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Maximum Independent Sets in Subcubic Graphs: New Results

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Abstract. We consider the complexity of the classical INDEPENDENT SET problem on classes of subcubic graphs characterized by a finite set of forbidden induced subgraphs. It is well-known that a necessary condition for INDEPENDENT SET to be tractable in such a class (unless $P=NP$) is that the set of forbidden induced subgraphs includes a subdivided star $S_{k,k,k}$, for some k . Here, $S_{k,k,k}$ is the graph obtained by taking three paths of length k and identifying one of their endpoints.

It is an interesting open question whether this condition is also sufficient: is INDEPENDENT SET tractable on all hereditary classes of subcubic graphs that exclude some $S_{k,k,k}$? A positive answer to this question would provide a complete classification of the complexity of INDEPENDENT SET on all classes of subcubic graphs characterized by a finite set of forbidden induced subgraphs. The best currently known result of this type is tractability for $S_{2,2,2}$ -free graphs. In this paper we generalize this result by showing that the problem remains tractable on $S_{2,k,k}$ -free graphs, for any fixed k . Along the way, we show that subcubic INDEPENDENT SET is tractable for graphs excluding a type of graph we call an “apple with a long stem”, generalizing known results for apple-free graphs.

Keywords: Independent Set · Sub-Cubic Graphs · Apple-Free Graphs

1 Introduction

In a graph, an *independent set* is a subset of vertices no two of which are adjacent. The maximum independent set problem asks to find in a graph G an independent set of maximum size. The size of a maximum independent set in G is called the *independence number* of G and is denoted $\alpha(G)$.

The maximum independent set problem is one of the first problems that were shown to be NP-hard. Moreover, the problem remains NP-hard under substantial restrictions. In particular, it is NP-hard for graphs of vertex degree at most 3, also known as *subcubic* graphs. In terms of vertex degree, this is the strongest possible restriction under which the problem remains NP-hard, since for graphs of vertex degree at most 2 the problem is solvable in polynomial time. However, with respect to other parameters the restriction to subcubic graphs is not best possible, as the problem remains NP-hard for subcubic graphs of girth at least

k for any fixed value of k [10], where the girth of a graph is the size of a smallest cycle. In other words, the problem is NP-hard for (C_3, \dots, C_k) -free subcubic graphs for each value of k , where C_k is a chordless cycle of length k . The idea behind this conclusion is quite simple: it is not difficult to see that a double subdivision of an edge increases the independence number of the graph by exactly one, and hence, by repeatedly subdividing the edges of a subcubic graph G we destroy all small cycles in G , i.e. we transform G into a graph of large girth.

Let us observe that by means of edge subdivisions we can also destroy small copies of some other graphs, in particular, graphs of the form H_k represented in Figure 1 (left). Therefore, the maximum independent set problem remains NP-hard for $(C_3, \dots, C_k, H_1, \dots, H_k)$ -free subcubic graphs for each value of k .

Let us denote by S_k the class of $(C_3, \dots, C_k, H_1, \dots, H_k)$ -free subcubic graphs and by $\kappa(G)$ the maximum k such that $G \in S_k$. If G belongs to no class S_k , then $\kappa(G)$ is defined to be 0, and if G belongs to all classes S_k , then $\kappa(G)$ is defined to be ∞ . Also, for a set of graphs M , $\kappa(M)$ is defined as $\kappa(M) = \sup\{\kappa(G) : G \in M\}$. With this notation, we can derive the following conclusion from the above discussion (see e.g. [7]).

Theorem 1. *Let M be a set of graphs. If $\kappa(M) < \infty$, then the maximum independent set problem is NP-hard in the class of M -free subcubic graphs.*

This theorem suggests that, unless $P = NP$, the maximum independent set problem is solvable in polynomial time in the class of M -free graphs only if the parameter κ is unbounded in the set M . There are three basic ways to unbind this parameter in M :

1. include in M a graph G with $\kappa(G) = \infty$;
2. include in M graphs with arbitrarily large induced cycles;
3. include in M graphs with arbitrarily large induced subgraphs of the form H_k .

To give an example of a polynomial-time result of the first type, let us observe that $\kappa(G) = \infty$ if and only if every connected component of G has the form $S_{i,j,k}$ represented in Figure 1 (right). We call any graph of the form $S_{i,j,k}$ a *tripod*.

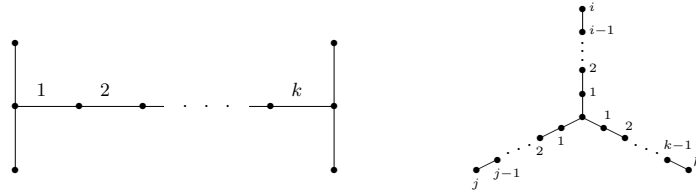


Fig. 1. The graphs H_k (left) and $S_{i,j,k}$ (right)

In other words, if the set M of forbidden induced subgraphs is finite, then M must contain a graph for which every component is a tripod for the maximum independent set problem in the class of M -free subcubic graphs to be polynomial-time solvable (assuming $P \neq NP$). In [5], it was conjectured that this condition is also sufficient. Moreover, for graphs of bounded vertex degree the problem can be easily reduced to connected forbidden induced graphs, in which case the conjecture can be restated as follows.

Conjecture 1. The maximum independent set problem is polynomial-time solvable for G -free subcubic graphs if and only if G is a tripod.

One of the minimal non-trivial tripods is the claw $S_{1,1,1}$. The problem can be solved for the claw-free graphs in polynomial time even without the restriction to bounded degree graphs [9]. In [6], the result for claw-free graphs was extended to $S_{1,1,2}$ -free graphs, also known as fork-free graphs, and again without the restriction to bounded degree graphs. However, any further extension becomes much harder even for bounded degree graphs, and only recently a solution was found for $S_{2,2,2}$ -free subcubic graphs [8]. Currently, this is one of the few maximal subclasses of subcubic graphs with polynomial-time solvable independent set problem.

Now we turn to polynomial-time solutions of the second type, i.e. classes of graphs where forbidden induced subgraphs contain arbitrarily large chordless cycles. Clearly, in this case the set of forbidden induced subgraphs must be infinite. A typical example of this type deals with classes of bounded chordality, i.e. classes excluding *all* chordless cycle of length at least k for a constant k . Without a restriction to bounded degree graphs a solution of this type is known only for $k = 4$, i.e. for chordal graphs [4], and is unknown for larger values of k . Together with the restriction to bounded degree graphs bounded chordality implies bounded tree-width [2] and hence polynomial-time solvability of the maximum independent set problem. In other words, the problem can be solved for (C_k, C_{k+1}, \dots) -free graphs of bounded vertex degree for each value of $k \geq 3$.

An *apple* A_k , $k \geq 4$, is a graph formed of a chordless cycle C_k and an additional vertex, called the *stem*, which has exactly one neighbour on the cycle C_k . The class of (A_4, A_5, \dots) -free graphs generalizes both chordal graphs and claw-free graphs, and a solution for the maximum independent set problem in this class was presented in [3]. In case of bounded degree graphs this solution can be extended to graphs without *large* apples, i.e. to (A_k, A_{k+1}, \dots) -free graphs of bounded vertex degree for any fixed value of k [7].

Generalizing both the subcubic graphs without large apples and $S_{2,2,2}$ -free subcubic graphs, in the present paper we prove polynomial-time solvability of the maximum independent set problem for subcubic graphs excluding large apples with a long stem. An *apple with a long stem* A_k^* is obtained from an apple A_k by adding one more vertex which is adjacent to the stem of A_k only. We show that for any fixed value of k , the maximum independent set problem in the class of $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graphs can be solved in polynomial time. Observe

that this class contains all $S_{2,p,p}$ -free subcubic graphs for any fixed $p < k$ (as long as $k \geq 6$) and hence our result brings us much closer to the proof of Conjecture 1.

2 Preliminaries

All graphs in this paper are simple, i.e. undirected, without loops and multiple edges. The vertex set and the edge set of a graph G is denoted by $V(G)$ and $E(G)$, respectively. The *neighbourhood* $N(v)$ of a vertex $v \in V(G)$ is the set of vertices of G adjacent to v . The *degree* of $v \in V(G)$ is the number of its neighbours, i.e. $|N(v)|$. As usual, P_n and C_n denote a chordless path and a chordless cycle with n vertices, respectively,

A subgraph of G induced by a subset $U \subseteq V(G)$ is denoted $G[U]$. If G contains no induced subgraph isomorphic to a graph H , we say that G is H -free.

Outline of the proof. To prove polynomial-time solvability of the maximum independent set problem in the class of $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graphs,

1. We start by checking if the input graph G has an induced copy of $S_{2,2,2}$. If G is $S_{2,2,2}$ -free, then the problem can be solved for G in polynomial time [8]. Otherwise, we proceed to checking whether G has an induced cycle of length at least $p = 300k$. This can be done in polynomial time, as shown in Lemma 1 below. If G does not contain induced cycles of length at least p , then the tree-width of G is bounded by a function of k [2] and hence the problem can be solved in polynomial time for G .
2. If G contains an induced copy of $S_{2,2,2}$ and a large induced cycle C , then in the absence of large induced apples with long stems we prove that it must contain a large extended cycle C^* , which is a graph obtained from C by adding two vertices that create a C_6 together with four consecutive vertices of C (see Figure 7 in Section 4). This is shown in Section 3. An important ingredient of this proof is the assumption that the input graph G is connected and has no separating cliques, i.e. cliques whose removal disconnects the graph. A polynomial-time reduction of the maximum independent set problem to graphs without separating cliques can be found in [11, 12].
3. After the previous two steps we can assume that our graph contains a large extended cycle. In Section 4 we show how to destroy such a large extended cycle by means of various local reductions. Each of them transforms G into a smaller graph G' in the same class with a fixed difference $\alpha(G) - \alpha(G')$. The set of reductions is described in Section 4.1 and their application to a graph G containing a large extended cycle is described in Section 4.2. By destroying the large extended cycle C^* , we destroy either the cycle C or the induced copy of $S_{2,2,2}$ (or both) and return to Step 1 to check if there are other copies of a large induced cycle or an induced $S_{2,2,2}$.

The first step of the proof outline above is rather straight-forward and relies on Lemma 1, stated below. The main difficulties lie in the second step (showing that if the graph has an $S_{2,2,2}$ and a large induced cycle, then it has a long

extended cycle), which is handled in Section 3; and in the third step (showing how to deal with a large extended cycle), which is handled in Section 4. Due to space constraints, some proofs have been moved to the appendix.

Lemma 1. *For each p there is an algorithm running in time $n^{O(p)}$ which decides if a given n -vertex graph contains an induced cycle of length at least p .*

3 From large cycles to extended large cycles

We recall that C^* denotes an *extended cycle*, i.e. the graph obtained from a cycle C by adding two vertices that create a C_6 together with four consecutive vertices of C (see Figure 7 in Section 4). Also, A_p^* denotes an *apple with a long stem*, where p stands for the size of the cycle in the apple. An apple with a long stem consisting of a cycle C and two stem vertices x, y will be denoted $C_{x,y}$.

The main goal of this section is to show that if G contains a large induced cycle and an induced copy of $S_{2,2,2}$, then it contains either a large induced extended cycle or a large induced apple with a long stem. This will be shown in two steps in Lemmas 2 and 3. Since we are dealing with graphs which do not contain large induced apples with long stems, the result of this section is that we may assume that our graph contains a large induced extended cycle. We note that throughout this section we will assume that our graph does not contain any separating cliques; in case it does, it is known how to reduce solving INDEPENDENT SET to smaller graphs that do not contain such cliques [11, 12].

Lemma 2. *Let G be a subcubic graph without separating cliques. If G has an induced cycle C of length p and an induced copy of $S_{2,2,2}$, then G has an induced cycle of length at least $p/12$ containing the center of an induced $S_{2,2,2}$.*

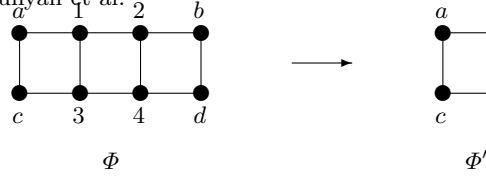
Lemma 3. *Let G be a subcubic graph without separating cliques. If G has an induced cycle C of length p containing the center of an induced $S_{2,2,2}$, then G has an induced extended cycle C_t^* or an induced apple with a long stem A_t^* with $t \geq p/8$.*

4 Destroying large extended cycles

According to the previous section, if an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G contains a large induced cycle and an induced copy of $S_{2,2,2}$, then it must contain a large extended cycle C^* . The goal of the present section is to show how to destroy large extended cycles by means of various local graph reductions. We describe these reductions in Section 4.1 and apply them to large extended cycles in Section 4.2.

4.1 Graph reductions

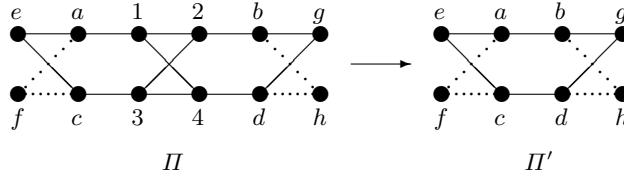
Φ -reduction and house-reduction We start with the Φ -reduction introduced in [8]. It applies to a graph G containing an induced copy of the graph Φ represented on the left of Figure 2 and consists in replacing Φ by the graph on the right of Figure 2.

Fig. 2. Φ -reduction

Lemma 4. *By applying the Φ -reduction to an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G , we obtain an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G' with $\alpha(G') = \alpha(G) - 2$.*

A *house* is the complement of a P_5 . If a graph G contains an induced *house*, the *house*-reduction consists in removing from G the vertices that form a triangle in the *house*. It was shown in [8] that if G is a subcubic graph, then the *house*-reduction reduces $\alpha(G)$ by exactly 1.

Π -reduction Now we introduce the Π -reduction illustrated in Figure 3. In a graph G , an induced Π is the graph represented on the left of Figure 3. We observe that vertex f can be missing, in which case vertices a and c have no other neighbours in G . However, if f exists, that is, if one of a, c has a neighbour outside of $\{1, 3, e\}$, then f is a common neighbour of a, c . Similarly, vertex h can be missing, in which case vertices b and d have no other neighbours in G .

Fig. 3. Π -reduction

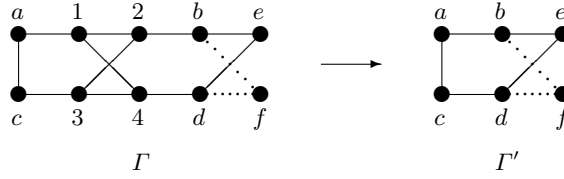
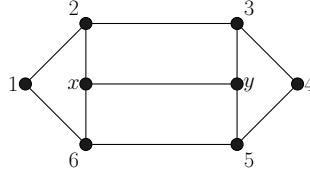
Lemma 5. *By applying the Π -reduction to an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G , we obtain an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G' with $\alpha(G') = \alpha(G) - 2$.*

Γ -reduction One more reduction is illustrated in Figure 4. We will refer to it as Γ -reduction. Again, vertex f can be missing, in which case vertices b and d have degree 2 in the graph, but if f exists it is a common neighbour of b, d .

Lemma 6. *By applying the Γ -reduction to an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G , we obtain an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph G' with $\alpha(G') = \alpha(G) - 2$.*

Θ -reduction

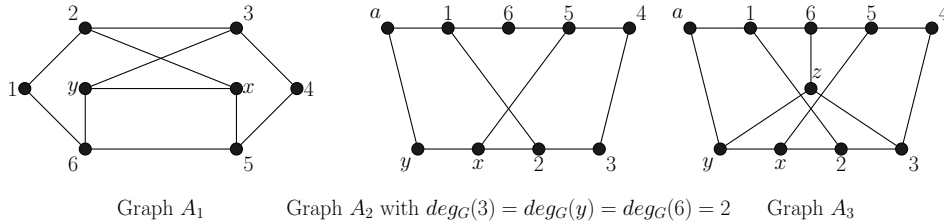
Lemma 7. *If a subcubic graph G contains an induced Θ (see Figure 5), then the deletion of vertices x, y reduces the independence number of G by exactly 1.*

Fig. 4. Γ -reductionFig. 5. Θ graph

Total struction and subgraph reduction Total struction is an operation that was introduced in [1]. Roughly speaking, this operation allows us to identify a part of the graph that can be replaced by an auxiliary graph in a way that decreases the size of the maximum independent set by a precise value. Even though this operation is quite powerful, in this paper we will only need to use two special cases of total struction, given by Corollaries 1 and 2.

Corollary 1. *For any graph $G = (V, E)$ and $H \subseteq V$ let $N[H]$ denote the set of vertices at distance at most 1 from H . Then, we have the following: if $\alpha(G[H]) = \alpha(G[N[H]])$, then $\alpha(G[V \setminus N[H]]) = \alpha(G) - \alpha(G[H])$.*

Informally, Corollary 1 gives rise to the following transformation: if we can find a set of vertices H such that $G[H]$ and $G[N[H]]$ have the same maximum independent set, then we simply select an independent set of H in our solution and delete all vertices of $N[H]$. The deletion of $N[H]$ in the case when $\alpha(G[H]) = \alpha(G[N[H]])$ was called in [8] the H -subgraph reduction.

Fig. 6. Graphs A_1 , A_2 and A_3

It is not difficult to check that if A_1 , A_2 , or A_3 (see Figure 6) is an induced subgraph of a subcubic graph, then we can use Corollary 1 as we have:

- $\alpha(A_1[\{2, 3, 5, 6, x, y\}]) = \alpha(A_1) = 3$,
- $\alpha(A_2[\{1, 2, 3, 5, 6, x, y\}]) = \alpha(A_2) = 4$,
- $\alpha(A_3[\{1, 2, 3, 5, 6, x, y\}]) = \alpha(A_3) = 4$.

Lemma 8. *If A_1 , A_2 , or A_3 is an induced subgraph of a subcubic graph G , then $\alpha(G - A_1) = \alpha(G) - 3$, $\alpha(G - A_2) = \alpha(G) - 4$, $\alpha(G - A_3) = \alpha(G) - 4$.*

Corollary 2. *Let $G = (V, E)$ be a subcubic graph and $K \subseteq V$ such that $G[K]$ induces a $K_{2,3}$. Then, if G' is the graph obtained from G by deleting the vertices of K and introducing a new vertex z connected to $N(K)$, we have (i) $\alpha(G') = \alpha(G) - 2$ and (ii) if G' contains an apple with a long stem A_p^* , then G also contains an apple with a long stem $A_{p'}^*$, with $p' \geq p$.*

4.2 Applying graph reductions to large extended cycles

Let G be an $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graph. For ease of terminology and notation we will refer to any A_t^* with $t \geq k$ simply as a large apple with a long stem. According to Section 3, we may assume that G contains a large extended cycle C_p^* , i.e. a graph that consists of an induced cycle of length p , plus two extra vertices which form a C_6 together with four consecutive vertices of the cycle and have no other neighbours in C_p^* . We denote the vertices of an extended cycle as shown in Figure 7, where we have given labels to the vertices of the C_6 , plus some other interesting vertices. In the remainder we use simply C^* to denote the extended cycle and C_6 to denote the set of vertices $\{1, 2, 3, 4, 5, 6\}$. Without loss of generality, we assume that $p \geq 3k$.

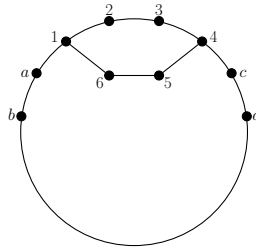


Fig. 7. An extended cycle

We will now go through a sequence of cases that covers all possible ways in which C^* may be connected to the rest of the graph.

Case 0: Vertices 2 and 3 both have degree 2 in G . In this case we delete 2, 3 from the graph and add the edge connecting 1 to 4. This decreases $\alpha(G)$

by exactly 1. Also, it is not difficult to check that this transformation does not create any new forbidden induced subgraphs.

Because of the above we can assume that the set $\{2, 3\}$ has a neighbour outside of C^* . We call this vertex x . Without loss of generality we assume that x is connected to 2. Let us consider how x is connected to the rest of C^* . The rest of the cases are defined as follows.

- *Case 1.1:* $N(x) \cap C^* = \{2\}$
- *Case 1.2:* $N(x) \cap C_6 = \{2\}$ and x has exactly one neighbour in $C^* \setminus C_6$
- *Case 1.3:* $N(x) \cap C_6 = \{2\}$ and x has two neighbours in $C^* \setminus C_6$

If we rule out the above cases we conclude that x has at least two neighbours in C_6 . Since the degrees of 1, 4 are already three in C^* , we conclude that x has at least two neighbours in $\{2, 3, 5, 6\}$. Let us also rule out two further cases.

- *Case 1.4:* $N(x) \cap C_6 = \{2, 3\}$;
- *Case 1.5:* $|N(x) \cap C_6| = 3$

Lemma 9. *If one of Cases 1.1-1.5 applies, then the instance can be simplified in polynomial time. If none of Cases 1.1-1.5 applies, then either $N(x) \cap C_6 = \{2, 5\}$ or $N(x) \cap C_6 = \{2, 6\}$.*

Thus, we may suppose: $N(x) \cap C_6 = \{2, 5\}$ or $N(x) \cap C_6 = \{2, 6\}$. We handle these two cases separately in the following subsections.

x is adjacent to 2 and 6

Lemma 10. *Let x be a vertex adjacent to 2 and 6 and assume x has a neighbour y not in C^* . Then G contains an induced Φ or an induced Π or an induced Γ or an induced Θ .*

Proof. If y is adjacent to 3, then by Lemma 9 (and symmetry) y is also adjacent to 5 and hence vertices 1, 2, 3, 4, 5, 6, x , y induce a Θ .

If y is adjacent to c , then vertices 2, 3, 4, x , y , c create a cycle of length 6 which, together with the path $1ab \dots d$ gives a second large extended cycle. Therefore, by Lemma 9 applied to this extended cycle, vertex 5 must be adjacent to y and hence vertices 1, 2, x , 6, y , 5, c , 4 induce a Φ .

If y is adjacent to a , then vertices a , y , 1, 2, x , 6, 3, 4, 5 induce a Γ with a possible missing common neighbour of 3 and 5 (any neighbour of these vertices must be common by Lemma 9).

If y is adjacent to b and not adjacent to a , then vertices a , b , y , 1, 2, x , 6, 3, 4, 5 induce a Π with a possible missing common neighbour of 3 and 5 (any neighbour of these vertices must be common by Lemma 9).

From now on, we assume y has no neighbours in $\{3, 5, a, b, c\}$. If y has neighbours on $C^* \setminus C_6$, then we can distinguish at most 3 cycles containing y as shown in Figure 8 (if y has only 1 neighbour on $C^* \setminus C_6$, the cycle C^2 is missing).

We observe that at least one of the cycles C^1 , C^2 , C^3 is large, i.e. has length at least $p/3$. Then G contains a large apple with a long stem

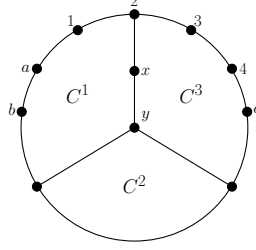


Fig. 8. Vertex y has neighbours on C

- $C^* \cup \{x, y\} \setminus \{5, 6\}$ if y has no neighbours on $C^* \setminus C_6$,
- $C^1 \cup \{3, 4\}$ if C^1 is large,
- $C^2 \cup \{x, 2\}$ if C^2 is large,
- $C^3 \cup \{1, a\}$ if C^3 is large.

A contradiction in all cases shows that y has a neighbour in $\{3, 5, a, b, c\}$ and hence G contains an induced Φ or an induced Π or an induced Γ or an induced Θ . \square

We therefore find ourselves in the following context: $N(x) \cap C_6 = \{2, 6\}$ and $N(x) \setminus C^* = \emptyset$. Before we proceed, let us identify another relevant vertex. If 3 has a neighbour outside C^* we call that vertex y . By Lemma 9 (and appropriate symmetry) y is also connected to 5. We have also argued that x and y are not adjacent. We will in the remainder assume that the degree of x is at least as large as the degree of y . This is without loss of generality, as the two vertices can be exchanged by an appropriate automorphism of C^* . In what follows, we analyze all possible adjacencies of x and y to the vertices of C^* .

Case 2.1: If x has degree 2 and y does not exist (therefore 3, 5 have degree 2), then we apply the H -subgraph reduction (Corollary 1) with $H = \{x, 3, 5\}$, in which case $\alpha(G[H]) = \alpha(G[N[H]]) = 3$ and hence the removal of $N[H]$ decreases $\alpha(G)$ by 3.

Case 2.2: Assume x has degree 2 and y exists (therefore, y is connected to 3, 5). We have assumed without loss of generality that x has at least as high degree as y , therefore y has no other neighbour. We delete from the graph vertices 2, 3, x , y . If G' is the new graph, we claim that $\alpha(G') = \alpha(G) - 2$. The inequality $\alpha(G') \geq \alpha(G) - 2$ is clear, since no independent set can take more than two of the deleted vertices. To see that $\alpha(G) \geq \alpha(G') + 2$, take a maximum independent set in G' . If it contains vertex 5, then it does not contain 4 or 6. Therefore, we can augment it with $x, 3$. If it contains 6, we can augment it similarly by adding $y, 2$. Finally, if it contains neither 5 nor 6, we augment it with x, y .

Case 2.3: If x is connected to a , $\{x, 1, a, 2, 6\}$ induces a $K_{2,3}$, we can therefore invoke Corollary 2 to simplify the graph.

Case 2.4: If x is connected to c , then $x61ab \dots cx$ together with 3, 4 form a large apple with a long stem.

Case 2.5: If x is connected to d , then $x21ab\dots dx$ together with $3, 4$ form a large apple with a long stem.

Case 2.6: If x is connected to a vertex f of C^* in the path from b to d (but not b or d), then: if f is closer to a than to c , we take the path $xf\dots dc432x$ plus $1, a$; otherwise we take $xf\dots ba12x$ plus $3, 4$. In both cases these form a large apple with a long stem.

Case 2.7: If x is connected to b and y does not exist, then we apply the H -subgraph reduction with $H = \{x, 1, 3, 5\}$. It is not hard to check that $\alpha(G[H]) = \alpha(G[N[H]]) = 4$ and hence the removal of $N[H]$ decreases $\alpha(G)$ by 4.

Case 2.8: Assume x is connected to b , y exists and it has degree 2 (that is, y is connected only to $3, 5$). We delete from the graph the vertices $\{x, y, 1, 2, 3, 5, 6\}$ and add a new vertex z adjacent to $a, b, 4$. We claim $\alpha(G') = \alpha(G) - 3$. To see that $\alpha(G) \geq \alpha(G') + 3$ take an independent set of the new graph. If it does not include z then we augment it with $\{2, 6, y\}$; if it does include z , it does not contain any of $a, b, 4$, so we replace z with $\{1, x, 3, 5\}$. To see that $\alpha(G') \geq \alpha(G) - 3$ take an independent set of G . If it contains at most three of the deleted vertices we are done. If it contains four, these must be $\{1, x, 3, 5\}$, therefore the set does not contain any of $a, b, 4$; in this case we replace the deleted vertices by z .

The new graph does not have a large apple with a long stem that uses z and both a, b , since that would induce a triangle. If, on the other hand, it has an apple with a long stem that uses z and at most two of its neighbours, then G also has a subdivided copy of the same subgraph if we replace z with $1, 2, 3$.

Case 2.9: Finally, suppose x is connected to b , y exists and y has degree 3. Since x and y have the same degree, we may exchange their roles, and by symmetry and the same case analysis that we did for x we conclude that y must be connected to d (otherwise one of the previous cases applies). We transform the graph as follows: we delete the vertices $1, 2, 3, 4, 5, 6, x, y$ and add two new vertices z, w such that z, w are connected to each other, z is connected to a, b , and w is connected to c, d . We claim that $\alpha(G') = \alpha(G) - 3$. First, to obtain $\alpha(G') \geq \alpha(G) - 3$, take a maximum independent set of G . If it contains a vertex from a, b and a vertex from c, d , then it contains at most three of the deleted vertices, since the six deleted vertices which are not adjacent to a vertex of the independent set induce a cycle of length 6. In all other cases, the independent set in G contains at most four of the deleted vertices. However, if the set does not contain any of a, b , we can augment it with z in G' , while if it does not contain any of c, d we can add to it w . To see that $\alpha(G) \geq \alpha(G') + 3$, take a maximum independent set in G' . If it is using z , then it does not contain a or b . In G we replace z with $1, x, 3, 5$. The situation is symmetric if the set contains w . Finally, if it does not contain either z or w , we observe that deleting the neighbours of the set among the removed vertices gives a cycle of length 6, of which we can select three vertices. The transformation does not introduce a new large apple with a long stem, since the closed neighbourhoods of z, w include a triangle, therefore if one or two of these vertices is used in the apple we can replace them with an appropriate induced path through the deleted vertices in G .

x is adjacent to 2 and 5

Lemma 11. *Let x be a vertex adjacent to 2 and 5 and assume x has a neighbour y not in C^* . Then G contains an induced A_1 or an induced A_2 or an induced A_3 (Figure 6).*

Proof. If y is adjacent to 3 or 6, then y is adjacent to both 3 and 6 (Lemma 9) and hence G contains an induced A_1 . Assume y is adjacent to a . Then, if all three vertices 3, 6, y have degree 2 in G , then G contains an induced A_2 . If vertex 3 has degree three, it has a common neighbour with 6 (by Lemma 9), call this neighbour z . We claim that z must also be connected to y , which will give an induced A_3 . To see this, consider the set of vertices $(C^* \setminus \{2, 3\}) \cup \{x, y\}$. This set induces an extended cycle, where the C_6 is now formed by $a, 1, 6, 5, x, y$. Since z is connected to 6, it must be connected to one of $\{x, y\}$ (Lemma 9). However, x already has three neighbours $(2, 5, y)$, therefore, z is connected to y .

If y is adjacent to c this is symmetric to y being adjacent to a . So, we suppose that y is adjacent to none of 3, 6, a, c . The rest of the proof is similar to that of Lemma 10 with the only difference that if y is adjacent only to b this time we can find a large apple with a long stem, where the stem is $\{1, 6\}$ and the cycle goes through $byx234cd \dots b$. \square

Lemma 12. *Let x be a vertex adjacent to 2 and 5 and assume x has a neighbour in $C^* \setminus C_6$. Then this neighbour is one of a and c .*

To complete the case analysis, we prove the following lemma.

Lemma 13. *Let x be a vertex adjacent to 2 and 5 and suppose that if x has a neighbour in $C^* \setminus C_6$, then this neighbour is a . Then we can in polynomial time reduce our instance to a smaller instance.*

5 Conclusion

Summarizing the discussion in the previous sections, we make the following conclusion, which extends several previously known results.

Theorem 2. *Maximum independent set can be solved in polynomial time in the class of $(A_k^*, A_{k+1}^*, \dots)$ -free subcubic graphs for any fixed value of k .*

Since A_t^* contains $S_{2,k,k}$ for any $t > 2k + 1$, we derive the following corollary

Corollary 3. *Maximum independent set can be solved in polynomial time in the class of $S_{2,k,k}$ -free subcubic graphs for any fixed value of k .*

This result brings us closer to the dichotomy of Conjecture 1. However, proving this conjecture in its whole generality remains a challenging open problem.

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