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Multivariate Hadamard self-similarity: Testing fractal connectivity

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HIGHLIGHTS

- A new multivariate Gaussian scale-free and fractal connectivity stochastic model.
- Asymptotic performance study of multivariate DWT estimators of scaling exponents.
- Approximate confidence interval construction for scaling exponent estimators.
- Statistical test for the presence of fractal connectivity from a single sample path.

Keywords:
Multivariate self-similarity
Hadamard self-similarity
Fractal connectivity
Wavelet analysis
Confidence intervals
Hypothesis testing

ABSTRACT

While scale invariance is commonly observed in each component of real world multivariate signals, it is also often the case that the inter-component correlation structure is not fractally connected, i.e., its scaling behavior is not determined by that of the individual components. To model this situation in a versatile manner, we introduce a class of multivariate Gaussian stochastic processes called Hadamard fractional Brownian motion (HfBm). Its theoretical study sheds light on the issues raised by the joint requirement of entry-wise scaling and departures from fractal connectivity. An asymptotically normal wavelet-based estimator for its scaling parameter, called the Hurst matrix, is proposed, as well as asymptotically valid confidence intervals. The latter are accompanied by original finite sample procedures for computing confidence intervals and testing fractal connectivity from one single and finite size observation. Monte Carlo simulation studies are used to assess the estimation performance as a function of the (finite) sample size, and to quantify the impact of omitting wavelet cross-correlation terms. The simulation studies are shown to validate the use of approximate confidence intervals, together with the significance level and power of the fractal connectivity test. The test performance and properties are further studied as functions of the HfBm parameters.

1. Introduction

1.1. Scale invariance

The relevance of the paradigm of scale invariance is evidenced by its successful use, over the last few decades, in the analysis of the dynamics in data obtained from a rather diverse spectrum of real world applications. The latter range from natural phenomena – physics (hydrodynamic turbulence [1], out-of-equilibrium physics), geophysics (rainfalls), biology (body rhythms [2], heart rate [3,4], neurosciences and genomics [5–8]) – to human activity – Internet traffic [9,10], finance [11], urban growth and art investigation [12–14].

In essence, scale invariance – also called scaling, or scale-free dynamics – implies that the phenomenical or phenomenological dynamics are driven by a large continuum of equally important time scales, rather than by a small number of characteristic scales. The investigation's focus is on identifying a relation amongst relevant scales rather than picking out characteristic scales.

Historically, self-similarity was one of the first proposed mathematical frameworks for the modeling of scale invariance (e.g., [15]). A random system is called self-similar when dilated copies of a single signal X are statistically indistinguishable, namely,

$$\{X(t)\}_{t\in\mathbb{R}} \overset{\text{fdd}}{=} \{a^H X(t/a)\}_{t\in\mathbb{R}}, \quad \forall a > 0, \tag{1.1}$$

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where $\overset{\text{fdd}}{=}$ stands for the equality of finite-dimensional distributions. An example of a stochastic process that satisfies the property (1.1) is fractional Brownian motion (fBm). Indeed, the latter is the only self-similar, Gaussian, stationary increment process, and it is the most widely used scaling model for real-world signals [16].

Starting from (1.1), the key parameter for quantifying scale-free dynamics is the scaling, or Hurst, exponent $0 < H < 1$. The estimation of H is the central task in scaling analysis, and it has received considerable effort and attention in the last three decades (see [17] for a review). The present contribution is about wavelet-based estimation [18,19]. It relies on the key scaling property

$$\frac{1}{T}\sum_t T_X^2(a,t) \simeq Ca^{\alpha}, \quad \alpha := 2H, \tag{1.2}$$

where $T_X(a,t)$ is the wavelet coefficient of an underlying self-similar stochastic process and T is the number of available coefficients. In other words, the sample wavelet variance of the stochastic process behaves like a power law with respect to the scale a.

1.2. Multivariate scaling

In many modern fields of application such as Internet traffic and neurology, data is collected in the form of multivariate time series. Univariate-like analysis in the spirit of (1.2) – i.e., independently on each component – does not account for the information stemming from correlations across components. The classical fBm parametric family, for example, provides at best a model for component-wise scaling, and thus cannot be used as the foundation for a multivariate modeling paradigm.

To model self-similarity in a multivariate setting, a natural extension of fBm, called Operator fractional Brownian motion (OfBm), was recently defined and studied (see [20–22]). An OfBm $\underline{X}$ satisfies the m-variate self-similarity relation

$$\{\underline{X}(t)\}_{t\in\mathbb{R}} \overset{\text{fdd}}{=} \{a^{\underline{\underline{H}}}\underline{X}(t/a)\}_{t\in\mathbb{R}}, \quad \forall a > 0, \tag{1.3}$$

where the scaling exponent is a $m \times m$ matrix $\underline{\underline{H}}$, and $a^{\underline{\underline{H}}}$ stands for the matrix exponential $\sum_{k=0}^{\infty}(H\log a)^k/k!$. Likewise, the wavelet spectrum of each individual component is not a single power law as in (1.2); instead, it behaves like a mixture of distinct univariate power laws. In its most general form, OfBm remains scarcely used in applications; recent efforts have tackled many difficulties that arise in the identification of its parameters [23,24].

1.3. Entry-wise multivariate scaling

We call an OfBm entry-wise scaling when the Hurst parameter is simply a diagonal matrix $\underline{\underline{H}} = \operatorname{diag}(H_1,\ldots,H_m)$. This instance of OfBm has been used in many applications (e.g., [6,25]) and its estimation is thoroughly studied in [20]. Since $\underline{\underline{H}}$ is diagonal, the relation (1.3) takes the form

$$\{X_1(t),\ldots,X_m(t)\}_{t\in\mathbb{R}} \overset{\text{fdd}}{=} \{a^{H_1}X_1(t/a),\ldots,a^{H_m}X_m(t/a)\}_{t\in\mathbb{R}}, \quad \forall a > 0, \tag{1.4}$$

which is reminiscent of the univariate case. This implies that the extension of (1.2) to all auto- and cross-components of m-variate data can be written as

$$\frac{1}{T}\sum_t T_{X_{q_1}}(a,t)T_{X_{q_2}}(a,t) \simeq Ca^{\alpha_{q_1q_2}}, \quad \alpha_{q_1q_2} := H_{q_1} + H_{q_2}, \quad q_1,q_2 = 1,\ldots,m, \tag{1.5}$$

where T is as in (1.2).

1.4. Fractal connectivity

Yet, entry-wise scaling OfBm is a restrictive model since the cross-scaling exponents $\alpha_{q_1q_2}$, $q_1 \neq q_2$, are determined by the auto-scaling exponents $\alpha_{q_1q_1}$ and $\alpha_{q_2q_2}$, i.e.,

$$\alpha_{q_1q_2} = H_{q_1} + H_{q_2} = (\alpha_{q_1q_1} + \alpha_{q_2q_2})/2. \tag{1.6}$$

In this situation, called *fractal connectivity* [6,25,26], no additional scaling information can be extracted from the analysis of *cross-components*. However, in real world applications, cross-components are expected to contain information on the dynamics underlying the data, e.g., cross-correlation functions. As an example, recent investigation of multivariate brain dynamics in [6] produced evidence of departures from fractal connectivity, notably for subjects carrying out prescribed tasks. Other fields where cross-correlation or non-fractally connected modeling has been pursued include physics [27–36] and econometrics [37–39]. This means that there is a clear need for more versatile models than entry-wise scaling OfBm (see also Remark 2.2). The covariance structure of the new model should satisfy the following two requirements:

1. all auto- and cross-components are (approximately) self-similar;
2. departures from fractal connectivity are allowed, i.e., the exponents of the cross-components are not necessarily determined by the exponents of the corresponding auto-components.

Hereinafter, a departure from fractal connectivity (1.6) on a given covariance structure entry (q_1,q_2) will be quantified by means of the parameter

$$\delta_{q_1q_2} = \frac{\alpha_{q_1q_1} + \alpha_{q_2q_2}}{2} - \alpha_{q_1q_2} \geq 0, \quad q_1,q_2 = 1,\ldots,m, \tag{1.7}$$

where nonnegativeness is a consequence of the Cauchy–Schwarz inequality (see (2.10)). It is clear that $\delta_{q_1q_2} = 0$ when $q_1 = q_2$.

Remark 1.1. Note that the precise theoretical definition of fractal connectivity includes the situation where the cross-correlation between two components q_1 and q_2 is identically zero. In this paper, we are mainly interested in situations where this does not occur (see (2.8)).

1.5. Goals, contributions and outline

Our contribution comprises four main components. First, we propose a new class of multivariate Gaussian stochastic processes, called *Hadamard fractional Brownian motion* (HfBm), that combines scale-free dynamics and potential departures from fractal connectivity. Moreover, we provide a precise discussion of the issues entailed by the presence of these two properties (Section 2). Second, we study the multivariate discrete wavelet transform (DWT) of HfBm, define wavelet-based estimators for the scaling exponents $\alpha_{q_1q_2}$ and the fractal connectivity parameter $\delta_{q_1q_2}$, mathematically establish their asymptotic performance (i.e., asymptotic normality and covariance structure), and computationally quantify finite sample size effects (Section 3). Third, starting from a single sample path, we construct approximate confidence intervals for the proposed estimators. The procedure is backed up by the mathematical identification of the approximation orders as a function of the sample path size and the limiting coarse scale. This is further investigated by means of Monte Carlo simulations, as well as by means of a study of the ubiquitous issue of the impact of (partially) neglecting the correlation amongst wavelet coefficients (Section 4). Beyond being of interest in multivariate modeling, the study sheds more light on the same issue for the univariate case. Fourth, we devise an efficient test for the presence of fractal connectivity from a single sample path. In addition, we assess the finite sample performance of the test in terms of targeted risk by means of Monte Carlo simulations (Section 5). Finally, routines for the synthesis of HfBm, as well as for estimation, computation of confidence intervals and testing will be made publicly available at time of publication. All proofs can be found in Appendix.

2. Hadamard fractional Brownian motion

For Hadamard fractional Brownian motion, defined next, the fractal connectivity relation (1.6) does not necessarily hold.

Definition 2.1. A Hadamard fractional Brownian motion $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ (HfBm) is a proper, Gaussian (stationary increment) process whose second moments can be written as

$$\mathbb{E}\big[B_H(s)B_H(t)^*\big] = \int_{\mathbb{R}} \Big(\frac{e^{isx}-1}{ix}\Big)\Big(\frac{e^{-itx}-1}{-ix}\Big) f_H(x)dx, \quad s,t\in\mathbb{R}. \tag{2.1}$$

For $0 < h_{\min} \le h_{\max} < 1$, the matrix exponent $H = \Big(h_{q_1q_2}\Big)_{q_1,q_2=1,\dots,m}$ satisfies the conditions

$$h_{q_1q_2} \in [h_{\min}, h_{\max}], \quad q_1,q_2 = 1,\dots,m. \tag{2.2}$$

The matrix-valued function f is a spectral density of the form

$$f_H(x)_{q_1q_2} = \Big(\rho_{q_1q_2}\sigma_{q_1}\sigma_{q_2}|x|^{-2(h_{q_1q_2}-1/2)}\Big) g_{q_1q_2}(x), \quad q_1,q_2 = 1,\dots,m, \tag{2.3}$$

i.e., the Hadamard scaling parameters are given by

$$\alpha_{q_1q_2} = 2h_{q_1q_2}, \quad q_1,q_2 = 1,\dots,m, \tag{2.4}$$

where $\rho_{q_1q_2} \in [-1,1]$, $\sigma_{q_1},\sigma_{q_2} \in \mathbb{R}^+$. The real-valued functions $g_{q_1q_2} \in C^2(\mathbb{R})$ satisfy

$$\max_{l=0,1,2}\sup_{x\in\mathbb{R}}\Big|\frac{d^l}{dx^l}g_{q_1q_2}(x)\Big| \le C, \tag{2.5}$$

$$\Big|\frac{d^l}{dx^l}(g_{q_1q_2}(x)-1)\Big| \le C'|x|^{\varpi_0-l}, \quad x\in(-\varepsilon_0,\varepsilon_0), \quad l=0,1,2, \tag{2.6}$$

for constants $C, C', \varepsilon_0 > 0$, where

$$2h_{\max} < \varpi_0 \le 2(1+h_{\min}). \tag{2.7}$$

In addition, throughout the paper we will assume that

$$\rho_{q_1q_2} \ne 0, \quad q_1,q_2 = 1,\dots,m, \quad q_1 \ne q_2. \tag{2.8}$$

Example 2.1. An HfBm with parameters $h_{q_1q_2} = (h_{q_1q_1}+h_{q_2q_2})/2$ (i.e., fractally connected), $g_{q_1q_2}(x) \equiv 1$, $q_1,q_2 = 1,\dots,m$, is an entry-wise scaling OfBm with diagonal Hurst matrix $H = \mathrm{diag}(h_1,\dots,h_m)$ (see [20–22]).

By the known properties of spectral densities, $f_H(x) = (f_H(x)_{q_1q_2})_{q_1,q_2=1,\dots,m} \in \mathcal{S}_{>0}(m,\mathbb{R})$ (symmetric positive semidefinite matrices) a.e. and satisfies

$$|f_H(x)_{q_1q_2}| \le \sqrt{f_H(x)_{q_1q_1}}\sqrt{f_H(x)_{q_2q_2}} \quad dx\text{-a.e.} \tag{2.9}$$

([40], p. 436). The relation (2.9) further implies that

$$h_{q_1q_2} \le \frac{h_{q_1q_1}+h_{q_2q_2}}{2}, \quad q_1,q_2 = 1,\dots,m. \tag{2.10}$$

Whenever convenient we also write

$$h_{qq} = h_q, \quad \alpha_{qq} = \alpha_q, \quad q = 1, \dots, m.$$

The name "Hadamard" comes from Hadamard (entry-wise) matrix products. If one rewrites HfBm componentwise as $B_H(t) = (B_{H,1}(t), \dots, B_{H,m}(t))^*$, then the conditions (2.1), (2.2) and (2.6) yield the asymptotic equivalence

$$\mathbb{E}\big[B_{H,q_1}(cs)B_{H,q_2}(ct)\big] \sim \frac{\varsigma}{2}\{|cs|^{2h_{q_1q_2}} + |ct|^{2h_{q_1q_2}} - |c(s-t)|^{2h_{q_1q_2}}\}, \quad c \to \infty,$$

$q_1, q_2 = 1, \dots, m$, for some $\varsigma \in \mathbb{R}$. In other words, over large scales, the covariance between each pair of entries of an HfBm approaches that of a univariate fBm, up to a change of sign (see also Proposition 3.1(iii)). In this sense, for large c, an HfBm behaves like its ideal counterpart $B_{H,\text{ideal}} = \{B_{H,\text{ideal}}(t)\}_{t\in\mathbb{R}}$, defined as a generally non-existent stochastic process satisfying the also ideal Hadamard (entry-wise) self-similarity relation

$$\mathbb{E}\big[B_{H,\text{ideal}}(cs)B_{H,\text{ideal}}(ct)^*\big] = c^{\circ H} \circ \mathbb{E}\big[B_{H,\text{ideal}}(s)B_{H,\text{ideal}}(t)^*\big], \quad c > 0, \tag{2.11}$$

where $\circ$ denotes the Hadamard (entry-wise) matrix product and $c^{\circ H} := \left(c^{h_{q_1q_2}}\right)_{q_1,q_2=1,\dots,m}$. The process $B_{H,\text{ideal}}$ can be viewed as a heuristic tool for developing intuition on multivariate self-similarity. Mathematically, though, it can only exist in fractally connected instances, the reason being that distinct (spectral) power laws cross over. Indeed, we must have $\alpha_{q_1q_2} \le \frac{\alpha_{q_1q_1}+\alpha_{q_2q_2}}{2}$ for x close to 0 and $\alpha_{q_1q_2} \ge \frac{\alpha_{q_1q_1}+\alpha_{q_2q_2}}{2}$ for large $|x|$, whence $\alpha_{q_1q_2} \equiv \frac{\alpha_{q_1q_1}+\alpha_{q_2q_2}}{2}$. This shows that HfBm is a perturbation of its virtual counterpart, where the regularization functions $g_{q_1q_2}(x)$ in (2.3) introduce high-frequency corrections.

Example 2.2. An illustrative subclass of HfBm is obtained by setting $g_{qq}(x) \equiv 1 = \sigma_q$, $q = 1, \dots, m$, and $g_{q_1q_2}(x) = e^{-x^2}$, $q_1 \neq q_2$. Note that $g_{q_1q_2}(\cdot)$ satisfies (2.7) with $\varpi_0 = 2$. In this case, the expression for the main diagonal spectral entry of an HfBm is identical to that of an ideal-HfBM, and the difference lies on the off-diagonal entries:

$$\rho^2_{q_1q_2}|x|^{-2\alpha_{q_1q_2}}e^{-x^2} \le \rho_{q_1q_1}\rho_{q_2q_2}|x|^{-(\alpha_{q_1q_1}+\alpha_{q_2q_2})} \quad dx\text{-a.e.} \tag{2.12}$$

In this case, each individual entry $\{B_H(t)_q\}_{t\in\mathbb{R}}$, $q = 1, \dots, m$, of HfBm B_H is by itself a fBm with Hurst parameter $0 < h_q < 1$. In particular,

$$\{B_H(ct)_q\}_{t\in\mathbb{R}} \overset{\text{fdd}}{=} \{c^{h_q}B_H(t)_q\}_{t\in\mathbb{R}}, \quad c > 0, \quad q = 1, \dots, m.$$

However, it is generally *not* true that

$$\{B_H(ct)\}_{t\in\mathbb{R}} \overset{\text{fdd}}{=} \{\text{diag}(c^{h_1}, \dots, c^{h_q}, \dots, c^{h_m})B_H(t)\}_{t\in\mathbb{R}}, \quad c > 0.$$

Otherwise, B_H would necessarily be fractally connected.

Remark 2.1. When simulating HfBm via Circulant Matrix Embedding, one verifies that regularization is rarely necessary for ensuring the positive definiteness of the covariance matrix for finite sample sizes. In other words, ideal-HfBM is also a useful approximation in practice.

Remark 2.2. Let $Y_H(t) = B_H(t) - B_H(t-1)$, where $\{B_H(t)\}_{t\in\mathbb{R}}$ is an HfBm. Then, $\{Y_H(t)\}_{t\in\mathbb{Z}}$ is a (discrete time) stationary process with spectral density

$$f_{Y_H}(x) = 2(1-\cos(x))\sum_{k=-\infty}^{\infty}\frac{f_H(x+2\pi k)}{|x+2\pi k|^2}, \quad x \in (-\pi, \pi],$$

where f_H is the HfBm spectral density (2.1). We call $\{Y_H(t)\}_{t\in\mathbb{Z}}$ Hadamard fractional Gaussian noise (HfGn). Under (2.5)–(2.7), we obtain the entry-wise limiting behavior

$$f_{Y_H}(x)_{q_1q_2}(x) \sim \rho_{q_1q_2}\sigma_{q_1}\sigma_{q_2}|x|^{-2(h_{q_1q_2}-1/2)}, \quad x \to 0^+, \quad q_1, q_2 = 1, \dots, m.$$

In particular, non-fractally connected instances of $\{Y_H(t)\}_{t\in\mathbb{Z}}$ can only satisfy the usual definitions of multivariate scaling behavior or long range dependence by setting limiting low frequency constants in the latter to zero [41–44].

3. Wavelet-based analysis of HfBm

In this section, we construct the wavelet analysis and estimation for HfBm. Due to the mathematical convenience of the notion of Hadamard (approximate) self-similarity, most of the properties of wavelet-based constructs resemble their univariate analogues.

3.1. Multivariate discrete wavelet transform

Throughout the rest of the paper, we will make the following assumptions on the underlying wavelet basis. Such assumptions will be omitted in the statements.

Assumption $(W1)$: $\psi \in L^1(\mathbb{R})$ is a wavelet function, namely,

$$\int_{\mathbb{R}}\psi^2(t)dt = 1, \quad \int_{\mathbb{R}}t^q\psi(t)dt = 0, \quad q = 0, 1, \dots, N_\psi - 1, \quad N_\psi \ge 2. \tag{3.1}$$

Assumption $(W2)$: the functions φ (a bounded scaling function) and ψ correspond to

$$\text{a MRA of } L^2(\mathbb{R}), \text{ and } \operatorname{supp}(\varphi) \text{ and } \operatorname{supp}(\psi) \text{ are compact intervals.} \tag{3.2}$$

Assumption $(W3)$: for some $\beta > 1$,

$$\widehat{\psi} \in C^2(\mathbb{R})$$

and

$$\sup_{x\in\mathbb{R}} |\widehat{\psi}(x)|(1+|x|)^{\beta} < \infty. \tag{3.3}$$

Under (3.1)–(3.3), ψ is continuous, $\widehat{\psi}(x)$ is everywhere differentiable and

$$\widehat{\psi}^{(l)}(0) = 0, \quad l = 0, \ldots, N_{\psi} - 1 \tag{3.4}$$

(see [45], Theorem 6.1 and the proof of Theorem 7.4).

Definition 3.1. Let $B_H = \{B_H(t)\}_{t\in\mathbb{R}} \in \mathbb{R}^m$ be an HfBm. For a scale parameter $j \in \mathbb{N}$ and a shift parameter $k \in \mathbb{Z}$, its (L^1-normalized) wavelet transform is defined by

$$D(2^j, k) = 2^{-j/2} \int_{\mathbb{R}} 2^{-j/2} \psi(2^{-j}t - k) B_H(t) dt =: \Big(d_q(j,k)\Big)_{q=1,\ldots,m}. \tag{3.5}$$

Under (3.1)–(3.3) and the continuity of the covariance function (2.1), the wavelet transform (3.5) is well-defined in the mean squared sense and $\mathbb{E}D(2^j, k) = 0$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$ (see [46], p. 86).

3.2. Multivariate wavelet spectrum

Fix $j_1 \le j_2$, $j_1, j_2 \in \mathbb{N}$. Because of the approximate nature of Hadamard self-similarity, analysis and estimation must be considered in the coarse scale limit, by means of a sequence of dyadic numbers $\{a(n)\}_{n\in\mathbb{N}}$ satisfying

$$1 \le \frac{n}{a(n)2^{j_2}} \le \frac{n}{a(n)2^{j_1}} \le n, \quad \frac{a(n)^{4(h_{\max}-h_{\min})+1}}{n} + \frac{n}{a(n)^{1+2\varpi_0}} \to 0, \quad n \to \infty. \tag{3.6}$$

Example 3.1. An example of a scaling sequence satisfying (3.6) for large enough n is

$$a(n) := 2\lfloor n^{\frac{\eta}{4(h_{\max}-h_{\min})+1}} \rfloor, \quad \frac{4(h_{\max}-h_{\min})+1}{1+2\varpi_0} < \eta < 1.$$

In other words, a wide parameter range $[h_{\min}, h_{\max}]$ or a low regularity parameter value ϖ_0 implies that $a(n)$ must grow slowly by comparison to n.

Remark 3.1. For a fixed octave range $\{j_1, \ldots, j_2\}$ associated with an initial scaling factor value $a(n_0) = 1$ and sample size n_0, define the scale range $\{2^{j_1(n)}, \ldots, 2^{j_2(n)}\} = \{a(n)2^{j_1}, \ldots, a(n)2^{j_2}\}$ for a general sample size n (where $a(n)$ is assumed dyadic). Then, under (3.6), for every n the range of useful octaves is constant and given by $j_2(n) - j_1(n) = j_2 - j_1$, where the new octaves are $j_l(n) = \log_2 a(n) + j_l \in \mathbb{N}$, $l = 1, 2$.

Definition 3.2. Let $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ be an HfBm. Let $\{D(a(n)2^j, k)\}_{k=1,\ldots,n_j,\, j=j_1,\ldots,j_2}$ be its wavelet coefficients, where

$$n_{a,j} = \frac{n}{a(n)2^j}$$

is the number of wavelet coefficients $D(a(n)2^j, \cdot)$ at scale $a(n)2^j$, of a total of n (wavelet) data points. The wavelet variance and the sample wavelet variance, respectively, at scale $a(n)2^j$ are defined by

$$\mathbb{E}\big[S_n(a(n)2^j)\big], \quad S_n(a(n)2^j) = \sum_{k=1}^{n_{a,j}} \frac{D(a(n)2^j, k) D(a(n)2^j, k)^*}{n_{a,j}}, \quad j = j_1, \ldots, j_2. \tag{3.7}$$

Let

$$\varrho^{(q_1 q_2)}(a(n)2^j) = \mathbb{E}\big[d_{q_1}(a(n)2^j, 0) d_{q_2}(a(n)2^j, 0)\big]. \tag{3.8}$$

The standardized counterparts of (3.7) are

$$\mathbb{E}\big[W_n(a(n)2^j)\big], \quad W_n(a(n)2^j) = \varrho(a(n)2^j)^{\circ -1} \circ \sum_{k=1}^{n_{a,j}} \frac{D(a(n)2^j, k) D(a(n)2^j, k)^*}{n_{a,j}}, \tag{3.9}$$

$j = j_1, \ldots, j_2$, where $\varrho(a(n)2^j) = \Big(\varrho^{(q_1 q_2)}(a(n)2^j)\Big)_{q_1, q_2 = 1,\ldots,m}$. Entry-wise, for $1 \le q_1, q_2 \le m$,

$$S_n^{(q_1 q_2)}(a(n)2^j) = \sum_{k=1}^{n_{a,j}} \frac{d_{q_1}(a(n)2^j, k) d_{q_2}(a(n)2^j, k)}{n_{a,j}}, \quad W_n^{(q_1 q_2)}(a(n)2^j) = \frac{S_n^{(q_1 q_2)}(a(n)2^j)}{\varrho^{(q_1 q_2)}(a(n)2^j)}. \tag{3.10}$$

Proposition 3.1, stated next, provides basic results on the moments and asymptotic distribution, in the coarse scale limit $a(n)2^j \to \infty$, of the wavelet transform (3.5) and variance (3.9). In particular, in regard to the limits in distribution, the vector of random matrices $\{S_n(a(n)2^j)\}_{j=j_1,\dots,j_2}$ can be intuitively interpreted as an asymptotically unbiased and Gaussian estimator of its population counterpart $\{\mathbb{E}\big[S_n(a(n)2^j,k)\big]\}_{j=j_1,\dots,j_2}$. The celebrated decorrelation property of the wavelet transform (e.g., [47], Proposition II.2) lies behind the Gaussian limits in the proposition, as well as of the fact that the random matrices $S_n(a(n)2^j)$ have weak inter-component, intra-scale and inter-scale correlations. Also, each entry $S_n^{(q_1q_2)}(a(n)2^j)$, $q_1, q_2 = 1,\dots,m$, displays asymptotic power law scaling. In Section 3.3, these properties are used to define estimators of the scaling exponents and to analytically establish their asymptotic performance.

In the statement of Proposition 3.1, we make use of the operator

$$\mathrm{vec}_{\mathcal{S}} S = (s_{11}, s_{12}, \dots, s_{1m}; s_{22}, \dots, s_{2m}; \dots; s_{m-1,m-1}, s_{m-1,m}; s_{m,m})^*. \tag{3.11}$$

In other words, $\mathrm{vec}_{\mathcal{S}}\cdot$ vectorizes the upper triangular entries of the symmetric matrix S.

Proposition 3.1. Let $B_H = \{B_H(t)\}_{t\in\mathbb{R}}$ be an HfBm. Consider $j, j' \in \mathbb{N}$ and $k, k' \in \mathbb{Z}$. Then,

(i) for fixed j, j', k, k', we can write

$$\mathbb{E}\Big[D(a(n)2^j,k)D(a(n)2^{j'},k')^*\Big] = \Big(\Xi^{jj',a}_{q_1q_2}(a(n)(2^jk-2^{j'}k'))\Big)_{q_1,q_2=1,\dots,m} \tag{3.12}$$

for some matrix-valued function $\Xi^{jj',a}(\cdot)$ that depends on $a(n)$. In particular, for fixed scales j, j', $\{D(a(n)2^j,k)\}_{k\in\mathbb{Z}}$ is a stationary stochastic process;

(ii) for $q_1, q_2 = 1,\dots,m$,

$$\varrho^{(q_1q_2)}(a(n)2^j) = \Xi^{jj,a}_{q_1q_2}(0) \neq 0, \quad j\in\mathbb{N}, \tag{3.13}$$

i.e., (3.9) is well-defined;

(iii) for $\Xi^{jj',a}(\cdot)$ as in (3.12), $q_1, q_2 = 1,\dots,m$, and $z\in\mathbb{Z}$, as $n\to\infty$,

$$\frac{\Xi^{jj',a}_{q_1q_2}(a(n)z)}{a(n)^{2h_{q_1q_2}}} \to \Phi^{jj'}_{q_1q_2}(z) = \rho_{q_1q_2}\sigma_{q_1}\sigma_{q_2}\int_{\mathbb{R}} e^{izx}|x|^{-(2h_{q_1q_2}+1)}\widehat{\psi}(2^jx)\overline{\widehat{\psi}(2^{j'}x)}dx. \tag{3.14}$$

In particular, the wavelet spectrum $\mathbb{E}S_n(a(n)2^j)$ (see (3.7)) can be approximated by that of an ideal-HfBm (see (2.11)) in the sense that

$$a(n)^{\circ -2H}\circ\mathbb{E}\big[S_n(a(n)2^j)\big] \to \mathbb{E}\big[S_{\mathrm{ideal}}(2^j)\big] = \Big(\Phi^{jj}_{q_1q_2}(0)\Big)_{q_1,q_2=1,\dots,m}; \tag{3.15}$$

(iv) for $\Xi^{jj'}$ as in (3.12) and $q_1, q_2 = 1,\dots,m$, as $n\to\infty$,

$$\begin{aligned}&\frac{1}{\sqrt{n_{a,j}}}\frac{1}{\sqrt{n_{a,j'}}}\sum_{k=1}^{n_{a,j}}\sum_{k'=1}^{n_{a,j'}}\frac{\Xi^{jj',a}_{q_1q_2}(a(n)(2^jk-2^{j'}k'))}{a(n)^{2h_{q_1q_2}}}\\ &\to 2^{-\frac{(j+j')}{2}}\gcd(2^j,2^{j'})\sum_{z=-\infty}^{\infty}\Phi^{jj'}_{q_1q_2}(z\gcd(2^j,2^{j'}));\end{aligned} \tag{3.16}$$

where $\Phi^{jj'}_{q_1q_2}(\cdot)$ is given by (3.14);

(v) for $1\le q_1\le q_2\le m$, $1\le q_3\le q_4\le m$, as $n\to\infty$,

$$\frac{\sqrt{n_{a,j}}}{a(n)^{\delta_{q_1q_2}}}\frac{\sqrt{n_{a,j'}}}{a(n)^{\delta_{q_3q_4}}}\,\mathrm{Cov}\Big[W_n^{(q_1q_2)}(a(n)2^j), W_n^{(q_3q_4)}(a(n)2^{j'})\Big] \to G^{jj'}(q_1,q_2,q_3,q_4), \tag{3.17}$$

for $\delta_{\cdot\cdot}$ as in (1.7), where

$$\begin{aligned}&\frac{\Phi^{jj}_{q_1q_2}(0)\Phi^{j'j'}_{q_3q_4}(0)}{2^{-\frac{(j+j')}{2}}}G^{jj'}(q_1,q_2,q_3,q_4)\\ &= \begin{cases}\phi^{jj'}(q_1,q_3,q_2,q_4), & \text{if } \delta_{q_1q_3}=0=\delta_{q_2q_4}\\ & \text{and } (\delta_{q_1q_4}>0 \text{ or } \delta_{q_2q_3}>0);\\ \phi^{jj'}(q_1,q_4,q_2,q_3), & \text{if } (\delta_{q_1q_3}>0 \text{ or } \delta_{q_2q_4}>0)\\ & \text{and } \delta_{q_1q_4}=0=\delta_{q_2q_3};\\ \phi^{jj'}(q_1,q_3,q_2,q_4)+\phi^{jj'}(q_1,q_4,q_2,q_3), & \text{if } \delta_{q_1q_3}=\delta_{q_2q_4}=\delta_{q_1q_4}=\delta_{q_2q_3}=0;\\ 0, & \text{otherwise,}\end{cases}\end{aligned}$$

and

$$\phi^{jj'}(q_1,q_2,q_3,q_4) := \gcd(2^j,2^{j'})\sum_{z=-\infty}^{\infty}\Phi^{jj'}_{q_1q_2}(z\gcd(2^j,2^{j'}))\Phi^{jj'}_{q_3q_4}(z\gcd(2^j,2^{j'})); \tag{3.18}$$

(vi) as $n\to\infty$,

$$\Big(\mathrm{vec}_{\mathcal{S}}\Big[\Big(\frac{\sqrt{n_{a,j}}}{a(n)^{\delta_{q_1q_2}}}\Big)_{q_1,q_2=1,\dots,m}\circ(W_n(a(n)2^j)-\mathbf{1})\Big]\Big)_{j=j_1,\dots,j_2} \xrightarrow{d} \mathcal{N}_{\frac{m(m+1)}{2}\times J}(0,G), \tag{3.19}$$

where

$$J = j_2 - j_1 + 1, \tag{3.20}$$

and **1** is a vector of ones. Each entry of the asymptotic covariance matrix G in (3.19) is given by the terms $G^{jj'}(q_1, q_2, q_3, q_4)$ in (3.17) for appropriate values of j, j', q_1, q_2, q_3 and q_4.

Remark 3.2. Note that, up to a change of sign, each entry $\Phi^{jj'}_{q_1q_2}(z)$ on the right-hand side of (3.14) corresponds to the covariance between the wavelet coefficients at octaves j and j' of a fBm with parameter $h_{q_1q_2}$.

Remark 3.3. In Proposition 3.1, (v), the asymptotic covariance between same-entry wavelet variance terms is always nontrivial, irrespective of the values of fractal connectivity parameters. For example, when $(1, 2) =: (q_1, q_2) = (q_3, q_4)$, it is given by

$$G^{jj'}(1, 2, 1, 2) = \frac{2^{-\frac{(j+j')}{2}}}{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{12}(0)} \cdot \begin{cases} \phi^{jj'}(1, 1, 2, 2), & \delta_{12} > 0; \\ \phi^{jj'}(1, 1, 2, 2) + \phi^{jj'}(1, 2, 2, 1), & \delta_{12} = 0. \end{cases}$$

This is not true for terms associated with different pairs of indices. For example, if $(q_1, q_2) = (1, 1) \neq (q_3, q_4) = (2, 2)$, then

$$G^{jj'}(1, 1, 2, 2) = \frac{2^{-\frac{(j+j')}{2}}}{\Phi^{jj}_{11}(0)\Phi^{j'j'}_{22}(0)} \cdot \begin{cases} 2\phi^{jj'}(1, 2, 1, 2), & \delta_{12} = 0; \\ 0, & \delta_{12} > 0. \end{cases}$$

In other words, the phenomenon of the asymptotic decorrelation of wavelet variance terms is only observed for instances involving departures from fractal connectivity.

3.3. Estimation of the scaling exponents

3.3.1. Definition of the estimators

As in the univariate case, the fact that the wavelet variance (3.7) satisfies the Hadamard scaling relation in the coarse scale limit (see (3.15)) points to the development of a log-regression regression method based on the sample wavelet variance across scales.

Estimators can be defined in a standard way by means of the log-regression relations

$$\widehat{\alpha}_q = \sum_{j=j_1}^{j_2} w_j \log_2 S_n^{(qq)}(a(n)2^j), \qquad 1 \le q \le m,$$

$$\widehat{\alpha}_{q_1q_2} = \sum_{j=j_1}^{j_2} w_j \log_2 |S_n^{(q_1q_2)}(a(n)2^j)|, \qquad 1 \le q_1 < q_2 \le m, \tag{3.21}$$

which are well-defined with probability 1, where $w_j, j = j_1, \ldots, j_2$, are linear regression weights satisfying

$$\sum_{j=j_1}^{j_2} w_j = 0, \quad \sum_{j=j_1}^{j_2} j w_j = 1 \tag{3.22}$$

(e.g., [48,49]). The derived estimator

$$\widehat{\delta}_{q_1q_2} = \frac{\widehat{\alpha}_{q_1q_1} + \widehat{\alpha}_{q_2q_2}}{2} - \widehat{\alpha}_{q_1q_2}, \qquad q_1 \neq q_2, \tag{3.23}$$

for the parameter (1.7) will be used in Section 5 in the construction of a hypothesis test for fractal connectivity (see also [18–20]). With the standard Gaussian asymptotics for the sample wavelet variances $\{S_n(a(n)2^j)\}_{j=j_1,\ldots,j_2}$ established in Proposition 3.1, the system of Eqs. (3.21) and (3.23) is expected to yield efficient estimators. In fact, this is proved in the next section.

3.3.2. Asymptotic distribution

In the next result, Theorem 3.1, we draw upon Proposition 3.1 to show that the vector $(\widehat{\alpha}_{q_1q_2})_{q_1,q_2=1,\ldots,m}$ is an asymptotically unbiased and Gaussian estimator of the scaling exponents $(\alpha_{q_1q_2})_{q_1,q_2=1,\ldots,m}$ (whence an analogous statement holds for the estimator (3.23)). As in the aforementioned proposition, the decorrelation property of the wavelet transform contributes to the Gaussianity of the asymptotic distribution.

Theorem 3.1. *Let $w_{j_1}, \ldots, w_{j_2}$ be the weight terms in (3.21) and let G be as in (3.19). Then, as $n \to \infty$,*

$$\mathrm{vec}_{\mathcal{S}}\left[\left(\frac{\sqrt{n}}{a(n)^{\delta_{q_1q_2}+1/2}}\,(\widehat{\alpha}_{q_1q_2} - \alpha_{q_1q_2})\right)_{q_1,q_2=1,\ldots,m}\right] \xrightarrow{d} \mathcal{N}_{\frac{m(m+1)}{2}}(0, MGM^*). \tag{3.24}$$

Remark 3.4. Condition (2.8) is needed to ensure the consistency of the estimator. Furthermore, it is clear that $\rho_{qq} > 0$ for $q = 1, \ldots, m$, since otherwise the process is not proper or its spectral density is not positive semidefinite a.e.

Remark 3.5. The convergence rate in (3.24) depends on the unknown fractal connectivity parameters $\delta_{\cdot\cdot}$. In practice, the latter can be replaced by their estimates or, in some cases, ignored, since they exponentiate the slow growth term $a(n)$.

3.3.3. Numerical simulation setting

We conducted Monte Carlo experiments over 1000 independent realizations of bivariate HfBm. Though several parameter settings were tested, results are reported only for two representative cases: fractally connected ideal-HfBm $(\alpha_{11}, \alpha_{22}, \rho_{12}) = (0.4, 0.8, 0.6)$; and non-fractally connected HfBm (see Example 2.2) with $(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho_{12}) = (0.4, 0.8, 0.2, 0.6)$. HfBm copies were synthesized using the multivariate process synthesis toolbox described in [50,51] and available at `www.hermir.org`. For the wavelet analysis, least asymmetric Daubechies wavelet with $N_\psi = 2$ vanishing moments were used [45]. Estimation by means of weighted linear regression was performed on the octave range $(j_1, j_2) = (3, \log_2 n - N_\psi)$. This choice of regression range (with fine scale j_1 fixed and $j_2 \sim \log_2 n$) is made here in order to, first, obtain results that are consistent and comparable with those reported in the literature for the estimation of univariate scaling exponents, cf., e.g., [48] and references therein, and, second, avoid the issue of fixing $h_{\min}$, $h_{\max}$ and ϖ_0 in (3.6).

Monte Carlo parameter estimates $(\widehat{\alpha}_{11}, \widehat{\alpha}_{22}, \widehat{\alpha}_{12}, \widehat{\delta}_{12})$ are reported in Figs. 1 and 2 in terms of bias, standard deviation, skewness and kurtosis, as functions of (the $\log_2$ of) sample sizes $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}, 2^{18}, 2^{20}\}$.

3.3.4. Finite sample performance

Figs. 1 and 2 show that for all four estimates $(\widehat{\alpha}_{11}, \widehat{\alpha}_{22}, \widehat{\alpha}_{12}, \widehat{\delta}_{12})$, the bias becomes negligible as the sample size grows. It can also be seen that the bias is slightly larger for non-fractally connected data, notably for the parameter δ_{12}.

The figures further show that standard deviations for all estimates decrease as $n^{-1/2}$ (the latter trend being plotted as superimposed dashed lines), with no significant difference between fractally and non-fractally connected instances. For $(\widehat{\alpha}_{11}, \widehat{\alpha}_{22}, \widehat{\alpha}_{12})$, there is no detectable dependence of the outcomes on the actual values of the parameters themselves (in accordance with theoretical calculations reported in Table 3). Accordingly, one observes that the variances for $\widehat{\alpha}_{11}$ and $\widehat{\alpha}_{22}$ are identical and depend on the sample size, but not on any parameter of the stochastic process. For $\widehat{\alpha}_{12}$, the variance does not depend on α_{12}, but results not shown demonstrate that it does vary with ρ_{12}, as confirmed by approximate calculations (see (4.9)). For $\widehat{\delta}$, further simulations not displayed indicate that $\operatorname{Var}\widehat{\delta}$ depends on δ roughly proportionally to $1+\delta$. Departures from fractal connectivity tend to imply a decrease in the variance of $\widehat{\delta}$; this is a counterintuitive phenomenon that has yet to be fully explained.

In regard to convergence in distribution, Figs. 1 and 2 show that the Monte Carlo skewness and kurtosis estimates for $(\alpha_{11}, \alpha_{22})$ are very close to 0 even at very small sample sizes, both with and without fractal connectivity. This characterizes a fast convergence to limiting normal distributions, and lies in agreement with the asymptotics presented in Theorem 3.1. However, the weak convergence is observed to be much slower in practice for the cross-exponents $(\alpha_{12}, \delta_{12})$, yet with no noticeable difference between fractally and non-fractally connected instances, i.e., different values of the parameter δ_{12}.

4. Confidence intervals

To complement the asymptotic and finite sample results in Sections 3.3.2 and 3.3.4, we now construct confidence intervals for the estimators (3.21) and (3.23). In particular, we investigate the effect of omitting the covariance among sample wavelet variance terms, as well as the dependence of the asymptotic variance in (3.24) and confidence intervals on the unknown parameter values. This will allow us to efficiently compute approximate confidence intervals in practice based on a single sample path.

4.1. Asymptotic covariance

In Theorem 4.1, we provide a finite sample approximation for the covariance between the logarithms of sample wavelet variances, coupled with the convergence rate of the covariance approximation to its asymptotic expression in terms of ideal-HfBm covariances (3.14). As in Theorem 3.1, in part due to the decorrelation property of the wavelet transform, the estimators' second order properties are comparable to those observed in the i.i.d. case (e.g., [52], p. 430). In particular, the finite sums provided in Theorem 4.1 mathematically justify heuristic expressions obtained from Taylor expansions (c.f. (4.5)), and the theorem also accurately quantifies the approximation error as a function of the sample size and scale.

In the theorem, we use indicator functions to regularize $\log_2 |S_n^{(q_1q_2)}(a(n)2^j)|$ around the origin. The truncation converges fast to zero and is of no impact in practice. Indeed, one can redefine the estimator (3.21) with the indicators and obtain the same asymptotic distribution (3.24).

Theorem 4.1. *Let $q_1, q_2, q_3, q_4 = 1, \ldots, m$ be generic indices, and suppose condition (2.8) holds. Consider $S_n(a(n)2^j) = \{S_n^{(q_1q_2)}(a(n)2^j)\}$, $W_n(a(n)2^j) = \{W_n^{(q_1q_2)}(a(n)2^j)\}$ as in (3.10) and $\Phi_{q_1q_2}^{jj'}(\cdot)$ as in (3.14). For notational simplicity, denote*

$$n_* = \frac{n}{a(n)2^{j+j'}} \tag{4.1}$$

and also recall the definition of ϖ_0 in (2.7). Consider a sequence $\{r_n\}_{n\in\mathbb{N}} \subseteq (0, 1/2)$ satisfying

$$\frac{1}{r_n} = O\Big(\exp\Big\{C\Big(\frac{n_*}{a(n)^{4h_{\max}-4h_{\min}}}\Big)^{1-\xi}\Big\}\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^{1/2}\Big) \tag{4.2}$$

for any $C > 0$ and any $0 < \xi < 1$. Then,

$$\operatorname{Cov}\Big[\log_2 |S_n^{(q_1q_2)}(a(n)2^j)| 1_{\{|W_n^{(q_1q_2)}(a(n)2^j)|>r_n\}}, \log_2 |S_n^{(q_3q_4)}(a(n)2^{j'})| 1_{\{|W_n^{(q_3q_4)}(a(n)2^{j'})|>r_n\}}\Big]$$
$$= \frac{(\log_2 e)^2\, 2^{-(j+j')}}{\Phi_{q_1q_2}^{jj}(0)\Phi_{q_3q_4}^{j'j'}(0)\{1 + O(a(n)^{-\varpi_0})\}^2}$$

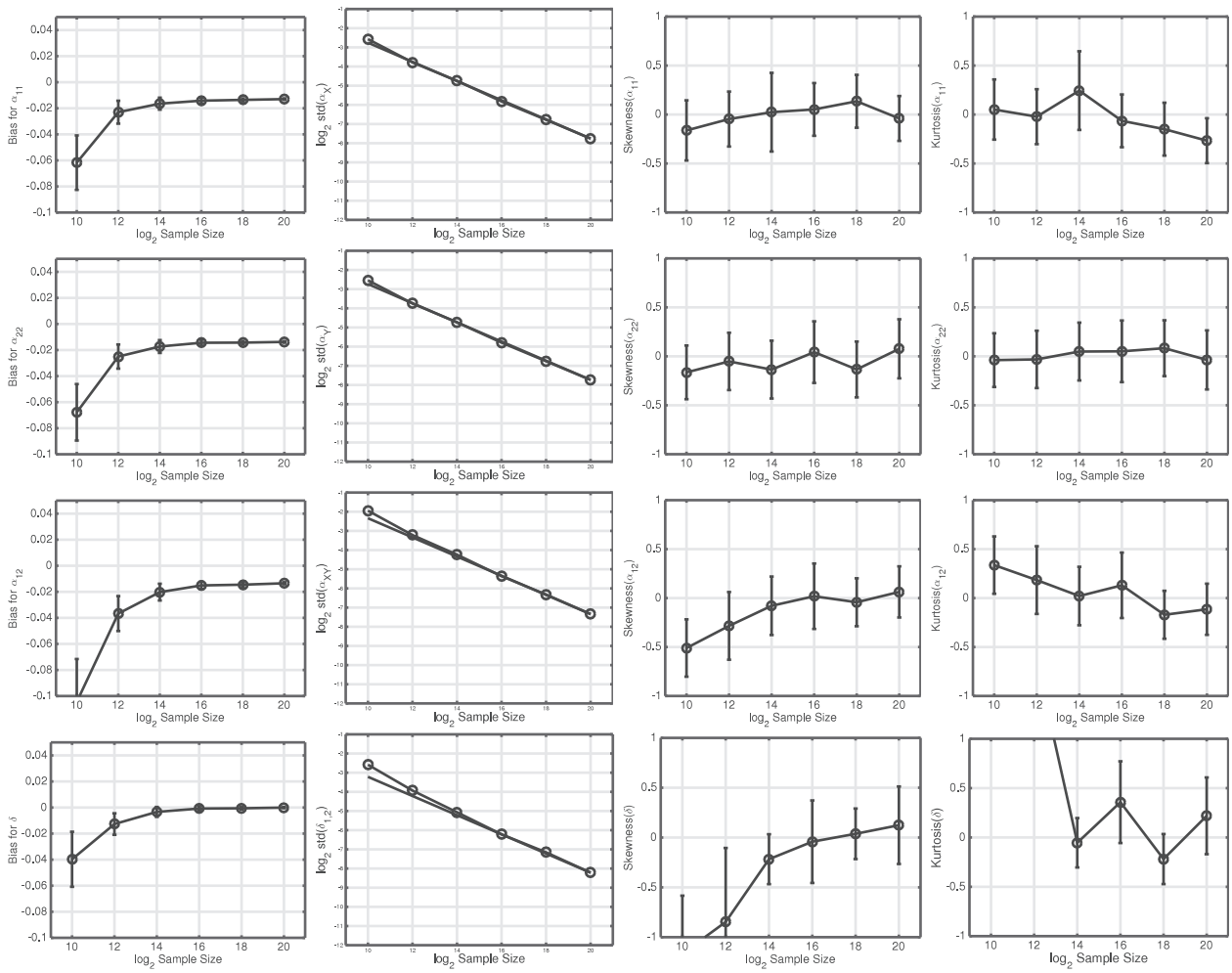

Fig. 1. Estimation performance and asymptotic normality. Study based on 1000 Monte Carlo realizations of (fractally connected) ideal-HfBm with parameters $(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho_{12}) = (0.4, 0.8, 0, 0.6)$.

$$\Big\{\frac{a(n)^{2(h_{q_1q_3}+h_{q_2q_4})-2(h_{q_1q_2}+h_{q_3q_4})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{q_1q_3}(2^jk-2^{j'}k')\Phi^{jj'}_{q_2q_4}(2^jk-2^{j'}k')\Big]$$
$$+\frac{a(n)^{2(h_{q_1q_4}+h_{q_2q_3})-2(h_{q_1q_2}+h_{q_3q_4})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{q_1q_4}(2^jk-2^{j'}k')\Phi^{jj'}_{q_2q_3}(2^jk-2^{j'}k')\Big]$$
$$+\,o\Big(\frac{a^{2\max\{h_{q_1q_3}+h_{q_2q_4},h_{q_1q_4}+h_{q_2q_3}\}-2(h_{q_1q_2}+h_{q_3q_4})}}{n_*}\Big)\Big\}+O\Big(\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^2\Big). \tag{4.3}$$

Remark 4.1. Note that the finite summations on the right-hand side of (4.3) converge as $n \to \infty$. For example,

$$\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^{j}n_*}\Phi^{jj'}_{q_1q_3}(2^jk-2^{j'}k')\Phi^{jj'}_{q_2q_4}(2^jk-2^{j'}k') \to \phi^{jj'}(q_1,q_2,q_3,q_4),$$

where $\phi^{jj'}(\cdot)$ is defined by (3.18) (this can be shown by the same argument for establishing (A.16) and (A.25) in the proof of Proposition 3.1(iv)).

Remark 4.2. In (4.3), the main residual term satisfies

$$O\Big(\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^2\Big) = o(1) \tag{4.4}$$

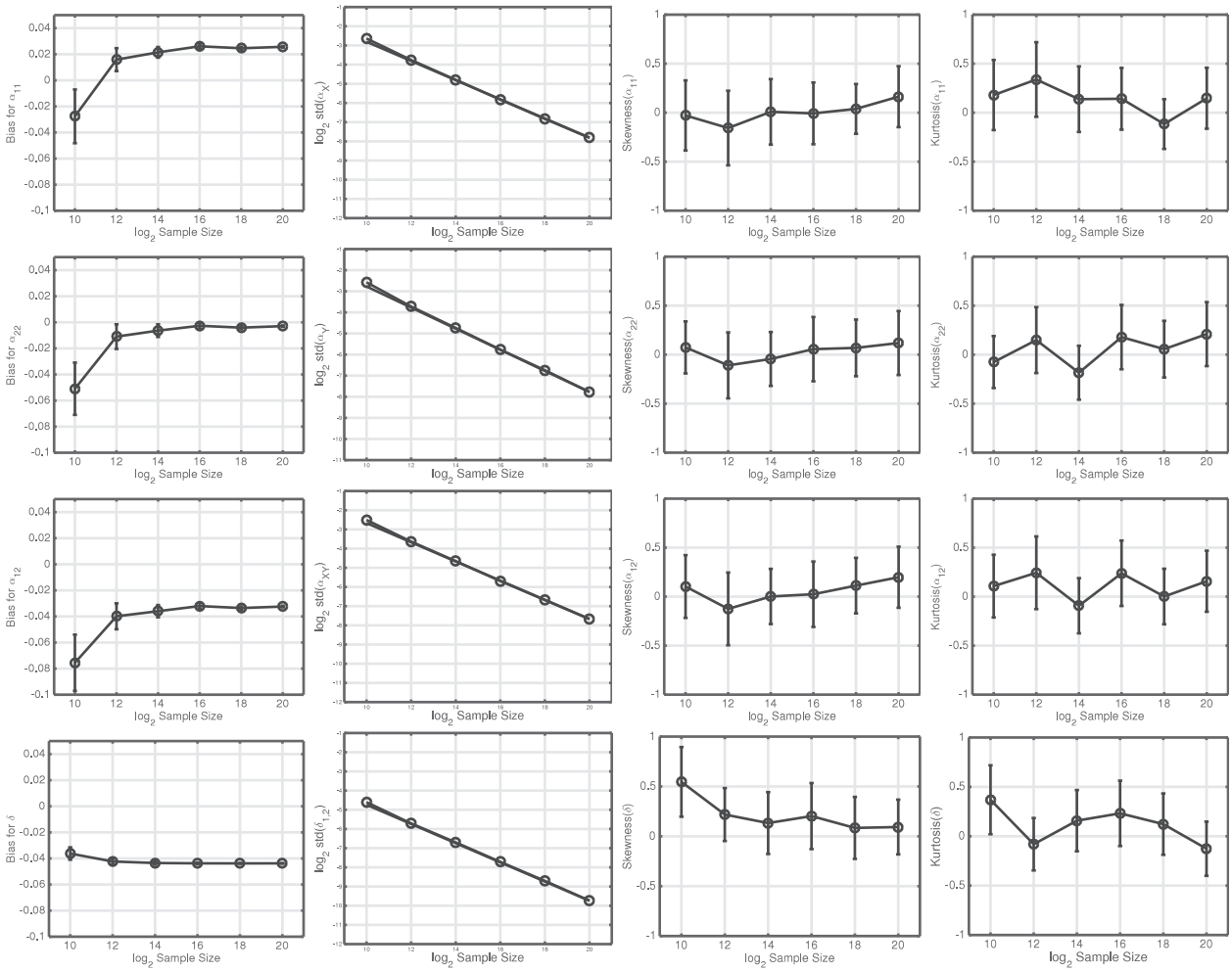

Fig. 2. Estimation performance and asymptotic normality. Study based on 1000 Monte Carlo realizations of (non-fractally connected) HfBm with parameters $(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho_{12}) = (0.4, 0.8, 0.2, 0.6)$.

under condition (3.6). In practice, the modeling of instances involving extreme deviations from fractal connectivity ($\delta_{\cdot\cdot}$ close to 1) may require greater regularity from the functions $g_{\cdot\cdot}$ (see (2.3)) or the wavelet basis. Otherwise, for conservative choices of the regularity parameters (e.g., $\varpi_0 = 2$ and $\beta = 1 + \varepsilon$ for small $\varepsilon > 0$), expression (4.4) may converge slower to zero than the main terms in (4.3).

Remark 4.3. In (3.5), we assume that an HfBm continuous time sample path is available. However, it is well known that, under mild assumptions, the availability of discrete time observations does not generally alter the nature of the asymptotic distribution for wavelet estimators (see, for instance, [47], Section III; [49], Section 3.2; [53], Section C). Establishing this for the case of HfBm is a topic for future work.

Remark 4.4. When $q = q_1 = q_2$, for the univariate case the expansion (4.3) appears implicitly in [54], expression (86). Also, when computing the moments of $W_n^{(qq)}(a(n)2^j)$, the truncation is unnecessary (see Remark C.5).

4.2. Variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\delta}_{q_1q_2}$

4.2.1. Closed-form approximations

Turning back to expression (4.3), by ignoring the scaling factor ($a(n) \equiv 1$), the truncation and setting all the convergence order terms to zero we obtain the approximation

$$\begin{aligned}
&\mathrm{Cov}\left[\log_2 |S_n^{(q_1q_2)}(2^j)|, \log_2 |S_n^{(q_3q_4)}(2^{j'})|\right] \\
&\approx \frac{(\log_2 e)^2 2^{-(j+j')}}{\Phi^{jj}_{q_1q_2}(0)\Phi^{j'j'}_{q_3q_4}(0)} \frac{1}{n_j}\left\{\frac{1}{n_{j'}}\sum_{k=1}^{n_j}\sum_{k'=1}^{n_{j'}} \Phi^{jj'}_{q_1q_3}(2^jk - 2^{j'}k')\Phi^{jj'}_{q_2q_4}(2^jk - 2^{j'}k')\right. \\
&\quad \left. + \frac{1}{n_{j'}}\sum_{k=1}^{n_j}\sum_{k'=1}^{n_{j'}} \Phi^{jj'}_{q_1q_4}(2^jk - 2^{j'}k')\Phi^{jj'}_{q_3q_2}(2^{j'}k' - 2^jk)\right\},
\end{aligned} \tag{4.5}$$

Table 1
Closed-form expression for $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right]$.

$\frac{\mathrm{Var}[\widehat{\alpha}_{qq}]}{(\log_2 \mathrm{e})^2}$	$2\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j, n_j'} \mathrm{r}_{qq}^2(j,k;j',k')$
$\frac{\mathrm{Var}[\widehat{\alpha}_{q_1q_2}]}{(\log_2 \mathrm{e})^2}$	$\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j, n_j'} \frac{\mathrm{r}_{q_1q_2}(j,k;j',k')\mathrm{r}_{q_1q_2}(j',k';j,k)+\mathrm{r}_{q_1q_1}(j,k;j',k')\mathrm{r}_{q_2q_2}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_1q_2}(j',k';j',k')}$
$\frac{\mathrm{Cov}[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}]}{(\log_2 \mathrm{e})^2}$	$2\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j, n_j'} \frac{\mathrm{r}_{q_1q_2}^2(j,k;j',k')}{\mathrm{r}_{q_1q_1}(j,k;j,k)\mathrm{r}_{q_2q_2}(j',k';j',k')}$
$\frac{\mathrm{Cov}[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}]}{(\log_2 \mathrm{e})^2}$	$2\sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k,k'=1}^{n_j, n_j'} \frac{\mathrm{r}_{q_1q_1}(j,k;j',k')\mathrm{r}_{q_1q_2}(j,k;j',k')}{\mathrm{r}_{q_1q_1}(j,k;j,k)\mathrm{r}_{q_1q_2}(j',k';j',k')}$

where

$$n_j := \frac{n}{2^j}.$$

Note that the approximation (4.5) is a finite sample one, and ignores the potential asymptotic decorrelation effect stemming from the shifting scaling factor $a(n)$ (see Proposition 3.1, (v), and (4.3)). Consider the normalization

$$\mathrm{r}_{q_1q_2}(j,k;j',k') := \frac{\Phi_{q_1q_2}^{jj'}(2^j k - 2^{j'} k')}{\sqrt{\Phi_{q_1q_1}^{jj}(0)\Phi_{q_2q_2}^{j'j'}(0)}} \in [-1, 1]. \tag{4.6}$$

Expressions (3.21), (4.5) and (4.6) yield the covariance approximation

$$\begin{aligned}\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right] \approx (\log_2 \mathrm{e})^2 \sum_{j,j'=j_1}^{j_2} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_{k=0}^{n_j-1} \sum_{k'=0}^{n_{j'}-1} \frac{\mathrm{r}_{q_1q_3}(j,k;j',k')\mathrm{r}_{q_2q_4}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j',k';j',k')} \\ + \frac{\mathrm{r}_{q_1q_4}(j,k;j',k')\mathrm{r}_{q_2q_3}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j',k';j',k')}.\end{aligned} \tag{4.7}$$

In particular, (4.7) further allows us to compute

$$\begin{aligned}\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right] = \frac{1}{4}\left(\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right] + \mathrm{Var}\left[\widehat{\alpha}_{q_2q_2}\right]\right) + \mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right] \\ + \frac{1}{2}\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right] - \mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right] - \mathrm{Cov}\left[\widehat{\alpha}_{q_2q_2}, \widehat{\alpha}_{q_1q_2}\right].\end{aligned} \tag{4.8}$$

The variances (4.8) will be used in Section 5 in the construction of a test for fractal connectivity, i.e., for the hypothesis $H_0 : \delta_{q_1q_2} \equiv 0$. Table 1 summarizes the closed-form approximations for $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$, $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right]$ and $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right]$ established in (4.7).

4.2.2. Impact of inter- and intra-scale correlations

The expression of $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right]$ can further be split into three terms, namely,

$$\begin{aligned}\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right] \approx (\log_2 \mathrm{e})^2 \sum_{j=j_1}^{j_2} \Bigg[\frac{w_j^2}{n_j}\left(1 + \frac{1}{\mathrm{r}_{q_1q_2}(j,0;j,0)\mathrm{r}_{q_3q_4}(j,0;j,0)}\right) \\ + \frac{w_j^2}{n_j^2} \sum_k \sum_{k' \neq k} \frac{\mathrm{r}_{q_1q_3}(j,k;j,k')\mathrm{r}_{q_2q_4}(j,k;j,k') + \mathrm{r}_{q_1q_4}(j,k;j,k')\mathrm{r}_{q_2q_3}(j,k;j,k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j,k';j,k')} \\ + \sum_{j' \neq j} \frac{w_j w_{j'}}{n_j n_{j'}} \sum_k \sum_{k'} \frac{\mathrm{r}_{q_1q_3}(j,k;j',k')\mathrm{r}_{q_2q_4}(j,k;j',k') + \mathrm{r}_{q_1q_4}(j,k;j',k')\mathrm{r}_{q_2q_3}(j,k;j',k')}{\mathrm{r}_{q_1q_2}(j,k;j,k)\mathrm{r}_{q_3q_4}(j',k';j',k')} \Bigg],\end{aligned} \tag{4.9}$$

where the first term in the sum over j reflects the variance only of wavelet coefficients, the second term the covariance of wavelet coefficients at a given scale, and the third term, the covariance of wavelet coefficients at different scales. In other words, if wavelet coefficients were independent, the second and third terms would equal zero.

The relative contributions of the three terms to the final variances are quantified by means of Monte Carlo simulations conducted following the same protocol and settings as those described in Section 3.3.3. Table 2, reporting the relative contributions of each of the three terms for various sample sizes under fractal connectivity (i.e., $\delta_{q_1q_2} \equiv 0$), clearly shows that the second and third terms (intra- and inter-scale covariances) cannot be neglected, namely, one cannot use only the first term (variance) in the construction of confidence intervals for fBm, as proposed in [9]. Identical conclusions, not shown here, are drawn under departures from fractal connectivity (i.e., $\delta_{q_1q_2} > 0$).

4.2.3. First order approximations for the variances and covariances of $\widehat{\alpha}_{q_1q_2}$

It is of interest to further examine the leading order approximations for the variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\delta_{q_1q_2}$, corresponding to neglecting all intra- and inter-scale correlations amongst wavelet coefficients. The first order approximations neglecting all inter- and intra- scale correlations amongst wavelet coefficients of $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right]$ and of $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ are summarized in Table 3.

The results show that $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$ and $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $q_1 \neq q_2$, do not depend on the actual values of the scaling exponents ($\alpha_{q_1q_1}, \alpha_{q_2q_2}, \alpha_{q_1q_2}$), which corroborates the numerical performance reported in Section 3.3.4.

Table 2
Relative contributions of the three terms in (4.9) to $\mathrm{Var}\left[\widehat{\alpha}_{(\cdot)}\right]$, $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_2}, \widehat{\alpha}_{q_3q_4}\right]$ and $\mathrm{Var}\left[\widehat{\delta}_{(\cdot)}\right]$ for various sample sizes. ($(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho) = (0.2, 0.6, 0, 0.9)$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$).

n	2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}
	$\mathrm{Var}\left[\widehat{\alpha}_{11}\right]$				$\mathrm{Var}\left[\widehat{\alpha}_{22}\right]$				$\mathrm{Var}\left[\widehat{\alpha}_{12}\right]$				$\mathrm{Var}\left[\widehat{\delta}_{12}\right]$			
var $\times 10^3$	65.33	9.51	1.87	0.42	69.27	10.05	1.97	0.44	18.76	2.73	0.54	0.12	12.42	1.81	0.35	0.08
term 1	0.74	0.68	0.66	0.66	0.69	0.65	0.63	0.62	0.72	0.67	0.65	0.64	0.71	0.66	0.65	0.64
term 2	0.15	0.14	0.14	0.14	0.20	0.19	0.18	0.18	0.17	0.16	0.16	0.16	0.18	0.17	0.16	0.16
term 3	0.11	0.17	0.20	0.20	0.11	0.17	0.19	0.19	0.11	0.17	0.19	0.20	0.11	0.17	0.19	0.2
co $\times 10^3$	$\mathrm{Cov}\left[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}\right]$				$\mathrm{Cov}\left[\widehat{\alpha}_{11}, \widehat{\alpha}_{12}\right]$				$\mathrm{Cov}\left[\widehat{\alpha}_{22}, \widehat{\alpha}_{12}\right]$							
	54.47	7.92	1.55	0.35	33.12	4.82	0.95	0.21	34.1	4.95	0.97	0.22				
term 1	0.72	0.66	0.65	0.64	0.73	0.67	0.66	0.65	0.71	0.66	0.64	0.63				
term 2	0.18	0.17	0.16	0.16	0.16	0.15	0.15	0.15	0.19	0.18	0.17	0.17				
term 3	0.11	0.17	0.19	0.20	0.11	0.17	0.19	0.20	0.11	0.17	0.19	0.2				

Table 3
First-order approximations in (4.7) for the variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\delta_{q_1q_2}$ neglecting all inter- and intra-scale correlations amongst wavelet coefficients.

$\frac{\mathrm{Var}\left[\widehat{\alpha}_{qq}\right]}{(\log_2 e)^2}$	$2\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}$
$\frac{\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]}{(\log_2 e)^2}$	$\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}\left(1 + \frac{1}{\mathrm{r}^2_{q_1q_2}(j,0;j,0)}\right)$
$\frac{\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right]}{(\log_2 e)^2}$	$2\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j} \mathrm{r}^2_{q_1q_2}(j, 0; j, 0)$
$\frac{\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_1q_2}\right]}{(\log_2 e)^2}$	$2\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}$
$\frac{\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]}{(\log_2 e)^2}$	$\sum_{j=j_1}^{j_2} \frac{w_j^2}{n_j}\left(\left(\mathrm{r}^2_{q_1q_2}(j, 0; j, 0) + \frac{1}{\mathrm{r}^2_{q_1q_2}(j,0;j,0)}\right) - 2\right)$

Note that for an ideal-HfBm with fractal connectivity, it is straightforward to show that $\mathrm{r}_{qq}(j, 0; j, 0) \equiv 1$ and $\mathrm{r}_{q_1q_2}(j, 0; j, 0) \equiv \rho_{q_1q_2}$ when $q_1 \neq q_2$. While $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$ does not depend on correlations $\rho_{q_1q_2}$, as expected, $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$, $q_1 \neq q_2$ does vary with $\rho_{q_1q_2}$ according to $1/\rho^2_{q_1q_2}$, showing that $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right] \to +\infty$ when $\rho_{q_1q_2} \to 0$. This can be interpreted as the fact that when $\rho_{q_1q_2} \to 0$, the scaling exponent $\alpha_{q_1q_2}$, $q_1 \neq q_2$, loses its meaning. Furthermore, $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right]$ depends on ρ as $\rho^2_{q_1q_2}$, not surprisingly indicating that when $\rho_{q_1q_2} \to 0$ (no correlation amongst components), $\mathrm{Cov}\left[\widehat{\alpha}_{q_1q_1}, \widehat{\alpha}_{q_2q_2}\right] \to 0$ (no correlation amongst estimates).

Moreover, the first order approximation of $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ is observed not to depend on the actual value of $\delta_{q_1q_2}$, while $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ clearly depends on $\delta_{q_1q_2}$ based on the numerical simulations reported in Section 3.3.4. This can be interpreted as the fact that, for $\widehat{\delta}_{q_1q_2}$, the first order approximation (neglecting all intra- and inter-scale correlations amongst wavelet coefficients) is not sufficient to approximate well $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$, as opposed to what is observed for $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_1}\right]$ and $\mathrm{Var}\left[\widehat{\alpha}_{q_1q_2}\right]$, $q_1 \neq q_2$.

To finish with, $\mathrm{Var}\left[\widehat{\delta}_{q_1q_2}\right]$ varies with $\rho_{q_1q_2}$ as $\rho^2_{q_1q_2} + 1/\rho^2_{q_1q_2} - 2$. This shows again that when $\rho_{q_1q_2} \to 0$, parameter $\delta_{q_1q_2}$ becomes irrelevant. Moreover, it also shows that when $\rho_{q_1q_2} \to \pm 1$, $\mathrm{Var}\left[\delta_{q_1q_2}\right] \to 0$. This can be understood as the fact that when $\rho_{q_1q_2} \to \pm 1$, a departure from fractal connectivity is no longer permitted, as indicated by (2.12). Thus, $\rho_{q_1q_2} \to \pm 1$ implies $\delta_{q_1q_2} \to 0$, which is then no longer a random variable.

4.3. Practical computation of the variances and covariances of $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\delta}_{q_1q_2}$

4.3.1. Computation of $\mathbb{E}\left[d_{q_1}\right](j, k)d_{q_2}(j', k')$ and $\mathrm{r}_{q_1q_2}(j, k; j', k')$

Evaluation of (4.7) requires knowledge of the covariance between wavelet coefficients (4.6), which will be developed here explicitly for the discrete and the dyadic wavelet transforms. Let $h(k)$ and $g(k)$, $k = 1, \ldots, L$, be the coefficients of the high pass and low pass filters of the discrete wavelet transform, respectively, and let $\uparrow_2[\cdot]$ and $\downarrow_2[\cdot]$ be the dyadic upsampling and decimation operators. For any $q = 1, \ldots, m$, the wavelet transform of the discrete time process component $B_H(k)_q$ yields, at each scale $j = 1, \ldots, J$, sequences of approximation coefficients $a_q(j, k)$ and detail coefficients $d_q(j, k)$, $q = 1, \ldots, m$. The corresponding dyadic coefficients are given by $\tilde{a}_q(j, k) = a_q(j, 2^j k)$ and $\tilde{d}_q(j, k) = d_q(j, 2^j k)$, respectively. Pick the initialization $a_q(0, k) = B_H(k)_q$ (see [55] for a discussion of the initialization of the discrete wavelet transform). At scale $j = 1$, $a_q(1, \cdot) = h * a_q(0, \cdot)$ and $d_q(1, \cdot) = g * a_q(0, \cdot)$, at scale $j = 2$, $a_q(2, \cdot) = \uparrow_2[h] * a_q(1, \cdot) = \uparrow_2[h] * h * a_q(0, \cdot)$ and $d_q(2, \cdot) = \uparrow_2[g] * a_q(1, \cdot) = \uparrow_2[g] * h * a_q(0, \cdot)$. By iteration, we obtain the sequences of detail coefficients at each scale $j = j'$, i.e.,

$$d_q(j', k) = \left(g_{j'} * a_q(0, \cdot)\right)(k), \quad d_q(j', k) = d_q(j', 2^{j'} k), \tag{4.10}$$

where

$$g_{j'} = \uparrow_{2^{j'-1}}[g] * \left(\mathop{*}_{j=0}^{j'-2} \uparrow_{2^{j'}}[h]\right).$$

Now let $\gamma_{h_{q_1q_2}}(s, t)$, $1 \leq q_1 \leq q_2 \leq m$, be (univariate) fBm covariance functions with indices $h_{q_1q_2} = \alpha_{q_1q_2}/2$, i.e.,

$$\gamma_{q_1q_2}(s, t) = \rho_{q_1q_2}\sigma_{q_1}\sigma_{q_2}\{|s|^{\alpha_{q_1q_2}} + |t|^{\alpha_{q_1q_2}} - |s - t|^{\alpha_{q_1q_2}}\}. \tag{4.11}$$

Then (cf. [18]; note that a change in time scale would only result in a multiplicative constant, which we assume to be absorbed in $\rho_{q_1q_2}$ and which cancels out in the final expression for $\mathrm{r}_{q_1q_2}$)

$$\begin{aligned}
&\mathbb{E}[d_{q_1}(j,k)d_{q_2}(j',k+\tau)]/(\sigma_{q_1}\sigma_{q_2})\\
&\quad=\sum_p\sum_q g_j(p)g_{j'}(q)\mathbb{E}[a_{q_1}(0,k-p)a_{q_2}(0,k+\tau-q)]\\
&\quad=-\rho_{q_1q_2}\sum_p\sum_q g_j(p)g_{j'}(q)|-\tau+q-p|^{\alpha_{q_1q_2}}\\
&\qquad+\rho_{q_1q_2}\sum_p g_j(p)\sum_q g_{j'}(q)|k-p|^{\alpha_{q_1q_2}}+\rho_{q_1q_2}\sum_q g_{j'}(q)\sum_p g_j(p)|k+\tau-q|^{\alpha_{q_1q_2}}\\
&\quad=-\rho_{q_1q_2}\sum_p\sum_q g_j(p)g_{j'}(q)|\tau-q+p|^{\alpha_{q_1q_2}} \qquad (p'=p-q)\\
&\quad=-\rho_{q_1q_2}\sum_{p'}\sum_q g_j(p'+q)g_{j'}(q)|\tau-p'|^{\alpha_{q_1q_2}}=-\rho_{q_1q_2}\sum_{p'}\sum_q g_j(p'-q)\check{g}_{j'}(q)|\tau-p'|^{\alpha_{q_1q_2}}\\
&\quad=-\rho_{q_1q_2}\sum_p(g_j*\check{g}_{j'})(p)|\tau-p|^{\alpha_{q_1q_2}}=-\rho_{q_1q_2}((g_j*\check{g}_{j'})*\eta_{q_1q_2})(\tau),
\end{aligned}$$

where

$$\begin{aligned}
\check{g}_j(k)&=g_j(L-k),\quad k=1,\dots,L,\\
\eta_{q_1q_2}(\tau)&=|\tau|^{\alpha_{q_1q_2}}.
\end{aligned}$$

Consequently,

$$\mathrm{r}_{q_1q_2}(j,k;j',k')=\rho_{q_1q_2}\frac{((g_j*\check{g}_{j'})*\eta_{q_1q_2})(k'-k)}{\sqrt{((g_j*\check{g}_j)*\eta_{ii})(0)\,((g_{j'}*\check{g}_{j'})*\eta_{ll})(0)}} \tag{4.12}$$

and, for the dyadic wavelet transform,

$$\tilde{\mathrm{r}}_{q_1q_2}(j,k;j',k')=\mathrm{r}_{q_1q_2}(j,2^jk;j',2^{j'}k'). \tag{4.13}$$

For given values of $\alpha_{q_1q_2}$ and $\rho_{q_1q_2}$, these expressions can be easily evaluated numerically.

4.3.2. Practical estimation of (co)variances of $\widehat{\alpha}_{q_1q_2}$ and of $\widehat{\delta}_{q_1q_2}$

Evaluating (4.7) and (4.8) for HfBm in practice requires the unknown parameter values $\alpha_{q_1q_2}$ and $\rho_{q_1q_2}$ in (4.11), and hence we replace them by their estimates $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\rho}_{q_1q_2}$. The former are defined in (3.21), and estimates $\widehat{\rho}_{q_1q_2}$ for $\rho_{q_1q_2}$ for $q_1\neq q_2$ can be readily obtained as the cross-correlation coefficients of the first difference processes $Y_H(t)_{q_1}$ and $Y_H(t)_{q_2}$ (HfGn; see Remark 2.2). However, note that the expressions for the (co)variances of $\widehat{\alpha}_{q_1q_2}$ in the previous sections are derived assuming knowledge of the true parameter values and can only be expected to be approximations when these are replaced by estimates. This will be studied numerically in the next section.

4.3.3. Assessment of the estimated (co)variances of $\widehat{\alpha}_{q_1q_2}$ and of $\widehat{\delta}_{q_1q_2}$ by means of Monte Carlo experiments

Monte Carlo studies were conducted following the protocol and settings described in Section 3.3.3, aiming to evaluate the quality of the estimated approximations (4.7) and (4.8) for the (co)variances of $\widehat{\alpha}_{q_1q_2}$ and of $\widehat{\gamma}_{q_1q_2}$. The simulations involved 1000 independent realizations of each of two general instances of HfBm with $m=2$ components, one using the true values of the parameters $\alpha_{q_1q_2}$ and $\rho_{q_1q_2}$, and the other, their estimates $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\rho}_{q_1q_2}$. Four different sample sizes, $n=\{2^{10},2^{12},2^{14},2^{16}\}$ and three different values $\rho_{12}=\{0.3,0.6,0.9\}$ are investigated for the set of exponents $[\alpha_{11},\alpha_{22},\alpha_{12}]=[0.2,0.6,0.4]$.

Table 4 summarizes the square roots of the ratios of the averages over realizations of (co)variance estimates and of the Monte Carlo (co)variances. The first four columns, labeled "theo/MC", report results obtained when using theoretical parameter values and yield the following conclusions. First, even for small sample size $n=2^{10}$ and weak correlation $\rho_{12}=0.3$, the quality of the approximations (4.7) and (4.8) is very good for the variances of exponents α_q, $q=1,2$, and satisfactory for the cross-exponent α_{12}, the covariance parameters and the connectivity parameter δ_{12}. Second, when the sample size n and correlation level ρ_{12} increase, the approximation of variances and covariances becomes excellent, with maximum errors of the order of 5% for $n=2^{16}$ and strong correlation $\rho_{12}=0.9$. Finally, the last four columns of Table 4, labeled "est/MC", report results obtained when using estimates $\widehat{\alpha}_{q_1q_2}$ and $\widehat{\rho}_{12}$. They indicate that replacing the true parameter values $\alpha_{q_1q_2}$ and ρ_{12} with estimates has very little impact on the quality of approximations (4.7) and (4.8). Indeed, the average values of the (co)variance estimates are essentially equal to those obtained when using true parameter values.

5. Statistical test for fractal connectivity

5.1. Procedure

The mathematical and computational results in Sections 3 and 4 enable us to construct component-wise fractal connectivity tests, i.e., for the hypotheses

$$H_0:\ \delta_{q_1q_2}=\frac{\alpha_{q_1q_1}+\alpha_{q_2q_2}}{2}-\alpha_{q_1q_2}=0,\qquad q_1\neq q_2.$$

Table 4
Estimation of $\mathrm{Var}\left[\alpha_{(\cdot)}\right]$. Square roots of ratios of mean of (co)variances computed using (4.7) and of Monte Carlo (co)variances: (4.7) evaluated using theoretical values $\alpha_{11}, \alpha_{22}, \alpha_{12}, \rho_{12}$ (left columns, labeled "theo/MC") and estimates $\widehat{\alpha}_{11}, \widehat{\alpha}_{22}, \widehat{\alpha}_{12}, \widehat{\rho}_{12}$ (right columns, labeled "est/MC"). ($(\alpha_{11}, \alpha_{22}, \delta_{12}, \rho_{12}) = (0.2, 0.6, 0, \rho_{12})$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$).

n		2^{10}	2^{12}	2^{14}	2^{16}	2^{10}	2^{12}	2^{14}	2^{16}
ρ_{12}	Ratio of $\sqrt{-/-}$	theo/MC				est/MC			
0.3	$\mathrm{Var}[\widehat{\alpha}_{11}]$	0.95	0.97	0.95	0.99	0.94	0.97	0.94	0.99
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	0.82	0.85	0.90	0.93	0.83	0.86	0.90	0.93
	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$	1.12	1.07	0.97	1.32	1.09	1.06	0.97	1.31
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	0.78	0.82	0.88	0.91	0.80	0.82	0.88	0.91
0.6	$\mathrm{Var}[\widehat{\alpha}_{11}]$	0.98	0.98	0.96	0.98	0.98	0.97	0.96	0.98
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	0.94	0.92	1.00	1.00	0.93	0.92	1.00	1.00
	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$	1.00	0.93	0.97	1.05	0.99	0.92	0.97	1.05
	$\mathrm{Var}[\widehat{\delta}_{12}]$	0.88	0.87	0.98	0.97	0.88	0.87	0.98	0.97
0.9	$\mathrm{Var}[\widehat{\alpha}_{11}]$	0.97	0.95	0.98	0.98	0.96	0.95	0.98	0.98
	$\mathrm{Var}[\widehat{\alpha}_{12}]$	1.00	0.98	1.00	1.01	0.99	0.98	1.00	1.01
	$\mathrm{Cov}[\widehat{\alpha}_{11}, \widehat{\alpha}_{22}]$	1.01	0.99	1.01	1.02	1.00	0.98	1.01	1.01
	$\mathrm{Var}[\widehat{\delta}_{12}]$	0.93	0.92	0.95	0.97	0.95	0.93	0.96	0.99

Recall that we assume throughout that $\rho_{q_1 q_2} \neq 0$ for $q_1 \neq q_2$ (see (2.8)). As a consequence of Theorem 3.1, the distribution of $\widehat{\delta}_{q_1 q_2}$ under H_0 can be approximated over finite samples by

$$\widehat{\delta}_{q_1 q_2} \sim \mathcal{N}\left(0, \mathrm{Var}\left[\widehat{\delta}_{q_1 q_2}\right]\right) \text{ under } H_0,$$

where, in turn, $\mathrm{Var}\left[\widehat{\delta}_{q_1 q_2}\right]$ can be approximated by (4.8). Therefore, a simple two-sided test with significance level $\mathrm{s} = P(\text{reject } H_0 | H_0 \text{ true})$ is given by

$$d_{\mathrm{s}} = \begin{cases} 1, & \text{if } |\widehat{\delta}_{q_1 q_2}| > \sqrt{\mathrm{Var}\left[\widehat{\delta}_{q_1 q_2}\right]}\phi^{-1}(1 - \mathrm{s}/2); \\ 0, & \text{otherwise,} \end{cases} \tag{5.1}$$

where $\phi^{-1}(\cdot)$ is the inverse cumulative distribution function of the standard Normal distribution. In addition, the p-value of the test statistic (i.e., the probability of observing an absolute value at least as large as $|\widehat{\delta}_{q_1 q_2}|$ for the test statistic under H_0) is given by

$$p(|\widehat{\delta}_{q_1 q_2}|) := 2\phi\left(-|\widehat{\delta}_{q_1 q_2}| \Big/ \sqrt{\mathrm{Var}\left[\widehat{\delta}_{q_1 q_2}\right]}\right). \tag{5.2}$$

This test can be performed by evaluating (5.1) with an estimate for $\mathrm{Var}\left[\widehat{\delta}_{q_1 q_2}\right]$ obtained from the procedure detailed in Section 4.1.

5.2. Monte Carlo assessment of the test performance

We assess the performance of the test by applying it to 1000 independent realizations of HfBm with exponent values $[\alpha_{11}, \alpha_{22}] = [0.2, 0.6]$ and exponent values α_{12} detailed below for sample sizes $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$ and correlation levels $\rho_{12} = \{0.5, 0.7, 0.9\}$. For simplicity of illustration and without loss of generality, we consider again HfBm with $m = 2$ components. For each realization, the test decision (5.1) and the p-value (5.2) are evaluated using (4.8) with approximations (4.7) to obtain an estimate of the $\mathrm{Var}\left[\widehat{\delta}_{12}\right]$. Estimates of the expected values of the test decisions and p-values, denoted by $\widehat{d}_{\mathrm{s}}$ and $\widehat{p}$, are then obtained as the averages over realizations of test decisions and p-values (5.1) and (5.2).

We now compare the performance of the proposed test, denoted hereinafter HFBM (not to be confused with the stochastic process HfBm), to that of the test put forward in [26] (cf., [56] for preliminary comparative results). The latter relies on the intuition that the wavelet coherence function of two components of a multivariate Gaussian scale invariant random process approximately behaves as

$$\Gamma_{q_1 q_2}(j) = S_{n_j}^{(q_1 q_1)}(j) / \sqrt{S_{n_j}^{(q_1 q_1)}(j) S_{n_j}^{(q_2 q_2)}(j)} \simeq \rho_{q_1 q_2} 2^{j(\alpha_{q_1 q_2} - \alpha_{q_1 q_1} - \alpha_{q_2 q_2})}.$$

The test itself, denoted WCF (for wavelet coherence function), is formulated without the rigorous statistical framework developed above. Rather, it is built on the observation that $\Gamma_{q_1 q_2}(j)$ is the Pearson product-moment correlation coefficient of the time series $d_{q_1}(j, \cdot)$ and $d_{q_2}(j, \cdot)$, and hence that the Fisher's z statistics of $\Gamma_{q_1 q_2}(j)$, $j = j_1, \ldots, j_2$, are approximately Gaussian, with known variances and, in the case of fractal connectivity, with equal means across scales. The test for fractal connectivity is then formulated as the UMPI test for the equality of means of Gaussian random variables, cf. [26] for details.

5.2.1. Performance under H_0

We first consider the case that H_0 is true, i.e., $(\alpha_{11} + \alpha_{22})/2 = \alpha_{12} = 0.4$. The significance level is set to $\mathrm{s} = 0.1$, and results are reported in Table 5 for the proposed test (top) and for the test in [26] (bottom). Note that, under H_0, averages of test decisions $\widehat{d}_{\mathrm{s}}$ should equal the preset significance level s, and averages of p-values $\widehat{p}$ should equal $\frac{1}{2}$. HFBM rejects H_0 with slightly larger probability than the prescribed value $\mathrm{s} = 0.1$, yet the differences between empirical significance levels $\widehat{d}_{\mathrm{s}}$ and s never exceed 5%; similarly, average p-values are slightly below 0.5. For large sample size and large ρ_{12}, average test decisions and p-values are very close to the theoretical values $\mathrm{s} = 0.1$ and $p = \frac{1}{2}$. These remarks are consistent with the results reported in Table 4, where a small but systematic underestimation of $\mathrm{Var}\left[\widehat{\delta}_{12}\right]$ for small sample sizes and ρ_{12} is observed.

In contrast, the empirical significances $\widehat{d}_{\mathrm{s}}$ of WCF strongly differ from the preset value s by values of up to 16%, and this difference is especially pronounced for large sample sizes for which one would expect the test to perform better. One reason for this poor performance

Table 5
Test significance. Mean test decisions and p-values for different values of n and ρ_{12} under $H_0: \delta_{12} = (\alpha_{11} + \alpha_{22})/2 - \alpha_{12} = 0$ ($[\alpha_{11}, \alpha_{22}] = [0.2, 0.6]$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$) for the proposed test (top) and for the test in [26] (bottom).

HFBM – $H_0: \delta_{12} \equiv 0$, s $= 0.1$						
	$\rho_{12} = 0.5$		$\rho_{12} = 0.7$		$\rho_{12} = 0.9$	
	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$
$n = 2^{10}$	13.2	0.45	14.0	0.48	14.5	0.45
$n = 2^{12}$	13.8	0.45	10.9	0.47	11.1	0.45
$n = 2^{14}$	13.8	0.45	11.2	0.46	11.0	0.46
$n = 2^{16}$	12.3	0.46	10.9	0.48	11.0	0.47
WCF – $H_0: \delta_{12} \equiv 0$, s $= 0.1$						
	$\rho_{12} = 0.5$		$\rho_{12} = 0.7$		$\rho_{12} = 0.9$	
	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$	$100\widehat{d}_s$	$\widehat{p}$
$n = 2^{10}$	14.7	0.44	15.3	0.46	16.0	0.44
$n = 2^{12}$	20.7	0.42	16.0	0.43	17.2	0.42
$n = 2^{14}$	20.9	0.40	18.6	0.41	22.0	0.39
$n = 2^{16}$	26.1	0.36	22.5	0.36	22.1	0.38

Table 6
Test power for adjusted significance $\widehat{d}_s = 0.1$. Mean test decisions and p-values for different values of n, ρ_{12} and α_{12}/alternative hypotheses $H_1: \delta_{12} = (\alpha_{11}+\alpha_{22})/2-\alpha_{12} \neq 0$ ($[\alpha_{11}, \alpha_{22}] = [0.2, 0.6]$, $j_1 = 2$ and $j_2 = \{5, 7, 9, 11\}$, $n = \{2^{10}, 2^{12}, 2^{14}, 2^{16}\}$) for the proposed test (top) and for the test in [26] (bottom).

HFBM – $H_1: \delta_{12} \neq 0, \widehat{d}_s = 0.1$												
δ_{12}	$\rho_{12} = 0.5$				$\rho_{12} = 0.7$				$\rho_{12} = 0.9$			
	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2
$n = 2^{10}$	16.5	29.9	40.6	50.0	25.4	43.5	66.1	84.5	74.8	98.6	100.0	99.9
$n = 2^{12}$	28.8	60.8	81.9	93.3	63.6	96.8	99.8	100.0	100.0	100.0	100.0	100.0
$n = 2^{14}$	67.1	97.9	99.9	100.0	99.2	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$n = 2^{16}$	99.4	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0
WCF – $H_1: \delta_{12} \neq 0, \widehat{d}_s = 0.1$												
δ_{12}	$\rho_{12} = 0.5$				$\rho_{12} = 0.7$				$\rho_{12} = 0.9$			
	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2	0.05	0.1	0.15	0.2
$n = 2^{10}$	10.5	15.4	16.6	22.3	14.1	21.7	36.9	52.9	43.7	81.0	94.2	99.2
$n = 2^{12}$	17.8	35.9	52.5	69.4	34.7	78.9	95.8	99.4	98.2	100.0	100.0	100.0
$n = 2^{14}$	43.4	86.2	99.0	100.0	89.9	100.0	100.0	100.0	100.0	100.0	100.0	100.0
$n = 2^{16}$	93.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0	100.0

may lie in the fact that the test in [26] was designed for fGn, rather than fBm. Note that the asymptotic calculations developed above can be adapted to the easier case of fGn (and, in principle, any other Gaussian process with stationary increments) without difficulty by simply changing the covariance function $\gamma_{q_1 q_2}(s, t)$ in the calculations leading to the expressions (4.12) and (4.13).

5.2.2. Test power

We assess the power of the test under the alternative hypotheses $H_1: \delta_{12} = (\alpha_{11}+\alpha_{22})/2-\alpha_{12} \neq 0$ with $\delta_{12} = \{0.05, 0.1, 0.15, 0.2\}$. Yet, a direct power comparison of HFBM and WCF is only meaningful for identical rejection probabilities under H_0, since a test for which $\widehat{d}_s > s$ under H_0 is expected to display an artificially large power. In view of the distinct performances of HFBM and WCF, as discussed above (cf. Table 5), for each sample size and correlation level we adjusted the prescribed significance to the value $\tilde{s}$ for which the average rejection rate under H_0 equals $\widehat{d}_{\tilde{s}} = s = 0.1$. Using this adjusted level of significance $\tilde{s}$, the power of the test is then estimated as the average of the test decisions $d_{\tilde{s}}$ when H_1 is true. Results are reported in Table 6 and yield the following conclusions. First, the power of each test systematically increases with the magnitudes of the deviation from $\delta_{12} = 0$, of the correlation level ρ_{12} and of the sample size n, as expected. Second, HFBM is systematically and significantly more powerful. Indeed, it enables us to detect a non-zero value for δ_{12} up to two times as often as WCF. For instance, for the small sample size of $n = 2^{10}$ and the low correlation level of $\rho_{12} = 0.5$, it permits the detection of a deviation of 0.2 from the null value $\delta_{12} = 0$ with probability 0.5, as compared to a probability of 0.22 for the test in [26].

Overall, these results confirm that the proposed methods can be relevantly applied in the assessment of scaling and fractal connectivity in multivariate time series.

6. Conclusion

The present contribution introduces a versatile class of multivariate stochastic processes called Hadamard fractional Brownian motion (HfBm). HfBm provides a stochastic framework for scale invariance modeling within which cross-component scaling laws are not directly controlled by the scaling laws along the main diagonal. In other words, HfBm is not necessarily fractally connected.

Interestingly, the theoretical study of HfBm reveals that exact entry-wise scaling on both auto- and cross-components and departures from fractal connectivity are mathematically incompatible. In other words, there is a dichotomy in multivariate scaling modeling: either there is exact entry-wise scaling in every component combined with fractal connectivity, or departures from fractal connectivity are allowed at the price of approximate (i.e., asymptotic) scaling on the cross-components.

Our main mathematical results consist of an asymptotically normal, wavelet-based linear regression estimator for the scaling exponents, as well as asymptotically valid confidence intervals with convenient mathematical expressions. Furthermore, the Taylor expansions used in the development of the asymptotic confidence intervals lead to the construction of practical procedures for the numerical calculation of the variance of the estimates. These approximate calculations enable the study of the ubiquitous issue of the impact of neglecting intra- or inter-scale correlations amongst wavelet coefficients in the computations of variances and covariances for the estimates. We also devised an asymptotically normal hypothesis test for fractal connectivity. Again, a major feature of the designed test procedure is the fact that it can be applied to a single observed HfBm data path.

For both fractally and non-fractally connected instances, simulations demonstrate the satisfactory performance of the estimators of the scaling and fractal connectivity parameters, even for small sample size data. The estimation bias is shown to be negligible, and the variance decreases according to the inverse of the sample size. In addition, the practical computations of approximated variances and covariances of the estimates are shown to be of excellent quality, irrespective of sample size, and the Monte Carlo significance levels and powers are very close to their theoretical counterparts.

The tools developed in the present contribution pave the way for novel analysis and modeling perspectives on multivariate scaling in real-world data, in the spirit of [6]. Routines for the synthesis of HfBm, as well as for estimation, computation of confidence intervals and fractal connectivity testing will be made publicly available at the time of publication.

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Appendix. Proofs

This appendix comprises three parts, Appendix A, Appendix B and Appendix C, which contain the proofs for Sections 3.2, 3.3 and 4.1, respectively. In Appendix B and Appendix C, we assume throughout that the assumptions of Theorems 3.1 and 4.1, respectively, hold.

In the proofs, whenever convenient we will use the shorthand

$$a = a(n). \tag{.1}$$

For notational simplicity, we will assume throughout that $\sigma_q = 1$, $q = 1, \dots, m$. Since the main diagonal entries of an HfBm behave like a perturbed (univariate) fBm, throughout the appendix we only provide proofs for cross-components, i.e., when the indices q_l are pairwise distinct, $l = 1, 2, 3, 4$. Whenever convenient we will use $l = 1, 2, 3, 4$ in place of q_l, respectively, and also write

$$h_{q_l q_l} = h_l, \quad h_{q_l q_p} = h_{lp}.$$

In addition, without loss of generality it will be assumed that

$$\varrho^{(12)}(a2^j) > 0, \quad n \in \mathbb{N}, \quad j \in \mathbb{N} \tag{.2}$$

(see (3.8)), since otherwise we can consider $-\varrho^{(12)}(a2^j)$ instead (see also (3.13) and Remark C.1). We will also write log instead of $\log_2$, for visual clarity. In the proofs, C represents a generic constant whose value may change from one line to the next.

Appendix A. Section 3.2

Proof of Proposition 3.1. For simplicity, we will write $\Xi^{jj'}(\cdot)$ instead of $\Xi^{jj',a}(\cdot)$ throughout the proof.

To show (i), the change of variable $s = 2^{-j}t - k$ in (3.5) and the harmonizable representation of HfBm yield

$$\Xi^{jj'}(a(2^j k - 2^{j'} k')) = \int_{\mathbb{R}} e^{ia(2^j k - 2^{j'} k')x} f(x) \widehat{\psi}(a2^j x) \overline{\widehat{\psi}(a2^{j'} x)} dx, \tag{A.1}$$

where $f(x) := \left(|x|^{-2(h_{q_1 q_2} + 1/2)} g_{q_1 q_2}(x) \right)_{q_1, q_2 = 1, \dots, m}$. Statement (ii) is a consequence of the formula (A.1) with

$$k = k' = 0, \quad j = j'. \tag{A.2}$$

Statement (iii) follows from Lemma C.1(ii), below under condition (3.6).

Turning to (iv), establishing (3.16) is equivalent to showing that

$$\frac{2^{-(j+j')/2}}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \left(\frac{\Xi_{12}^{jj'}(a(2^j k - 2^{j'} k'))}{a^{2h_{12}}} - \Phi_{12}^{jj'}(2^j k - 2^{j'} k') \right)$$
$$+ \frac{2^{-(j+j')/2}}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi_{12}^{jj'}(2^j k - 2^{j'} k') \to 2^{-(j+j')/2} \gcd(2^j, 2^{j'}) \sum_{z=-\infty}^{\infty} \Phi_{12}^{jj'}(z \gcd(2^j, 2^{j'})), \tag{A.3}$$

$n \to \infty$.

Consider the entry $q_1 = 1$, $q_2 = 2$ of the matrix-valued function $\Xi^{jj'}$ as in (A.1). Based on a change of variable $y = ax$, recast the first term of the sum on the left-hand side of (A.3) as

$$\int_{\mathbb{R}} e^{iyz} |y|^{-(2h_{12}+1)} \rho_{12} \left\{ g_{12}\left(\frac{y}{a}\right) - 1 \right\} \widehat{\psi}(2^j y) \overline{\widehat{\psi}(2^{j'} y)} dy =: \int_{\mathbb{R}} e^{iyz} h_a(y) dy. \tag{A.4}$$

Fix $A > 0$ and denote $\widehat{\psi}^{(l)}(x) = \frac{d^l}{dx^l}(\Re\widehat{\psi}(x) + i\Im\widehat{\psi}(x))$, $l \geq 0$. By (3.1) and (3.4) and a Taylor expansion with Lagrange residual of the real and imaginary parts of $\widehat{\psi}$, there exist functions λ_1, λ_2 on $[-A, A]$ such that

$$\widehat{\psi}(x) = \left(\frac{d^{N_\psi}}{dx^{N_\psi}}\Re\widehat{\psi}(x)\Big|_{\lambda_1(x)} + i\frac{d^{N_\psi}}{dx^{N_\psi}}\Im\widehat{\psi}(x)\Big|_{\lambda_2(x)}\right)\frac{x^{N_\psi}}{N_\psi!}.$$

Therefore, and extending this reasoning to $\widehat{\psi}'(x)$, $\widehat{\psi}''(x)$,

$$|\widehat{\psi}^{(l)}(x)| = O(|x|^{N_\psi - l}), \quad x \to 0, \quad l = 0, 1, 2. \tag{A.5}$$

For h_a as in (A.4), we now show that

$$h_a, h_a', h_a'' \text{ are differentiable and } h_a(0) = h_a'(0) = h_a''(0) = 0. \tag{A.6}$$

We will only develop expressions for $y > 0$, since analogous developments hold for $y < 0$. For mathematical convenience, rewrite h_a as in (A.4) as

$$h_a(y) = y^{-(2h_{12}+1)}\rho_{12}\,\vartheta_a(y). \tag{A.7}$$

Hence,

$$h_a'(y) = \rho_{12}\{-(2h_{12}+1)y^{-(2h_{12}+2)}\vartheta_a(y) + y^{-(2h_{12}+1)}\vartheta_a'(y)\}, \tag{A.8}$$

$$h_a''(y) = \rho_{12}\{(2h_{12}+1)(2h_{12}+2)y^{-(2h_{12}+3)}\vartheta_a(y) - 2(2h_{12}+1)y^{-(2h_{12}+2)}\vartheta_a'(y) + y^{-(2h_{12}+1)}\vartheta_a''(y)\}. \tag{A.9}$$

Note that, by conditions (3.1) and (2.7),

$$2N_\psi + \varpi_0 - (2h_{12} + 1 + l) - 1 \geq 2 + \varpi_0 - 2h_{\max} - l > 0, \quad l = 0, 1, 2. \tag{A.10}$$

Then, h_a is also smooth around zero and $h_a(0) = 0$. Moreover, by (A.5) and (A.10) with $l = 0$, for fixed n and $|y| \leq a\varepsilon_0$,

$$\frac{|h_a(y)|}{y} \leq \frac{C}{y}y^{-(2h_{12}+1)}\left(\frac{y}{a}\right)^{\varpi_0}y^{2N_\psi} = \frac{C}{a^{\varpi_0}}y^{2N_\psi+\varpi_0-(2h_{12}+2)} \to 0, \tag{A.11}$$

as $y \to 0^+$, where the constant $C > 0$ does not depend on n. Similarly, by (2.6), (A.5) and (A.10) with $l = 1$,

$$\frac{|h_a'(y)|}{y} \leq \frac{C'}{y}\left\{y^{-(2h_{12}+2)}\left(\frac{y}{a}\right)^{\varpi_0}y^{2N_\psi} + y^{-(2h_{12}+1)}\left[\left(\frac{y}{a}\right)^{\varpi_0-1}\frac{1}{a}y^{2N_\psi} + \left(\frac{y}{a}\right)^{\varpi_0}y^{2N_\psi-1}\right]\right\} \leq C'\frac{y^{2N_\psi-(2h_{12}+2)+\varpi_0-1}}{a^{\varpi_0}} \to 0. \tag{A.12}$$

This proves (A.6), as desired. Next, note that

$$\frac{\partial}{\partial x}e^{itx}\psi(t) = ite^{itx}\psi(t), \quad \frac{\partial^2}{\partial x^2}e^{itx}\psi(t) = -t^2e^{itx}\psi(t)$$

and $it\psi(t)$, $t^2\psi(t) \in L^1(\mathbb{R})$ by the continuity of ψ and condition (3.2). Therefore, by the dominated convergence theorem, $\widehat{\psi}'(x) = C\int_{\mathbb{R}} e^{itx}it\psi(t)dt$, $\widehat{\psi}''(x) = C'\int_{\mathbb{R}} e^{itx}(-t^2)\psi(t)dt$ for appropriate constants $C, C' \in \mathbb{R}$. Consequently, by (3.2),

$$\max_{l=0,1,2}\sup_{x\in\mathbb{R}}|\widehat{\psi}^{(l)}(x)| \leq C\int_{\mathbb{R}}|t|^l|\psi(t)|dt < \infty. \tag{A.13}$$

So, fix $z \neq 0$. By (2.5) and (A.13),

$$\lim_{|y|\to\infty}\left|h_a(y)\right| = 0 = \lim_{|y|\to\infty}\left|h_a'(y)\right|. \tag{A.14}$$

Thus, in view of (A.4), (A.6), (A.8), (A.9), (A.10) (with $l = 2$) and (A.14), by integrating by parts twice we obtain

$$\begin{aligned}\left|\frac{\Xi_{12}^{jj'}(az)}{a^{2h_{12}}} - \Phi_{12}^{jj'}(z)\right| &= \left|\frac{1}{z^2}\int_{\mathbb{R}} e^{izy}h_a''(y)dy\right| \\ &\leq \frac{C}{z^2}\int_{\mathbb{R}}\{|y|^{-(2h_{12}+3)}|\vartheta_a(y)| + |y|^{-(2h_{12}+2)}|\vartheta_a'(y)| + |y|^{-(2h_{12}+1)}|\vartheta_a''(y)|\}dy \leq \frac{C'}{z^2},\end{aligned} \tag{A.15}$$

where the last inequality is a consequence of (2.5), (A.5) and (A.13).

Now consider the first summation term in (A.3). We proceed as in the proof of proposition 3.3, (i), in [53] to establish that

$$\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\left(\frac{\Xi_{12}^{jj'}(a(2^jk - 2^{j'}k'))}{a^{2h_{12}}} - \Phi_{12}^{jj'}(2^jk - 2^{j'}k')\right) \to 0, \quad n \to \infty. \tag{A.16}$$

We outline the main steps for the reader's convenience. By Theorem 1.8 in Jones and Jones [57], p. 10, the set of values $r \in \mathbb{Z}$ to the equation $a2^jk - a2^{j'}k' = r$, $k, k' \in \mathbb{Z}$, is given by $\gcd(a2^j, a2^{j'})\mathbb{Z} =: \mathcal{R}$. Therefore, a pair $(k, k') \in \mathbb{Z}^2$ is a solution to

$$a2^jk - a2^{j'}k' = \gcd(a2^j, a2^{j'})w \tag{A.17}$$

for some $w \in \mathbb{Z}$ if and only if it is a solution to

$$2^j k - 2^{j'} k' = \gcd(2^j, 2^{j'}) w \tag{A.18}$$

for the same w. Therefore, we can replace n with n_* in Lemmas B.2 and B.3, [53], and reexpress the first summation term on the left-hand side of (A.16) as

$$\sum_{r \in \mathcal{R} \cap B_{jj'}(n_*)} \frac{\xi_r(n_*)}{n_*} \left(\frac{\Xi_{12}^{jj'}(ar) - \Phi_{12}^{jj'}(ar)}{a^{2h_{12}}} \right). \tag{A.19}$$

In (A.19), $B_{jj'}(n_*)$ is the range for r such that the pairs (k, k') satisfying (A.18) lie in the region

$$1 \le k \le 2^{j'} n_*, \quad 1 \le k' \le 2^j n_*, \tag{A.20}$$

and $\xi_r(n_*)$ is the number of such solution pairs (k, k') given some r. Moreover, for any sufficiently large n, let $k_0 \in \{1, \dots, 2^{j'} n_*\}$ be the smallest number such that $(k_0, k'(k_0)) \in \mathbb{N}^2$ solves (A.17) (for some $w \in \mathbb{Z}$), where

$$k'(k) := \frac{2^j}{2^{j'}} k - \frac{\gcd(2^j, 2^{j'}) w}{2^{j'}}. \tag{A.21}$$

From the proof of Lemma B.2, [53], the set $\mathcal{A}$ of such solutions to (A.18) has the form

$$\mathcal{A} = \left\{ (k, k') \in \mathbb{Z}^2 : k = k_0 + \frac{2^{j'}}{\gcd(2^j, 2^{j'})} \mathbb{Z}, \quad k' \text{ is given by (A.21)} \right\}. \tag{A.22}$$

In light of (A.22), define the function $k(z) = k_0 + \frac{2^{j'}}{\gcd(2^j, 2^{j'})} z$, $z \in \mathbb{Z}$. In particular, $(k(0), k'(k(0)))$ is a solution pair for (A.18). Let $\mathbb{R} \ni x = \gcd(2^j, 2^{j'})(n_* - k_0/2^{j'})$. Then, by (A.22), $(k(\lfloor x \rfloor), k'(k(\lfloor x \rfloor)))$ is the rightmost solution for (A.17) within the first-entry range $k = 1, \dots, 2^{j'} n_*$. Moreover, given r, the number of solution pairs in the region (A.20) is $\xi_r(n_*) = \lfloor x \rfloor + 1$, where

$$\lfloor x \rfloor n_*^{-1} \to \gcd(2^j, 2^{j'}), \quad n \to \infty. \tag{A.23}$$

In addition, by (A.15), and (3.14),

$$\left| \frac{\Xi_{12}^{jj'}(ar) - \Phi_{12}^{jj'}(ar)}{a^{2h_{12}}} \right| \le \frac{C}{r^2}, \quad r \neq 0, \quad \lim_{n \to \infty} \left| \frac{\Xi_{12}^{jj'}(ar) - \Phi_{12}^{jj'}(ar)}{a^{2h_{12}}} \right| = 0, \quad r \in \mathbb{Z}. \tag{A.24}$$

By expression (A.19), the dominated convergence theorem and (A.23), the limit (A.16) holds.

Next recall that, up to a change of sign, $\Phi_{12}^{jj'}(z)$ corresponds to the cross moment of the wavelet transform of a fBm. Thus, by an analogous procedure, we also obtain

$$\frac{1}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi_{12}^{jj'}(2^j k - 2^{j'} k') \to \gcd(2^j, 2^{j'}) \sum_{z=-\infty}^{\infty} \Phi_{12}^{jj'}(z \gcd(2^j, 2^{j'})), \quad n \to \infty. \tag{A.25}$$

This establishes (A.3).

To show statement (v), first note that

$$o\left(\frac{a^{2\max\{h_{13}+h_{24}, h_{14}+h_{23}\} - 2(h_{12}+h_{34})}}{n_*} \right) \Big/ \frac{a^{(h_1+h_2+h_3+h_4) - 2(h_{12}+h_{34})}}{n_*} = o(1),$$

which follows from (3.6) and the fact that

$$2\max\{h_{13} + h_{24}, h_{14} + h_{23}\} \le h_1 + h_2 + h_3 + h_4. \tag{A.26}$$

Thus, by expression (C.56) for $\mathrm{Cov}[W_n^{(12)}(a2^j), W_n^{(34)}(a2^{j'})]$ (established in the proof of Lemma C.5),

$$\begin{aligned}
\frac{\sqrt{n_{a,j}}}{a^{\delta_{12}}} \frac{\sqrt{n_{a,j'}}}{a^{\delta_{34}}} \mathrm{Cov}\left[W_n^{(12)}(a2^j), W_n^{(34)}(a2^{j'}) \right] &= \frac{2^{\frac{j+j'}{2}} n_*}{a^{\delta_{12}+\delta_{34}}} \mathrm{Cov}\left[W_n^{(12)}(a2^j), W_n^{(34)}(a2^{j'}) \right] \\
&= \frac{2^{-\frac{(j+j')}{2}}}{[\Phi_{12}^{jj}(0)\Phi_{34}^{j'j'}(0)(1 + O(a^{-\varpi_0}))^2]} \\
&\quad \cdot \Bigg\{ \frac{a^{2(h_{13}+h_{24})}}{a^{h_1+h_2+h_3+h_4}} \frac{1}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi_{13}^{jj'}(2^j k - 2^{j'} k') \Phi_{24}^{jj'}(2^j k - 2^{j'} k') \\
&\quad + \frac{a^{2(h_{14}+h_{23})}}{a^{h_1+h_2+h_3+h_4}} \frac{1}{n_*} \sum_{k=1}^{2^{j'} n_*} \sum_{k'=1}^{2^j n_*} \Phi_{14}^{jj'}(2^j k - 2^{j'} k') \Phi_{23}^{jj'}(2^j k - 2^{j'} k') \Bigg\} + o(1).
\end{aligned} \tag{A.27}$$

By (A.26) and (A.27), statement (v) holds.

The argument for showing (vi) is an adaptation of the proof of Theorem 3.1 in [53] (see also [58], pp. 510-513; [47], p. 997; and [59], Lemma 2). For notational simplicity, we only write the proof for $m = 2$, where the entries are indexed 1 and 2, and $q_1, q_2 = 1, 2$ denote generic entries. The proof is by means of the Cramér–Wold device. Under (.2), form the vector of wavelet coefficients

$$\mathbf{Y}_n := \Big(d_1(a2^{j_1}, 1), d_2(a2^{j_1}, 1), \ldots, d_1(a2^{j_1}, n_{a,j_1}), d_2(a2^{j_1}, n_{a,j_1}); \ldots;$$

$$d_1(a2^{j_2}, 1), d_2(a2^{j_2}, 1), \ldots, d_1(a2^{j_2}, n_{a,j_2}), d_2(a2^{j_2}, n_{a,j_2})\Big) \in \mathbb{R}^{\Upsilon(n)}, \tag{A.28}$$

where

$$\Upsilon(n) := 2 \sum_{j=j_1}^{j_2} n_{a,j}. \tag{A.29}$$

Let

$$\boldsymbol{\theta} = (\boldsymbol{\theta}_{j_1}, \ldots, \boldsymbol{\theta}_{j_2}) \in \mathbb{R}^{3J}, \tag{A.30}$$

where $J = j_2 - j_1 + 1$ and $\boldsymbol{\theta}_j = (\theta_{j,1}, \theta_{j,12}, \theta_{j,2})^* \in \mathbb{R}^3, j = j_1, \ldots, j_2$. Now form the block-diagonal matrix D_n defined by

$$\operatorname{diag}\Big(\underbrace{\frac{1}{n_{a,j_1}}\sqrt{\frac{1}{2^{j_1}}}\Omega_{n,j_1}, \ldots, \frac{1}{n_{a,j_1}}\sqrt{\frac{1}{2^{j_1}}}\Omega_{n,j_1}}_{n_{a,j_1}}; \ldots; \underbrace{\frac{1}{n_{a,j_2}}\sqrt{\frac{1}{2^{j_2}}}\Omega_{n,j_2}, \ldots, \frac{1}{n_{a,j_2}}\sqrt{\frac{1}{2^{j_2}}}\Omega_{n,j_2}}_{n_{a,j_2}}\Big), \tag{A.31}$$

where

$$\Omega_{n,j} = \begin{pmatrix} \dfrac{\theta_{j,1}}{\mathbb{E}\big[d_1^2(a2^j, 0)\big]} & \dfrac{1}{a^{\delta_{12}}}\dfrac{\theta_{j,12}}{2\mathbb{E}\big[d_1(a2^j, 0)d_2(a2^j, 0)\big]} \\ \dfrac{1}{a^{\delta_{12}}}\dfrac{\theta_{j,12}}{2\mathbb{E}\big[d_1(a2^j, 0)d_2(a2^j, 0)\big]} & \dfrac{\theta_{j,2}}{\mathbb{E}\big[d_2^2(a2^j, 0)\big]} \end{pmatrix}, \quad j = j_1, \ldots, j_2. \tag{A.32}$$

In (A.32), it can be understood that the slow growth factors for the main diagonal terms are $\frac{1}{a^{2\delta_{11}}} = \frac{1}{a^{2\delta_{22}}} \equiv 1$. We would like to establish the limiting distribution of the statistic

$$\begin{aligned} T_n &= \sum_{j=j_1}^{j_2} \frac{\boldsymbol{\theta}_j^*}{\sqrt{2^j}} \operatorname{vec}_{\mathcal{S}}\Big[\Big(\frac{1}{a^{\delta_{q_1q_2}}}\Big)_{q_1,q_2=1,2} \circ W_n(a2^j)\Big] \\ &= \sum_{j=j_1}^{j_2} \frac{\boldsymbol{\theta}_j^*}{\sqrt{2^j}} \frac{1}{n_{a,j}} \Big(\sum_{k=1}^{n_{a,j}} \frac{d_1^2(a2^j, k)}{\mathbb{E}\big[d_1^2(a2^j, 0)\big]}, \frac{1}{a^{\delta_{12}}} \sum_{k=1}^{n_{a,j}} \frac{d_1(a2^j, k)d_2(a2^j, k)}{\mathbb{E}\big[d_1(a2^j, 0)d_2(a2^j, 0)\big]}, \sum_{k=1}^{n_{a,j}} \frac{d_2^2(a2^j, k)}{\mathbb{E}\big[d_2^2(a2^j, 0)\big]}\Big)^* \\ &= \mathbf{Y}_n^* D_n \mathbf{Y}_n, \end{aligned}$$

where it suffices to consider $\boldsymbol{\theta}$ in (A.30) such that

$$\boldsymbol{\theta}^* \Sigma(H) \boldsymbol{\theta} > 0 \tag{A.33}$$

(see [40], pp. 211 and 214). The matrix $\Sigma(H)$ in (A.33) is obtained from (3.17) and can be written in block form as $\Sigma(H) =: (G^{jj'})_{j,j'=j_1,\ldots,j_2}$, corresponding to block entries of the vector $\boldsymbol{\theta} = (\theta_{j_1}, \ldots, \theta_{j_2})^*$. Let $\Gamma_{\mathbf{Y}_n} = \operatorname{Cov}[\mathbf{Y}_n, \mathbf{Y}_n]$, and consider the spectral decomposition $\Gamma_{\mathbf{Y}_n}^{1/2} D_n \Gamma_{\mathbf{Y}_n}^{1/2} = O\Lambda O^*$, where Λ is diagonal with real, and not necessarily positive, eigenvalues

$$\xi_i(a2^j), \quad i = 1, \ldots, \Upsilon(n), \tag{A.34}$$

and O is an orthogonal matrix. Now let $\mathbf{Z} \sim N(0, I_{\Upsilon(n)})$. Then,

$$T_n \stackrel{d}{=} \mathbf{Z}^* \Gamma_{\mathbf{Y}_n}^{1/2} D_n \Gamma_{\mathbf{Y}_n}^{1/2} \mathbf{Z} = \mathbf{Z}^* O \Lambda O^* \mathbf{Z} \stackrel{d}{=} \mathbf{Z}^* \Lambda \mathbf{Z} =: \sum_{i=1}^{\Upsilon(n)} \xi_i(a2^j) Z_i^2.$$

Assume for the moment that

$$\max_{i=1,\ldots,\Upsilon(n)} |\xi_i(a2^j)| = o\Big(\Big(\frac{a}{n}\Big)^{1/2}\Big). \tag{A.35}$$

By (A.33) and (3.17),

$$\begin{aligned} \frac{n}{a}\operatorname{Var}[T_n] = \sum_{j=j_1}^{j_2}\sum_{j'=j_1}^{j_2} \boldsymbol{\theta}_j^* \Big\{\sqrt{\frac{n}{a2^j}}\sqrt{\frac{n}{a2^{j'}}} \operatorname{Cov}\Big[\operatorname{vec}_{\mathcal{S}}\Big[\Big(\frac{1}{a^{\delta_{q_1q_2}}}\Big)_{q_1,q_2=1,2} \circ W_n(a2^j)\Big], \\ \operatorname{vec}_{\mathcal{S}}\Big[\Big(\frac{1}{a^{\delta_{q_1q_2}}}\Big)_{q_1,q_2=1,2} \circ W_n(a2^{j'})\Big]\Big]\Big\}\boldsymbol{\theta}_{j'} \to \sum_{j=j_1}^{j_2}\sum_{j'=j_1}^{j_2} \boldsymbol{\theta}_j^* \, G^{jj'} \, \boldsymbol{\theta}_{j'} > 0. \end{aligned}$$

Therefore, there exists a constant $C > 0$ such that, for large enough n, $\frac{n}{a}\mathrm{Var}\,[T_n] \geq C > 0$. In view of condition (A.35),

$$\frac{\max_{i=1,\dots,\Upsilon(n)}|\xi_i(a2^j)|}{\sqrt{\mathrm{Var}\,[T_n]}} \leq C'\Big(\frac{n}{a}\Big)^{1/2} \max_{i=1,\dots,\Upsilon(n)}|\xi_i(a2^j)| \to 0, \quad n \to \infty.$$

The claim (3.19) is now a consequence of Lemma B.4 in [53].

So, we need to show (A.35). The first step is to establish the bound

$$\sup_{\mathbf{u}\in S^{\Upsilon(n)-1}} |\mathbf{u}^*\Gamma_{\mathbf{Y}_n}^{1/2}D_n\Gamma_{\mathbf{Y}_n}^{1/2}\mathbf{u}| \leq C \max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}}\|\Omega_{n,j}\| \sup_{\mathbf{u}\in S^{\Upsilon(n)-1}} \mathbf{u}^*\Gamma_{\mathbf{Y}_n}\mathbf{u}. \tag{A.36}$$

Let $\mathbf{u} \in S^{\Upsilon(n)-1}$ and let $\mathbf{v} = \Gamma^{1/2}\mathbf{u}$. We can break up the vector $\mathbf{v}$ into two-dimensional subvectors $v_{\cdot,\cdot}$ to reexpress $\mathbf{v} = (v_{j_1,1},\dots,v_{j_1,n_{a,j_1}};\dots;v_{j_2,1},\dots,v_{j_2,n_{a,j_2}})^*$. Based on the block-diagonal structure of D_n expressed in (A.31),

$$|\mathbf{u}^*\Gamma_{\mathbf{Y}_n}^{1/2}D_n\Gamma_{\mathbf{Y}_n}^{1/2}\mathbf{u}| = |\mathbf{v}^*D_n\mathbf{v}| = \Big|\sum_{j=j_1}^{j_2}\sum_{l=1}^{n_{a,j}} v_{j,l}^*\,\frac{\Omega_{n,j}}{n_{a,j}\sqrt{2^j}}\,v_{j,l}\Big| \leq C\sum_{j=j_1}^{j_2}\sum_{l=1}^{n_{a,j}}\frac{1}{n_{a,j}\sqrt{2^j}}\|\Omega_{n,j}\|\,\|v_{j,l}\|^2$$

$$\leq C\Big(\max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}\sqrt{2^j}}\|\Omega_{n,j}\|\Big)\sum_{j=j_1}^{j_2}\sum_{l=1}^{n_{a,j}}\|v_{j,l}\|^2 = C\Big(\max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}\sqrt{2^j}}\|\Omega_{n,j}\|\Big)\mathbf{u}^*\Gamma_{\mathbf{Y}_n}\mathbf{u}, \tag{A.37}$$

where the constant C comes from a change of matrix norms and only depends on the fixed dimension $m = 2$. By taking $\sup_{\mathbf{u}\in S^{\Upsilon(n)-1}}$ on both sides of (A.37), we arrive at (A.36).

The second step towards showing (A.35) consists of analyzing the asymptotic behavior of the right-hand side of (A.36), as $n \to \infty$. For this, we will assume the result of Lemma C.1. So, note that

$$\max_{j=j_1,\dots,j_2}\frac{1}{n_{a,j}}\|\Omega_{n,j}\| \leq C\,\frac{a}{n}\,\frac{1}{a^{2\min\{h_1,h_2,h_{12}\}}}\,\frac{1}{a^{\delta_{12}}}, \quad n \to \infty. \tag{A.38}$$

Moreover, by relation (C.14), the maximum eigenvalue of $\Gamma_{\mathbf{Y}_n}$ is bounded by $\|\Gamma_{\mathbf{Y}_n}\| \leq Ca^{2\max\{h_1,h_2\}}$, where $\|\cdot\|$ is the matrix Euclidean norm. Therefore, in view of (A.38) and (C.14), the right-hand side of (A.36) is bounded by $C\frac{a^{2(h_{\max}-h_{\min})+1}}{n}\frac{1}{a^{\delta_{12}}}$. In turn, the latter expression divided by $\sqrt{\frac{a}{n}}$ is bounded by $C\Big(\frac{a^{4(h_{\max}-h_{\min})+1}}{n}\Big)^{1/2}\frac{1}{a^{\delta_{12}}}$. By condition (3.6), this implies (A.35), and as a result, also (3.19). □

Appendix B. Section 3.3

Proof of Theorem 3.1. Fix $q_1, q_2 = 1,\dots,m$. Based on (3.22), rewrite

$$\begin{aligned}
&\frac{1}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}(\widehat{\alpha}_{q_1q_2} - \alpha_{q_1q_2})\\
&= \sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\,\log|W_n^{(q_1q_2)}(a2^j)| + \sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\Big\{\log\frac{\Big|\mathbb{E}\Big[S_n^{(q_1q_2)}(a2^j)\Big]\Big|}{a^{\alpha_{q_1q_2}}} - \log|\Phi_{q_1q_2}^{jj}(0)|\Big\}\\
&\quad + \frac{1}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\Big(\sum_{j=j_1}^{j_2} w_j\,\log|\Phi_{q_1q_2}^{jj}(0)| - \alpha_{q_1q_2}\Big).
\end{aligned} \tag{B.1}$$

By Lemma C.1(ii), and an application of the mean value theorem, for some $\theta(n) > 0$ between $\Big|\mathbb{E}\Big[S_n^{(q_1q_2)}(a2^j)\Big]\Big|/a^{\alpha_{q_1q_2}}$ and $|\Phi_{q_1q_2}^{jj}(0)|$ the second term in the sum (B.1) can be bounded in absolute value by

$$\begin{aligned}
&\sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\Big\{\frac{1}{\theta(n)}\Big|\frac{\Big|\mathbb{E}\Big[S_n^{(q_1q_2)}(a2^j)\Big]\Big|}{a^{\alpha_{q_1q_2}}} - |\Phi_{q_1q_2}^{jj}(0)|\Big|\Big\}\\
&= \sum_{j=j_1}^{j_2}\frac{w_j}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\,\frac{1}{|\Phi_{q_1q_2}^{jj}(0)| + o(1)}\,\frac{C_j}{a^{\varpi_0}} \leq C'\sqrt{\frac{n}{a^{2\delta_{q_1q_2}+1+2\varpi_0}}} \to 0,
\end{aligned}$$

as $n \to \infty$, where the limit is a consequence of condition (3.6). Also note that, after a change of variable $2^jx = y$ in the expression for $\Phi_{q_1q_2}^{jj}(0)$ (see (3.14)),

$$\Phi_{q_1q_2}^{jj}(0) = 2^{j\alpha_{q_1q_2}}\rho_{q_1q_2}\int_{\mathbb{R}}|y|^{-(\alpha_{q_1q_2}+1)}|\widehat{\psi}(y)|^2dy =: 2^{j\alpha_{q_1q_2}}c_{q_1q_2} \in \mathbb{R}.$$

Therefore, by (3.22), the third term in the sum (B.1) can be written as

$$\frac{1}{a^{\delta_{q_1q_2}}}\sqrt{\frac{n}{a}}\Big(\sum_{j=j_1}^{j_2} w_j\,(j\alpha_{q_1q_2} + \log|c_{q_1q_2}|) - \alpha_{q_1q_2}\Big) = 0.$$

So, in regard to the first term in the sum (B.1), consider the weight matrix $M \in M(\frac{m(m+1)}{2}, \frac{m(m+1)}{2}J, \mathbb{R})$ defined by

$$\left(2^{j_1/2} w_{j_1} I_{\frac{m(m+1)}{2}} \ ; 2^{(j_1+1)/2} w_{j_1+1} I_{\frac{m(m+1)}{2}} \ ; \quad \dots \quad ; 2^{j_2/2} w_{j_2} I_{\frac{m(m+1)}{2}}\right),$$

where $I_{\frac{m(m+1)}{2}}$ is a $\frac{m(m+1)}{2} \times \frac{m(m+1)}{2}$ identity matrix and J is given by (3.20). We would like to show that

$$M\left(\mathrm{vec}_{\mathcal{S}}\left[\left(\frac{\sqrt{n_{a,j}}}{a^{\delta_{q_1q_2}}}\right)_{q_1,q_2=1,\dots,m} \circ \ \log \circ |W_n(a2^j)|\right]\right)_{j=j_1,\dots,j_2} \xrightarrow{d} \mathcal{N}_{\frac{m(m+1)}{2}}(0, MGM^*), \tag{B.2}$$

where $\log \circ |A| := \left(\log |A_{q_1q_2}|\right)_{q_1,q_2=1,\dots,m}$ for any $m \times m$ real matrix A, and the term post-multiplying the matrix M in (B.2) is a $\frac{m(m+1)}{2}J$-dimensional random vector.

For any $0 < r < 1$, fix a pair q_1, q_2 and an octave j, which specifies one of the entries of the random vector on the left-hand side of (B.2). Define the set $A_n = \{\omega : \min_{q_1,q_2} W_n^{(q_1q_2)}(a2^j) > r\}$. Under (.2), Proposition 3.1, (vi), implies that $P(A_n) \to 1$ as $n \to \infty$. Thus, for large enough n, in the set A_n the mean value theorem gives the almost sure expression

$$\mathbb{R} \ni 2^{j/2} w_j \frac{\sqrt{n_{a,j}}}{a^{\delta_{q_1q_2}}} \log|W_n^{(q_1q_2)}(a2^j)| = 2^{j/2} w_j \frac{\sqrt{n_{a,j}}}{a^{\delta_{q_1q_2}}} \frac{(W_n^{(q_1q_2)}(a2^j) - 1)}{\theta_+(W_n^{(q_1q_2)}(a2^j))} \tag{B.3}$$

for a random variable $\theta_+(W_n^{(q_1q_2)}(a2^j))$ between $W_n^{(q_1q_2)}(a2^j)$ and 1. Since $W_n^{(q_1q_2)}(a2^j) \xrightarrow{P} 1$, then $\theta_+(W_n^{(q_1q_2)}(a2^j)) \xrightarrow{P} 1$. By considering (B.3) for all $1 \le q_1 \le q_2 \le m$ and $j = j_1, \dots, j_2$, relation (B.2) is now a consequence of Proposition 3.1, (vi), and Slutsky's theorem. □

Appendix C. Section 4.1

C.1. General results

Fix $q_1, q_2 = 1, \dots, m$. For $j = j_1, \dots, j_2$ and $n \in \mathbb{N}$, consider the jointly Gaussian vector

$$\mathcal{Y}_n = \Big(\frac{d_{q_1}(a2^{j_1}, 1)}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_1})}}, \dots, \frac{d_{q_1}(a2^{j_1}, n_{a,j_1})}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_1})}}, \dots, \frac{d_{q_1}(a2^{j_2}, 1)}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_2})}}, \dots, \frac{d_{q_1}(a2^{j_2}, n_{a,j_2})}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_2})}},$$
$$\frac{d_{q_2}(a2^{j_1}, 1)}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_1})}}, \dots, \frac{d_{q_2}(a2^{j_1}, n_{a,j_1})}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_1})}}, \dots, \frac{d_{q_2}(a2^{j_2}, 1)}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_2})}}, \dots, \frac{d_{q_2}(a2^{j_2}, n_{a,j_2})}{\sqrt{\varrho^{(q_1q_2)}(a2^{j_2})}}\Big)^* \in \mathbb{R}^{\Upsilon(n)} \tag{C.1}$$

(see (A.29) for the definition of $\Upsilon(n)$). Let

$$\Gamma_{\mathcal{Y}_n} = \mathbb{E}\big[\mathcal{Y}_n \mathcal{Y}_n^*\big] = O\Lambda_{\mathcal{Y}} O^* = O \ \mathrm{diag}(\lambda_{1,\mathcal{Y}}, \dots, \lambda_{\Upsilon(n),\mathcal{Y}}) \ O^*, \quad O \in O(\Upsilon(n)), \tag{C.2}$$

be the associated covariance matrix and its matrix spectral decomposition. The following lemma provides the finite-sample and asymptotic properties of the covariance structure of wavelet coefficients, both from $\Xi^{jj'}$ and $\mathcal{Y}_n$. Note that such covariance structure does not in general correspond to a multivariate stationary stochastic process when multiple octaves j are considered.

Lemma C.1. *For $j = j_1, \dots, j_2$, $q_1, q_2 = 1, \dots, m$, and $n \in \mathbb{N}$, the following statements hold.*

(i) *Consider $\Xi^{jj',a}(a(n)(2^jk - 2^{j'}k')) = \mathbb{E}\Big[D(a(n)2^j, k)D(a(n)2^{j'}, k')^*\Big]$ (see (3.12)). For $j = j'$ and $n \in \mathbb{N}$, there is a continuous spectral density $f_{\psi,n}(x)_{q_1q_2}$ such that we can write*

$$\frac{\Xi^{jj,a}_{q_1q_2}(a(n)2^j(k-k'))}{(a(n)2^j)^{2h_{q_1q_2}}} = \int_{-\pi}^{\pi} e^{i(k-k')x} f_{\psi,n}(x)_{q_1q_2} \, dx, \quad k, k' \in \mathbb{Z}. \tag{C.3}$$

Moreover,

$$|f_{\psi,n}(x)_{q_1q_2}| \le C, \quad x \in (-\pi, \pi], \tag{C.4}$$

for a constant $C > 0$ that does not depend on n.

(ii) *Let $\Phi^{jj'}_{\cdot\cdot}(z)$ be as in (3.14), $z \in \mathbb{Z}$, and fix any $0 < \xi < 1$. Then,*

$$\left|\frac{\Xi^{jj',a}_{q_1q_2}(za(n))}{a(n)^{2h_{q_1q_2}}} - \Phi^{jj'}_{q_1q_2}(z)\right| \le \frac{C}{a(n)^{\varpi_0}}, \quad n \to \infty, \tag{C.5}$$

where ϖ_0 and β are defined in expressions (2.6) and (3.3), respectively.

(iii) *Let $\mathcal{Y}_n$ be as in (C.1), and let $\lambda_{i,\mathcal{Y}}$, $i = 1, \dots, \Upsilon(n)$, be the eigenvalues of the covariance matrix $\Gamma_{\mathcal{Y}_n}$ (see (A.29) and (C.2)). Then, for some $C > 0$,*

$$\max_{i=1,\dots,\Upsilon(n)} \lambda_{i,\mathcal{Y}} \le Ca(n)^{2(\max\{h_{q_1}, h_{q_2}\} - h_{q_1q_2})}. \tag{C.6}$$

Proof. We will use the shorthand $q_1 = 1$ and $q_2 = 2$ throughout the proof.

We first show (i). By making the change of variable $y = a2^j x$ in (A.1), we obtain (C.3) with

$$f_{\psi,n}(x)_{12} := \sum_{l=-\infty}^{\infty} \frac{|\widehat{\psi}(x+2\pi l)|^2}{|x+2\pi l|^{2h_{12}+1}}\ \rho_{12} g_{12}\Big(\frac{x+2\pi l}{a2^j}\Big). \tag{C.7}$$

Moreover, by (3.1)–(3.3), $\widehat{\psi}$ is continuous, and hence, by (2.5), (3.3), (3.4) and the dominated convergence theorem, the periodic function $f_{\psi,n}(x)_{12}$ in (C.7) is also continuous on $[-\pi,\pi]$, and hence, bounded. This shows (C.3) and (C.4).

We now turn to (ii). It suffices to show that, for any $0<\xi<1$,

$$\Big|\frac{\Xi_{12}^{jj',a}(za)}{a^{2h_{12}}} - \Phi_{12}^{jj'}(z)\Big| \le \frac{C}{a^{\min\{\varpi_0,(1-\xi)(2h_{12}+2\beta)\}}}, \quad n\to\infty, \tag{C.8}$$

since for small enough ξ, conditions (2.7) and (3.3) imply that

$$(1-\xi)(2h_{12}+2\beta) > 2h_{\min}+2 \ge \varpi_0.$$

In fact, by (A.1), (3.14) and a change of variable $y = ax$,

$$\begin{aligned}
&\frac{\Xi_{12}^{jj',a}(za)}{a^{2h_{12}}} - \Phi_{12}^{jj'}(z)\\
&= \frac{1}{a^{2h_{12}}}\int_{\mathbb{R}} e^{izax}|x|^{-(2h_{12}+1)}\rho_{12}\{g_{12}(x)-1\}\widehat{\psi}(a2^jx)\overline{\widehat{\psi}(a2^{j'}x)}dx\\
&= \Big\{\int_{|y|\le a^{1-\xi}} + \int_{|y|>a^{1-\xi}}\Big\} e^{izy}|y|^{-(2h_{12}+1)}\rho_{12}\Big\{g_{12}\Big(\frac{y}{a}\Big)-1\Big\}\widehat{\psi}(2^jy)\overline{\widehat{\psi}(2^{j'}y)}dy
\end{aligned} \tag{C.9}$$

for any $0<\xi<1$. For large enough n, by (2.6), (2.7) and (3.1)–(3.3), the absolute value of the first integral in the sum (C.9) can be bounded by

$$\frac{C}{a^{\varpi_0}}\int_{|y|\le a^{1-\xi}} |y|^{-(2h_{12}+1)+\varpi_0}|\widehat{\psi}(2^jy)\overline{\widehat{\psi}(2^{j'}y)}|dy \le \frac{C'}{a^{\varpi_0}}. \tag{C.10}$$

On the other hand, by (3.2) the absolute value of the second term in the sum (C.9) can be bounded by

$$C\int_{|y|>a^{1-\xi}} |y|^{-(2h_{12}+1)-2\beta}dy \le \frac{C'}{a^{(1-\xi)(2h_{12}+2\beta)}}. \tag{C.11}$$

Expressions (C.10) and (C.11) yield (C.5).

We turn to (iii). For notational simplicity, consider only two octaves j, j', whence $J = j_2 - j_1 + 1 = 2$. Then, from (A.29),

$$\Upsilon(n) = 2(n_{a,j_1} + n_{a,j_2}).$$

Fix $\mathbf{v}\in\mathbb{C}^{\Upsilon(n)}$. For notational simplicity, divide the summation range $k = 1,\dots,\Upsilon(n)$ into the subranges

$$\begin{aligned}
&K_1 = \{1,\dots,n_{a,j}\}, \quad K_2 = \{n_{a,j}+1,\dots,n_{a,j}+n_{a,j'}\},\\
&K_3 = \{n_{a,j}+n_{a,j'}+1,\dots,2n_{a,j}+n_{a,j'}\}, \quad K_4 = \{2n_{a,j}+n_{a,j'}+1,\dots,2(n_{a,j}+n_{a,j'})\}.
\end{aligned}$$

Define the octave and index functions

$$j(k) = j1_{\{K_1\cup K_3\}}(k) + j'1_{\{K_2\cup K_4\}}(k), \quad q(k) = 1\ 1_{\{K_1\cup K_2\}}(k) + 2\ 1_{\{K_3\cup K_4\}}(k),$$

respectively, which reflects the order of appearance of different octave and index values in the vector (C.1). By (A.1),

$$\begin{aligned}
\mathbf{v}^*\Gamma_{\mathfrak{Y}_n}\mathbf{v} &= \sum_{k_1=1}^{\Upsilon(n)}\sum_{k_2=1}^{\Upsilon(n)} v_{k_1}\overline{v}_{k_2}\Big(\Gamma_{\mathfrak{Y}_n}\Big)_{k_1,k_2}\\
&= \int_{\mathbb{R}}\sum_{k_1=1}^{\Upsilon(n)}\sum_{k_2=1}^{\Upsilon(n)} \frac{v_{k_1}\overline{v}_{k_2}e^{ia(2^{j(k_1)}k_1-2^{j(k_2)}k_2)x}\widehat{\psi}(a2^{j(k_1)}x)\overline{\widehat{\psi}(a2^{j(k_2)}x)}}{\sqrt{\varrho^{(12)}(a2^{j(k_1)})\varrho^{(12)}(a2^{j(k_2)})}} f_{q(k_1)q(k_2)}(x)dx.
\end{aligned} \tag{C.12}$$

By expanding the double summation in (C.12) into each pair in the Cartesian product $\{K_1,K_2,K_3,K_4\}^2$, we end up with 16 double summation terms under the sign of the integral, i.e., 8 pairs of conjugates. To the cross terms, namely, those involving distinct summation ranges in the index k, we can apply the elementary inequality $|x\overline{y}+\overline{x}y| \le |x|^2+|y|^2$. One such pair, associated with the summation regions $K_2\times K_3$ and $K_3\times K_2$, is

$$\begin{aligned}
&\Big|\int_{\mathbb{R}}\Big(\sum_{k_1\in K_3}\frac{v_{k_1}e^{ia2^jk_1x}\widehat{\psi}(a2^jx)}{\sqrt{\varrho^{(12)}(a2^j)}}\sum_{k_2\in K_2}\frac{\overline{v}_{k_2}e^{-ia2^{j'}k_2x}\overline{\widehat{\psi}(a2^{j'}x)}}{\sqrt{\varrho^{(12)}(a2^{j'})}}\\
&\quad+\sum_{k_1\in K_2}\frac{v_{k_1}e^{ia2^{j'}k_1x}\widehat{\psi}(a2^{j'}x)}{\sqrt{\varrho^{(12)}(a2^{j'})}}\sum_{k_2\in K_3}\frac{\overline{v}_{k_2}e^{-ia2^jk_2x}\overline{\widehat{\psi}(a2^jx)}}{\sqrt{\varrho^{(12)}(a2^j)}}\Big)f_{12}(x)dx\Big|\\
&\le \int_{\mathbb{R}}\Big\{\Big|\sum_{k\in K_3}\frac{v_ke^{ia2^jkx}\widehat{\psi}(a2^jx)}{\sqrt{\varrho^{(12)}(a2^j)}}\Big|^2 + \Big|\sum_{k\in K_2}\frac{\overline{v}_ke^{-ia2^{j'}kx}\overline{\widehat{\psi}(a2^{j'}x)}}{\sqrt{\varrho^{(12)}(a2^{j'})}}\Big|^2\Big\}|f_{12}(x)|dx,
\end{aligned}$$

since $f_{12}(x) = f_{21}(x)$, and analogous bounds hold for the remaining terms. Therefore, (C.12) is bounded by

$$\int_{\mathbb{R}} \Big| \sum_{k\in K_1} v_k e^{ia2^j kx} \Big|^2 \frac{|\widehat{\psi}(a2^j x)|^2}{\varrho^{(12)}(a2^j)} (f_{11}(x) + 3|f_{12}(x)|)dx + \int_{\mathbb{R}} \Big| \sum_{k\in K_2} v_k e^{ia2^{j'} kx} \Big|^2 \frac{|\widehat{\psi}(a2^{j'} x)|^2}{\varrho^{(12)}(a2^{j'})} (f_{11}(x) + 3|f_{12}(x)|)dx$$

$$+ \int_{\mathbb{R}} \Big| \sum_{k\in K_3} v_k e^{ia2^j kx} \Big|^2 \frac{|\widehat{\psi}(a2^j x)|^2}{\varrho^{(12)}(a2^j)} (f_{22}(x) + 3|f_{12}(x)|)dx + \int_{\mathbb{R}} \Big| \sum_{k\in K_4} v_k e^{ia2^{j'} kx} \Big|^2 \frac{|\widehat{\psi}(a2^{j'} x)|^2}{\varrho^{(12)}(a2^{j'})} (f_{22}(x) + 3|f_{12}(x)|)dx.$$

By the change of variable $y = a2^{j(k)}x$ in each integral, breaking them up (in $\mathbb{R}$) into subregions of length 2π and using the periodicity of Fourier sums, we obtain

$$\begin{aligned}
&\int_{-\pi}^{\pi} \Big| \sum_{k\in K_1} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_1} |y + 2\pi l|^{-(2h_1+1)} \rho_1 g_1\Big(\frac{y+2\pi l}{a2^j}\Big) \\
&\quad + 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^j}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^j)} dy \\
&\quad + \int_{-\pi}^{\pi} \Big| \sum_{k\in K_2} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_1} |y + 2\pi l|^{-(2h_1+1)} \rho_1 g_1\Big(\frac{y+2\pi l}{a2^{j'}}\Big) \\
&\quad + 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^{j'}}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^{j'})} dy \\
&\quad + \int_{-\pi}^{\pi} \Big| \sum_{k\in K_3} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_2} |y + 2\pi l|^{-(2h_2+1)} \rho_2 g_2\Big(\frac{y+2\pi l}{a2^j}\Big) \\
&\quad + 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^j}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^j)} dy \\
&\quad + \int_{-\pi}^{\pi} \Big| \sum_{k\in K_4} v_k e^{iky} \Big|^2 \sum_{l=-\infty}^{\infty} \Big\{ (a2^j)^{2h_2} |y + 2\pi l|^{-(2h_2+1)} \rho_2 g_2\Big(\frac{y+2\pi l}{a2^{j'}}\Big) \\
&\quad + 3(a2^j)^{2h_{12}} |y + 2\pi l|^{-(2h_{12}+1)} |\rho_{12}| \Big| g_{12}\Big(\frac{y+2\pi l}{a2^{j'}}\Big)\Big| \Big\} \frac{|\widehat{\psi}(y+2\pi l)|^2}{\varrho^{(12)}(a2^{j'})} dy \\
&\le C a^{2(\max\{h_1,h_2\}-h_{12})} \int_{-\pi}^{\pi} \Big\{ \Big| \sum_{k\in K_1} v_k e^{iky} \Big|^2 + \Big| \sum_{k\in K_2} v_k e^{iky} \Big|^2 + \Big| \sum_{k\in K_3} v_k e^{iky} \Big|^2 + \Big| \sum_{k\in K_4} v_k e^{iky} \Big|^2 \Big\} dy \\
&= C a^{2(\max\{h_1,h_2\}-h_{12})} \mathbf{v}^* \mathbf{v}
\end{aligned} \tag{C.13}$$

for some $C > 0$. The inequality (C.13) is a consequence of (C.3) (from Lemma C.1(i)) and Proposition 3.1(iii), as applied to $\varrho^{(12)}(a2^{j(k)})$. Thus, the claim (C.6) holds. □

Remark C.1. Fix $q_1, q_2 = 1, \dots, m$, and recall that $\rho^{(q_1q_2)}(a(n)2^j) = \Xi^{jj,a}_{q_1q_2}(0)$ (see (3.13)). Condition (2.8), the expression for the constant $\Phi^{jj}_{q_1q_2}(0)$ (see the right-hand side of (3.14)) and Lemma C.1(ii), imply that $\rho^{(q_1q_2)}(a(n)2^j)$ is bounded away from zero for large n.

Remark C.2. For $q_1 = 1$ and $q_2 = 2$, let $\mathbf{Y}_n$ and $\mathcal{Y}_n$ be as in (A.28) and (C.1), respectively. Then,

$$\begin{aligned}
\mathbf{Y}_n &= \mathcal{P}_n \mathrm{diag}\Big(\underbrace{\sqrt{\varrho^{(12)}(a2^{j_1})}, \dots, \sqrt{\varrho^{(12)}(a2^{j_1})}}_{n_{a,j_1}}, \dots, \underbrace{\sqrt{\varrho^{(12)}(a2^{j_2})}, \dots, \sqrt{\varrho^{(12)}(a2^{j_2})}}_{n_{a,j_2}}, \\
&\quad \underbrace{\sqrt{\varrho^{(12)}(a2^{j_1})}, \dots, \sqrt{\varrho^{(12)}(a2^{j_1})}}_{n_{a,j_1}}, \dots, \underbrace{\sqrt{\varrho^{(12)}(a2^{j_2})}, \dots, \sqrt{\varrho^{(12)}(a2^{j_2})}}_{n_{a,j_2}} \Big) \mathcal{Y}_n \\
&=: \mathcal{P}_n N_n \mathcal{Y}_n
\end{aligned}$$

for some permutation matrix $\mathcal{P}_n$. Moreover, since $\Gamma_{\mathbf{Y}_n} = \mathcal{P}_n N_n \Gamma_{\mathcal{Y}_n} N_n \mathcal{P}_n^*$ is a real symmetric matrix, by Lemma C.1(ii) and (iii), we obtain the bound

$$\|\Gamma_{\mathbf{Y}_n}\| = \|\mathcal{P}_n N_n \Gamma_{\mathcal{Y}_n} N_n \mathcal{P}_n^*\| \le C a(n)^{2\max\{h_1,h_2\}} \tag{C.14}$$

for the maximum eigenvalue of $\Gamma_{\mathbf{Y}_n}$, where $\|\cdot\|$ is the matrix Euclidean norm.

For any $n \in \mathbb{N}$, consider the Gaussian vector $\mathcal{Y}_n$ as in (C.1) but with only one octave j. It will be convenient to reexpress the sum $W_n^{(12)}(a2^j)$ as in (3.10) (with $q_1 = 1$ and $q_2 = 2$) based on a quadratic form. Define the permutation matrix

$$\Pi_n = \begin{pmatrix} \mathbf{0} & I_{n_{a,j}} \\ I_{n_{a,j}} & \mathbf{0} \end{pmatrix} \in O(2n_{a,j}). \tag{C.15}$$

Consider the spectral decomposition (C.2). Then,

$$W_n^{(12)}(a2^j) = \frac{1}{2n_{a,j}} \mathcal{Y}_n^* \Pi_n \mathcal{Y}_n \quad \text{(for one octave } j\text{)}$$

$$\stackrel{d}{=} \frac{1}{n_{a,j}} \mathbf{Z}^* \frac{(O\Lambda_{\mathcal{Y}}^{1/2}O^*)\Pi_n(O\Lambda_{\mathcal{Y}}^{1/2}O^*)}{2} \mathbf{Z} \stackrel{d}{=} \frac{1}{n_{a,j}} \mathbf{Z}^* \frac{\Lambda_{\mathcal{Y}}^{1/2}O^*\Pi_n O\Lambda_{\mathcal{Y}}^{1/2}}{2} \mathbf{Z}, \tag{C.16}$$

where $\mathbf{Z} \sim \mathcal{N}_{2n_{a,j}}(0, I_{2n_{a,j}})$. Let

$$S_n = \frac{\Lambda_{\mathcal{Y}}^{1/2}O^*\Pi_n O\Lambda_{\mathcal{Y}}^{1/2}}{2}, \tag{C.17}$$

which is a real symmetric matrix. Write out its spectral decomposition

$$S_n = O_S \Lambda_S O_S^*, \quad O \in O(2n_{a,j}), \quad \Lambda_S = \operatorname{diag}(\lambda_{1,S}, \ldots, \lambda_{2n_{a,j},S}). \tag{C.18}$$

Expression (C.16) becomes

$$W_n^{(12)}(a2^j) = \frac{1}{n_{a,j}} \mathbf{Z}^* O_S \Lambda_S O_S^* \mathbf{Z} \stackrel{d}{=} \frac{1}{n_{a,j}} \mathbf{Z}^* \Lambda_S \mathbf{Z} = \sum_{i \in i_+(n)} \frac{\lambda_{i,S}}{n_{a,j}} Z_i^2 + \sum_{i \in i_-(n)} \frac{\lambda_{i,S}}{n_{a,j}} Z_i^2 =: \sum_{i \in i_+(n)} \eta_{i,n} Z_i^2 - \sum_{i \in i_-(n)} \eta_{i,n} Z_i^2, \tag{C.19}$$

where $i_+(n)$ and $i_-(n)$ are the indices for which $\lambda_{i,S}$ is nonnegative or negative, respectively. In particular, note that

$$\operatorname{Var}\left[W_n^{(12)}(a(n)2^j) \right] = \|\boldsymbol{\eta}_n\|_2^2,$$

where

$$\boldsymbol{\eta}_n := (\eta_{1,n}, \ldots, \eta_{2n_{a,j},n})^* \tag{C.20}$$

and $\eta_{i,n}$, $i = 1, \ldots, 2n_{a,j}$, are as in expression (C.19).

Remark C.3. The following bounds on $\|\boldsymbol{\eta}_n\|_2$ and $\|\boldsymbol{\eta}_n\|_\infty$ will be useful throughout this section.

As a consequence of expression (C.56) and the fact that $\frac{a^{\delta_{12}}}{\sqrt{n_{a,j}}} \to 0$ under (3.6),

$$\frac{\|\boldsymbol{\eta}_n\|_2}{a(n)^{\delta_{12}}} = \sqrt{\mathbb{E}\left[\left(\frac{W_n^{(12)}(a(n)2^j) - 1}{a(n)^{\delta_{12}}} \right)^2 \right]} \sim \sqrt{\frac{a(n)}{n}}\, C, \quad n \to \infty, \tag{C.21}$$

for some $C \geq 0$. In particular,

$$W_n^{(12)}(a2^j) \stackrel{L^2(P)}{\longrightarrow} 1, \quad n \to \infty. \tag{C.22}$$

Moreover, expression (C.6) in Lemma C.1(iii), implies that

$$\|\boldsymbol{\eta}_n\|_\infty = \frac{\|\Lambda_S\|}{n_{a,j}} = \frac{1}{n_{a,j}} \|\Lambda_{\mathcal{Y}}^{1/2}O^*\Pi_n O\Lambda_{\mathcal{Y}}^{1/2}\| \leq \frac{\|\Lambda_{\mathcal{Y}}^{1/2}\|^2}{n_{a,j}} = \max_{i=1,\ldots,\Upsilon(n)} \frac{\lambda_{i,\mathcal{Y}}}{n_{a,j}} \leq C' \frac{a(n)^{2(\max\{h_1,h_2\}-h_{12})+1}}{n} \tag{C.23}$$

for some $C' \in \mathbb{R}$.

Remark C.4. Note that any moment of $\frac{W_n^{(12)}(a(n)2^j)-1}{a^{\delta_{12}}}$ is bounded in n, i.e.,

$$\left| \mathbb{E}\left[\frac{W_n^{(12)}(a(n)2^j) - 1}{a(n)^{\delta_{12}}} \right]^\kappa \right| = O(1), \quad \kappa \in \mathbb{N}. \tag{C.24}$$

In fact, for even $\kappa \in \mathbb{N}$, expressions (C.19) and (C.23) imply that the left-hand side of (C.24) is bounded by

$$\sum_{i_1,\ldots,i_\kappa \in i_+(n) \cup i_-(n)} \frac{\eta_{i_1,n} \ldots \eta_{i_\kappa,n}}{a(n)^{\kappa\delta_{12}}} \left| \mathbb{E}\left[(Z_{i_1}^2 - 1) \ldots (Z_{i_\kappa}^2 - 1) \right] \right| \leq \frac{C(\mathbb{E}\left[(Z_1^2 - 1)^2\right])^{\kappa/2}}{a(n)^{\kappa\delta_{12}}} \sum_{i_1,\ldots,i_{\kappa/2} \in i_+(n) \cup i_-(n)} \eta_{i_1,n}^2 \ldots \eta_{i_{\kappa/2},n}^2$$

$$\leq \frac{C(\mathbb{E}\left[(Z_1^2 - 1)^2\right])^{\kappa/2}}{a(n)^{\kappa\delta_{12}}} n_{a,j}^{\kappa/2} \|\boldsymbol{\eta}_n\|_\infty^\kappa \leq \frac{C'}{a(n)^{\kappa\delta_{12}}} \left(\frac{n}{a}\right)^{\kappa/2} \left(\frac{a^{2(\max\{h_1,h_2\}-h_{12})+1}}{n} \right)^\kappa = \frac{C'}{a(n)^{\kappa\delta_{12}}} \left(\frac{a^{4(\max\{h_1,h_2\}-h_{12})+1}}{n} \right)^{\kappa/2} = O(1),$$

where the last equality is a consequence of (3.6) (a similar derivation holds for odd κ). Expression (C.24) will be used in the proof of Lemma C.4.

The following lemma provides a concentration inequality for centered quadratic forms, and corresponds to Lemma 1 in [60] (see also [61], Lemma 8, and [62], p. 39). It will be used in the ensuing Lemma C.3 to establish a bound on the rate of convergence to zero of the probabilities $P(W_n^{(12)}(a2^j) \leq r)$ and $P(W_n^{(12)}(a(n)2^j) \geq r')$, where $r < 1/2 < 3/2 < r'$ (under (.2)).

Lemma C.2 ([60]). *Let $Z_1, \dots, Z_n \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1)$ and $\eta_1, \dots, \eta_n \geq 0$, not all zero. Let $\|\boldsymbol{\eta}\|_2$ and $\|\boldsymbol{\eta}\|_\infty$ be the Euclidean square and sup norms of the vector $\boldsymbol{\eta} = (\eta_1, \dots, \eta_n)^*$. Also, define the random variable $X = \sum_{i=1}^n \eta_{i,n}(Z_i^2 - 1)$. Then, for every $x > 0$,*

$$P(X \geq 2\|\boldsymbol{\eta}\|_2 \sqrt{x} + 2\|\boldsymbol{\eta}\|_\infty x) \leq \exp(-x), \tag{C.25}$$

$$P(X \leq -2\|\boldsymbol{\eta}\|_2 \sqrt{x}) \leq \exp(-x). \tag{C.26}$$

Lemma C.3. *Let $W_n^{(12)}(a2^j)$ be as in (C.19), and fix $r < 1/2 < 3/2 < r'$. Then, for any $0 < \xi < 1$,*

$$P(W_n^{(12)}(a(n)2^j) \leq r) \leq \exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}, \tag{C.27}$$

$$P(W_n^{(12)}(a(n)2^j) \geq r') \leq \exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}, \tag{C.28}$$

for large enough n.

Proof. Expression (C.19) implies that

$$P(W_n^{(12)}(a2^j) \leq r) = P\Big(\sum_{i\in i_+(n)} \eta_{i,n}(Z_i^2 - 1) \leq -1 + r + \sum_{i\in i_-(n)} \eta_{i,n}(Z_i^2 - 1)\Big).$$

For notational simplicity, let $X_{n_{a,j}} = \sum_{i\in i_+(n)} \eta_{i,n}(Z_i^2 - 1)$ and $Y_{n_{a,j}} = \sum_{i\in i_-(n)} \eta_{i,n}(Z_i^2 - 1)$, which are zero mean random variables. Let

$$\boldsymbol{\eta}_{1,n} = (\eta_{i,n})_{i\in i_+(n)}, \quad \boldsymbol{\eta}_{2,n} = (\eta_{i,n})_{i\in i_-(n)}.$$

By (C.21),

$$\max\{\|\boldsymbol{\eta}_{1,n}\|_2^2, \|\boldsymbol{\eta}_{2,n}\|_2^2\} \leq \|\boldsymbol{\eta}_n\|_2^2 \sim C\frac{a^{2(h_1+h_2)-4h_{12}+1}}{n},$$
$$\max\{\|\boldsymbol{\eta}_{1,n}\|_\infty, \|\boldsymbol{\eta}_{2,n}\|_\infty\} \leq \|\boldsymbol{\eta}_n\|_\infty \leq \|\boldsymbol{\eta}_n\|_2, \tag{C.29}$$

for a constant $C \geq 0$. Moreover, in view of (.2), $\|\boldsymbol{\eta}_{1,n}\|_2 > 0$, $n \in \mathbb{N}$. Suppose, without loss of generality, that

$$\|\boldsymbol{\eta}_{2,n}\|_2 > 0, \quad n \in \mathbb{N}$$

(otherwise, $Y_{n_{a,j}} = 0$ a.s. for n such that $\|\boldsymbol{\eta}_{2,n}\|_2 = 0$). Fix a constant $r < \zeta < 1$. By the independence of $X_{n_{a,j}}$ and $Y_{n_{a,j}}$,

$$P(X_{n_{a,j}} \leq -1 + r + Y_{n_{a,j}}) = \Big\{\int_{-\infty}^{1-\zeta} + \int_{1-\zeta}^{\infty}\Big\} P(X_{n_{a,j}} \leq -1 + r + y) f_{Y_{n_{a,j}}}(y)dy, \tag{C.30}$$

where $f_{Y_{n_{a,j}}}$ is the density function of $Y_{n_{a,j}}$. The first integral in (C.30) is bounded from above by $P(X_{n_{a,j}} \leq -\zeta + r)P(Y_{n_{a,j}} \leq 1 - \zeta)$. Moreover, since $-\zeta + r < 0$, then

$$\begin{aligned} P(X_{n_{a,j}} \leq -\zeta + r) &= P\Big(\sum_{i\in i_+(n)} \eta_{i,n}(Z_i^2 - 1) \leq -\zeta + r\Big) \\ &= P\Big(\sum_{i\in i_+(n)} \eta_{i,n}(Z_i^2 - 1) \leq -2\|\boldsymbol{\eta}_{1,n}\|_2 \sqrt{\frac{(\zeta - r)^2}{4\|\boldsymbol{\eta}_{1,n}\|_2^2}}\Big) \leq \exp\Big\{-\frac{(\zeta - 1/2)^2}{4\|\boldsymbol{\eta}_{1,n}\|_2^2}\Big\} \\ &\leq \exp\Big\{-C\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big\}. \end{aligned} \tag{C.31}$$

In (C.31), C does not depend on r and the last two inequalities are a consequence of (C.26) and (C.29), respectively, where

$$2\max\{h_{q_1} + h_{q_2}, 2h_{q_1q_2}\} = 2(h_{q_1} + h_{q_2}), \quad q_1, q_2 = 1, \dots, m, \tag{C.32}$$

stems from (2.10). On the other hand, for any $0 < \xi < 1$ and large enough n the second integral in (C.30) is bounded from above by

$$\int_{1-\zeta}^{\infty} f_{Y_{n_{a,j}}}(y)dy = P(Y_{n_{a,j}} > 1 - \zeta) = P\Big(\sum_{i\in i_-(n)} \eta_{i,n}(Z_i^2 - 1) > 1 - \zeta\Big). \tag{C.33}$$

Note that, by (C.29),

$$\|\boldsymbol{\eta}_{2,n}\|_2\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{\frac{1-\xi}{2}} + \|\boldsymbol{\eta}_{2,n}\|_\infty\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi} \to 0,$$

for any $0 < \xi < 1$, as $n \to \infty$. Consequently, for large enough n, (C.33) is bounded by

$$\begin{aligned} &P\Big(\sum_{i\in i_-(n)} \eta_{i,n}(Z_i^2 - 1) > 2\|\boldsymbol{\eta}_{2,n}\|_2\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{\frac{1-\xi}{2}} + 2\|\boldsymbol{\eta}_{2,n}\|_\infty\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big) \\ &\quad \leq \exp\Big\{-\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}, \end{aligned} \tag{C.34}$$

where the last inequality follows from (C.25). From (C.31) and (C.34), since C does not depend on r, we obtain (C.28).

We now turn to (C.28). Fix $\frac{1}{2} < \zeta' < 1 < r'$. Then,

$$P(W_n^{(12)}(a2^j) \geq r') = P(X_{n_{a,j}} \geq r' - 1 + Y_{n_{a,j}})$$
$$= \left\{ \int_{-\infty}^{\zeta'-1} + \int_{\zeta'-1}^{\infty} \right\} P(X_{n_{a,j}} \geq r' - 1 + y) f_{Y_{n_{a,j}}}(y) dy. \tag{C.35}$$

The first and second integral terms in (C.35) are bounded by, respectively,

$$\int_{-\infty}^{\zeta'-1} f_{Y_{n_{a,j}}}(y) dy = P(Y_{n_{a,j}} \leq \zeta' - 1)$$

and

$$P(X_{n_{a,j}} \geq r' + \zeta' - 2),$$

where $\zeta' - 1 < 0 < r' + \zeta' - 2$. Therefore, an argument similar to that for (C.27) can be applied to conclude that (C.28) also holds. □

C.2. Asymptotic covariances

We will establish Theorem 4.1 at the end of this section, after proving Lemmas C.4–C.7. The latter establish the asymptotic behavior of the first four moments of the $\mathbb{R}$-valued random variables $W_n^{(q_1q_2)}(a(n)2^j)$, $W_n^{(q_3q_4)}(a(n)2^j)$. So, consider the function

$$\log |S_n^{(q_1q_2)}(a(n)2^j)| \, 1_{\{|W_n^{(q_1q_2)}(a(n)2^j)|>r\}}, \qquad 1 \leq q_1 \leq q_2 \leq m, \quad 0 < r < \frac{1}{2}, \tag{C.36}$$

where the truncation works as a regularization of the log function around the origin. In the event $|W_n^{(q_1q_2)}(a(n)2^j)| \leq r$, we interpret that

$$\log |S_n^{(q_1q_2)}(a(n)2^j)| \, 1_{\{|W_n^{(q_1q_2)}(a(n)2^j)|>r\}} = 0 \quad \text{a.s.}$$

Throughout this section, we will make use of the Isserlis theorem (e.g., [63]). For a zero mean, Gaussian random vector $\mathbf{Z} = (Z_1, \ldots, Z_m)^* \in \mathbb{R}^m$, the theorem states that

$$\mathbb{E}[Z_1 \ldots Z_{2k}] = \sum \prod \mathbb{E}\big[Z_i Z_j\big], \quad \mathbb{E}[Z_1 \ldots Z_{2k+1}] = 0, \quad k = 1, \ldots, \lfloor m/2 \rfloor. \tag{C.37}$$

The notation $\sum \prod$ stands for adding over all possible k-fold products of pairs $\mathbb{E}\big[Z_i Z_j\big]$, where the indices partition the set $1, \ldots, 2k$.

The following lemma shows that the high order centered cross moments of $W_n^{(12)}(a(n)2^j)$, $W_n^{(34)}(a(n)2^j)$ are negligible with respect to their low order counterparts.

Lemma C.4. *Let $\kappa_1, \kappa_2 \in \mathbb{N} \cup \{0\}$, $\kappa_1 + \kappa_2 \geq 3$, and fix $0 < r < 1/2$. Then, as $n \to \infty$,*

$$\mathbb{E}\left[\left(\frac{W_n^{(12)}(a(n)2^j) - 1}{a(n)^{\delta_{12}}} \right)^{\kappa_1} \left(\frac{W_n^{(34)}(a(n)2^{j'}) - 1}{a(n)^{\delta_{34}}} \right)^{\kappa_2} 1_{\{\min\{|W_n^{(12)}(a(n)2^{j'})|, |W_n^{(34)}(a(n)2^{j'})|\}>r\}} \right] = O\Big[\Big(\frac{a(n)}{n} \Big)^2 \Big]. \tag{C.38}$$

Proof. For each case $\kappa_1, \kappa_2 \in \mathbb{N}$, $\kappa_1 + \kappa_2 \geq 3$, we first show that

$$\mathbb{E}\left[\left(\frac{W_n^{(12)}(a2^j) - 1}{a^{\delta_{12}}} \right)^{\kappa_1} \left(\frac{W_n^{(34)}(a2^{j'}) - 1}{a^{\delta_{34}}} \right)^{\kappa_2} \right] = O\Big[\Big(\frac{a}{n} \Big)^2 \Big], \tag{C.39}$$

i.e., without indicators, and then extend the claim to the full expression (C.38).

Consider the case where $\kappa_1 = 1$ and $\kappa_2 = 2$. Then, the left-hand side of (C.39) can be rewritten as

$$\frac{1}{a^{\delta_{12}+2\delta_{34}}} \frac{1}{n_{a,j} n_{a,j'}^2} \mathbb{E}\Big[\sum_{k=1}^{n_{a,j}} \sum_{k_1'=1}^{n_{a,j'}} \sum_{k_2'=1}^{n_{a,j'}} \Big(\frac{d_1(a2^j, k) d_2(a2^j, k)}{\varrho^{(12)}(a2^j)} - 1 \Big) \Big(\frac{d_3(a2^{j'}, k_1') d_4(a2^{j'}, k_1')}{\varrho^{(34)}(a2^{j'})} - 1 \Big) \Big(\frac{d_3(a2^{j'}, k_2') d_4(a2^{j'}, k_2')}{\varrho^{(34)}(a2^{j'})} - 1 \Big) \Big]. \tag{C.40}$$

Starting from assumption (.2), for notational simplicity relabel the generic terms under the summation sign in (C.49) as $X_1 = d_1(a2^j, k)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_2 = d_2(a2^j, k)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_3 = d_3(a2^{j'}, k_1')/\sqrt{\varrho^{(34)}(a2^{j'})}$, $X_4 = d_4(a2^{j'}, k_1')/\sqrt{\varrho^{(34)}(a2^{j'})}$, $X_5 = d_3(a2^{j'}, k_2')/\sqrt{\varrho^{(34)}(a2^{j'})}$, $X_6 = d_4(a2^{j'}, k_2')/\sqrt{\varrho^{(34)}(a2^{j'})}$. Then,

$$\mathbb{E}[(X_1X_2 - 1)(X_3X_4 - 1)(X_5X_6 - 1)] = \mathbb{E}[X_1X_2X_3X_4X_5X_6] - \{\mathbb{E}[X_1X_2X_3X_4] + \mathbb{E}[X_1X_2X_5X_6] + \mathbb{E}[X_3X_4X_5X_6]\} + 2. \tag{C.41}$$

By applying the Isserlis relation (C.37) to the six-fold and four-fold products in (C.41), the latter expression becomes

$$\mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_5]\,\mathbb{E}[X_4X_6] + \mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_6]\,\mathbb{E}[X_4X_5] + \mathbb{E}[X_1X_4]\,\mathbb{E}[X_2X_5]\,\mathbb{E}[X_3X_6] + \mathbb{E}[X_1X_4]\,\mathbb{E}[X_2X_6]\,\mathbb{E}[X_3X_5]$$
$$+ \mathbb{E}[X_1X_5]\,\mathbb{E}[X_2X_3]\,\mathbb{E}[X_4X_6] + \mathbb{E}[X_1X_5]\,\mathbb{E}[X_2X_4]\,\mathbb{E}[X_3X_6] + \mathbb{E}[X_1X_6]\,\mathbb{E}[X_2X_3]\,\mathbb{E}[X_4X_5] + \mathbb{E}[X_1X_6]\,\mathbb{E}[X_2X_4]\,\mathbb{E}[X_3X_5]. \tag{C.42}$$

The asymptotic behavior of the summation of each term in the seven-fold sum (C.42) can be established in the same way, so we only study the first one. Up to a constant, the latter is asymptotically equivalent to

$$\frac{a^{(2h_{13}+2h_{23}+2h_4)-(2h_{12}+4h_{34})}}{a^{\delta_{12}+2\delta_{34}}\ n_{a,j}}\frac{1}{n_{a,j'}^2}\sum_{k=1}^{n_{a,j}}\sum_{k_1'=1}^{n_{a,j'}}\sum_{k_2'=1}^{n_{a,j'}}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k_1')\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]}{a^{2h_{23}}}\frac{\mathbb{E}\Big[d_4(a2^{j'},k_1')d_4(a2^{j'},k_2')\Big]}{a^{2h_4}}$$

$$=\frac{a^{2(h_{13}+h_{23}+h_4)}}{a^{h_1+h_2+2(h_3+h_4)}\ n_{a,j}}\frac{1}{n_{a,j'}^2}\sum_{k_1'=1}^{n_{a,j'}}\sum_{k_2'=1}^{n_{a,j'}}\frac{\mathbb{E}\Big[d_4(a2^{j'},k_1')d_4(a2^{j'},k_2')\Big]}{a^{2h_4}}\cdot\Big\{\sum_{k=1}^{n_{a,j}}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k_1')\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]}{a^{2h_{23}}}\Big\}. \tag{C.43}$$

However, the summation in k in expression (C.43) is bounded by

$$\Big|\sum_{k=1}^{n_{a,j}}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k_1')\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]}{a^{2h_{23}}}\Big|$$

$$\leq\sum_{k=1}^{n_{a,j}}\Big(\Big|\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k_1')\Big]-\Phi_{13}^{jj'}(a(2^jk-2^{j'}k_1'))}{a^{2h_{13}}}\Big|+\Big|\frac{\Phi_{13}^{jj'}(a(2^jk-2^{j'}k_1'))}{a^{2h_{13}}}\Big|\Big)$$

$$\Big(\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]-\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|+\Big|\frac{\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|\Big)$$

$$\leq\sum_{k=1}^{n_{a,j}}\Big\{\Big|\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k_1')\Big]-\Phi_{13}^{jj'}(a(2^jk-2^{j'}k_1'))}{a^{2h_{13}}}\Big|\cdot\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]-\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|$$

$$+\Big|\frac{\Phi_{13}^{jj'}(a(2^jk-2^{j'}k_1'))}{a^{2h_{13}}}\Big|\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]-\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|$$

$$+\Big|\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k_1')\Big]-\Phi_{13}^{jj'}(a(2^jk-2^{j'}k_1'))}{a^{2h_{13}}}\Big|\Big|\frac{\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|$$

$$+\Big|\frac{\Phi_{13}^{jj'}(a(2^jk-2^{j'}k_1'))}{a^{2h_{13}}}\Big|\Big|\frac{\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|\Big\}\leq C, \tag{C.44}$$

where C does not depend on k. To justify the last inequality, we only look at the second term in (C.44), since the remaining terms can be analyzed in a similar way. It suffices to proceed as in [53], in particular around expression (B.31) in the latter reference. Indeed, starting from (3.2), suppose without loss of generality that

$$\mathrm{supp}(\psi)=[0,1].$$

For $0<\varepsilon<1/2$, decompose

$$\sum_{k=1}^{n_{a,j}}\Big|\Phi_{13}^{jj'}(2^jk-2^{j'}k_1')\Big|\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]-\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|$$

$$=\sum_{k=1}^{n_{a,j}}\Big(1_{\left\{\frac{\max\{2^j,2^{j'}\}}{|2^jk-2^{j'}k_1'|}>\varepsilon\right\}}+1_{\left\{\frac{\max\{2^j,2^{j'}\}}{|2^jk-2^{j'}k_1'|}\leq\varepsilon\right\}}\Big)\Big|\Phi_{13}^{jj'}(2^jk-2^{j'}k_1')\Big|\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]-\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|. \tag{C.45}$$

The first sum term in (C.45) only contains a finite number of terms, where such number does not depend on k or k_1'. Moreover,

$$\max\Big\{|\Phi_{13}^{jj'}(2^jk-2^{j'}k_1')|,\Big|\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k_2')\Big]-\Phi_{23}^{jj'}(a(2^jk-2^{j'}k_2'))}{a^{2h_{23}}}\Big|\Big\}\leq C,$$

where C does not depend k, k_1' or k_2''. The latter statement follows from the fact that $\Phi_{13}^{jj'}$ is, up to a change of sign, the wavelet variance of a fBm and from (A.24). Moreover, by Proposition II.2 in [47] and again by (A.24), the second sum term in (C.45) is bounded by $C\sum_{z\in\mathbb{Z}\setminus\{0\}}z^{-4}$. This shows that (C.45) is bounded by a constant not depending on k, k_1' or k_2'. Hence, (C.44) holds.

Consequently, up to a constant the absolute value of (C.43) is bounded from above by

$$C'\frac{a^{2(h_{13}+h_{23}+h_4)+2}}{a^{h_1+h_2+2(h_3+h_4)}\ n^2}\frac{1}{n_{a,j'}}\sum_{k_1'=1}^{n_{a,j'}}\sum_{k_2'=1}^{n_{a,j'}}\Big|\frac{\mathbb{E}\Big[d_4(a2^{j'},k_1')d_4(a2^{j'},k_2')\Big]}{a^{2h_4}}\Big|\leq C''\Big(\frac{a}{n}\Big)^2, \tag{C.46}$$

where we used the fact that $\frac{1}{a^{\delta_{13}+\delta_{23}}}\leq 1$. This establishes (C.39).

So, rewrite

$$\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2\Big]-\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)1_{\{W_n^{(12)}(a2^j)>r\}}\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2 1_{\{W_n^{(34)}(a2^{j'})>r\}}\Big]$$

$$=\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2\Big\{1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})>r\}}$$

$$+1_{\{W_n^{(12)}(a2^j)>r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}+1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}\Big\}\Big]. \tag{C.47}$$

The asymptotic behavior of every term on the right-hand side of (C.47) can be established in the same way, so we only look at the first one. For any $0<\xi<1$, the Cauchy–Schwarz inequality, Lemma C.3 and expression (C.39) yield

$$\Big|\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2 1_{\{W_n^{(34)}(a2^{j'})>r\}}\Big]\Big|$$

$$\le\sqrt{\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^2\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^4\Big]}\sqrt{P(W_n^{(12)}(a2^j)\le r)}$$

$$\le C\Big(\frac{a}{n}\Big)\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big\}, \tag{C.48}$$

where C does not depend on r. Moreover, under (3.6),

$$\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big\}=o\Big(\frac{a}{n}\Big).$$

Therefore, (C.38) holds. The remaining cases where $\kappa_1+\kappa_2=3$ can be handled similarly.

Now consider the case where $\kappa_1=4$ and $\kappa_2=0$. Then, the left-hand side of (C.39) can be rewritten as

$$\frac{1}{a^{4\delta_{12}}}\frac{1}{n_{a,j}^4}\mathbb{E}\Big[\sum_{k_1=1}^{n_{a,j}}\sum_{k_2=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\Big(\frac{d_1(a2^j,k_1)d_2(a2^j,k_1)}{\varrho^{(12)}(a2^j)}-1\Big)\Big(\frac{d_1(a2^j,k_2')d_2(a2^j,k_2)}{\varrho^{(12)}(a2^j)}-1\Big)$$

$$\Big(\frac{d_1(a2^j,k_3)d_2(a2^j,k_3)}{\varrho^{(12)}(a2^j)}-1\Big)\Big(\frac{d_1(a2^j,k_4)d_2(a2^j,k_4)}{\varrho^{(12)}(a2^j)}-1\Big)\Big]. \tag{C.49}$$

Starting from the assumption (.2), for notational simplicity relabel the generic terms under the summation sign in (C.49) as $X_1=d_1(a2^j,k_1)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_2=d_2(a2^j,k_1)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_3=d_1(a2^j,k_2)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_4=d_2(a2^j,k_2)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_5=d_1(a2^j,k_3)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_6=d_2(a2^j,k_3)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_7=d_1(a2^j,k_4)/\sqrt{\varrho^{(12)}(a2^j)}$, $X_8=d_2(a2^j,k_4)/\sqrt{\varrho^{(12)}(a2^j)}$. Then,

$$\begin{aligned}
&\mathbb{E}[(X_1X_2-1)(X_3X_4-1)(X_5X_6-1)(X_7X_8-1)]\\
&=\Big\{\mathbb{E}[X_1X_2X_3X_4X_5X_6X_7X_8]\Big\}\\
&\quad-\Big\{\mathbb{E}[X_3X_4X_5X_6X_7X_8]+\mathbb{E}[X_1X_2X_5X_6X_7X_8]+\mathbb{E}[X_1X_2X_3X_4X_7X_8]+\mathbb{E}[X_1X_2X_3X_4X_5X_6]\Big\}\\
&\quad+\Big\{\mathbb{E}[X_5X_6X_7X_8]+\mathbb{E}[X_3X_4X_7X_8]+\mathbb{E}[X_1X_2X_7X_8]\\
&\quad+\mathbb{E}[X_1X_2X_3X_4]+\mathbb{E}[X_1X_2X_5X_6]+\mathbb{E}[X_3X_4X_5X_6]\Big\}\\
&\quad-\Big\{\mathbb{E}[X_1X_2]+\mathbb{E}[X_3X_4]+\mathbb{E}[X_5X_6]+\mathbb{E}[X_7X_8]-1\Big\}.
\end{aligned} \tag{C.50}$$

Namely, we arrive at an expression with four terms between braces. By relabeling them A, B, C and D, respectively, we can write

$$\mathbb{E}[(X_1X_2-1)(X_3X_4-1)(X_5X_6-1)(X_7X_8-1)]=A-B+C-D. \tag{C.51}$$

We claim that, by an application of the Isserlis theorem, the expression (C.51) can be written as a sum of products of the form

$$\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\,\mathbb{E}\big[X_{l_5}X_{l_6}\big]\,\mathbb{E}\big[X_{l_7}X_{l_8}\big], \tag{C.52}$$

where $l_1<l_3<l_5<l_7$ and no pair $\mathbb{E}[X_{\cdot}X_{\cdot}]$ has consecutive indices (the latter condition implies that no pair is identically 1 with respect to summation in k_1, k_2, k_3 and k_4 in expression (C.49)).

In fact, recall that $\mathbb{E}[X_1X_2]=\mathbb{E}[X_3X_4]=\mathbb{E}[X_5X_6]=\mathbb{E}[X_7X_8]=1$. Note that the Isserlis theorem breaks up any term in the original sum (C.50) into a product of expectations of the form $\mathbb{E}[X_{\cdot}X_{\cdot}]$, where each product can only be identically 1 (with respect to summation in k_1, k_2, k_3 and k_4) if each odd index l is paired with $l+1$ (e.g., after decomposing A by Isserlis, the only term which is identically 1 is $\mathbb{E}[X_1X_2]\,\mathbb{E}[X_3X_4]\,\mathbb{E}[X_5X_6]\,\mathbb{E}[X_7X_8]$). Hence, after applying the Isserlis theorem to A, B and C, each if the 11 terms contained in the sum $A-B+C$ ends up with exactly one term identically equal to 1. Therefore, we obtain $(1-4+6-3)\times 1=0$ the right-hand side of (C.51). Hence, the full expression (C.51) contains no term identically 1, and, in addition, we no longer need to account for D. Next, note that, after applying the Isserlis theorem to (C.50), no resulting term can be of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\times 1^3$, where 1 stands for "identically 1" with respect to summation in some index $k_{\cdot}$ (and $\mathbb{E}\big[X_{l_1}X_{l_2}\big]$ is not identically 1). So, consider terms of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\times 1^2$, where, again, 1 stands for

"identically 1" (and $\mathbb{E}\big[X_{l_1}X_{l_2}\big]$ and $\mathbb{E}\big[X_{l_3}X_{l_4}\big]$ are not identically 1). For the sake of clarity, consider the particular term $\mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_4]\times 1^2$. After applying the Isserlis theorem, the latter ends up appearing (once) in the expansions of A, of $\mathbb{E}[X_1X_2X_3X_4X_7X_8]$ and $\mathbb{E}[X_1X_2X_3X_4X_5X_6]$ in B, and of $\mathbb{E}[X_1X_2X_3X_4]$ in C. Thus, on the right-hand side of (C.51), we obtain $(1-2+1)\times(\mathbb{E}[X_1X_3]\,\mathbb{E}[X_2X_4]\times 1^2)=0$. Since the same is true for every other term of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\times 1^2$, the full expression (C.51) contains no such terms and, in addition, we no longer need to account for C. Now consider terms of the form $\mathbb{E}\big[X_{l_1}X_{l_2}\big]\,\mathbb{E}\big[X_{l_3}X_{l_4}\big]\,\mathbb{E}\big[X_{l_5}X_{l_6}\big]\times 1$, where 1 stands for "identically 1" (and $\mathbb{E}\big[X_{l_1}X_{l_2}\big]$, $\mathbb{E}\big[X_{l_3}X_{l_4}\big]$ and $\mathbb{E}\big[X_{l_5}X_{l_6}\big]$ are not identically 1). Note that, after applying the Isserlis theorem, B ends up containing the sum of all possible terms of this form, and each one appears only once. Since they also all appear only once in the expansion of A, the terms stemming from A and B cancel. Therefore, (C.50) is made up of the sum of terms of the form (C.52), as claimed.

We are now in a position to establish the asymptotic behavior of (C.49). For notational simplicity, we focus on the particular term of the form $\mathbb{E}[X_1X_5]\,\mathbb{E}[X_2X_6]\,\mathbb{E}[X_3X_7]\,\mathbb{E}[X_4X_8]$. After taking summations, the resulting expression is asymptotically equivalent to

$$
\begin{aligned}
&C\frac{a^{4h_1+4h_2}}{a^{4\delta_{12}+8h_{12}}}\frac{1}{n_{a,j}^4}\sum_{k_1=1}^{n_{a,j}}\sum_{k_2=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\frac{\mathbb{E}\big[d_1(a2^j,k_1)d_1(a2^j,k_3)\big]}{a^{2h_1}}\frac{\mathbb{E}\big[d_2(a2^j,k_1)d_2(a2^j,k_3)\big]}{a^{2h_2}}\\
&\quad\frac{\mathbb{E}\big[d_1(a2^j,k_2)d_1(a2^j,k_4)\big]}{a^{2h_1}}\frac{\mathbb{E}\big[d_2(a2^j,k_2)d_2(a2^j,k_4)\big]}{a^{2h_2}}\\
&=\frac{C}{n_{a,j}^2}\Bigg\{\frac{1}{n_{a,j}}\sum_{k_1=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\frac{\mathbb{E}\big[d_1(a2^j,k_1)d_1(a2^j,k_3)\big]}{a^{2h_1}}\frac{\mathbb{E}\big[d_2(a2^j,k_1)d_2(a2^j,k_3)\big]}{a^{2h_2}}\\
&\quad\times\frac{1}{n_{a,j}}\sum_{k_2=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\frac{\mathbb{E}\big[d_1(a2^j,k_2)d_1(a2^j,k_4)\big]}{a^{2h_1}}\frac{\mathbb{E}\big[d_2(a2^j,k_2)d_2(a2^j,k_4)\big]}{a^{2h_2}}\Bigg\}.
\end{aligned}
$$

The terms $|\mathbb{E}\big[d_2(a2^j,k_1)d_2(a2^j,k_3)\big]/a^{2h_2}|$, $|\mathbb{E}\big[d_2(a2^j,k_2)d_2(a2^j,k_4)\big]/a^{2h_2}|$ are bounded in k_1, k_2, k_3 and k_4. Moreover, by a reasoning analogous to the one leading to (C.46), the averages $n_{a,j}^{-1}\sum_{k_1=1}^{n_{a,j}}\sum_{k_3=1}^{n_{a,j}}\mathbb{E}\big[d_1(a2^j,k_1)d_1(a2^j,k_3)\big]/a^{2h_1}$ and $n_{a,j}^{-1}\sum_{k_2=1}^{n_{a,j}}\sum_{k_4=1}^{n_{a,j}}\mathbb{E}\big[d_1(a2^j,k_2)d_1(a2^j,k_4)\big]/a^{2h_1}$ converge to constants as $n\to\infty$. This shows (C.39) for $\kappa_1=4$ and $\kappa_2=0$.

A simple adaptation of the argument for the case $\kappa_1=1$ and $\kappa_2=2$ (see (C.47)) shows that (C.38) also holds when $\kappa_1=4$ and $\kappa_2=0$, as claimed.

We now turn to the case where $\kappa_1>2$ and $\kappa_2>2$. Let $\zeta_0>1/2$ be a fixed constant. Then,

$$
\begin{aligned}
&\Bigg|\mathbb{E}\Bigg[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^{\kappa_1}\Big(\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big)^{\kappa_2}\Bigg]\Bigg|\\
&\le\mathbb{E}\Bigg[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^{\kappa_1}\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}\Big(1_{\{|W_n^{(12)}(a2^j)-1|\ge\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}+1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}\\
&\quad+1_{\{|W_n^{(12)}(a2^j)-1|\ge\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|<\zeta_0\}}+1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|<\zeta_0\}}\Big)\Bigg].
\end{aligned}
\tag{C.53}
$$

However, by the Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\mathbb{E}\Bigg[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^{\kappa_1}\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}\Bigg]\\
&\le\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1}\mathbb{E}\Bigg[\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}1_{\{|W_n^{(34)}(a2^j)-1|\ge\zeta_0\}}\Bigg]\\
&\le\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1}\sqrt{\mathbb{E}\Big(\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big)^{2\kappa_2}P(|W_n^{(34)}(a2^j)-1|\ge\zeta_0)}\\
&\le C\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1}\sqrt{P(|W_n^{(34)}(a2^j)-1|\ge\zeta_0)}=O\Big[\Big(\frac{a}{n}\Big)^2\Big],
\end{aligned}
$$

where the last inequality and the equality are consequences of (C.24) and Lemma C.3, respectively. The same bound applies to the first and third terms in the sum (C.53). Therefore, the latter is bounded by

$$
\begin{aligned}
&O\Big[\Big(\frac{a}{n}\Big)^2\Big]+\mathbb{E}\Bigg[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^{\kappa_1}\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^{\kappa_2}1_{\{|W_n^{(12)}(a2^j)-1|<\zeta_0\}\cap\{|W_n^{(34)}(a2^j)-1|<\zeta_0\}}\Bigg]\\
&\le O\Big[\Big(\frac{a}{n}\Big)^2\Big]+\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1-2}\Big(\frac{\zeta_0}{a^{\delta_{34}}}\Big)^{\kappa_2-2}\mathbb{E}\Bigg[\Big|\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big|^2\Big|\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big|^2\Bigg]\\
&\le O\Big[\Big(\frac{a}{n}\Big)^2\Big]+\Big(\frac{\zeta_0}{a^{\delta_{12}}}\Big)^{\kappa_1-2}\Big(\frac{\zeta_0}{a^{\delta_{34}}}\Big)^{\kappa_2-2}\sqrt{\mathbb{E}\Big[\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big]^4}\sqrt{\mathbb{E}\Big[\frac{W_n^{(34)}(a2^j)-1}{a^{\delta_{34}}}\Big]^4}\\
&=O\Big[\Big(\frac{a}{n}\Big)^2\Big].
\end{aligned}
\tag{C.54}
$$

In (C.54), the first inequality is a consequence of the Cauchy–Schwarz inequality, and the second inequality follows from (C.38) for $\kappa_1 = 4$ and $\kappa_2 = 0$ or $\kappa_1 = 0$ and $\kappa_2 = 4$. In addition, by a simple adaptation of the argument for the case $\kappa_1 = 1$ and $\kappa_2 = 2$ (see (C.47)), (C.38) also holds for $\kappa_1 > 2$ and $\kappa_2 > 2$.

The cases where $\kappa_1 = 1$ and $\kappa_2 \geq 4$ or $\kappa_1 \geq 4$ and $\kappa_2 = 1$ can be tackled by a similar procedure. □

The following lemma expresses, up to a residual term, the cross-covariance (first cross-moment) between sample wavelet variances in terms of the functions (3.14).

Lemma C.5. *Let $\Phi^{jj'}_{\cdot\cdot}(z)$ and n_* be as in (3.14) and (4.1), respectively. For $0 < r < 1/2$,*

$$\mathbb{E}\Big[(W_n^{(12)}(a(n)2^j) - 1)1_{\{|W_n^{(12)}(a(n)2^j)|>r\}}(W_n^{(34)}(a(n)2^{j'}) - 1)1_{\{|W_n^{(34)}(a(n)2^{j'})|>r\}}\Big]$$
$$= \frac{2^{-(j+j')}}{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)\{1+O(a(n)^{-\varpi_0})\}^2}\Big\{\frac{a(n)^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big]$$
$$+\frac{a(n)^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Phi^{jj'}_{23}(2^jk-2^{j'}k')\Big]+o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{12}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big)\Big\}, \tag{C.55}$$

as $n \to \infty$.

Proof. As in the proof of Lemma C.4, we first drop the indicator functions on the left-hand side of (C.55) and investigate the limit. We will show that

$$\mathrm{Cov}\Big[W_n^{(12)}(a2^j), W_n^{(34)}(a2^{j'})\Big] = \frac{2^{-(j+j')}}{[\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1+O(a^{-\varpi_0}))^2]}$$
$$\cdot\Big\{\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Phi^{jj'}_{24}(2^jk-2^{j'}k')$$
$$+\frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Phi^{jj'}_{23}(2^jk-2^{j'}k')$$
$$+o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{14}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big)\Big\}. \tag{C.56}$$

In fact, the left-hand side of (C.56) can be written as

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)\Big] = \frac{1}{n_{a,j}n_{a,j'}}\sum_{k=1}^{n_{a,j}}\sum_{k'=1}^{n_{a,j'}}\Big\{\mathbb{E}\Big[\frac{d_1(a2^j,k)d_2(a2^j,k)}{\varrho^{(12)}(a2^j)}\frac{d_3(a2^{j'},k')d_4(a2^{j'},k')}{\varrho^{(34)}(a2^{j'})}\Big]-1\Big\}. \tag{C.57}$$

By the Isserlis relation (C.37), the first term in the argument of the sum (C.57) can be reexpressed as

$$\frac{\mathbb{E}\Big[d_1(a2^j,k)d_2(a2^j,k)d_3(a2^{j'},k')d_4(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}$$
$$=\Big\{1+\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]\ \mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}+\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]\ \mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}\Big\}. \tag{C.58}$$

By Lemma C.1(ii), and (C.58), we can rewrite (C.57) as

$$\frac{1}{n_{a,j}n_{a,j'}}\sum_{k=1}^{n_{a,j}}\sum_{k'=1}^{n_{a,j'}}\Big\{\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]\ \mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}+\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]\ \mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{\varrho^{(12)}(a2^j)\varrho^{(34)}(a2^{j'})}\Big\}$$
$$=\frac{2^{-(j+j')}}{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1+O(a^{-\varpi_0}))^2}$$
$$\Big\{\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}}\Big]$$
$$+\frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^j n_*}\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{14}}}\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{23}}}\Big]\Big\}. \tag{C.59}$$

By adding and subtracting the counterparts $\Phi^{jj'}_{\cdot\cdot}(2^jk-2^{j'}k')$ for each term, up to the factor $2^{-(j+j')}/\{\Phi^{jj}_{12}(0)\Phi^{j'j'}_{34}(0)(1+O(a^{-\varpi_0}))^2\}$ the expression (C.59) can be written as

$$\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}}-\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Big)\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}}-\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big)$$

$$+\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}}-\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big)$$

$$+\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}}-\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Big)+\{\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Phi^{jj'}_{24}(2^jk-2^{j'}k')\}\Big]$$

$$+\frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\Big[\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\Big\{\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{14}}}-\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Big)$$

$$\cdot\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{23}}}-\Phi^{j'j}_{23}(2^jk-2^{j'}k')\Big)$$

$$+\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{14}}}-\Phi^{jj'}_{23}(2^jk-2^{j'}k')\Big)$$

$$+\Phi^{jj'}_{23}(2^jk-2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{23}}}-\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Big)+\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Phi^{jj'}_{23}(2^jk-2^{j'}k')\Big\}\Big]$$

$$=\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Phi^{jj'}_{24}(2^jk-2^{j'}k')$$

$$+\frac{a^{2(h_{14}+h_{23})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\Phi^{jj'}_{14}(2^jk-2^{j'}k')\Phi^{jj'}_{23}(2^jk-2^{j'}k')+o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{14}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big). \tag{C.60}$$

It remains to justify the order of the error term in (C.60). So, by adapting the proof of (A.16), we obtain

$$\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\Big|\sum_{k=1}^{2^jn_*}\sum_{k'=1}^{2^{j'}n_*}\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}}-\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big)\Big|$$
$$=\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\,o(1)=o\Big(\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\Big).$$

The same bound holds for

$$\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\Big|\sum_{k=1}^{2^jn_*}\sum_{k'=1}^{2^{j'}n_*}\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}}-\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Big)\Big|$$

and

$$\frac{a^{2(h_{13}+h_{24})-2(h_{12}+h_{34})}}{n_*}\frac{1}{n_*}\Big|\sum_{k=1}^{2^{j'}n_*}\sum_{k'=1}^{2^jn_*}\Big(\frac{\mathbb{E}\Big[d_1(a2^j,k)d_3(a2^{j'},k')\Big]}{a^{2h_{13}}}-\Phi^{jj'}_{13}(2^jk-2^{j'}k')\Big)\Big(\frac{\mathbb{E}\Big[d_2(a2^j,k)d_4(a2^{j'},k')\Big]}{a^{2h_{24}}}-\Phi^{jj'}_{24}(2^jk-2^{j'}k')\Big)\Big|.$$

By extending this analysis to the remaining terms of (C.60), we obtain analogous bounds and the error term $o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{14}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big)$, as claimed.

In view of (C.59) and (C.60), it suffices to show that the indicator functions on the left-hand side of (C.55) do not affect the approximation order. In fact,

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)\Big]-\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}(W_n^{(34)}(a2^{j'})-1)1_{\{W_n^{(34)}(a2^{j'})>r\}}\Big]$$
$$=\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)\cdot\Big(1_{\{W_n^{(12)}(a2^j)>r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}$$
$$+\,1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})>r\}}+1_{\{W_n^{(12)}(a2^j)\le r\}}1_{\{W_n^{(34)}(a2^{j'})\le r\}}\Big)\Big]. \tag{C.61}$$

Define

$$h^*=\max_{q_1,q_2=1,2,3,4}h_{q_1q_2},\quad h_*=\min_{q_1,q_2=1,2,3,4}h_{q_1q_2}. \tag{C.62}$$

For $0<\xi'<1$, by the Cauchy–Schwarz inequality, expression (C.39) (from Lemma C.4) and Lemma C.3, the first term on the right-hand side of (C.61) is bounded by

$$\Big|\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}(W_n^{(34)}(a2^{j'})-1)1_{\{W_n^{(34)}(a2^{j'})\le r\}}\Big]\Big|$$
$$\le a^{\delta_{12}+\delta_{34}}\sqrt{\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^2\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^2\Big]}\sqrt{P(W_n^{(34)}(a2^j)\le r)}$$
$$\le C\Big(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\Big)\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi'}\Big\}$$
$$=o\Big(\frac{a^{2\max\{h_{13}+h_{24},h_{12}+h_{23}\}-2(h_{12}+h_{34})}}{n_*}\Big),$$

where the constant C does not depend on r and the last equality is a consequence of (3.6). Similar bounds hold for the remaining terms on the right-hand side of (C.61). Therefore, the expression (C.55) follows. □

The following lemma describes the decay rate of the first individual truncated moment of the wavelet variance.

Lemma C.6. *For any $0<\xi<1$ and $0<r<1/2$,*

$$\max\Big\{\Big|\mathbb{E}\Big[\{W_n^{(12)}(a(n)2^j)-1\}1_{\{W_n^{(12)}(a(n)2^j)\le r\}}\Big]\Big|;\ \Big|\mathbb{E}\Big[\{W_n^{(12)}(a(n)2^j)-1\}1_{\{W_n^{(12)}(a(n)2^j)>r\}}\Big]\Big|\Big\}$$
$$=O\Big(\frac{a(n)^{h_1+h_2-2h_{12}+1/2}}{\sqrt{n}}\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}\Big). \tag{C.63}$$

Proof. Notice that

$$0=\mathbb{E}\big[W_n^{(12)}(a2^j)-1\big]=\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}\Big]+\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big].$$

Hence,

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)>r\}}\Big]=-\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big].$$

However, by the Cauchy–Schwarz inequality,

$$\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)1_{\{W_n^{(12)}(a2^j)\le r\}}\Big]\Big|\le\sqrt{\mathrm{Var}\Big[W_n^{(12)}(a2^j)\Big]}\sqrt{P(W_n^{(12)}(a2^j)\le r)}. \tag{C.64}$$

By expressions (C.64), (C.22) and Lemma C.3, the expression (C.63) follows. □

The following lemma establishes the decay of the covariances between truncated terms (C.36) and indicators involving wavelet variances, or between indicators only.

Lemma C.7. *For any $0<\xi<1$ and $0<r<1/2$,*

(i)

$$\mathrm{Cov}\Big[1_{\{|W_n^{(12)}(a2^{j'})|>r\}},1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\Big]\le\exp\Big\{-\frac{1}{2}\Big[\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}+\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big]\Big\}; \tag{C.65}$$

(ii)

$$\mathrm{Var}\Big[\log|W_n^{(12)}(a(n)2^j)|1_{\{W_n^{(12)}(a(n)2^j)<-r\}}\Big]\le(C+\log^2(r))\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a(n)^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\} \tag{C.66}$$

and

$$\mathrm{Var}\Big[\log W_n^{(12)}(a(n)2^j)1_{\{W_n^{(12)}(a(n)2^j)>r\}}\Big]\le\log^2(r)\exp\Big\{-\Big(\frac{n}{a(n)^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}\Big\}+o(1); \tag{C.67}$$

(iii)

$$\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r\}},\,1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\right]$$
$$\leq\sqrt{\log^2(r)\exp\left\{-\left(\frac{n}{a(n)^{2(h_1+h_2)-4h_{12}+1}}\right)^{1-\xi}+o(1)\right\}}\cdot\exp\left\{-\frac{1}{2}\left(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\right)^{1-\xi}\right\}. \tag{C.68}$$

In (C.66), $C>0$ does not depend on r.

Proof. We first show (C.65). By the Cauchy–Schwarz inequality, the left-hand side of (C.65) is bounded from above by

$$\sqrt{\mathrm{Var}\left[1_{\{|W_n^{(12)}(a2^j)|>r\}}\right]}\sqrt{\mathrm{Var}\left[1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\right]}. \tag{C.69}$$

Moreover, for $0<\xi<1$ and the octave j, Lemma C.3 implies that

$$\mathrm{Var}\left[1_{\{|W_n^{(12)}(a2^j)|>r\}}\right]=P(|W_n^{(12)}(a2^j)|>r)P(|W_n^{(12)}(a2^j)|\leq r)\leq\exp\left\{-\left(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\right)^{1-\xi}\right\}. \tag{C.70}$$

The same bound holds for the octave j' in (C.69). Thus, (C.65) holds.

To prove (C.66), note that $\mathrm{Var}\left[\log|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)<-r\}}\right]$ is bounded by

$$\begin{aligned}
&\mathbb{E}\left[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)<-r\}}\right]\\
&=\mathbb{E}\left[\log^2|W_n^{(12)}(a2^j)|\left(1_{\{W_n^{(12)}(a2^j)\leq-1/2\}}+1_{\{-1/2<W_n^{(12)}(a2^j)<-r\}}\right)\right]\\
&\leq C\mathbb{E}\left[|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\leq-1/2\}}\right]+\log^2(r)P(-1/2<W_n^{(12)}(a2^j)<-r)\\
&\leq C\sqrt{\mathbb{E}\left[W_n^{(12)}(a2^j)^2\right]P(W_n^{(12)}(a2^j)\leq-1/2)}+\log^2(r)P(-1/2<W_n^{(12)}(a2^j)<-r)\\
&\leq C'\exp\left\{-\frac{1}{2}\left(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\right)^{1-\xi}\right\}+\log^2(r)\exp\left\{-\left(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\right)^{1-\xi}\right\}.
\end{aligned}$$

This establishes (C.66).

To prove (C.67), note that $\mathrm{Var}\left[\log W_n^{(12)}(a2^j)1_{\{W_n^{(12)}(a2^j)>r\}}\right]$ is bounded by

$$\begin{aligned}
&\mathbb{E}\left[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)>r\}}\right]\\
&=\mathbb{E}\left[\log^2|W_n^{(12)}(a2^j)|\left(1_{\{r<W_n^{(12)}(a2^j)<1/2\}}+1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\right)\right]\\
&\leq\log^2(r)P(r<W_n^{(12)}(a2^j)<1/2)+\mathbb{E}\left[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\right].
\end{aligned}$$

However, for some $C>0$,

$$\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\leq CW_n^{(12)}(a2^j)^2,$$

where, by (C.22), $W_n^{(12)}(a2^j)\xrightarrow{P}1$ and

$$\mathbb{E}\left[W_n^{(12)}(a2^j)^2\right]=\mathrm{Var}W_n^{(12)}(a2^j)+1\to1,\quad n\to\infty.$$

Therefore, by the dominated convergence theorem for convergence in probability,

$$\lim_{n\to\infty}\mathbb{E}\left[\log^2|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)\geq1/2\}}\right]=0.$$

This establishes (C.67).

To show (C.68), again by applying the Cauchy–Schwarz inequality, the left-hand side of (C.68) is bounded from above by

$$\sqrt{\mathrm{Var}\left[\log|W_n^{(12)}(a2^j)|1_{\{|W_n^{(12)}(a2^j)|>r\}}\right]}\sqrt{\mathrm{Var}\left[1_{\{|W_n^{(34)}(a2^{j'})|>r\}}\right]}. \tag{C.71}$$

Hence, the bound follows from (C.65) and (C.67). □

We are now in a position to establish Theorem 4.1.

Proof of Theorem 4.1. Fix $0<\xi<1$ and recall that $n_*=\frac{n}{a2^{j+j'}}$. Then,

$$\begin{aligned}
&\mathrm{Cov}\left[\log|S_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\log|S_n^{(34)}(a2^{j'})|\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]\\
&=\mathrm{Cov}\left[\log|W_n^{(12)}(a2^j)|\,1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\log|W_n^{(34)}(a2^{j'})|\,1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\right]
\end{aligned}$$

$$
\begin{aligned}
&+ \log\left|\mathbb{E}\Big[d_3(a2^j,0)d_4(a2^{j'},0)\Big]\right| \operatorname{Cov}\Big[\log|W_n^{(12)}(a2^j)|\, 1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\, 1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\Big] \\
&+ \log\left|\mathbb{E}\Big[d_1(a2^j,0)d_2(a2^j,0)\Big]\right| \operatorname{Cov}\Big[1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\, \log|W_n^{(34)}(a2^{j'})|\, 1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\Big] \\
&+ \log\left|\mathbb{E}\Big[d_1(a2^j,0)d_2(a2^j,0)\Big]\right| \log\left|\mathbb{E}\Big[d_3(a2^{j'},0)d_4(a2^{j'},0)\Big]\right| \operatorname{Cov}\Big[1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\, 1_{\{|W_n^{(34)}(a2^{j'})|>r_n\}}\Big] \\
&= \operatorname{Cov}\Big[\log|W_n^{(12)}(a2^j)|1_{\{|W_n^{(12)}(a2^j)|>r_n\}},\, \log|W_n^{(34)}(a(n)2^{j'})|1_{\{|W_n^{(34)}(a(n)2^{j'})|>r_n\}}\Big] \\
&+ o\Big(\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^2\Big).
\end{aligned}
\tag{C.72}
$$

The last two equalities in (C.72) are a consequence of Lemma C.1, (*ii*), as applied to $\mathbb{E}\Big[d_1(a2^j,0)d_2(a2^j,0)\Big]$ and $\mathbb{E}\Big[d_3(a2^{j'},0)d_4(a2^{j'},0)\Big]$, and of the bound (C.68) (from Lemma C.7(iii)) under the condition (4.2).

Therefore, it suffices to show that the main term on the right-hand side of (C.72) is equal to the main term on the right-hand side of (4.3). By accounting for absolute values, the covariance term in the former can be broken up into a sum of four terms, namely,

$$
\operatorname{Cov}\Big[\log W_n^{(12)}(a2^j)1_{\{W_n^{(12)}(a2^j)>r_n\}},\, \log W_n^{(34)}(a2^{j'})1_{\{W_n^{(34)}(a2^{j'})>r_n\}}\Big] \tag{C.73}
$$

plus the remainder

$$
\begin{aligned}
&\operatorname{Cov}\Big[\log W_n^{(12)}(a2^j)\, 1_{\{W_n^{(12)}(a2^j)>r_n\}},\, \log|W_n^{(34)}(a2^j)|\, 1_{\{W_n^{(34)}(a2^j)<-r_n\}}\Big] \\
&+ \operatorname{Cov}\Big[\log|W_n^{(12)}(a2^j)|\, 1_{\{W_n^{(12)}(a2^j)<-r_n\}},\, \log W_n^{(34)}(a2^j)\, 1_{\{W_n^{(34)}(a2^j)>r_n\}}\Big] \\
&+ \operatorname{Cov}\Big[\log|W_n^{(12)}(a2^j)|\, 1_{\{W_n^{(12)}(a2^j)<-r_n\}},\, \log|W_n^{(34)}(a2^j)|\, 1_{\{W_n^{(34)}(a2^j)<-r_n\}}\Big].
\end{aligned}
\tag{C.74}
$$

By the Cauchy–Schwarz inequality, the bounds (C.66) and (C.67) (from Lemma C.7, (*ii*)) and condition (4.2), the absolute value of the second term in the sum (C.74) is bounded by

$$
\sqrt{\operatorname{Var}\Big[\log|W_n^{(12)}(a2^j)|1_{\{W_n^{(12)}(a2^j)<-r_n\}}\Big]\operatorname{Var}\Big[\log W_n^{(34)}(a2^j)1_{\{W_n^{(34)}(a2^j)>r_n\}}\Big]} = o\Big(\Big(\frac{a(n)^{4h_{\max}-4h_{\min}}}{n_*}\Big)^2\Big).
$$

By a similar argument, the same bound holds for the remaining terms in the sum (C.74). Thus, it suffices to focus on (C.73). In the following derivations, expressions involving individual sample wavelet variance terms will be expressed in terms of $W_n^{(12)}(a2^j)$, but analogous expressions hold when substituting $W_n^{(34)}(a2^{j'})$ for $W_n^{(12)}(a2^j)$.

Fix $0 < \xi < 1$. For any given j, write out the almost sure Taylor expansion

$$
\log W_n^{(12)}(a2^j)\, 1_{\{W_n^{(12)}(a2^j)>r_n\}} = \Big\{(W_n^{(12)}(a2^j)-1) - \frac{1}{2}\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2\Big\}1_{\{W_n^{(12)}(a2^j)>r_n\}}, \tag{C.75}
$$

where $\theta_+(W_n^{(12)}(a2^j)) \in [\min\{W_n^{(12)}(a2^j),1\}, \max\{W_n^{(12)}(a2^j),1\}]$. Then,

$$
\begin{aligned}
&\mathbb{E}\Big[\log W_n^{(12)}(a2^j)\ \log W_n^{(34)}(a2^{j'})\ 1_{\min\{\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\Big] \\
&= \mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)(W_n^{(34)}(a2^{j'})-1)1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\Big] \\
&\quad - \frac{1}{2}\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)\Big(\frac{W_n^{(34)}(a2^{j'})-1}{\theta_+(W_n^{(34)}(a2^{j'}))}\Big)^2 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\Big] \\
&\quad - \frac{1}{2}\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2(W_n^{(34)}(a2^{j'})-1)\ 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\Big] \\
&\quad + \frac{1}{4}\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2\Big(\frac{W_n^{(34)}(a2^{j'})-1}{\theta_+(W_n^{(34)}(a2^{j'}))}\Big)^2 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\Big].
\end{aligned}
\tag{C.76}
$$

For $0 < r_n < 1/2$, recast

$$
\begin{aligned}
&\Big(\frac{W_n^{(12)}(a2^j)-1}{\widehat{\theta}_+(W_n^{(12)}(a2^j))}\Big)^2 1_{\{W_n^{(12)}(a2^j)>r_n\}} \\
&= \Big(\frac{W_n^{(12)}(a2^j)-1}{\widehat{\theta}_+(W_n^{(12)}(a2^j))}\Big)^2\Big(1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}} + 1_{\{W_n^{(12)}(a2^j)\geq 1/2\}}\Big) \\
&\leq \Big(\frac{W_n^{(12)}(a2^j)-1}{r_n}\Big)^2 1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}} + \Big(\frac{W_n^{(12)}(a2^j)-1}{1/2}\Big)^2 1_{\{W_n^{(12)}(a2^j)\geq 1/2\}}.
\end{aligned}
\tag{C.77}
$$

Therefore, up to a constant, we can bound the fourth term in (C.76) by

$$\begin{aligned}
&\mathbb{E}\Bigg[\Bigg|\Big(\frac{W_n^{(12)}(a2^j)-1}{\theta_+(W_n^{(12)}(a2^j))}\Big)^2\Big(\frac{W_n^{(34)}(a2^j)-1}{\theta_+(W_n^{(34)}(a2^j))}\Big)^2 1_{\{\min\{W_n^{(12)}(a2^j),W_n^{(34)}(a2^{j'})\}>r_n\}}\Bigg|\Bigg]\\
&\le \frac{1}{r_n^4}\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)^2 1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\Big]\\
&\quad+\frac{1}{(r_n/2)^2}\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)^2 1_{\{W_n^{(12)}(a2^j)\ge 1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\Big]\\
&\quad+\frac{1}{(r_n/2)^2}\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)^2 1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{W_n^{(34)}(a2^j)\ge 1/2\}}\Big]\\
&\quad+\frac{1}{(1/2)^4}\mathbb{E}\Big[(W_n^{(12)}(a2^j)-1)^2 1_{\{W_n^{(12)}(a2^j)\ge 1/2\}}(W_n^{(34)}(a2^{j'})-1)^2 1_{\{W_n^{(34)}(a2^{j'})\ge 1/2\}}\Big].
\end{aligned}\tag{C.78}$$

By Lemma C.4, the fourth term in the sum on the right-hand side of (C.78) is bounded by

$$O\Big(\Big(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\Big)^2\Big).\tag{C.79}$$

By the Cauchy–Schwarz inequality, Lemma C.4 and condition (4.2), the first term in the sum (C.78) is bounded by

$$\begin{aligned}
&\frac{a^{2(\delta_{12}+\delta_{34})}}{r_n^4}\sqrt{\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^4\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^4\Big]}\cdot\sqrt{\mathbb{E}\Big[1_{\{r_n<W_n^{(12)}(a2^j)<1/2\}}1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\Big]}\\
&\le \frac{a^{2(\delta_{12}+\delta_{34})}}{r_n^4}O\Big(\frac{a}{n}\Big)\sqrt{P(r_n<W_n^{(12)}(a2^j)<1/2)P(r_n<W_n^{(34)}(a2^j)<1/2)}\\
&\le \frac{a^{2(\delta_{12}+\delta_{34})}}{r_n^4}O\Big(\frac{a}{n}\Big)\exp\Big\{-\frac{1}{2}\Big[\Big(\frac{n}{a^{2(h_1+h_2)-4h_{12}+1}}\Big)^{1-\xi}+\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big]\Big\}\\
&= O\Big(\Big(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\Big)^2\Big),
\end{aligned}\tag{C.80}$$

since $2(\delta_{12}+\delta_{34})\le 8h_{\max}-8h_{\min}$. The second term in the sum (C.78) is bounded by

$$\begin{aligned}
&\frac{Ca^{2(\delta_{12}+\delta_{34})}}{r_n^2}\sqrt{\mathbb{E}\Big[\Big(\frac{W_n^{(12)}(a2^j)-1}{a^{\delta_{12}}}\Big)^4\Big(\frac{W_n^{(34)}(a2^{j'})-1}{a^{\delta_{34}}}\Big)^4\Big]}\cdot\sqrt{\mathbb{E}\Big[1_{\{W_n^{(12)}(a2^j)\ge 1/2\}}1_{\{r_n<W_n^{(34)}(a2^j)<1/2\}}\Big]}\\
&\le \frac{Ca^{2(\delta_{12}+\delta_{34})}}{r_n^2}O\Big(\frac{a}{n}\Big)\sqrt{P(W_n^{(12)}(a2^j)\ge 1/2)P(r_n<W_n^{(34)}(a2^j)<1/2)}\\
&\le \frac{Ca^{2(\delta_{12}+\delta_{34})}}{r_n^2}O\Big(\frac{a}{n}\Big)\exp\Big\{-\frac{1}{2}\Big(\frac{n}{a^{2(h_3+h_4)-4h_{34}+1}}\Big)^{1-\xi}\Big\}=O\Big(\Big(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\Big)^2\Big).
\end{aligned}\tag{C.81}$$

An analogous bound holds for the third term in the sum (C.78). Therefore, by (C.79), (C.80) and (C.81), the fourth term in (C.76) is of the order

$$O\Big(\Big(\frac{a^{4h_{\max}-4h_{\min}+1}}{n}\Big)^2\Big).$$

By a similar reasoning, the same conclusion holds for the second and third terms in (C.76). Therefore, by (C.76) and Lemma C.5, we conclude that (C.72) is equal to the right-hand side of (4.3), as claimed. □

Remark C.5. For $q=q_1=q_2$, $W_{n,-}^{(qq)}(a2^j)=0$ a.s. (see (C.19)). Then, the existence of the moment $\mathbb{E}\Big[\log^l|W_n^{(qq)}(a2^j)|\Big]$, $l\in\mathbb{N}$, can be directly established by applying relation (96) in [54]. Moreover, the analysis of moments in this section can be extended without the truncation based on the sequence (4.2).

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