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HITCHIN HAMILTONIANS IN GENUS 2

VIKTORIA HEU AND FRANK LORAY

ABSTRACT. We give an explicit expression of the Hitchin Hamiltonian system for rank two vector bundles with trivial determinant bundle over a curve of genus two.

1. INTRODUCTION

We are interested in rank two vector bundles $E \rightarrow X$ with trivial determinant bundle $\det(E) = \mathcal{O}_X$ over a Riemann surface X of genus 2. The moduli space \mathcal{M}_{NR} of semistable such vector bundles up to S -equivalence has been constructed by Narasimhan and Ramanan in [16]. If $E \in \mathcal{M}_{NR}$ is stable (and is therefore the unique vector bundle S -equivalent to E), the cotangent space of \mathcal{M}_{NR} at E is canonically isomorphic to the moduli space of trace free holomorphic Higgs fields $\theta : E \rightarrow E \otimes \Omega_X^1$ on E :

$$T_E^\vee \mathcal{M}_{NR} \simeq \text{Higgs}(X)|_E.$$

Since $\mathcal{M}_{NR} \simeq \mathbb{P}^3$ and the locus of semistable but non stable bundles (up to S -equivalence) there is given by a singular quartic hypersurface, we have

$$T^\vee \mathcal{M}_{NR} \simeq \text{Higgs}(X)$$

in restriction to a Zariski open subset of \mathcal{M}_{NR} , where $\text{Higgs}(X)$ denotes the moduli space of tracefree holomorphic Higgs bundles (E, θ) .

In [13], Hitchin considered the map

$$(1.1) \quad \text{Hitch} : \left\{ \begin{array}{ll} \text{Higgs}(X) & \rightarrow \text{H}^0(X, \Omega_X^1 \otimes \Omega_X^1) \\ (E, \theta) & \mapsto \det(\theta) \end{array} \right\}$$

and established that it defines an algebraically completely integrable Hamiltonian system: the Liouville form on \mathcal{M}_{NR} induces a symplectic structure on $\text{Higgs}(X)$ and any set of (three) generators of $\text{H}^0(X, \Omega_X^1 \otimes \Omega_X^1)$ commutes for the induced Poisson structure. Moreover, fibers of the Hitchin map are open sets of abelian varieties whose compactification is given by the Jacobian of the spectral curve. A broad field of applications has been deduced from the various algebraic and geometric properties of the Hitchin system and its generalizations since then.

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Of course the Hitchin system in [13] is defined in a more general setting, but in the present paper, we focus on the special case as above (rank 2 vector bundles with trivial determinant over curves of genus 2) and announce results of a forthcoming paper [12]:

- We describe the moduli space $\text{Bun}(X)$ of (not necessarily semistable) vector bundles E equivariant under the hyperelliptic involution ι on X . On the categorical quotient we have a birational morphism

$$\text{Bun}(X) \dashrightarrow \mathcal{M}_{NR}.$$

- It is well-known that there is no universal bundles on a Zariski-open subset of \mathcal{M}_{NR} . Yet from the dictionary between equivariant bundles and parabolic bundles on the quotient established in [3] (see also [2]) we obtain a rational two-cover

$$\text{Bun}(X/\iota) \xrightarrow{2:1} \text{Bun}(X)$$

and we construct a universal bundle over affine charts of $\text{Bun}(X/\iota)$ which can be identified with the universal bundle in [4] obtained from different methods.

- We deduce a universal family of Higgs bundles on affine charts of $\text{Higgs}(X/\iota) := T^\vee \text{Bun}(X/\iota)$. Note that in restriction to the stable locus, $\text{Higgs}(X/\iota) \rightarrow \text{Bun}(X/\iota)$ is a principal \mathbb{C}^3 -bundle.
- These explicit universal families allow us to calculate the determinant map on $\text{Higgs}(X/\iota)$ explicitly, which by construction factors through the Hitchin map. We deduce an explicit expression of the Hitchin map (1.1) completing partial results in [7].

2. THE NARASIMHAN-RAMANAN MODULI SPACE \mathcal{M}_{NR}

Let us first briefly recall the classical Narasimhan-Ramanan construction. Let $E \rightarrow X$ be a semistable rank 2 vector bundle with trivial determinant bundle over a Riemann surface X of genus 2. The subset

$$C_E := \{L \in \text{Pic}^1(X) \mid \dim H^0(X, E \otimes L) > 0\}$$

of $\text{Pic}^1(X)$ defines a divisor D_E on $\text{Pic}^1(X)$ which is linearly equivalent to the divisor 2Θ on $\text{Pic}^1(X)$, where Θ denotes the theta-divisor defined by the canonical embedding of X in $\text{Pic}^1(X)$. In that way, we associate to the vector bundle E an element D_E of the Narasimhan-Ramanan moduli space

$$\mathcal{M}_{NR} := \mathbb{P}H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1(X)}(2\Theta)) \simeq \mathbb{P}^3.$$

For each smooth (analytic or algebraic) family $\mathcal{E} \rightarrow X \times T$ of semistable rank 2 vector bundle with trivial determinant bundle on X , the Narasimhan-Ramanan classifying map

$$T \rightarrow \mathcal{M}_{NR}; \quad t \rightarrow D_{\mathcal{E}|_{X \times \{t\}}}$$

is an (analytic or algebraic) morphism. Moreover, if $D_E = D_{E'}$, then the vector bundles E and E' are S -equivalent:

- either E is stable and then $E' = E$

- or E is strictly semistable (*i.e.* semistable but not stable) and then there are line subbundles L and L' of degree 0 of E and E respectively, such that $L = L'$ or $L = L'^{\otimes -1}$

The strictly semistable locus in \mathcal{M}_{NR} , which we shall denote by $\text{Kum}(X)$ is thus defined by an embedding

$$\left\{ \begin{array}{ccc} \text{Pic}^0(X)/\iota & \hookrightarrow & \mathcal{M}_{NR} \\ L \bmod \iota & \mapsto & D_{L \oplus L^{\otimes -1}} = L \cdot \Theta + \iota^* L \cdot \Theta \end{array} \right\}.$$

Note that if ι denotes the hyperelliptic involution on X , then $\iota^* L = L^{\otimes -1}$ by a classical argument that will be recalled in Section 2.1.

2.1. Straightforward coordinates on \mathcal{M}_{NR} . Any compact connected Riemann surface X of genus 2 can be embedded into $\mathbb{P}^1 \times \mathbb{P}^1$ and is given, in a convenient affine chart, by an equation of the form

$$(2.1) \quad X : y^2 = F(x) \quad \text{with} \quad F(x) = x(x-1)(x-r)(x-s)(x-t).$$

The hyperelliptic involution on X then writes $\iota : (x, y) \mapsto (x, -y)$ and the induced projection on the Riemann sphere \mathbb{P}^1 is given by $\pi : (x, y) \mapsto x$. Denote by

$$W := \{w_0, w_1, w_r, w_s, w_t, w_\infty\}$$

the six Weierstrass points on X invariant under the hyperelliptic involution, given by $w_i = (i, 0)$ for $i \neq \infty$ and $w_\infty = (\infty, \infty)$. We will write \underline{W} for $\pi(W)$.

Recall that the rational map

$$\left\{ \begin{array}{ccc} X^2 & \dashrightarrow & \text{Pic}^2(X) \\ \{P, Q\} & \mapsto & [P] + [Q] \end{array} \right\}$$

is surjective. More precisely, it is a blow-up of the canonical divisor

$$K_X \sim [P] + [\iota(P)] \quad \text{for all} \quad P \in X.$$

Moreover, $\text{Pic}^2(X) \simeq \text{Pic}^1(X)$; $D \mapsto D - [w_\infty]$ is an isomorphism. Global sections of $\mathcal{O}_{\text{Pic}^1}(2\Theta)$ thus correspond bijectively to symmetric meromorphic functions on $X \times X$ with polar divisor at most $2\Delta + 2\infty_1 + 2\infty_2$, where $\Delta := \{(P, \iota(P)) \mid P \in X\}$ and $\infty_i := \{(P_1, P_2) \mid P_i = w_\infty\}$. Any set of (four) generators of the vector space $H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1}(2\Theta))$ can be expressed as such meromorphic functions, the simplest one being certainly the set $\{1, \text{Sum}, \text{Prod}, \text{Diag}\}$ of functions in $(P_1, P_2) \in X \times X$ given, for $P_i = (x_i, y_i)$, by

$$(2.2) \quad \begin{aligned} 1 & : (P_1, P_2) \mapsto 1 \\ \text{Sum} & : (P_1, P_2) \mapsto x_1 + x_2 \\ \text{Prod} & : (P_1, P_2) \mapsto x_1 x_2, \\ \text{Diag} & : (P_1, P_2) \mapsto \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - (x_1 + x_2)^3 + (1 + r + s + t)(x_1 + x_2)^2 + \\ & \quad + x_1 x_2 (x_1 + x_2) - (r + s + t + rs + st + tr)(x_1 + x_2) \end{aligned}$$

We obtain coordinates on $\mathcal{M}_{NR} = \mathbb{P}H^0(\text{Pic}^1(X), \mathcal{O}_{\text{Pic}^1}(2\Theta)) \simeq \mathbb{P}^3$, where we identify a point $(v_0 : v_1 : v_2 : v_3)$ with the push-forward D_E on $\text{Pic}^1(X)$ of the zero-divisor of the meromorphic function

$$(2.3) \quad v_0 \cdot 1 + v_1 \cdot \text{Sum} + v_2 \cdot \text{Prod} + v_3 \text{Diag}$$

on $X^{(2)}$.

For example if $E = L \oplus L^{-1}$, where

$$L = \mathcal{O}_X([Q_1] + [Q_2] - 2[w_\infty]) = \mathcal{O}_X([Q_1] - [l(Q_2)])$$

and $Q_1, Q_2 \in X$, we can calculate the (unique up to scalar) meromorphic function as in (2.3) whose 0-divisor corresponds to $D_E = L \cdot \Theta + \iota^* L \cdot \Theta$ and we obtain

$$(2.4) \quad (v_0 : v_1 : v_2 : v_3) = (-\text{Diag}(Q_1, Q_2) : \text{Prod}(Q_1, Q_2) : -\text{Sum}(Q_1, Q_2) : 1).$$

The strictly semistable locus $\text{Kum}(X)$ in \mathcal{M}_{NR} is parametrized by $\{Q_1, Q_2\} \in X^{(2)}$ according to formula (2.4). We deduce an equation for $\text{Kum}(X)$ in our coordinates $(v_0 : v_1 : v_2 : v_3)$ of \mathcal{M}_{NR} :

$$(2.5) \quad \begin{aligned} \text{Kum}(X) : \\ 0 = & \quad (v_0 v_2 - v_1^2)^2 \cdot 1 \\ & - 2 [(\sigma_1 + \sigma_2)v_1 + (\sigma_2 + \sigma_3)v_2](v_0 v_2 - v_1^2) \\ & + 2(v_0 + \sigma_1 v_1)(v_0 + v_1)v_1 + 2(\sigma_2 v_1 + \sigma_3 v_2)(v_1 + v_2)v_1] \cdot v_3 \\ & - 2\sigma_3(v_0 v_2 - v_1^2) + [(\sigma_1 + \sigma_2)^2 v_1 + (\sigma_2 + \sigma_3)^2 v_2](v_1 + v_2) \\ & - (\sigma_1 + \sigma_3)^2 v_1 v_2 + 4[(\sigma_2 + \sigma_3)v_0 - \sigma_3 v_2]v_1] \cdot v_3^2 \\ & - 2\sigma_3 [(\sigma_1 + \sigma_2)v_1 - (\sigma_2 + \sigma_3)v_2] \cdot v_3^3 \\ & + \sigma_3^2 \cdot v_3^4. \end{aligned}$$

Since the strictly semistable locus $\text{Kum}(X)$ is a quartic with sixteen conic singularities it is usually referred to as *the Kummer surface* in the context of \mathcal{M}_{NR} .

Let $\mathcal{O}_X(\tau)$ be a 2-torsion line bundle on X , *i.e.* $\mathcal{O}_X(2\tau) \simeq \mathcal{O}_X$. Then

$$(2.6) \quad \tau \sim [w_i] - [w_j] \quad \text{with } w_i, w_j \in W$$

and the group of 2-torsion line bundles on X with respect to the tensor product is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$. If E is a rank two vector bundle with trivial determinant bundle over X , then its *twist* $E \otimes \mathcal{O}_X(\tau)$ also has the trivial determinant line bundle. Moreover, by construction of the Narasimhan-Ramanan moduli space, the action of the group of 2-torsion line bundles by twist is linear and free on \mathcal{M}_{NR} and preserves $\text{Kum}(X)$. By formula (2.4) we can explicitly calculate the coordinates $(v_0 : v_1 : v_2 : v_3)$ of the trivial bundle $E_0 = \mathcal{O}_X \oplus \mathcal{O}_X$ and its twists

$$E_\tau := E_0 \otimes \mathcal{O}_X(\tau).$$

The trivial bundle for example is given by $E_0 : (1 : 0 : 0 : 0)$. Note that these sixteen bundles E_τ correspond to the sixteen singularities of the Kummer surface $\text{Kum}(X)$.

The fact that we know the action (by permutation) of the 2-torsion group on the set of bundles E_τ and we also know the coordinates of these bundles in the Narasimhan-Ramanan moduli space is sufficient to calculate explicitly the linear action of the 2-torsion group on \mathcal{M}_{NR} : for any τ as in (2.6), there is a matrix $M_\tau \in \mathrm{SL}_4\mathbb{C}$ such that if the image of E under the Narasimhan-Ramanan classifying map is given by $(v_0 : v_1 : v_2 : v_3)$, then $E' = E \otimes \mathcal{O}_X(\tau)$ is given by $(v'_0 : v'_1 : v'_2 : v'_3)$ with

$$\begin{pmatrix} v'_0 \\ v'_1 \\ v'_2 \\ v'_3 \end{pmatrix} = M_\tau \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

The equivalence classes in $\mathrm{PGL}_4\mathbb{C}$ of these matrices (with respect to a set of generators of the 2-torsion group) then are given by the following:

$$\begin{aligned} M_{[w_0]-[w_\infty]} &\sim \begin{pmatrix} 0 & rs + st + rt + rst & rst & 0 \\ 0 & 0 & 0 & rst \\ 1 & 0 & 0 & -(rs + st + rt + rst) \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ M_{[w_1]-[w_\infty]} &\sim \begin{pmatrix} 1 & r + s + t + rst & rs + st + rt & 0 \\ -1 & -1 & 0 & rs + st + rt \\ 1 & 0 & -1 & -(r + s + t + rst) \\ 0 & 1 & 1 & 1 \end{pmatrix} \\ M_{[w_r]-[w_\infty]} &\sim \begin{pmatrix} r^2 & r^2(1 + s + t) + st & r^2(s + t + st) & 0 \\ -r & -r^2 & 0 & r^2(s + t + st) \\ 1 & 0 & -r^2 & -r^2(1 + s + t) - st \\ 0 & 1 & r & r^2 \end{pmatrix} \\ M_{[w_s]-[w_\infty]} &\sim \begin{pmatrix} s^2 & s^2(1 + r + t) + rt & s^2(r + t + rt) & 0 \\ -s & -s^2 & 0 & s^2(r + t + rt) \\ 1 & 0 & -s^2 & -s^2(1 + r + t) - rt \\ 0 & 1 & s & s^2 \end{pmatrix} \end{aligned}$$

2.2. Nice coordinates on \mathcal{M}_{NR} . A quick calculation shows that the character of the representation

$$\rho : \left\{ \begin{array}{ll} (\mathbb{Z}/2\mathbb{Z})^4 & \rightarrow \mathrm{SL}_4\mathbb{C} \\ \mathcal{O}_X(\tau) & \mapsto M_\tau \end{array} \right\}$$

introduced above is the regular one : it vanishes on all elements of the group except \mathcal{O}_X . Hence ρ is conjugated for example to the regular representation $\tilde{\rho} : \mathcal{O}_X(\tau) \mapsto \tilde{M}_\tau$ given

by

$$\begin{aligned} \widetilde{M}_{[w_0]-[w_\infty]} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & \widetilde{M}_{[w_1]-[w_\infty]} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \\ \widetilde{M}_{[w_r]-[w_\infty]} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & \widetilde{M}_{[w_s]-[w_\infty]} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \end{aligned}$$

Any conjugation matrix $M \in \mathrm{SL}_4\mathbb{C}$ such that $\tilde{\rho} = M\rho M^{-1}$ is given, up to a scalar, as follows: Choose square-roots $\omega_0, \omega_1, \omega_r, \omega_s$ such that

$$\omega_0^2 = F'(0), \quad \omega_1^2 = -F'(1), \quad \omega_r^2 = F'(r), \quad \omega_s^2 = F'(s),$$

where $F(x)$ is given in (2.1) and $F'(x)$ is its derivative with respect to x . Then

$$(2.7) \quad M = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & -\omega_0 \\ 0 & \omega_1 & 0 & 0 \\ 0 & \omega_0 & \omega_0 & \omega_0 \\ 0 & 0 & 0 & \omega_0\omega_1 \end{pmatrix},$$

$$\begin{aligned} \text{where} \quad a &= rst(r-s)\omega_1 + t\omega_r\omega_s - rt(r-1)\omega_s - st\omega_1\omega_r \\ b &= -st(s-1)\omega_r + rt\omega_1\omega_s \\ c &= t(r-s)\omega_0\omega_1 - t(r-1)\omega_0\omega_s \\ d &= -t(r-1)(s-1)(r-s)\omega_0 + t(s-1)\omega_0\omega_r. \end{aligned}$$

After the coordinate-change $(v_0 : v_1 : v_2 : v_3) \mapsto (u_0 : u_1 : u_2 : u_3)$ on \mathcal{M}_{NR} defined by

$$\begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix} = M \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

the action of the 2-torsion group is then normalized to $\tilde{\rho}$. In particular, the equation of the Kummer surface with respect to the coordinates $(u_0 : u_1 : u_2 : u_3)$ is invariant under double-transpositions and double-changes of signs. Calculation shows

$$(2.8) \quad \begin{aligned} \text{Kum}(X) : \\ 0 &= (u_0^4 + u_1^4 + u_2^4 + u_3^4) - 8 \frac{rs-rt+r-s}{t(s-1)} u_0 u_1 u_2 u_3 - 2 \frac{st+t-2s}{t(s-1)} (u_0^2 u_3^2 + u_1^2 u_2^2) \\ &\quad - 2 \frac{2r-t}{t} (u_1^2 u_3^2 + u_0^2 u_2^2) + 2 \frac{2r-s-1}{s-1} (u_2^2 u_3^2 + u_0^2 u_1^2). \end{aligned}$$

In summary, the straightforward coordinates $(v_0 : v_1 : v_2 : v_3)$ of \mathcal{M}_{NR} introduced in the previous section have the advantage that

- a given divisor D_E on $\mathrm{Pic}^1(X)$ linearly equivalent to 2Θ can rather easily be expressed in terms of $(v_0 : v_1 : v_2 : v_3)$,

- and we are going to use this property when we describe the universal family on a 2-cover of \mathcal{M}_{NR} ,

whereas the new coordinates $(u_0 : u_1 : u_2 : u_3)$ of \mathcal{M}_{NR} defined above have the advantages that

- the action of the 2-torsion group is simply expressed by double-transpositions and double-changes of signs of $(u_0 : u_1 : u_2 : u_3)$ and
- the equation of the Kummer surface is rather symmetric. As pointed out in [7], the classical line geometry for Kummer surfaces in \mathbb{P}^3 is related to certain symmetries of the Hitchin Hamiltonians. For this geometrical reason, the explicit Hitchin Hamiltonians we are going to establish have a much simpler expression with respect to (dual) coordinates $(u_0 : u_1 : u_2 : u_3)$ when compared to $(v_0 : v_1 : v_2 : v_3)$.
- Moreover, the five u_i -polynomials in (2.8) invariant under the action of the 2-torsion group define a natural map $\mathcal{M}_{NR} \rightarrow \mathbb{P}^4$. The image is a quartic hyper surface [5, Proposition 10.2.7] and can be seen as the coarse moduli space of semistable \mathbb{P}^1 -bundles over X .

3. HOW TO CONSTRUCT A BUNDLE FROM A POINT IN \mathcal{M}_{NR}

Given a stable rank 2 vector bundle E with trivial determinant bundle on X , the Narasimhan-Ramanan divisor $D_E \in |2\Theta|$ can be seen as space of line subbundles L of E of degree -1 . Whilst we know that for any $D \in |2\Theta|$ there is a semistable vector bundle E with $D_E = D$, it is not obvious how to construct it. We provide such a construction by considering the moduli space of rank 2 vector bundles E with trivial determinant bundle on X equivariant under the hyperelliptic involution. For the present exposition however, we restrict our attention to the space $\text{Bun}(X)$ of rank 2 vector bundles E with trivial determinant bundle on X such that

- E is stable but *off the odd Gunning planes*, which means that no line subbundle $L \subset E$ is isomorphic to $\mathcal{O}_X(-[w_i])$ for some $w_i \in W$, or
- E is strictly semistable but undecomposable, or
- $E = L \oplus \iota^*L$ where $L = \mathcal{O}_X([P] - [Q])$ satisfies $P, Q \notin W$, or
- E is an odd Gunning bundle, *i.e* given by the unique non-trivial extension

$$0 \longrightarrow \mathcal{O}_X([w_i]) \longrightarrow E \longrightarrow \mathcal{O}_X(-[w_i]) \longrightarrow 0$$

for a Weierstrass point $w_i \in W$.

We construct $\text{Bun}(X)$ as an algebraic stack whose categorical quotient is birational to the Narasimhan-Ramanan moduli space

$$\text{Bun}(X) \dashrightarrow^{1:1} \mathcal{M}_{NR}.$$

For convenience of notation let us for now denote by $\text{Bun}(X)$ the set of vector bundles E as in the above list, before we put an algebraic structure on $\text{Bun}(X)$. We use the fact that any bundle $E \in \text{Bun}(X)$ is equivariant under the hyperelliptic involution:

Proposition 3.1. *Let E be a vector bundle in $\text{Bun}(X)$. Then there is a bundle isomorphism h such that the following diagram commutes*

$$\begin{array}{ccc} E & \xrightarrow{\sim} & \iota^* E & \xrightarrow{\sim} & E \\ & \searrow h & & \nearrow \iota^* h & \\ & & \text{id}_E & & \end{array}$$

and such that for each Weierstrass point $w_i \in W$, the induced automorphism of the Weierstrass fibre

$$h|_{E_{w_i}} : E_{w_i} \rightarrow \iota^* E_{w_i} \simeq E_{w_i}$$

possesses two opposite eigenvalues $+1$ and -1 .

Now, hyperelliptic descent [3]

$$\pi_* E = (\underline{E}^+, \underline{\mathbf{p}}^+) \oplus (\underline{E}^-, \underline{\mathbf{p}}^-)$$

produces two rank 2 vector bundles \underline{E}^\pm with determinant bundle $\det(\underline{E}^\pm) = \mathcal{O}_{\mathbb{P}^1}(-3)$ over the Riemann sphere, each endowed with a natural quasi-parabolic structure $\underline{\mathbf{p}}^\pm$ with support $\underline{W} = \pi(W)$. Moreover,

Proposition 3.2. *Consider (E, h) as in Proposition 3.1. Denote by \mathbf{p}^+ and \mathbf{p}^- the quasi-parabolic structure with support W on E induced by the $+1$ and -1 eigendirections of h respectively. Then*

$$(E, \mathbf{p}^\pm) = \text{elm}_W^+(\pi^*(\underline{E}^\pm, \underline{\mathbf{p}}^\pm)),$$

where elm_W^+ denotes the composition of six positive elementary transformations, one over each Weierstrass point, given by the corresponding quasi-parabolic direction of $\pi^*(\underline{E}^\pm, \underline{\mathbf{p}}^\pm)$.

In convenient local coordinates $(\zeta, Y) \in U \times \mathbb{C}^2$ of $E \rightarrow X$ near a Weierstrass point $w_i : \{\zeta = 0\}$, the map $\text{elm}_W^+ \circ \pi^*$ can be understood as follows:

$$\begin{array}{ccc} \mathbf{p}|_{w_i} : \{Y \in \text{Vect}_{\mathbb{C}}(\begin{smallmatrix} 1 & \\ & 0 \end{smallmatrix})\} & & (\zeta, Y) \xrightarrow{h} (-\zeta, (\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix})Y) \\ & \text{elm}_{\mathbf{p}}^+ \uparrow & \text{elm}_{\mathbf{p}}^+ \uparrow \\ \hat{\mathbf{p}}|_{w_i} : \{\hat{Y} \in \text{Vect}_{\mathbb{C}}(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix})\} & & (\zeta, \hat{Y}) = (\zeta, (\begin{smallmatrix} 1 & 0 \\ 0 & \zeta \end{smallmatrix})Y) \xrightarrow{h} (-\zeta, \hat{Y}) \\ & \pi^* \uparrow & \pi^* \uparrow \\ \underline{\mathbf{p}}|_{\pi(w_i)} : \{\underline{Y} \in \text{Vect}_{\mathbb{C}}(\begin{smallmatrix} 0 & \\ & 1 \end{smallmatrix})\} & & (\underline{\zeta}, \underline{Y}) = (\zeta^2, \hat{Y}) \xrightarrow{\text{id}} (\underline{\zeta}, \underline{Y}) \end{array}$$

Let μ be a real number in $[0, 1]$. Denote by $\text{Bun}_\mu(X/\iota)$ the moduli space of pairs $(\underline{E}, \underline{\mathbf{p}})$, where \underline{E} is a rank 2 vector bundle of degree -3 over \mathbb{P}^1 and $\underline{\mathbf{p}}$ is a quasi-parabolic structure with support \underline{W} such that $(\underline{E}, \underline{\mathbf{p}})$ is a stable parabolic bundle if to each quasi-parabolic direction $\underline{\mathbf{p}}|_{w_i}$ we associate the parabolic weight μ . For each choice of μ , this moduli space is either empty or birational to \mathbb{P}^3 [15]. Moreover, for any $\mu \in [0, 1]$, the map

$$\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+$$

is a canonical birational isomorphism between $\text{Bun}_\mu(X/\iota)$ and $\text{Bun}_{1-\mu}(X/\iota)$.

Note further that for $\mu = \frac{1}{5}$, the space $\text{Bun}_\mu(X/\iota)$ is precisely the moduli space of those quasi-parabolic bundles $(\underline{E}, \underline{p})$, where \underline{E} is a vector bundle on \mathbb{P}^1 and \underline{p} is a quasi parabolic structure with support \underline{W} on \underline{E} such that

- $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$,
- the quasi-parabolic directions $\underline{p}_{\underline{w}_i}$ are all disjoint from the total space of the destabilizing subbundle $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ and
- the quasi-parabolic directions $\underline{p}_{\underline{w}_i}$ are not all contained in the total space of a same subbundle $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$

Consider the following affine chart $(R, S, T) \in \mathbb{C}^3$ of $\text{Bun}_{\frac{1}{5}}(X/\iota)$, which we shall call *the canonical chart* : Recall that $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. Let σ_1 be a meromorphic section of some line subbundle $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$ with only one (double) pole over $x = \infty$. Let σ_{-1} be a meromorphic section of the unique line subbundle $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ with only one pole over $x = \infty$. In the total space of \underline{E} restricted to $\mathbb{P}^1 \setminus \{\infty\}$, we consider coordinates $(x, (\frac{z_1}{z_2}))$ given by $(x, z_1\sigma_{-1} + z_2\sigma_1)$. To $(R, S, T) \in \mathbb{C}^3$ we then associate the following normalized quasi-parabolic structure on \underline{E} :

$$(3.1) \quad \begin{array}{cccccc} x = 0 & x = 1 & x = r & x = s & x = t & x = \infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} R \\ 1 \end{pmatrix} & \begin{pmatrix} S \\ 1 \end{pmatrix} & \begin{pmatrix} T \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1) \end{array}.$$

Here the first line indicates the Weierstrass point \underline{w}_i we are considering, whereas the second line defines a generator of the corresponding quasi-parabolic direction. Note that (3.1) already defines a universal quasi-parabolic bundle over the canonical chart. The lifting map $\text{elm}_{\underline{W}}^+ \circ \pi^*$ is well-defined and algebraic and provides a universal rank 2 vector bundle with trivial determinant bundle over the canonical chart of $\text{Bun}_{\frac{1}{5}}(X/\iota)$.

Proposition 3.3. *The Narasimhan-Ramanan classifying map $\mathbb{C}_{(R,S,T)}^3 \dashrightarrow \mathcal{M}_{NR}$ is explicitly given by $(R, S, T) \mapsto (v_0 : v_1 : v_2 : v_3)$ where*

$$\begin{aligned} v_0 &= s^2t^2(r^2 - 1)(s - t)R - r^2t^2(s^2 - 1)(r - t)S + s^2r^2(t^2 - 1)(r - s)T + \\ &\quad + t^2(t - 1)(r^2 - s^2)RS - s^2(s - 1)(r^2 - t^2)RT + r^2(r - 1)(s^2 - t^2)ST \\ v_1 &= rst [((r - 1)(s - t)R - (s - 1)(r - t)S + (t - 1)(r - s)T + \\ &\quad + (t - 1)(r - s)RS - (s - 1)(r - t)RT + (r - 1)(s - t)ST] \\ v_2 &= -st(r^2 - 1)(s - t)R + rt(s^2 - 1)(r - t)S - rs(t^2 - 1)(r - s)T - \\ &\quad - t(t - 1)(r^2 - s^2)RS + s(s - 1)(r^2 - t^2)RT - r(r - 1)(s^2 - t^2)ST \\ v_3 &= st(r - 1)(s - t)R - rt(s - 1)(r - t)S + sr(t - 1)(r - s)T + \\ &\quad + t(t - 1)(r - s)RS - s(s - 1)(r - t)RT + r(r - 1)(s - t)ST \end{aligned}$$

The indeterminacy points

$$(R, S, T) = (0, 0, 0), \quad (1, 1, 1) \quad \text{and} \quad (r, s, t)$$

of this map correspond to the odd Gunning bundles $E_{[w_1]}$, $E_{[w_0]}$ and $E_{[w_\infty]}$ respectively. Conversely, a generic point $(v_0 : v_1 : v_2 : v_3) \in \mathcal{M}_{NR}$ has precisely two preimages in

$\mathbb{C}_{(R,S,T)}^3$ given by

$$R = \frac{r(t-1)(v_0+rv_1-r(s+t+st)v_3)T}{t(r-1)(v_0+tv_1-t(r+s+rs)v_3)-(r-t)(v_0+v_1-(rs+st+rt)v_3)T}$$

$$S = \frac{s(t-1)(v_0+sv_1-s(r+t+rt)v_3)T}{t(s-1)(v_0+tv_1-t(r+s+rs)v_3)-(s-t)(v_0+v_1-(rs+st+rt)v_3)T},$$

where T is any solution of $aT^2 + bT + ct^2 = 0$ with

$$a = (v_1 + v_2t + v_3t^2)(v_0 + v_1 - (rs + st + rt)v_3)$$

$$b = -(1+t)(v_0v_2 + v_1^2 + tv_1v_3) - 2(v_0v_1 + tv_0v_3 + tv_1v_2) \\ + (rs + st + rt)(tv_1 + v_2 + tv_3)v_3 + (r + s + rs)(v_1 + t^2v_2 + t^2v_3)v_3$$

$$c = (v_1 + v_2 + v_3)(v_0 + tv_1 - t(r + s + rs)v_3).$$

The discriminant of this polynomial leads again to the equation (2.5) of the Kummer surface.

By construction, $\text{Bun}_{\frac{1}{5}}(X/\iota)$ is covered by affine charts similar to the canonical chart, where we just permute the role of the Weierstrass points in (3.1). The (birational) transition maps between affine charts are obvious. The Galois-involution $\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+$ is given in the canonical chart by the birational map $(R, S, T) \mapsto (\tilde{R}, \tilde{S}, \tilde{T})$, where

$$\tilde{R} = \lambda(R, S, T) \cdot \frac{(s-t)+(t-1)S-(s-1)T}{-t(s-1)S+s(t-1)T+(s-t)ST}$$

$$\tilde{S} = \lambda(R, S, T) \cdot \frac{(r-t)+(t-1)R-(r-1)T}{-t(r-1)R+r(t-1)T+(r-t)RT}$$

$$\tilde{T} = \lambda(R, S, T) \cdot \frac{(r-s)+(s-1)R-(r-1)S}{-s(r-1)R+r(s-1)S+(r-s)RS}$$

$$\text{and } \lambda(R, S, T) = \frac{t(r-s)RS-s(r-t)RT+r(s-t)ST}{(s-t)R-(r-t)S+(r-s)T}.$$

For the Galois-involution to be everywhere well defined, we need to consider the smooth (non separated) scheme $\text{Bun}(X/\iota)$ obtained by canonically gluing $\text{Bun}_{\frac{1}{5}}(X/\iota)$ and $\text{Bun}_{\frac{4}{5}}(X/\iota)$. From an exhaustive case-by case study, one can show that $\text{Bun}(X)$ corresponds precisely to the isomorphism classes of the Galois-involution on $\text{Bun}(X/\iota)$. In terms of parabolic bundles, the Galois involution is given by $\text{elm}_{\underline{W}}^+ \circ \pi^*$. In other words,

Proposition 3.4. *The map $\text{elm}_{\underline{W}}^+ \circ \pi^* : \text{Bun}(X/\iota) \xrightarrow{2:1} \text{Bun}(X)$ is an algebraic 2 cover.*

Moreover, the lift of the Kummer surface in \mathcal{M}_{NR} defines a dual Weddle surface in $\text{Bun}_{\frac{1}{5}}(X/\iota) \simeq \mathbb{P}^3 \subset \text{Bun}(X/\iota)$ which is given with respect to the canonical chart by the equation

$$\text{Wed}(X) \quad :$$

$$0 = ((s-t)R + (t-r)S + (r-s)T)RST + t((r-1)S - (s-1)R)RS \\ + r((s-1)T - (t-1)S)ST + s((t-1)R - (r-1)T)RT \\ - t(r-s)RS - r(s-t)ST - s(t-r)RT.$$

4. APPLICATION TO HIGGS BUNDLES

Let E again be a rank 2 vector bundle over X . By definition, the moduli space of tracefree Higgs fields on E is given by $H^0(X, \mathfrak{sl}(E) \otimes \Omega_X^1)$, where $\mathfrak{sl}(E)$ denotes the vector bundle of trace-free endomorphisms of E . By Serre duality, we have

$$H^0(X, \mathfrak{sl}(E) \otimes \Omega_X^1) \simeq H^1(X, \mathfrak{sl}(E)^\vee \otimes \Omega_X^1)^\vee.$$

If $\det(E) = \mathcal{O}_X$, then $\mathfrak{sl}(E)^\vee = \mathfrak{sl}(E)$. If E is stable, then $\mathfrak{sl}(E)$ possesses no non-trivial global sections and then $H^1(X, \mathfrak{sl}(E)^\vee \otimes \Omega_X^1)$ canonically identifies with $\text{Higgs}(X) := T^\vee \mathcal{M}_{NR}$. We can calculate explicitly the Hitchin map

$$\text{Hitch} : \left\{ \begin{array}{ll} \text{Higgs}(X) & \rightarrow H^0(X, \Omega_X^1 \otimes \Omega_X^1) \\ (E, \theta) & \mapsto \det(\theta) \end{array} \right\}$$

from the following idea : The complement in $\text{Bun}(X)$ of the image of the Weddle surface is embedded into $\mathcal{M} \setminus \text{Kum}(X)$ (we obtain all stable bundles except those on the odd Gunning planes). Since we have a universal vector bundle in each affine chart of the two-cover $\text{Bun}(X/\iota)$ of $\text{Bun}(X)$, we can expect to find a universal family of Higgs bundles there as well. Then we calculate a Hitchin map for $\text{Bun}(X/\iota)$ and push it down to \mathcal{M}_{NR} .

More precisely, we will calculate the Hitchin map in the following steps :

- Provided that $H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{\mathbf{p}})) = \{0\}$, we have a canonical isomorphism

$$T_{(\underline{E}, \underline{\mathbf{p}})} \text{Bun}(X/\iota) = H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{\mathbf{p}})),$$

where $\mathfrak{sl}(\underline{E}, \underline{\mathbf{p}})$ denotes trace free endomorphisms of \underline{E} leaving $\underline{\mathbf{p}}$ invariant. We work out how the hyperelliptic descent $\phi : \text{elm}^+ \circ \pi^*$ defines an algebraic 2-cover

$$\text{Higgs}(X/\iota) := T^\vee \text{Bun}(X/\iota) \xrightarrow{\phi} T^\vee \text{Bun}(X)$$

- The Liouville form on $\text{Bun}(X/\iota)$ is given with respect to coordinates (R, S, T) of the canonical chart by $dR + dS + dT$. We work out Serre duality for the generators

$$\frac{\partial}{\partial R}, \frac{\partial}{\partial S}, \frac{\partial}{\partial T} \in T \text{Bun}(X/\iota)$$

and deduce an explicit universal Higgs bundle on an affine chart of $\text{Higgs}(X/\iota)$.

- We calculate the determinant map

$$\text{Higgs}(X/\iota) \longrightarrow H^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1 \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})) \simeq H^0(X, \Omega_X^1 \otimes \Omega_X^1)$$

and show that it factors through the Hitchin map. Then we deduce the explicit Hitchin map from the formulas in Proposition 3.3.

4.1. Hyperelliptic descent, again. One can show that if E is a stable rank 2 vector bundle with trivial determinant on X and h is a lift of the hyperlliptic involution as in Proposition 3.1, then any trace free Higgs field θ on E is h -equivariant, that is, the

following diagram commutes :

$$\begin{array}{ccc} E & \xrightarrow{\theta} & E \otimes \Omega_X^1 \\ \downarrow h & & \uparrow \iota^* h \\ \iota^* E & \xrightarrow{\iota^* \theta} & \iota^* E \otimes \Omega_X^1. \end{array}$$

Hyperelliptic decent of the pair (E, θ) then produces two triples $(\underline{E}, \underline{\theta}, \underline{p})$, where

$$\underline{\theta} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$$

is a logarithmic Higgs field with at most apparent singularities over \underline{W} : the residue at any $\underline{w}_i \in \underline{W}$ is either zero or conjugated to

$$\text{Res}_{\underline{w}_i}(\underline{\theta}) \sim \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

such that the quasi-parabolic $\underline{p}_{\underline{w}_i}$ corresponds precisely to the 0-eigendirection of the residue. Conversely, consider a logarithmic Higgs field

$$\tilde{\theta} \in H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \tilde{\underline{p}}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$$

that lies in the image of the canonical embedding

$$H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}) \otimes \Omega_{\mathbb{P}^1}^1) \hookrightarrow H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})).$$

In other words, $\tilde{\theta}$ has only trivial residues:

$$\text{Res}_{\underline{w}_i}(\tilde{\theta}) \sim \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then the logarithmic Higgs field $\underline{\theta}$ obtained from $\tilde{\theta}$ by applying the meromorphic gauge transformation

$$\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+$$

has (at most) apparent singularities over each Weierstrass point, and the quasi-parabolic structure \underline{p} obtained from $\tilde{\underline{p}}$ corresponds to 0-eigendirections of $\underline{\theta}$. Note that here we have to choose a meromorphic section $\sigma : \mathbb{P}^1 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-3)$. As long as we are not on the Kummer, respectively Weddle surface, we obtain an isomorphism

$$\begin{array}{ccc} T_{(\underline{E}, \tilde{\underline{p}})}^\vee \text{Bun}(X/\iota) = H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \tilde{\underline{p}}))^\vee & \xrightarrow[\sim]{\text{Serre}} & H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \tilde{\underline{p}}) \otimes \Omega_{\mathbb{P}^1}^1) \\ \downarrow \sim & & \sim \downarrow \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+ \\ & & H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{apparent}} \\ & & \sim \downarrow \text{elm}_{\underline{W}}^+ \circ \pi^* \\ T_E^\vee \text{Bun}(X) = H^1(X, \mathfrak{sl}(E))^\vee & \xleftarrow[\text{Serre}]{\sim} & H^0(X, \mathfrak{sl}(E) \otimes \Omega_X^1) \end{array}$$

and deduce an algebraic 2-cover

$$\text{Higgs}(X/\iota) \xrightarrow{2:1} \text{Higgs}(X).$$

4.2. Universal Higgs bundles. Serre duality gives us a perfect pairing

$$\langle \cdot, \cdot \rangle : \left\{ \begin{array}{l} \mathrm{H}^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{\mathbf{p}})) \times \mathrm{H}^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}, \underline{\mathbf{p}}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{apparent}} \rightarrow \mathbb{C} \\ (\phi, \underline{\theta}) \mapsto \sum \text{Res}(\text{trace}(\phi \cdot \underline{\theta})) \end{array} \right\}.$$

Let $(\underline{E}, \underline{\mathbf{p}})$ be an element of $\text{Bun}(X/\iota)$ given with respect to the canonical chart by $(R_0, S_0, T_0) \in \mathbb{C}^3$. The vector field $\frac{\partial}{\partial R} \in \mathrm{T}_{(R_0, S_0, T_0)} \text{Bun}(X/\iota)$ is given in $\mathrm{H}^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{\mathbf{p}}))$ by the cocycle

$$\phi_{01} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

with respect to trivialization charts $U_0 \times \mathbb{C}^2$ with $U_0 := \mathbb{P}^1 \setminus \{r\}$ and $U_1 \times \mathbb{C}^2$ with $U_1 := D_\varepsilon(r)$ of $\mathfrak{sl}(\underline{E})$. Indeed, if we consider $\exp(\zeta\phi) = \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix}$ as applied from the left over $U_0 \cap U_1 \subset U_1$ to $U_0 \cap U_1 \subset U_0$ for a quasi-parabolic vector bundle $(\underline{E}, \underline{\mathbf{p}})$ with parabolic structure normalized as in (3.1), we obtain the quasi parabolic structure corresponding to $(R + \zeta, S, T)$. The dual basis in $\mathrm{H}^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}, \underline{\mathbf{p}}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{apparent}}$ with respect to $\langle \cdot, \cdot \rangle$ of the basis

$$\left(\frac{\partial}{\partial R}, \frac{\partial}{\partial S}, \frac{\partial}{\partial T} \right)$$

of $\mathrm{T}_{(R_0, S_0, T_0)} \text{Bun}(X/\iota)$ then is given by $(\underline{\theta}_r, \underline{\theta}_s, \underline{\theta}_t)$ with

$$\begin{aligned} \underline{\theta}_r &:= \begin{pmatrix} 0 & 0 \\ 1 - R & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} R & -R \\ R & -R \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -R & R^2 \\ -1 & R \end{pmatrix} \frac{dx}{x-r} \\ \underline{\theta}_s &:= \begin{pmatrix} 0 & 0 \\ 1 - S & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} S & -S \\ S & -S \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -S & S^2 \\ -1 & S \end{pmatrix} \frac{dx}{x-s} \\ \underline{\theta}_t &:= \begin{pmatrix} 0 & 0 \\ 1 - T & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} T & -T \\ T & -T \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -T & T^2 \\ -1 & T \end{pmatrix} \frac{dx}{x-t}. \end{aligned}$$

We obtain the universal Higgs bundle $\tilde{\underline{\theta}} = \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+(\underline{\theta})$ defined by

$$(4.1) \quad \underline{\theta} = c_r \underline{\theta}_r + c_s \underline{\theta}_s + c_t \underline{\theta}_t$$

on the canonical chart $(R, S, T, c_r, c_s, c_t) \in \mathbb{C}^6$ of $\mathrm{T}^\vee \text{Bun}(X/\iota)$. Recall that all other charts are obtained up to Galois involution by permuting the role of the Weierstrass points.

Corollary 4.1. *The Liouville form on $\text{Bun}(X/\iota)$, given with respect to the canonical chart $(R, S, T) \in \mathbb{C}^3$ by*

$$dR + dS + dT$$

defines a holomorphic symplectic 2-form on $\text{Higgs}(X/\iota)$ given with respect to the canonical chart $(R, S, T, c_r, c_s, c_t) \in \mathbb{C}^6$ of $\mathrm{T}^\vee \text{Bun}(X/\iota)$ by

$$dR \wedge dc_r + dS \wedge dc_s + dT \wedge dc_t.$$

4.3. **The Hitchin fibration.** The determinant map

$$(4.2) \quad \left\{ \begin{array}{ccc} \mathbb{C}^6 & \rightarrow & \mathrm{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1 \otimes \Omega_{\mathbb{P}^1}^1(-6)) \\ (R, S, T, c_r, c_s, c_t) & \mapsto & \det(\underline{\theta}) = (h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)} \end{array} \right\},$$

where $\underline{\theta}$ is the Higgs bundle in (4.1) is given by

$$h_0 = (c_r(R-1) + c_s(S-1) + c_t(T-1))(c_r s t(R-r)R + c_s r t(S-s)S + c_t r s(T-t)T)$$

$$\begin{aligned} h_1 = & +c_r (c_r(s+t)(r+1) + c_s s(t+1) + c_t t(s+1)) R^2 - c_r^2 (t+s) R^3 \\ & +c_s (c_s(r+t)(s+1) + c_r r(t+1) + c_t t(r+1)) S^2 - c_s^2 (t+r) S^3 \\ & +c_t (c_t(r+s)(t+1) + c_r r(s+1) + c_s s(r+1)) T^2 - c_t^2 (r+s) T^3 \\ & -c_r c_s (t(R-1+S-1) + r(S-s) + s(R-r)) RS \\ & -c_r c_t (s(R-1+T-1) + r(T-t) + t(R-r)) RT \\ & -c_s c_t (r(S-1+T-1) + s(T-t) + t(S-s)) ST \\ & - (c_t t(r+s) + c_r r(s+t) + c_s s(r+t)) (c_r R + c_s S + c_t T) \end{aligned}$$

$$h_2 = (c_r(R-1)R + c_s(S-1)S + c_t(T-1)T)(c_r(R-r) + c_s(S-s) + c_t(T-t))$$

Table 1: Explicit Hitchin Hamiltonians for the canonical coordinates (R, S, T) on $\mathrm{Bun}(X/\iota)$

It is easy to check that the Hitchin Hamiltonians h_0, h_1, h_2 do Poisson-commute as expected : for any $f, g \in \{h_0, h_1, h_2\}$, we have

$$\sum_{i=r,s,t} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} = 0$$

in Darboux notation $(p_r, p_s, p_t, q_r, q_s, q_t) := (R, S, T, c_r, c_s, c_t)$.

Since the determinant is invariant under meromorphic gauge transformations (and in particular elementary transformations), we can immediately deduce the Hitchin map

$$\underline{\mathrm{Hitch}} : \left\{ \begin{array}{ccc} \mathrm{T}^\vee \mathrm{Bun}(X/\iota) & \rightarrow & \mathrm{H}^0(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1 \otimes \Omega_{\mathbb{P}^1}^1) \\ (R, S, T, c_r, c_s, c_t) & \mapsto & \det(\tilde{\theta}) = (h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{\sigma^2 \cdot x(x-1)(x-r)(x-s)(x-t)} \end{array} \right\},$$

where σ is our previously chosen meromorphic section $\sigma : \mathbb{P}^1 \rightarrow \mathcal{O}_{\mathbb{P}^1}(-3)$ and $\tilde{\theta}$ is the universal Higgs bundle in the canonical chart of $\mathrm{T}^\vee \mathrm{Bun}(X/\iota)$.

More importantly, again since the determinant map does is invariant under meromorphic Gauge transformations, the map in (4.2) factors through the Hitchin map $\mathfrak{Higgs}(X) \simeq \mathrm{T}^\vee \mathfrak{Bun}(X) \rightarrow \mathrm{H}^0(X, \Omega_X^1 \otimes \Omega_X^1)$ by construction. Consider the natural rational map $\phi^* : \mathrm{T}^\vee \mathcal{M}_{NR} \dashrightarrow \mathrm{T}^\vee \mathrm{Bun}(X/\iota)$ induced by the map $\phi : \mathrm{Bun}(X/\iota) \dashrightarrow \mathcal{M}_{NR}$ stated explicitly with respect to the canonical chart in Proposition 3.3. The general section $c_r dR + c_s dS + c_t dT$ then lifts to a general section $\mu_0 d\left(\frac{v_0}{v_3}\right) + \mu_1 d\left(\frac{v_1}{v_3}\right) + \mu_2 d\left(\frac{v_2}{v_3}\right)$. Moreover, from the explicit coordinate change $(v_0 : v_1 : v_2 : v_3) \leftrightarrow (u_0 : u_1 : u_2 : u_3)$ to the nice

coordinates, given in (2.7), we know how to identify general sections

$$\eta_0 d\left(\frac{u_0}{u_3}\right) + \eta_1 d\left(\frac{u_1}{u_3}\right) + \eta_2 d\left(\frac{u_2}{u_3}\right) = \mu_0 d\left(\frac{v_0}{v_3}\right) + \mu_1 d\left(\frac{v_1}{v_3}\right) + \mu_2 d\left(\frac{v_2}{v_3}\right).$$

The Hamiltonians h_0, h_1, h_2 of the Hitchin map on \mathcal{M}_{NR} then can be explicitly deduced from (4.2). We get

$$\text{Hitch} : \left\{ \begin{array}{ccc} \text{T}^\vee \mathcal{M}_{NR} & \rightarrow & \text{H}^0(X, \Omega_X^1 \otimes \Omega_X^1) \\ ((u_0 : u_1 : u_2 : u_3), \eta_0, \eta_1, \eta_2) & \mapsto & (h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)} \end{array} \right\},$$

where

$$\begin{aligned} h_0 &= \frac{1}{4u_3^4} \cdot \begin{cases} rst \cdot [\eta_0(u_0^2 - u_3^2) + \eta_1(u_0 u_1 + u_2 u_3) + \eta_2(u_0 u_2 + u_1 u_3)]^2 \\ -st \cdot [\eta_0(u_0 u_1 - u_2 u_3) + \eta_1(u_1^2 + u_3^2) + \eta_2(u_0 u_3 + u_1 u_2)]^2 \\ +4rs \cdot (\eta_0 u_0 + \eta_1 u_1)^2 u_3^2 \\ -rt \cdot [\eta_0(u_0^2 + u_3^2) + \eta_1(u_0 u_1 + u_2 u_3) + \eta_2(u_0 u_2 - u_1 u_3)]^2 \end{cases} \\ h_1 &= \frac{1}{4u_3^4} \cdot \begin{cases} t \cdot (u_0^2 + u_1^2 + u_2^2 + u_3^2) [(\eta_0^2 + \eta_1^2 + \eta_2^2) u_3^2 + (\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2)^2] \\ +st \cdot (u_0^2 - u_1^2 + u_2^2 - u_3^2) [(\eta_0^2 - \eta_1^2 + \eta_2^2) u_3^2 - (\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2)^2] \\ +4r \cdot (u_0 u_2 - u_1 u_3) u_3 [\eta_0 \eta_2 u_3 + (\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2) \eta_1] \\ +4sr \cdot (u_0 u_2 + u_1 u_3) u_3 [\eta_0 \eta_2 u_3 - (\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2) \eta_1] \\ +4s \cdot (u_0 u_3 + u_1 u_2) u_3 [\eta_1 \eta_2 u_3 - (\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2) \eta_0] \\ +4rt \cdot (u_0 u_1 + u_2 u_3) u_3 [\eta_0 \eta_1 u_3 - (\eta_0 u_0 + \eta_1 u_1 + \eta_2 u_2) \eta_2] \end{cases} \\ h_2 &= \frac{1}{4u_3^4} \cdot \begin{cases} s \cdot [\eta_0(u_0 u_2 + u_1 u_3) + \eta_1(u_0 u_3 + u_1 u_2) + \eta_2(u_2^2 - u_3^2)]^2 \\ -1 \cdot [\eta_0(u_0 u_2 - u_1 u_3) + \eta_1(u_0 u_3 + u_1 u_2) + \eta_2(u_2^2 + u_3^2)]^2 \\ -t \cdot [\eta_0(u_0 u_1 + u_3 u_3) - \eta_2(u_0 u_3 - u_1 u_2) + \eta_1(u_2^2 + u_3^2)]^2 \\ +4r \cdot (\eta_1 u_1 + \eta_2 u_2)^2 u_3^2 \end{cases} \end{aligned}$$

Table 2: Explicit Hitchin Hamiltonians for the coordinates $(u_0 : u_1 : u_2 : u_3)$ of \mathcal{M}_{NR} .

Note that in [7], B. van Geemen and E. Previato conjectured a projective version of explicit Hitchin Hamiltonians, which has been confirmed in [6]. These Hamiltonians H_1, \dots, H_6 can be seen as evaluations, up to functions in the base, of the explicit Hitchin map at the Weierstrass points. More precisely, if we denote

$$h(x) := h_2 x^2 + h_1 x + h_0,$$

where h_i for $i \in \{0, 1, 2\}$ then

$$\begin{aligned} H_1 &= \frac{4h(0)}{rst} & H_4 &= \frac{4h(s)}{s(s-1)(s-r)(s-t)} \\ H_2 &= -\frac{4h(t)}{t(t-1)(t-r)(t-s)} & H_5 &= \frac{4h(r)}{r(r-1)(r-s)(r-t)} \\ H_3 &= \frac{4h(1)}{(r-1)(s-1)(t-1)} & H_6 &= 0. \end{aligned}$$

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