|_{1+\delta}$ is the norm in $H^{1+\delta}(0,1)$. We will not prove this formula here, it could be proved by similar arguments as in section 3.

Integrating with respect to ν over H and taking into account the invariance of ν, yields

$$\begin{aligned} \int_H (D_x\varphi(x)\cdot h)d\nu \;&=\; \int_H (D_xP_t\varphi(x)\cdot h)\,d\nu \\ &\quad - \int_0^t\int_H (\langle Ah + b'(x)h, D_xP_s\varphi(x)\rangle)d\nu\,ds. \end{aligned} \tag{9}$$

Using identity (9) we arrive at the main result of the paper, proved in Section 2.

Theorem 1. *For any $p > 1$, $\delta > 0$, there exists $C > 0$ such that for all $\varphi \in \mathscr{B}_b(H)$ and all $h \in H^{1+\delta}(0,1)$, we have*

$$\int_H D_x\varphi(x) \cdot h \, \nu(dx) \leq C\|\varphi\|_{L^p(H,\nu)} \, |h|_{1+\delta}, \tag{10}$$

for $t > 0$, where $|\cdot|_{1+\delta}$ is the norm in $H^{1+\delta}(0,1)$.

For a gaussian measure, it is easy to obtain such estimate. In fact, if μ is the invariant measure of the stochastic heat equation on $(0,1)$, *i.e.* equation (2) without the nonlinear term, then the same formula holds with $\delta = 0$. Thus our result is not totally optimal and we except that it can be extended to $\delta = 0$.

Also, identity (9) is general and we believe that it can be used in many other situations. For instance, we will investigate the generalization of our results to other SPDEs such as reaction–diffusion and 2D- Navier–Stokes equations. This will be the object of a future work.

In section 2, we show that Theorem 1 can be used to derive an integration by part formula for the measure ν. Theorem 1 is proved in section 3.

We end this section with some notations. We shall denote by (e_k) an orthonormal basis in H and by (α_k) a sequence of positive numbers such that

$$Ae_k = -\alpha_k e_k, \quad k \in \mathbb{N}.$$

For any $k \in \mathbb{N}$, D_k will represent the directional derivative in the direction of e_k.

The norm of $L^2(0,1)$ is denoted by $|\cdot|$. For $p \geq 1$, $|\cdot|_{L^p}$ is the norm of $L^p(0,1)$. The operator A is self–adjoint negative. For any $\alpha \in \mathbb{R}$, $(-A)^\alpha$ denotes the α power of the operator $-A$ and $|\cdot|_\alpha$ is the norm of $D((-A)^{\alpha/2})$ which is equivalent to the norm of the Sobolev space $H^\alpha(0,1)$. We have $|\cdot|_0 = |\cdot| = |\cdot|_{L^2}$. We shall use the interpolatory estimate

$$|x|_\beta \leq |x|_\alpha^{\frac{\gamma-\beta}{\gamma-\alpha}} \, |x|_\gamma^{\frac{\beta-\alpha}{\gamma-\alpha}}, \quad \alpha < \beta < \gamma, \tag{11}$$

and the Agmon's inequality

$$|x|_{L^\infty} \leq |x|^{\frac{1}{2}} \, |x|_1^{\frac{1}{2}}. \tag{12}$$

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2 Integration by part formula for ν

This section is devoted to some consequences of Theorem 1. Here we take $p = 2$ for simplicity. In this case (10) can be rewritten as

$$\int_H ((-A)^{-\alpha} D_x\varphi(x) \cdot h)\, \nu(dx) \le C\|\varphi\|_{L^2(H,\nu)}\, |h|, \quad \forall\, h \in H, \tag{13}$$

where $\alpha = \frac{1+\delta}{2}$.

Proposition 2. *Let $\alpha > \frac{1}{2}$, then for any $h \in H$ the linear operator*

$$\varphi \in C_b^1(H) \mapsto ((-A)^{-\alpha} D_x\varphi(x) \cdot h) \in C_b(H)$$

is closable in $L^2(H,\nu)$.

Proof. Let $(\varphi_n) \subset C_b^1(H)$ and $f \in L^2(H,\nu)$ such that

$$\varphi_n \to 0 \ \text{ in } L^2(H,\nu), \quad ((-A)^{-\alpha} D_x\varphi_n(x) \cdot h) \to f \ \text{ in } L^2(H,\nu).$$

Let $\psi \in C_b^1(H)$, then by (13) it follows that

$$\left| \int_H [\psi(x)((-A)^{-\alpha} D_x\varphi_n(x) \cdot h) + \varphi_n(x)((-A)^{-\alpha} D_x\psi(x) \cdot h)]\, \nu(dx) \right|$$

$$\le \|\varphi_n\psi\|_{L^2(H,\nu)}\, |h|_H \le \|\psi\|_0\, \|\varphi_n\|_{L^2(H,\nu)}\, |h|_H.$$

Letting $n \to \infty$, yields

$$\int_H \psi(x) f(x)\, \nu(dx) = 0,$$

which yields $f = 0$ by the arbitrariness of ψ. □

We can now define the Sobolev space $W_\alpha^{1,2}(H,\nu)$. First we improve Proposition 2.

Corollary 3. *Let $\alpha > \frac{1}{2}$, then the linear operator*

$$\varphi \in C_b^1(H) \mapsto (-A)^{-\alpha} D_x\varphi \in C_b(H;H)$$

is closable in $L^2(H,\nu)$.

Proof. By Proposition 2 taking $h = e_k$ we see that D_k is a closed operator on $L^2(H,\nu)$ for any $k \in \mathbb{N}$. Set

$$(-A)^{-\alpha} D_x\varphi(x) = \sum_{k=1}^{\infty} \alpha_k^{-\alpha} D_k\varphi(x)\ e_k, \quad \forall\, h \in H,$$

the series being convergent in $L^2(H,\nu)$. Then

$$|(-A)^{-\alpha}D_x\varphi(x)|^2 = \sum_{k=1}^{\infty} \alpha_k^{-2\alpha}|D_k\varphi(x)|^2$$

Let $(\varphi_n) \subset C_b^1(H)$ and $F \in L^2(H,\nu;H)$ such that

$$\varphi_n \to 0 \ \text{ in } L^2(H,\nu), \quad (-A)^{-\alpha}D_x\varphi \to F \ \text{ in } L^2(H,\nu;H).$$

We have to show that $F = 0$.

Now for any $k \in \mathbb{N}$ we have $D_k\varphi_n(x) \to \alpha_k^{\alpha}\langle F(x), e_k\rangle$ in $L^2(H,\nu)$. So, $\langle F, e_k\rangle = 0$ and the conclusion follows.

□

Let us denote by $W_{\alpha}^{1,2}(H,\nu)$ the domain of the closure of $(-A)^{-\alpha}D_x$. Then if M^* denotes the adjoint of $(-A)^{-\alpha}D_x$ we have

$$\int_H ((-A)^{-\alpha}D_x\varphi(x)\cdot F(x))\,\nu(dx) = \int_H \varphi(x)\,M^*(F)(x)\,\nu(dx). \tag{14}$$

Set now $F_h(x) = h$ where $h \in H$. By Theorem 1 F_h belongs to the domain of M^*. Setting $M^*(F_h) = v_h$ we obtain the following integration by part formula.

Proposition 4. *Let $\alpha > \frac{1}{2}$, then for any $h \in H$ there exists a function $v_h \in L^2(H,\nu)$ such that*

$$\int_H ((-A)^{-\alpha}D_x\varphi(x)\cdot h)\,\nu(dx) = \int_H \varphi(x)\,v_h(x)\,\nu(dx), \tag{15}$$

for any $\varphi \in W_{\alpha}^{1,2}(H,\nu)$.

By (15) it follows that the measure ν possesses the Fomin derivative in all directions $(-A)^{-\alpha}h$ for $h \in H$, see e.g. [Pu98].

If, in (2), $b = 0$ then the gaussian measure $\mu = N_Q$, where $Q = -\frac{1}{2}\,A^{-1}$, is the invariant measure and $v_h(x) = \sqrt{2}\langle Q^{-1/2}x, h\rangle$. Then (15) reduces to the usual integration by parts formula for the Gaussian measure μ. Note that it follows that, as already mentionned, Theorem 1 is true with $\delta = 0$ in this case.

We recall the importance of formula (15) for different topics as Malliavin calculus [Ma97], definition of integral on infinite dimensional surfaces of H [AiMa88], [FePr92], [Bo98], definition of BV functions in abstract Wiener spaces [AmMiMaPa10], infinite dimensional generalization of DiPerna-Lions theory [AmFi09], [DaFlRo14] and so on.

We think that Theorem 1 open the possibility to study these topics in the more general situations of non Gaussian measures.

3 Proof of Theorem 1

For $h \in H$, $\eta^h(t,x)$ is the differential of $X(t,x)$ in the direction h and $(\eta^h(t,x))_{\geq 0}$ satisfies the equation

$$\begin{cases} \dfrac{d\eta^h(t,x)}{dt} = A\eta^h(t,x) + b'(X(t,x))\eta^h(t,x), \\ \\ \eta^h(0,x) = h. \end{cases} \tag{16}$$

Note that this equation as well as the computations below are done at a formal level. They could easily be justified rigorously by an approximation argument, such as Galerkin approximation for instance. The following result is proved in [DaDe07], see Proposition 2.2.

Lemma 5. *For any $\alpha \in [-1,0]$ there exists $c = c(\alpha) > 0$ such that*

$$e^{-c\int_0^t |X(s,x)|_{L^4}^{\frac{8}{3}}\,ds}\,|\eta^h(s,x)|_\alpha^2 + \int_0^t e^{-c\int_s^t |X(\tau,x)|_{L^4}^{\frac{8}{3}}\,d\tau}\,|\eta^h(s,x)|_{1+\alpha}^2\,ds \leq |h|_\alpha^2. \tag{17}$$

We introduce the following Feynman-Kac semigroup

$$S_t\varphi(x) = \mathbb{E}\left[\varphi(X(t,x))e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\right]$$

Next lemma is a slight generalization of Lemma 3.2 in [DaDe07].

Lemma 6. *For any $\alpha \in [0,1]$ and $p > 1$, if K is chosen large enough then for any φ Borel and bounded we have*

$$|D_x S_t\varphi(x)|_\alpha \leq c\,e^{ct}(1 + t^{-\frac{1+\alpha}{2}})(1 + |x|_{L^6}^3)\,\left[\mathbb{E}\left(\varphi^p(X(t,x))\right]^{1/p}, \tag{18}$$

where c depends on p, K, α.

Proof. It is clearly suficient to prove the result for $p \leq 2$. We proceed as in [DaDe07] and write

$$D_x S_t\varphi(x) \cdot h = I_1 + I_2.$$

where

$$I_1 = \frac{1}{t}\mathbb{E}\left(e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\varphi(X(t,x))\int_0^t (\eta^h(s,x), dW(s))\right)$$

and

$$I_2 = -4K\mathbb{E}\left(e^{-K\int_0^t |X(s,x)|_{L^4}^4\,ds}\varphi(X(t,x))\int_0^t (X^3(s,x), \eta^h(s,x))ds\right).$$

For I_1 we have with $\frac{1}{p}+\frac{1}{q}=1$:

$$I_1 \leq \frac{1}{t}\left[\mathbb{E}\left(\varphi^p(X(t,x))\right)\right]^{1/p}\left[\mathbb{E}\left(e^{-Kq\int_0^t |X(s,x)|^4_{L^4}\,ds}\left|\int_0^t (\eta^h(s,x),dW(s)\right|^q\right)\right]^{1/q}$$

Using Itô's formula for $|z(t)|^q = e^{-Kq\int_0^t |X(s,x)|^4_{L^4}\,ds}\left|\int_0^t(\eta^h(s,x),dW(s))\right|^q$, we get:

$$\begin{aligned}|z(t)|^q &= -4Kq\int_0^t |X(s,x)|^4_{L^4}|z(s)|^q\,ds\\ &+q\int_0^t e^{-K\int_0^s |X(s,x)|^4_{L^4}\,ds}|z(s)|^{q-2}z(s)(\eta^h(s,x),dW(s))\\ &+\frac{1}{2}q(q-1)\int_0^t e^{-2K\int_0^t |X(s,x)|^4_{L^4}\,ds}|z(s)|^{q-2}|\eta^h(s,x)|^2ds.\end{aligned}$$

We deduce:

$$\begin{aligned}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right) &\leq q\mathbb{E}\left(\sup_{r\in[0,t]}\left|\int_0^r e^{-K\int_0^s |X(s,x)|^4_{L^4}\,ds}|z(s)|^{q-2}z(s)(\eta^h(s,x),dW(s))\right|\right)\\ &+\frac{1}{2}q(q-1)\mathbb{E}\left(\int_0^t e^{-2K\int_0^t |X(s,x)|^4_{L^4}\,ds}|z(s)|^{q-2}|\eta^h(s,x)|^2ds\right)\\ &= A_1+A_2.\end{aligned}$$

By a standard martingale inequality, (11) and Lemma 5, we have

$$\begin{aligned}A_1 &\leq 3q\mathbb{E}\left(\left|\int_0^t e^{-2K\int_0^s |X(s,x)|^4_{L^4}\,ds}|z(s)|^{2(q-1)}|\eta^h(s,x)|^2ds\right|^{1/2}\right)\\ &\leq 3q\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q-1}\left(\int_0^t e^{-2K\int_0^s |X(s,x)|^4_{L^4}\,ds}|\eta^h(s,x)|^2ds\right)^{1/2}\right)\\ &\leq 3q\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q-1}\left(\int_0^t e^{-2K\int_0^s |X(s,x)|^4_{L^4}\,ds}\,|\eta^h(s,x)|^{2(1-\alpha)}_{-\alpha}\,|\eta^h(s,x)|^{2\alpha}_{1-\alpha}\,ds\right)^{1/2}\right)\\ &\leq 3qt^{\frac{1-\alpha}{2}}\,|h|_{-\alpha}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q-1}\right)\\ &\leq 3qt^{\frac{1-\alpha}{2}}\,|h|_{-\alpha}\left[\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right)\right]^{(q-1)/q}\\ &\leq \frac{1}{4}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^q\right)+ct^{\frac{q(1-\alpha)}{2}}\,|h|^q_{-\alpha}.\end{aligned}$$

Similarly:

$$\begin{aligned}
A_2 &\leq \frac{1}{2}q(q-1)\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q-2}\int_0^t e^{-2K\int_0^s|X(s,x)|^4_{L^4}\,ds}|\eta^h(s,x)|^2ds\right)\\
&\leq \frac{1}{2}q(q-1)\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q-2}\int_0^t e^{-2K\int_0^t|X(s,x)|^4_{L^4}\,ds}\,|\eta^h(s,x)|^{2(1-\alpha)}_{-\alpha}\,|\eta^h(s,x)|^{2\alpha}_{1-\alpha}\,ds\right)\\
&\leq \frac{1}{2}q(q-1)t^{1-\alpha}\,|h|^2_{-\alpha}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q-2}\right)\\
&\leq \frac{1}{2}q(q-1)t^{1-\alpha}\,|h|^2_{-\alpha}\left[\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q}\right)\right]^{(q-2)/q}\\
&\leq \frac{1}{4}\mathbb{E}\left(\sup_{r\in[0,t]}|z(r)|^{q}\right)+ct^{\frac{q(1-\alpha)}{2}}\,|h|^q_{-\alpha}.
\end{aligned}$$

We deduce:

$$I_1 \leq ct^{-\frac{1+\alpha}{2}}\,|h|_{-\alpha}\left[\mathbb{E}\left(\varphi^p(X(t,x)\right)\right]^{1/p}$$

For I_2 we write

$$\begin{aligned}
I_2 &= 4K\mathbb{E}\left(e^{-K\int_0^t|X(s,x)|^4_{L^4}\,ds}\varphi(X(t,x))\int_0^t(X^3(s,x),\eta^h(s,x))ds\right)\\
&\leq 4K\left[\mathbb{E}\left(\varphi^p(X(t,x)\right)\right]^{1/p}\left[\mathbb{E}\left(e^{-Kq\int_0^t|X(s,x)|^4_{L^4}\,ds}\left(\int_0^t|X(s,x)|^3_{L^6}\,|\eta^h(s,x)|\,ds\right)^q\right)\right]^{1/q}.
\end{aligned}$$

By Lemma 5 and Proposition 2.2 in [DaDe07]

$$I_2 \leq c_q(1+|x|^3_{L^6})\left[\mathbb{E}\left(\varphi^p(X(t,x)\right)\right]^{1/p}\,|h|_{-1}.$$

Gathering the estimates on I_1 and I_2 gives the result. □

Lemma 7. *For any $\alpha\in[0,1)$, $p>1$, $q>1$ satisfying $\frac{1}{p}+\frac{1}{q}<1$, if K is chosen large enough there exists a constants c depending on α, p, q such that for any φ Borel bounded and $h: H\to D((-A)^{-\alpha/2})$ Borel such that $\int_H|h(x)|^q_{-\alpha}\nu(dx)<\infty$ we have*

$$\left|\int_H D_xP_t\varphi(x)\cdot h(x)\nu(dx)\right|\leq ce^{ct}(1+t^{-\frac{1+\alpha}{2}})\|\varphi\|_{L^p(H,\nu)}\left(\int_H|h(x)|^q_{-\alpha}\nu(dx)\right)^{1/q}. \quad (19)$$

Proof. We first prove a similar estimate for S_t. Using Lemma 6 we have by Hölder

inequality

$$\begin{aligned}
&\left|\int_H D_x S_t\varphi(x)\cdot h(x)\,\nu(dx)\right| \\
&\leq c\,e^{ct}(1+t^{-\frac{1+\alpha}{2}})\int_H (1+|x|_{L^6}^3)\,\left[\mathbb{E}\left(\varphi^p(X(t,x))\right)\right]^{1/p}|h(x)|_{-\alpha}\nu(dx) \\
&\leq c\,e^{ct}(1+t^{-\frac{1+\alpha}{2}})\left[\int_H (1+|x|_{L^6}^3)^r\nu(dx)\right]^{1/r} \\
&\times\left[\int_H \mathbb{E}\left(\varphi^p(X(t,x))\,\nu(dx)\right]^{1/p}\left[\int_H |h(x)|_{-\alpha}^q\nu(dx)\right]^{1/q}.
\end{aligned}$$

with $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=1$. Thus by Proposition 2.3 in [DaDe07] and the invariance of ν:

$$\left|\int_H D_x S_t\varphi(x)\cdot h(x)\,\nu(dx)\right| \leq ce^{ct}(1+t^{-\frac{1+\alpha}{2}})\|\varphi\|_{L^p(H,\nu)}\left(\int_H |h(x)|_{-\alpha}^q\nu(dx)\right)^{1/q}.$$

We then proceed as in [DaDe07] to get a similar estimate on P_t. We write

$$P_t\varphi(x) = S_t\varphi(x) + K\int_0^t S_{t-s}(|x|_{L^4}^4 P_s\varphi)ds.$$

It follows that, using the estimate above with $p>\tilde{p}>1$ such that $\frac{1}{\tilde{p}}+\frac{1}{q}<1$:

$$\begin{aligned}
&\left|\int_H D_x P_t\varphi(x)\cdot h(x)\nu(dx)\right| \leq ce^{ct}(1+t^{-\frac{1+\alpha}{2}})\|\varphi\|_{L^p(H,\nu)}\left(\int_H |h(x)|_{-\alpha}^q\nu(dx)\right)^{1/q} \\
&+K\int_0^t ce^{c(t-s)}(1+(t-s)^{-\frac{1+\alpha}{2}})\left(\int_H |x|_{L^4}^4|P_s\varphi(x)|^{\tilde{p}}d\nu(dx)\right)^{1/\tilde{p}} \\
&\times\left(\int_H |h(x)|_{-\alpha}^q\nu(dx)\right)^{1/q} ds.
\end{aligned}$$

The result follows by Hölder inequality and the invariance of ν. □

Theorem 1 follows directly from the following result thanks to the invariance of ν and taking for instance $t=1$.

Proposition 8. *For all $p>1$, $\delta>0$, there exists a constant such that for φ Borel bounded, and all $h\in H^{1+\delta}(0,1)$, we have*

$$\left|\int_H P_t(D_x\varphi\cdot h)(x)\,\nu(dx)\right| \leq ce^{ct}(1+t^{-1/2})\|\varphi\|_{L^p(H,\nu)}|h|_{1+\delta} \tag{20}$$

Proof. By Poincaré inequality, it is no loss of generality to assume $\delta < \min\{2(1-\frac{1}{p}), \frac{1}{2}\}$.

We start by integrating (7) on H:

$$\begin{aligned}\int_H P_t(D_x\varphi\cdot h)(x)\nu(dx) &= \int_H (D_xP_t\varphi(x)\cdot h)\,\nu(dx)\\ &\quad -\int_0^t\int_H P_{t-s}[(\langle Ah+b'(x)h, D_xP_s\varphi(x)\rangle)]ds\,\nu(dx).\end{aligned} \tag{21}$$

Then by Lemma 7 we deduce

$$\begin{aligned}\left|\int_H P_t(D_x\varphi\cdot h)(x)\,\nu(dx)\right| &\le ce^{ct}(1+t^{-\frac{1}{2}})\|\varphi\|_{L^p(H,\nu)}|h|\\ &\quad +\left|\int_H\int_0^t P_{t-s}[(Ah+b'(\cdot)\cdot h, D_xP_s\varphi)]ds\,\nu(dx)\right|.\end{aligned}$$

By the invariance of ν:

$$\int_H\int_0^t P_{t-s}[(Ah+b'(\cdot)\cdot h, D_xP_s\varphi)]ds\,\nu(dx) = \int_H\int_0^t (Ah+b'(\cdot)\cdot h, D_xP_s\varphi)ds\,\nu(dx).$$

Therefore, by Lemma 7 with $\alpha = 1-\delta$ and $q = \frac{2}{\delta}$:

$$\begin{aligned}&\left|\int_H\int_0^t P_{t-s}[(Ah+b'(\cdot)\cdot h, D_xP_s\varphi)]ds\,\nu(dx)\right|\\ &\le \int_0^t ce^{c(t-s)}(1+s^{-1+\frac{\delta}{2}})\|\varphi\|_{L^p(H,\nu)}\left(\int_H |Ah+b'(\cdot)\cdot h|^{\delta/2}_{-1+\delta}\nu(dx)\right)^{\delta/2}.\end{aligned}$$

Note that

$$|b'(x)\cdot h|_{-1+\delta} = |\partial_\xi\,(xh)|_{-1+\delta} \le c\,|xh|_\delta$$

Then, we have:

$$|xh| \le c|x|\,|h|_1$$

by the embedding $H^1\subset L^\infty$ and

$$|xh|_1 \le c|x|_1\,|h|_1,$$

since H^1 is an algebra. We deduce by interpolation

$$|xh|_\delta \le c|x|_\delta|h|_1.$$

It follows

$$\left|\int_H \int_0^t P_{t-s}[(Ah + b'(\cdot)\cdot h, D_x P_s\varphi)]ds\,\nu(dx)\right|$$
$$\leq c_\delta e^{ct}\|\varphi\|_{L^p(H,\nu)}\left(1+\int_H |x|_\delta^{\delta/2}\nu(dx)\right)^{2/\delta}|h|_{1+\delta}.$$

We need to estimate $\int_H |x|_\delta^{\delta/2}\nu(dx)$. We use the notation of [DaDe07, Proposition 2.2]

$$|X(s,x)|_\delta \leq |Y(s,x)|_\delta + |z_\alpha(s)|_\delta$$

$$\leq |Y(s,x)|^{1-\delta}\,|Y(s,x)|_1^\delta + |z_\alpha(s)|_\delta.$$

Using computation in [DaDe07, Proposition 2.2], we obtain

$$\sup_{t\in[0,1]}|Y(t,x)|^2 + \int_0^1 |Y(s,x)|_1^2 ds \leq c(|x|^2+\kappa)$$

where κ is a random variable with all moments finite. It follows by (11):

$$\mathbb{E}\left(\int_0^1 |Y(s,x)|_\delta^{2/\delta}\,ds\right) \leq \mathbb{E}\left(\int_0^1 |Y(s,x)|^{2(1-\delta)/\delta}|Y(s,x)|_1^2\,ds\right) \leq c(|x|^2+1)^{1/\delta}$$

Genelarizing slightly Proposition 2.1 in [DaDe07], we have:

$$\mathbb{E}\left(|z_\alpha(t)|_\delta^p\right) \leq c_{\delta,p}$$

for $t\in[0,1]$, $\delta<1/2$, $\alpha\geq 1$, $p\geq 1$. We deduce:

$$\mathbb{E}\left(\int_0^1 |X(s,x)|_\delta^{2/\delta}\,ds\right) \leq c(|x|^2+1)^{1/\delta}.$$

Integrating with respect to ν and using Proposition 2.3 in [DaDe07] we deduce:

$$\int_H |x|_\delta^{\delta/2}\nu(dx) \leq c_\delta$$

Then (20) follows. □

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