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OPETOPIC ALGEBRAS I: ALGEBRAIC STRUCTURES ON OPETOPIC SETS

CÉDRIC HO THANH AND CHAITANYA LEENA SUBRAMANIAM

ABSTRACT. We define a family of structures called “opetopic algebras”, which are algebraic structures with an underlying opetopic set. Examples of such are categories, planar operads, and Loday’s combinads over planar trees. Opetopic algebras can be defined in two ways, either as the algebras of a “free pasting diagram” parametric right adjoint monad, or as models of a small projective sketch over the category of opetopes. We define an opetopic nerve functor that fully embeds each category of opetopic algebras into the category of opetopic sets. In particular, we obtain fully faithful opetopic nerve functors for categories and for planar coloured Set-operads.

This paper is the first in a series aimed at using opetopic spaces as models for higher algebraic structures.

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1. INTRODUCTION

This paper deals with algebraic structures whose operations have *higher dimensional “tree-like” arities*. As an example in lieu of a definition, a category \mathcal{C} is an algebraic structure whose operation of *composition* has as its inputs, or “arities”, sequences of composable morphisms of the category. These sequences can be seen as *filiform* or *linear* trees. Moreover, each morphism of \mathcal{C} can itself be seen as an operation whose arity is a single point (i.e. an object of \mathcal{C}). A second example, one dimension above, is that of a planar coloured Set-operad \mathcal{P} (a.k.a. a nonsymmetric multicategory), whose operation of composition has planar trees of operations (a.k.a. multimorphisms) of \mathcal{P} as arities. Moreover, the arity of an operation of \mathcal{P} is an ordered list (a filiform tree) of colours (a.k.a. objects) of \mathcal{P} . Heuristically extending this pattern leads one to presume that such an algebraic structure one dimension above planar operads should have an operation of composition whose arities are trees of things that can themselves be seen as operations whose arities are planar trees. Indeed, such algebraic structures are precisely the *PT-combinads in Set* (combinads over the combinatorial pattern of planar trees) of Loday [Lod12].

| Structure | Sets =0-algebras | Categories =1-algebras | Operads =2-algebras | PT-combinads =3-algebras | ... |
|-----------|---------------------|---------------------------|------------------------|-----------------------------|-----|
| Arity | {*} | Lists | Trees | 4-opetopes | ... |
| | =1-opetopes | =2-opetopes | =3-opetopes | | |

The goal of this article is to give a precise definition of the previous sequence of algebraic structures.

1.1. Context. It is well-known that higher dimensional tree-like arities are encoded by the opetopes (*operation polytopes*) of Baez and Dolan [BD98], which were originally introduced in order to give a definition of weak n -categories and a precise formulation of the “microcosm” principle.

The fundamental definitions of [BD98] are those of \mathcal{P} -opetopic sets and n -coherent \mathcal{P} -algebras (for a coloured symmetric Set-operad \mathcal{P}), the latter being \mathcal{P} -opetopic sets along with certain “horn-filling” operations that are “universal” in a suitable sense. When \mathcal{P} is the identity monad on Set (i.e. the unicolour symmetric Set-operad with a single unary operation), \mathcal{P} -opetopic sets are simply called *opetopic sets*, and n -coherent \mathcal{P} -algebras are the authors’ proposed definition of *weak n -categories*.

While the coinductive definitions in [BD98] of \mathcal{P} -opetopic sets and n -coherent \mathcal{P} -algebras are straightforward and general, they have the disadvantage of not defining a *category* of \mathcal{P} -opetopes such that presheaves over it are precisely \mathcal{P} -opetopic sets, even though this is ostensibly the case. Directly defining the category of \mathcal{P} -opetopes turns out to be a tedious and non-trivial task, and was worked out explicitly by Cheng in [Che03, Che04a] for the particular case of the identity monad on Set, giving the category \mathbb{O} of opetopes.

The complexity in the definition of a category $\mathbb{O}_{\mathcal{P}}$ of \mathcal{P} -opetopes has its origin in the difficulty of working with a suitable notion of *symmetric tree*. Indeed, the objects of $\mathbb{O}_{\mathcal{P}}$ are trees of trees of ... of trees of operations of the symmetric operad \mathcal{P} , and their automorphism groups are determined by the action of the (“coloured”) symmetric groups on the sets of operations of \mathcal{P} .

However, when the action of the symmetric groups on the sets of operations of \mathcal{P} is free, it turns out that the objects of the category $\mathbb{O}_{\mathcal{P}}$ are rigid, i.e. have no non-trivial automorphisms (this follows from [Che04a, proposition 3.2]). The identity monad on Set is of course such an operad, and this vastly simplifies the definition of \mathbb{O} . Indeed, such operads are precisely the (finitary) *polynomial monads* in Set, and the machinery of polynomial endofunctors and polynomial monads developed in [Koc11, KJBM10, GK13] gives a very satisfactory definition of \mathbb{O} [HT18, CHTM19] which we review in sections 2 and 3.

1.2. Contributions. The main contribution of the present article is to show how the polynomial definition of \mathbb{O} allows, for all $k, n \in \mathbb{N}$ with $k \leq n$, a definition of (k, n) -opetopic algebras, which constitute a full subcategory of the category $\mathcal{Psh}(\mathbb{O})$ of opetopic sets. More precisely, we show that the polynomial monad whose set of operations is the set \mathbb{O}_{n+1} of $(n+1)$ -dimensional opetopes can be extended to a parametric right adjoint monad whose algebras are the (k, n) -opetopic algebras. Important particular cases are the categories of $(1, 1)$ - and $(1, 2)$ -opetopic algebras, which are the categories \mathcal{Cat} and $\mathcal{Op}_{\text{col}}$ of small categories and coloured planar Set-operads respectively. Loday’s combinads over the combinatorial pattern of planar trees [Lod12] are also recovered as $(1, 3)$ -opetopic algebras.

We further show that each category of (k, n) -opetopic algebras admits a fully faithful *opetopic nerve functor* to $\mathcal{Psh}(\mathbb{O})$. As a direct consequence of this framework, we obtain commutative triangles of adjunctions

$$\begin{array}{ccc}
 & \mathcal{Cat} & \\
 h \swarrow & & \nwarrow u \\
 \mathcal{Psh}(\mathbb{O}) & \xrightarrow{N} & \mathcal{Psh}(\Delta) \\
 \xleftarrow{h^*} & \xrightarrow{h_!} & \xleftarrow{N_u} \\
 & \perp & \\
 & \xrightarrow{h^*} &
 \end{array}
 \qquad
 \begin{array}{ccc}
 & \mathcal{Op}_{\text{col}} & \\
 h \swarrow & & \nwarrow u \\
 \mathcal{Psh}(\mathbb{O}_{\geq 1}) & \xrightarrow{N} & \mathcal{Psh}(\Omega) \\
 \xleftarrow{h^*} & \xrightarrow{h_!} & \xleftarrow{N_u} \\
 & \perp & \\
 & \xrightarrow{h^*} &
 \end{array}
 \tag{1.2.1}$$

where Δ is the category of simplices, Ω is the planar version of Moerdijk and Weiss’s category of *dendrices* and $\mathbb{O}_{\geq 1}$ is the full subcategory of \mathbb{O} on opetopes of dimension > 0 . This gives a direct comparison between the opetopic nerve of a category (resp. a planar operad) and its corresponding well-known simplicial (resp. dendroidal) nerve.

This formalism seems to provide infinitely many types of (k, n) -opetopic algebras. However this is not really the case, as the notion stabilises at the level of combinads. Specifically, we show a phenomenon we call *algebraic trompe-l’œil*, where an (k, n) -opetopic algebra is entirely specified by its underlying opetopic set and by a $(1, 3)$ -opetopic algebra. In other words, its algebraic data can be “compressed” into a $(1, 3)$ -algebra (a combinad). The intuition behind this is that fundamentally, opetopes are just trees whose nodes are themselves trees, and that once this is obtained at the level of combinads, opetopic algebras can encode no further useful information.

1.3. Outline. We begin by recalling elements of the theory of polynomial functors and polynomial monads in section 2. This formalism is the basis for the modern definition of opetopes and of the category of opetopes [KJBM10, CHTM19] that we survey in section 3. Section 4 contains the central constructions of this article, namely those of opetopic algebras and coloured opetopic algebras, as well as the definition of the opetopic nerve functor, which is a full embedding of (coloured) opetopic algebras into opetopic sets. Section 5 is devoted to showing how the algebraic information carried by opetopes turns out to be limited.

1.4. Related work. The (k, n) -opetopic algebras that we obtain are related to the n -coherent \mathcal{P} -algebras of [BD98] as follows: for $n \geq 1$, $(1, n)$ -opetopic algebras are precisely 1-coherent \mathcal{P} -algebras for \mathcal{P} the polynomial monad $\mathbb{O}_{n-1} \leftarrow E_n \rightarrow \mathbb{O}_n \rightarrow \mathbb{O}_{n-1}$. We therefore do not obtain all n -coherent \mathcal{P} -algebras with our framework, and this means in particular that we cannot capture all the weak n -categories of [BD98] (except for $n = 1$, which are just usual 1-categories). This is not unexpected, as weak n -categories are not defined just by *equations* on the opetopes of an opetopic set, but by its more subtle *universal* opetopes.

However, we are able to promote the triangles of (1.2.1) to Quillen equivalences of simplicial model categories. This, along with a proof that opetopic spaces (i.e. simplicial presheaves on \mathbb{O}) model $(\infty, 1)$ -categories and planar ∞ -operads, will be the subject of the second paper of this series.

1.5. Acknowledgments. We are grateful to Pierre-Louis Curien for his patient guidance and reviews. The second named author would like to thank Paul-André Melliès for introducing him to nerve functors

and monads with arities, Mathieu Anel for discussions on Gabriel–Ulmer localisations, and Eric Finster for a discussion related to proposition 2.6.5. The first named author has received funding from the European Union’s Horizon 2020 research and innovation program under the Marie Skłodowska–Curie grant agreement No. 665850.

1.6. Preliminary category theory. We review relevant notions and basic results from the theory of locally presentable categories. Original references are [GU71, SGA72], and most of this material can be found in [AR94].

1.6.1. Presheaves and nerve functors. For $\mathcal{C} \in \text{Cat}$, we write $\mathcal{Psh}(\mathcal{C})$ for the category of Set-valued presheaves over \mathcal{C} , i.e. the category of functors $\mathcal{C}^{\text{op}} \rightarrow \text{Set}$ and natural transformations between them. If $X \in \mathcal{Psh}(\mathcal{C})$ is a presheaf, then its *category of elements* \mathcal{C}/X is the comma category y/X , where $y: \mathcal{C} \hookrightarrow \mathcal{Psh}(\mathcal{C})$ is the Yoneda embedding.

Let $\mathcal{C} \in \text{Cat}$ and $F: \mathcal{C} \rightarrow \mathcal{D}$ be a functor to a (not necessarily small) category \mathcal{D} . Then the *nerve functor associated to F* (also called the *nerve of F*) is the functor $N_F: \mathcal{D} \rightarrow \mathcal{Psh}(\mathcal{C})$ mapping $d \in \mathcal{D}$ to $\mathcal{D}(F-, d)$. The functor F is said to be *dense* if for all $d \in \mathcal{D}$, the colimit of $F/d \rightarrow \mathcal{C} \xrightarrow{F} \mathcal{D}$ exists in \mathcal{D} , and is canonically isomorphic to d . Equivalently, F is dense if and only if N_F is fully faithful.

Let $i: \mathcal{C} \rightarrow \mathcal{D}$ be a functor between small categories. Then the precomposition functor $i^*: \mathcal{Psh}(\mathcal{D}) \rightarrow \mathcal{Psh}(\mathcal{C})$ has a left adjoint $i_!$ and a right adjoint i_* , given by left and right Kan extension along i^{op} respectively. If i has a right adjoint j , then $i^* \dashv j^*$, or equivalently, $i^* \cong j_!$. Note that the nerve of i is the functor $N_i = i^*y_{\mathcal{D}}$, where $y_{\mathcal{D}}: \mathcal{D} \hookrightarrow \mathcal{Psh}(\mathcal{D})$ is the Yoneda embedding. Recall that i^* is the nerve of the functor $y_{\mathcal{D}}i$ and that i_* is the nerve of the functor $N_i = i^*y_{\mathcal{D}}$, i.e. it is the nerve of the nerve of i .

1.6.2. Orthogonality. Let \mathcal{C} be a category, and $l, r \in \mathcal{C}^{\rightarrow}$. We say that l is *left orthogonal to r* (equivalently, r is *right orthogonal to l*), written $l \perp r$, if for any solid commutative square as follows, there exists a *unique* dashed arrow making the two triangles commute (the relation \perp is also known as the *unique lifting property*):

$$\begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow l & \dashrightarrow & \downarrow r \\ \cdot & \xrightarrow{\quad} & \cdot \end{array}$$

If \mathcal{C} has a terminal object 1 , then for all $X \in \mathcal{C}$, we write $l \perp X$ if l is left orthogonal to the unique map $X \rightarrow 1$. Let L and R be two classes of morphisms of \mathcal{C} . We write $\mathsf{L} \perp \mathsf{R}$ if for all $l \in \mathsf{L}$ and $r \in \mathsf{R}$ we have $l \perp r$. The class of all morphisms f such that $\mathsf{L} \perp f$ (resp. $f \perp \mathsf{R}$) is denoted L^{\perp} (resp. ${}^{\perp}\mathsf{R}$).

1.6.3. Localisations. Let J be a class of morphisms of a category \mathcal{C} . Recall that the *localisation of \mathcal{C} at J* is a functor $\gamma_{\mathsf{J}}: \mathcal{C} \rightarrow \mathsf{J}^{-1}\mathcal{C}$ such that $\gamma_{\mathsf{J}}f$ is an isomorphism for every $f \in \mathsf{J}$, and such that γ_{J} is universal for this property. We say that J has the *3-for-2* property when for every composable pair $\cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot$ of morphisms in \mathcal{C} , if any two of f , g and gf are in J , then so is the third.

Assume now that \mathcal{C} is a small category, and that J is a *set* (rather than a proper class) of morphisms of \mathcal{C} , and consider the full subcategory $\mathcal{C}_{\mathsf{J}} \hookrightarrow \mathcal{Psh}(\mathcal{C})$ of all those $X \in \mathcal{Psh}(\mathcal{C})$ such that $\mathsf{J} \perp X$. A category is *locally presentable* if and only if it is equivalent to one of the form \mathcal{C}_{J} . The pair $({}^{\perp}(\mathsf{J}^{\perp}), \mathsf{J}^{\perp})$ forms an *orthogonal factorisation system*, meaning that any morphism f in $\mathcal{Psh}(\mathcal{C})$ can be factored as $f = pi$, where $p \in \mathsf{J}^{\perp}$ and $i \in {}^{\perp}(\mathsf{J}^{\perp})$. Applied to the unique arrow $X \rightarrow 1$, this factorisation provides a left adjoint (i.e. a reflection) $a_{\mathsf{J}}: \mathcal{Psh}(\mathcal{C}) \rightarrow \mathcal{C}_{\mathsf{J}}$ to the inclusion $\mathcal{C}_{\mathsf{J}} \hookrightarrow \mathcal{Psh}(\mathcal{C})$. Furthermore, a_{J} is the localisation of $\mathcal{Psh}(\mathcal{C})$ at J .¹

With \mathcal{C} and J as in the previous paragraph, the class of *J -local isomorphisms* W_{J} is the class of all morphisms $f \in \mathcal{Psh}(\mathcal{C})^{\rightarrow}$ such that for all $X \in \mathcal{C}_{\mathsf{J}}$, $f \perp X$ (that is, $\mathsf{W}_{\mathsf{J}} = {}^{\perp}\mathcal{C}_{\mathsf{J}}$). It is the smallest

¹The results of this paragraph still hold when $\mathcal{Psh}(\mathcal{C})$ is replaced by a locally κ -presentable category \mathcal{E} and a set K of κ -small morphisms of \mathcal{E} . We call a localisation of the form $a_{\kappa}: \mathcal{E} \rightarrow \mathcal{E}_{\kappa}$ the *Gabriel–Ulmer localisation of \mathcal{E} at K* .

class of morphisms that contains \mathbf{J} , that satisfies the 3-for-2 property, and that is closed under colimits in $\mathcal{P}\text{sh}(\mathcal{C})^\rightarrow$ [GU71, theorem 8.5]. Thus the localisation $a_{\mathbf{J}}$ is also the localisation of $\mathcal{P}\text{sh}(\mathcal{C})$ at $\mathbf{W}_{\mathbf{J}}$. Furthermore, $\mathbf{W}_{\mathbf{J}}$ is closed under pushouts.

1.6.4. *Projective sketches.* A *projective sketch* is the data of a $\mathcal{C} \in \text{Cat}$ and a set K of cones in \mathcal{C} . For \mathcal{B} a category with all limits, the category of *models in \mathcal{B} of (\mathcal{C}, K)* is the category of functors $\mathcal{C} \rightarrow \mathcal{B}$ that take each cone in K to a limit cone, and natural transformations between them. If (\mathcal{C}, K) is a projective sketch, then equivalently, K can be seen as a set of cocones in \mathcal{C}^{op} , or as a set of subrepresentables (subobjects of representables) in $\mathcal{P}\text{sh}(\mathcal{C}^{\text{op}})$. Let $\mathcal{C}_K^{\text{op}} \hookrightarrow \mathcal{P}\text{sh}(\mathcal{C}^{\text{op}})$ be the full subcategory of all $X \in \mathcal{P}\text{sh}(\mathcal{C}^{\text{op}})$ such that $K \perp X$. Then $\mathcal{C}_K^{\text{op}}$ is precisely the category of models in Set of the projective sketch (\mathcal{C}, K) . Let κ be a regular cardinal, and let (\mathcal{C}, K) be a projective sketch in which each cone in K is a κ -small diagram. Then $\mathcal{C}_K^{\text{op}}$ is a locally κ -presentable category, and the inclusion $\mathcal{C}_K^{\text{op}} \hookrightarrow \mathcal{P}\text{sh}(\mathcal{C}^{\text{op}})$ preserves κ -filtered colimits.

2. POLYNOMIAL FUNCTORS AND POLYNOMIAL MONADS

We survey elements of the theory of polynomial functors, trees, and monads. For more comprehensive references, see [Koc11, GK13].

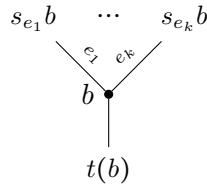
2.1. Polynomial functors.

Definition 2.1.1 (Polynomial functor). A *polynomial (endo)functor P over I* is a diagram in Set of the form

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I. \quad (2.1.2)$$

P is said to be *finitary* if the fibres of $p : E \rightarrow B$ are finite sets. We will always assume polynomial functors to be finitary.

We use the following terminology for a polynomial functor P as in equation (2.1.2), which is motivated by the intuition that a polynomial functor encodes a multi-sorted signature of function symbols. The elements of B are called the *nodes* or *operations* of P , and for every node b , the elements of the fibre $E(b) := p^{-1}(b)$ are called the *inputs* of b . The elements of I are called the *colours* or *sorts* of P . For every input e of a node b , we denote its colour by $s_e(b) := s(e)$.



Definition 2.1.3 (Morphism of polynomial functor). A morphism from a polynomial functor P over I (as in equation (2.1.2)) to a polynomial functor P' over I' (on the second row) is a commutative diagram of the form

$$\begin{array}{ccccccc} I & \xleftarrow{s} & E & \xrightarrow{p} & B & \xrightarrow{t} & I \\ f_0 \downarrow & & f_2 \downarrow & \lrcorner & f_1 \downarrow & & f_0 \downarrow \\ I' & \xleftarrow{s'} & E' & \xrightarrow{p'} & B' & \xrightarrow{t} & I' \end{array}$$

where the middle square is cartesian (i.e. is a pullback square). If P and P' are both polynomial functors over I , then a morphism from P to P' over I is a commutative diagram as above, but where f_0 is required to be the identity. Let PolyEnd denote the category of polynomial functors and morphisms of polynomial functors, and $\text{PolyEnd}(I)$ the category of polynomial functors over I and morphisms of polynomial functors over I .

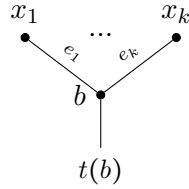
Remark 2.1.4 (Polynomial functors really are functors!). The term “polynomial (endo)functor” is due to the association of P to the composite endofunctor

$$\mathbf{Set}/I \xrightarrow{s^*} \mathbf{Set}/E \xrightarrow{p_*} \mathbf{Set}/B \xrightarrow{t_!} \mathbf{Set}/I$$

where we have denoted $a_!$ and a_* the left and right adjoints to the pullback functor a^* along a map of sets a . Explicitly, for $(X_i \mid i \in I) \in \mathbf{Set}/I$, $P(X)$ is given by the “polynomial”

$$P(X) = \left(\sum_{b \in B(j)} \prod_{e \in E(b)} X_{s(e)} \mid j \in I \right), \quad (2.1.5)$$

where $B(i) := t^{-1}(i)$ and $E(b) = p^{-1}(b)$. Visually, elements of $P(X)_j$ are nodes $b \in B$ such that $tb = j$, and whose inputs are decorated by elements of $(X_i \mid i \in I)$ in a manner compatible with their colours. Graphically, an element of PX_i can be represented as



with $b \in B$ such that $t(b) = i$, and $x_j \in X_{s_{e_j}b}$ for $1 \leq j \leq k$. Moreover, the endofunctor $P : \mathbf{Set}/I \rightarrow \mathbf{Set}/I$ preserves connected limits: s^* and p_* preserve all limits (as right adjoints), and $t_!$ preserves and reflects connected limits.

This construction extends to a fully faithful functor $\mathcal{P}\text{oly}\mathcal{E}\text{nd}(I) \rightarrow \mathcal{C}\text{art}(\mathbf{Set}/I)$, the latter being the category of endofunctors of \mathbf{Set}/I and cartesian natural transformations². In fact, the image of this full embedding consists precisely of those endofunctors that preserve connected limits [GK13, section 1.18]. The composition of endofunctors gives $\mathcal{C}\text{art}(\mathbf{Set}/I)$ the structure of a monoidal category, and $\mathcal{P}\text{oly}\mathcal{E}\text{nd}(I)$ is stable under this monoidal product [GK13, proposition 1.12]. The identity polynomial functor $I \leftarrow I \rightarrow I \rightarrow I$ is associated to the identity endofunctor; thus $\mathcal{P}\text{oly}\mathcal{E}\text{nd}(I)$ is a monoidal subcategory of $\mathcal{C}\text{art}(\mathbf{Set}/I)$.

2.2. Trees.

Definition 2.2.1 (Polynomial tree). A polynomial functor T given by

$$T_0 \xleftarrow{s} T_2 \xrightarrow{p} T_1 \xrightarrow{t} T_0$$

is a (*polynomial*) *tree* [Koc11, section 1.0.3] if

- (1) the sets T_0 , T_1 and T_2 are finite (in particular, each node has finitely many inputs);
- (2) the map t is injective;
- (3) the map s is injective, and the complement of its image $T_0 - \text{im } s$ has a single element, called the *root*;
- (4) let $T_0 = T_2 + \{r\}$, with r the root, and define the *walk-to-root* function σ by $\sigma(r) = r$, and otherwise $\sigma(e) = tp(e)$; then we ask that for all $x \in T_0$, there exists $k \in \mathbb{N}$ such that $\sigma^k(x) = r$.

We call the colours of a tree its *edges* and the inputs of a node the *input edges* of that node.

Let $\mathcal{T}\text{ree}$ be the full subcategory of $\mathcal{P}\text{oly}\mathcal{E}\text{nd}$ whose objects are trees. Note that it is the category of *symmetric* or *non-planar* trees (the automorphism group of a tree is in general non-trivial) and that its morphisms correspond to inclusions of non-planar subtrees. An *elementary tree* is a tree with at most one node. Let $\text{el}\mathcal{T}\text{ree}$ be the full subcategory of $\mathcal{T}\text{ree}$ spanned by elementary trees.

²We recall that a natural transformation is *cartesian* if all its naturality squares are cartesian.

Definition 2.2.2 (P -tree). For $P \in \mathcal{PolyEnd}$, the category $\text{tr}P$ of P -trees is the slice $\mathcal{T}\text{ree}/P$. The fundamental difference between $\mathcal{T}\text{ree}$ and $\text{tr}P$ is that the latter is always rigid i.e. it has no non-trivial automorphisms [Koc11, proposition 1.2.3]. In particular, this implies that $\mathcal{PolyEnd}$ does not have a terminal object.

Notation 2.2.3. Every P -tree $T \in \text{tr}P$ corresponds to a morphism from a tree (which we shall denote by $\langle T \rangle$) to P , so that $T : \langle T \rangle \rightarrow P$. We point out that $\langle T \rangle_1$ is the set of nodes of $\langle T \rangle$, while $T_1 : \langle T \rangle_1 \rightarrow P_1$ is a decoration of the nodes of $\langle T \rangle$ by nodes of P , and likewise for edges.

Definition 2.2.4 (Category of elements). For $P \in \mathcal{PolyEnd}$, its *category of elements*³ $\text{elt}P$ is the slice $\text{el}\mathcal{T}\text{ree}/P$. Explicitly, for P as in equation (2.1.2), the set of objects of $\text{elt}P$ is $I + B$, and for each $b \in B$, there is a morphism $t : t(b) \rightarrow b$, and a morphism $s_e : s_e(b) \rightarrow b$ for each $e \in E(b)$. Remark that there is no non-trivial composition of arrows in $\text{elt}P$.

Proposition 2.2.5 ([Koc11, proposition 2.1.3]). *There is an equivalence of categories $\mathcal{P}\text{sh}(\text{elt}P) \simeq \mathcal{PolyEnd}/P$.*

Proof. For $X \in \mathcal{P}\text{sh}(\text{elt}P)$, construct the following polynomial functor over P :

$$\begin{array}{ccccccc} \sum_{i \in I} X_i & \longleftarrow & E_X & \longrightarrow & \sum_{b \in B} X_b & \longrightarrow & \sum_{i \in I} X_i \\ & & \downarrow & \lrcorner & \downarrow & & \downarrow \\ & & I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & I, \end{array}$$

where $E_X \rightarrow \sum_{i \in I} X_i$ is given by the maps $X_{s_e} : X_b \rightarrow X_{s_e b}$, for $b \in B$ and $e \in E(b)$. In the other direction, note that the full inclusion $\text{el}\mathcal{T}\text{ree} \hookrightarrow \mathcal{PolyEnd}$ induces a full inclusion $\iota : \text{elt}P \hookrightarrow \mathcal{PolyEnd}/P$ whose nerve functor $\mathcal{PolyEnd}/P \rightarrow \mathcal{P}\text{sh}(\text{elt}P)$ maps $Q \in \mathcal{PolyEnd}/P$ to $\mathcal{PolyEnd}/P(\iota-, Q)$. The two constructions are easily seen to define the required equivalence of categories. \square

2.3. Addresses.

Definition 2.3.1 (Address). Let $T \in \mathcal{T}\text{ree}$ be a polynomial tree and σ be its walk-to-root function (definition 2.2.1). We define the *address* function $\&$ on edges inductively as follows:

- (1) if r is the root edge, let $\&r := []$,
- (2) if $e \in T_0 - \{r\}$ and if $\&\sigma(e) = [x]$, define $\&e := [xe]$.

The address of a node $b \in T_1$ is defined as $\&b := \&t(b)$. Note that this function is injective since t is. Let T^\bullet denote its image, the set of *node addresses* of T , and let $T^|$ be the set of addresses of leaf edges, i.e. those not in the image of t .

Assume now that $T : \langle T \rangle \rightarrow P$ is a P -tree. If $b \in \langle T \rangle_1$ has address $\&b = [p]$, write $s_{[p]}T := T_1(b)$. For convenience, we let $T^\bullet := \langle T \rangle^\bullet$, and $T^| := \langle T \rangle^|$.

Remark 2.3.2. The formalism of addresses is a useful bookkeeping syntax for the operations of grafting and substitution on trees. The syntax of addresses will extend to the category of opetopes and will allow us to give a precise description of the composition of morphisms in the category of opetopes (see definition 3.3.2) as well as certain constructions on opetopic sets.

Notation 2.3.3. We denote by $\text{tr}^|P$ the set of P -trees with a marked leaf, i.e. endowed with the address of one of its leaves. Similarly, we denote by $\text{tr}^\bullet P$ the set of P -trees with a marked node.

³Not to be confused with the category of elements of a presheaf over some category.

2.4. Grafting.

Definition 2.4.1 (Elementary P -trees). Let P be a polynomial endofunctor as in equation (2.1.2). For $i \in I$, define $\mathfrak{l}_i \in \text{tr } P$ as having underlying tree

$$\{i\} \longleftarrow \emptyset \longrightarrow \emptyset \longrightarrow \{i\}, \quad (2.4.2)$$

along with the obvious morphism to P , that which maps i to $i \in I$. This corresponds to a tree with no nodes and a unique edge decorated by i . Define $\mathfrak{Y}_b \in \text{tr } P$, the *corolla* at b , as having underlying tree

$$s(E(b)) + \{*\} \xleftarrow{s} E(b) \longrightarrow \{b\} \longrightarrow s(E(b)) + \{*\}, \quad (2.4.3)$$

where the right map sends b to $*$, and where the morphism $\mathfrak{Y}_b \longrightarrow P$ is the identity on $s(E(b)) \subseteq I$, maps $*$ to $t(b) \in I$, is the identity on $E(b) \subseteq E$, and maps b to $b \in B$. This corresponds to a P -tree with a unique node, decorated by b . Observe that for $T \in \text{tr } P$, giving a morphism $\mathfrak{l}_i \longrightarrow T$ is equivalent to specifying the address $[p]$ of an edge of T decorated by i . Likewise, morphisms of the form $\mathfrak{Y}_b \longrightarrow T$ are in bijection with addresses of nodes of T decorated by b .

Definition 2.4.4 (Grafting). For $S, T \in \text{tr } P$, $[l] \in S^{\downarrow}$ such that the leaf of S at $[l]$ and the root edge of T are decorated by the same $i \in I$, define the *grafting* $S \circ_{[l]} T$ of S and T on $[l]$ by the following pushout (in $\text{tr } P$):

$$\begin{array}{ccc} \mathfrak{l}_i & \xrightarrow{\square} & T \\ \mathfrak{[l]} \downarrow & & \downarrow \\ S & \longrightarrow & S \circ_{[l]} T. \end{array} \quad (2.4.5)$$

Note that if S (resp. T) is a trivial tree, then $S \circ_{[l]} T = T$ (resp. S). We assume, by convention, that the grafting operator \circ associates to the right.

Proposition 2.4.6 ([Koc11, proposition 1.1.21]). *Every P -tree is either of the form \mathfrak{l}_i , for some $i \in I$, or obtained by iterated graftings of corollas (i.e. P -trees of the form \mathfrak{Y}_b for $b \in B$).*

Notation 2.4.7 (Total grafting). Take $T, U_1, \dots, U_k \in \text{tr } P$, where $T^{\downarrow} = \{[l_1], \dots, [l_k]\}$, and assume the grafting $T \circ_{[l_i]} U_i$ is defined for all i . Then the *total grafting* will be denoted concisely by

$$T \bigcirc_{[l_i]} U_i = (\dots (T \circ_{[l_1]} U_1) \circ_{[l_2]} U_2 \dots) \circ_{[l_k]} U_k. \quad (2.4.8)$$

It is easy to see that the result does not depend on the order in which the graftings are performed.

Definition 2.4.9 (Substitution). Let $[p] \in T^{\bullet}$ and $b = s_{[p]} T$. Then T can be decomposed as

$$T = A \circ_{[p]} \mathfrak{Y}_b \bigcirc_{[e_i]} B_i, \quad (2.4.10)$$

where $E(b) = \{e_1, \dots, e_k\}$, and $A, B_1, \dots, B_k \in \text{tr } P$. For U a P -tree with a bijection $\wp : U^{\downarrow} \longrightarrow E(b)$ over I , we define the *substitution* $T \square_{[p]} U$ as

$$T \square_{[p]} U := A \circ_{[p]} U \bigcirc_{\wp^{-1}e_i} B_i. \quad (2.4.11)$$

In other words, the node at address $[p]$ in T has been replaced by U , and the map \wp provided “rewiring instructions” to connect the leaves of U to the rest of T .

2.5. Polynomial monads.

Definition 2.5.1 (Polynomial monad). A *polynomial monad over I* is a monoid in $\text{PolyEnd}(I)$. Note that a polynomial monad over I is thus necessarily a cartesian monad on Set/I .⁴ Let $\text{PolyMnd}(I)$ be the category of monoids in $\text{PolyEnd}(I)$. That is, $\text{PolyMnd}(I)$ is the category of polynomial monads over I and morphisms of polynomial functors over I that are also monad morphisms.

Definition 2.5.2 ($(-)^*$ construction). Given a polynomial endofunctor P as in equation (2.1.2), we define a new polynomial endofunctor P^* as

$$I \xleftarrow{s} \text{tr}^{\downarrow} P \xrightarrow{p} \text{tr} P \xrightarrow{t} I \quad (2.5.3)$$

where s maps a P -tree with a marked leaf to the decoration of that leaf, p forgets the marking, and t maps a tree to the decoration of its root. Remark that for $T \in \text{tr} P$ we have $p^{-1}T = T^{\downarrow}$.

Theorem 2.5.4 ([Koc11, section 1.2.7], [KJBM10, sections 2.7 to 2.9]). *The polynomial functor P^* has a canonical structure of a polynomial monad. Furthermore, the functor $(-)^*$ is left adjoint to the forgetful functor $\text{PolyMnd}(I) \rightarrow \text{PolyEnd}(I)$, and the adjunction is monadic.*

Definition 2.5.5 (Readdressing function). We abuse notation slightly by letting $(-)^*$ denote the associated monad on $\text{PolyEnd}(I)$. Let M be a polynomial monad as on the left below. By theorem 2.5.4, M is a $(-)^*$ -algebra, and we will write its structure map $M^* \rightarrow M$ as on the right:

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I, \quad \begin{array}{ccccc} I & \xleftarrow{\quad} & \text{tr}^{\downarrow} M & \xrightarrow{\quad} & \text{tr} M & \xrightarrow{\quad} & I \\ \parallel & & \downarrow \wp & \lrcorner & \downarrow \mathfrak{t} & & \parallel \\ I & \xleftarrow{\quad} & E & \xrightarrow{\quad} & B & \xrightarrow{\quad} & I. \end{array} \quad (2.5.6)$$

where rT is the decoration of the root edge of a tree $T \in \text{tr} M$. We call $\wp_T : T^{\downarrow} \xrightarrow{\cong} E(rT)$ the *readdressing* function of T , and $\mathfrak{t}T \in B$ is called the *target* of T . If we think of an element $b \in B$ as the corolla Y_b , then the target map \mathfrak{t} “contracts” a tree to a corolla, and since the middle square is a pullback, the number of leaves is preserved. The map \wp_T establishes a coherent correspondence between the set T^{\downarrow} of leaf addresses of a tree T and the elements of $E(T)$.

2.6. The Baez–Dolan construction.

Definition 2.6.1 (Baez–Dolan $(-)^+$ construction). Let M be a polynomial monad as in equation (2.1.2), and define its *Baez–Dolan construction* M^+ to be

$$B \xleftarrow{\mathfrak{s}} \text{tr}^{\bullet} M \xrightarrow{p} \text{tr} M \xrightarrow{\mathfrak{t}} B \quad (2.6.2)$$

where \mathfrak{s} maps an M -tree with a marked node to the label of that node, p forgets the marking, and \mathfrak{t} is the target map. If $T \in \text{tr} M$, remark that $p^{-1}T = T^{\bullet}$ is the set of node addresses of T . If $[p] \in T^{\bullet}$, then $\mathfrak{s}[p] := \mathfrak{s}_{[p]} T$.

Theorem 2.6.3 ([KJBM10, section 3.2]). *The polynomial functor M^+ has a canonical structure of a polynomial monad.*

Remark 2.6.4. The $(-)^+$ construction is an endofunctor on PolyMnd whose definition is motivated as follows. If we begin with a polynomial monad M , then the colours of M^+ are the operations of M . The operations of M^+ , along with their output colour, are given by the monad multiplication of M : they are the *relations* of M , i.e. the reductions of trees of M to operations of M . The monad multiplication

⁴We recall that a monad is *cartesian* if its endofunctor preserves pullbacks and its unit and multiplication are cartesian natural transformations.

on M^+ is given as follows: the reduction of a tree of M^+ to an operation of M^+ (which is a tree of M) is obtained by *substituting* trees of M into nodes of trees of M .

Let M be a finitary (i.e. the fibers of p below are finite, or equivalently, p_* , and therefore $M = t_!p_*s^*$, preserves filtered colimits) polynomial monad whose underlying polynomial functor is

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} I.$$

The Baez–Dolan construction gives the polynomial monad M^+ whose underlying polynomial functor is

$$B \xleftarrow{s} \mathrm{tr}^\bullet M \xrightarrow{p} \mathrm{tr} M \xrightarrow{t} B.$$

Recall also from theorem 2.5.4 that the category $\mathcal{P}\mathrm{olyMnd}(I)$ is the category of $(-)^*$ -algebras. The following fact is analogous to proposition 2.2.5 and is at the heart of the Baez–Dolan construction (indeed, it is even the original *definition* of the construction, see [BD98, definition 15]).

Proposition 2.6.5. *For M a polynomial monad, there is an equivalence of categories $\mathcal{A}\mathrm{lg}(M^+) \simeq \mathcal{P}\mathrm{olyMnd}(I)/M$.*

Proof. Given a M^+ -algebra $M^+X \xrightarrow{x} X$ in Set/B , define $\Phi X \in \mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)/M$ as

$$\begin{array}{ccccc} I & \longleftarrow & E_X & \longrightarrow & X & \longrightarrow & I \\ \parallel & & \downarrow & \lrcorner & \downarrow & & \parallel \\ I & \longleftarrow & E & \longrightarrow & B & \longrightarrow & I \end{array}$$

There is an evident bijection $\mathrm{tr} \Phi X \cong M^+X$ in Set/I , and the structure map x extends by pullback along $E_X \rightarrow X$ to a map $(\Phi X)^* \rightarrow \Phi X$ in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$. It is easy to verify that this determines a $(-)^*$ -algebra structure on ΦX , and that the map $\Phi X \rightarrow M$ in $\mathcal{P}\mathrm{oly}\mathcal{E}\mathrm{nd}(I)$ is a morphism of $(-)^*$ -algebras. Conversely, given an $N \in \mathcal{P}\mathrm{olyMnd}(I)/M$ whose underlying polynomial functor is

$$I \longleftarrow E' \longrightarrow B' \longrightarrow I,$$

then the bijection $\mathrm{tr} N \cong M^+B'$ in Set/I and the $(-)^*$ -algebra map $N^* \rightarrow N$ provide a map $M^+B' \xrightarrow{\Psi N} B'$ in Set/I . It is easy to verify that ΨN is the structure map of a M^+ -algebra and that the constructions Φ and Ψ are functorial and mutually inverse. \square

Remark 2.6.6. The previous result provides an equivalence between $\mathcal{P}\mathrm{olyMnd}(I)/M$ and the category of M^+ -algebras. A “coloured” version of this result can be (informally) stated as follows: for $\mathcal{A}\mathrm{lg}^{\mathrm{col}}(M^+)$ a suitable category of coloured M^+ -algebras, there is an equivalence $\mathcal{A}\mathrm{lg}^{\mathrm{col}}(M^+) \simeq \mathcal{P}\mathrm{olyMnd}/M$, where $\mathcal{P}\mathrm{olyMnd}$ is the category of all polynomial monads (for all I in Set).

3. OPETOPES

In this section, we use the formalism of polynomial functors and polynomial monads of section 2 to define opetopes and morphisms between them. This gives us a category \mathbb{O} of opetopes and a category $\mathcal{P}\mathrm{sh}(\mathbb{O})$ of opetopic sets. Our construction of opetopes is precisely that of [KJBM10], and by [KJBM10, theorem 3.16], also that of [Lei04, Lei98], and by [Che04b, corollary 2.6], also that of [Che03]. As we will see, the category \mathbb{O} is rigid, i.e. it has no non-trivial automorphisms (it is in fact a *direct* category).

3.1. Polynomial definition of opetopes.

Definition 3.1.1 (The \mathfrak{Z}^n monad). Let \mathfrak{Z}^0 be the identity polynomial monad on Set , as depicted on the left below, and let $\mathfrak{Z}^n := (\mathfrak{Z}^{n-1})^+$. Write \mathfrak{Z}^n as on right:

$$\{\ast\} \longleftarrow \{\ast\} \longrightarrow \{\ast\} \longrightarrow \{\ast\}, \quad \mathbb{O}_n \xleftarrow{\mathfrak{s}} E_{n+1} \xrightarrow{\mathfrak{p}} \mathbb{O}_{n+1} \xrightarrow{\mathfrak{t}} \mathbb{O}_n. \quad (3.1.2)$$

Definition 3.1.3 (Opetope). An n -dimensional opetope (or simply n -opetope) ω is by definition an element of \mathbb{O}_n , and we write $\dim \omega = n$. An opetope $\omega \in \mathbb{O}_n$ with $n \geq 2$ is called *degenerate* if its underlying tree has no nodes (thus consists of a unique edge); it is *non degenerate* otherwise.

Following (2.5.6), for $\omega \in \mathbb{O}_{n+2}$, the structure of polynomial monad $(\mathfrak{Z}^n)^\ast \rightarrow \mathfrak{Z}^n$ gives a bijection $\wp_\omega : \omega^\downarrow \rightarrow (\mathfrak{t}\omega)^\bullet$ between the leaves of ω and the nodes of $\mathfrak{t}\omega$, preserving the decoration by n -opetopes.

Example 3.1.4. (1) The unique 0-opetope is denoted \blacklozenge and called the *point*.

(2) The unique 1-opetope is denoted \blacksquare and called the *arrow*.

(3) If $n \geq 2$, then $\omega \in \mathbb{O}_n$ is a \mathfrak{Z}^{n-2} -tree, i.e. a tree whose nodes are labeled in $(n-1)$ -opetopes, and edges are labeled in $(n-2)$ -opetopes. In particular, 2-opetopes are \mathfrak{Z}^0 -trees, i.e. linear trees, and thus in bijection with \mathbb{N} . We will refer to them as *opetopic integers*, and write \mathbf{n} for the 2-opetope having exactly n nodes.

3.2. Higher addresses. By definition, an opetope ω of dimension $n \geq 2$ is a \mathfrak{Z}^{n-2} -tree, and thus the formalism of tree addresses (definition 2.3.1) can be applied to designate nodes of ω , also called its *source faces* or simply *sources*. In this section, we iterate this formalism into the concept of *higher dimensional address*, which turns out to be more convenient. This material is largely taken from [CHTM19, section 2.2.3] and [HT18, section 4].

Definition 3.2.1 (Higher address). Start by defining the set \mathbb{A}_n of n -addresses as follows:

$$\mathbb{A}_0 = \{\ast\}, \quad \mathbb{A}_{n+1} = \text{lists } \mathbb{A}_n,$$

where $\text{lists } X$ is the set of finite lists (or words) on the alphabet X .

Explicitly, the unique 0-address is \ast (also written $[]$ by convention), while an $(n+1)$ -address is a sequence of n -addresses. Such sequences are enclosed by brackets. Note that the address $[]$, associated to the empty word, is in \mathbb{A}_n for all $n \geq 0$. However, the surrounding context will almost always make the notation unambiguous.

Here are examples of higher addresses:

$$[] \in \mathbb{A}_1, \quad [\ast \ast \ast] \in \mathbb{A}_1, \quad [[][\ast][[]]] \in \mathbb{A}_2, \quad [[[[\ast]]]] \in \mathbb{A}_4.$$

For $\omega \in \mathbb{O}$ an opetope, nodes of ω can be specified uniquely using higher addresses, as we now show. Recall that E_{n-1} is the set of arities of \mathfrak{Z}^{n-2} (equation (3.1.2)). In \mathfrak{Z}^0 , set $E_1(\blacksquare) = \{\ast\}$, so that the unique node address of \blacksquare is $\ast \in \mathbb{A}_0$. For $n \geq 2$, recall that an opetope $\omega \in \mathbb{O}_n$ is a \mathfrak{Z}^{n-2} -tree $\omega : \langle \omega \rangle \rightarrow \mathfrak{Z}^{n-2}$ (definition 2.2.2), and write $\langle \omega \rangle$ as

$$I_\omega \longleftarrow E_\omega \longrightarrow B_\omega \longrightarrow I_\omega.$$

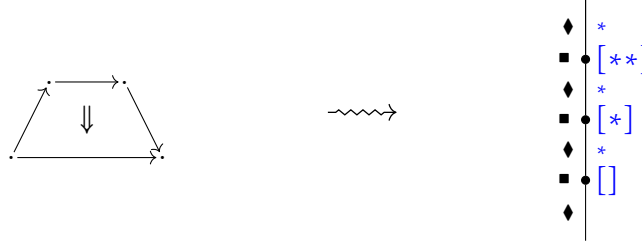
A node $b \in B_\omega$ has an address $\&b \in \text{lists } E_\omega$, which by $\omega_2 : E_\omega \rightarrow E_{n-1}$ is mapped to an element of $\text{lists } E_{n-1}$. By induction, elements of E_{n-1} are $(n-2)$ -addresses, whence $\omega_2(\&b) \in \text{list } \mathbb{A}_{n-2} = \mathbb{A}_{n-1}$. For the induction step, elements of $E_n(\omega)$ are nodes of $\langle \omega \rangle$, which we identify by their aforementioned $(n-1)$ -addresses. Consequently, for all $n \geq 1$ and $\omega \in \mathbb{O}_n$, elements of $E_n(\omega) = \omega^\bullet$ can be seen as the set of $(n-1)$ -addresses of the nodes of ω , and similarly, ω^\downarrow can be seen as the set of $(n-1)$ -addresses of edges of ω .

We now identify the nodes of $\omega \in \mathbb{O}_{n+2}$ with their addresses. In particular, for $[p] \in \omega^\bullet$ a node address of ω , we make use of the notation $\mathfrak{s}_{[p]}\omega$ of section 2.2 to refer to the decoration of the node at address $[p]$,

which is an $(n + 1)$ -opetope. Let $[l] = [p[q]] \in \mathbb{A}_{n-1}$ be an address such that $[p] \in \omega^\bullet$ and $[q] \in (s_{[p]}\omega)^\bullet$. Then as a shorthand, we write

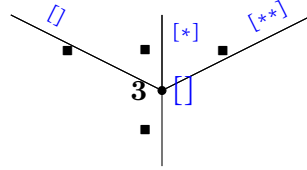
$$e_{[l]}\omega := s_{[q]}s_{[p]}\omega. \quad (3.2.2)$$

Example 3.2.3. Consider the 2-opetope on the left, called **3**:

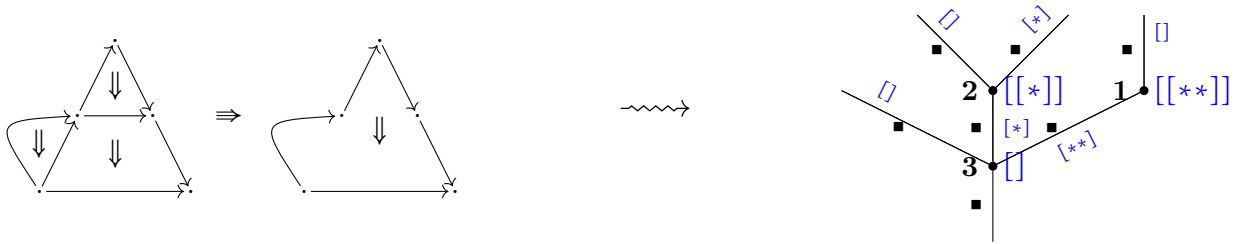


Its underlying pasting diagram consists of 3 arrows \blacksquare grafted linearly. Since the only node address of \blacksquare is $* \in \mathbb{A}_0$, the underlying tree of **3** can be depicted as on the right. On the left of that three are the decorations: nodes are decorated with $\blacksquare \in \mathbb{O}_1$, while the edges are decorated with $\blacklozenge \in \mathbb{O}_0$. For each node in the tree, the set of input edges of that node is in bijective correspondence with the node addresses of the decorating opetope, and that address is written on the right of each edges. In this low dimensional example, those addresses can only be $*$. Finally, on the right of each node of the tree is its 1-address, which is just a sequence of 0-addresses giving “walking instructions” to get from the root to that node.

The 2-opetope **3** can then be seen as a corolla in some 3-opetope as follows:



As previously mentioned, the set of input edges is in bijective correspondence with the set of node addresses of **3**. Here is now an example of a 3-opetope, with its annotated underlying tree on the right (the 2-opetopes **1** and **2** are analogous to **3**):



3.3. The category of opetopes. In this subsection, we define the category \mathbb{O} of planar opetopes introduced in [HT18], following the work of [Che03].

Proposition 3.3.1 (Opetopic identities, [HT18, theorem 4.1]). *Let $\omega \in \mathbb{O}_n$ with $n \geq 2$.*

- (1) (Inner edge) For $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), we have $\text{ts}_{[p[q]]}\omega = s_{[q]}s_{[p]}\omega$.
- (2) (Globularity 1) If ω is non degenerate, we have $\text{ts}_{[]} \omega = \text{tt}\omega$.
- (3) (Globularity 2) If ω is non degenerate, and $[p[q]] \in \omega^\dagger$, we have $s_{[q]}s_{[p]}\omega = s_{\wp\omega[p[q]]}\text{tt}\omega$.
- (4) (Degeneracy) If ω is degenerate, we have $s_{[]} \text{tt}\omega = \text{tt}\omega$.

Definition 3.3.2 ([HT18, section 4.2]). With these identities in mind, we define the category \mathbb{O} of opetopes by generators and relations as follows.

- (1) (Objects) We set $\text{ob } \mathbb{O} = \sum_{n \in \mathbb{N}} \mathbb{O}_n$.
- (2) (Generating morphisms) Let $\omega \in \mathbb{O}_n$ with $n \geq 1$. We introduce a generator $\mathfrak{t}\omega \xrightarrow{\mathfrak{t}} \omega$, called the *target embedding*. If $[p] \in \omega^\bullet$, then we introduce a generator $\mathfrak{s}_{[p]}\omega \xrightarrow{\mathfrak{s}_{[p]}} \omega$, called a *source embedding*. A *face embedding* is either a source or the target embedding.
- (3) (Relations) We impose 4 relations described by the following commutative squares, that are well defined thanks to proposition 3.3.1. Let $\omega \in \mathbb{O}_n$ with $n \geq 2$
- (a) (**Inner**) for $[p[q]] \in \omega^\bullet$ (forcing ω to be non degenerate), the following square must commute:

$$\begin{array}{ccc} \mathfrak{s}_{[q]}\mathfrak{s}_{[p]}\omega & \xrightarrow{\mathfrak{s}_{[q]}} & \mathfrak{s}_{[p]}\omega \\ \mathfrak{t} \downarrow & & \downarrow \mathfrak{s}_{[p]} \\ \mathfrak{s}_{[p[q]]}\omega & \xrightarrow{\mathfrak{s}_{[p[q]]}} & \omega \end{array}$$

- (b) (**Glob1**) if ω is non degenerate, the following square must commute:

$$\begin{array}{ccc} \mathfrak{t}\mathfrak{t}\omega & \xrightarrow{\mathfrak{t}} & \mathfrak{t}\omega \\ \mathfrak{t} \downarrow & & \downarrow \mathfrak{t} \\ \mathfrak{s}_{[]} \omega & \xrightarrow{\mathfrak{s}_{[]}} & \omega. \end{array}$$

- (c) (**Glob2**) if ω is non degenerate, and for $[p[q]] \in \omega^\bullet$, the following square must commute:

$$\begin{array}{ccc} \mathfrak{s}_{\wp_\omega[p[q]]}\mathfrak{t}\omega & \xrightarrow{\mathfrak{s}_{\wp_\omega[p[q]]}} & \mathfrak{t}\omega \\ \mathfrak{s}_{[q]} \downarrow & & \downarrow \mathfrak{t} \\ \mathfrak{s}_{[p]}\omega & \xrightarrow{\mathfrak{s}_{[p]}} & \omega. \end{array}$$

- (d) (**Degen**) if ω is degenerate, the following square must commute:

$$\begin{array}{ccc} \mathfrak{t}\mathfrak{t}\omega & \xrightarrow{\mathfrak{t}} & \mathfrak{t}\omega \\ \mathfrak{s}_{[]} \downarrow & & \downarrow \mathfrak{t} \\ \mathfrak{t}\omega & \xrightarrow{\mathfrak{t}} & \omega. \end{array}$$

See [HT18] for a graphical explanation of those relations.

Notation 3.3.3. For $n \in \mathbb{N}$, we let $\mathbb{O}_{\leq n}$ be the full subcategory of \mathbb{O} spanned by opetopes of dimension at most n . The subcategories $\mathbb{O}_{< n}$, $\mathbb{O}_{\geq n}$, $\mathbb{O}_{> n}$, and $\mathbb{O}_{=n}$ are defined similarly. Note that the latter is simply the set \mathbb{O}_n .

3.4. Opetopic sets. Recall from section 1.6 that $\mathcal{P}\text{sh}(\mathbb{O})$ is the category of opetopic sets, i.e. Set-valued presheaves over \mathbb{O} . For $X \in \mathcal{P}\text{sh}(\mathbb{O})$ and $\omega \in \mathbb{O}$, we will refer to the elements of the set X_ω as the *cells* of X of *shapeshape* ω .

Definition 3.4.1. (1) The representable presheaf at $\omega \in \mathbb{O}_n$ is denoted $O[\omega]$. Its cells are morphisms of \mathbb{O} of the form $\mathfrak{f} : \psi \rightarrow \omega$, for \mathfrak{f} a sequence of face embeddings, which we write $\mathfrak{f}\omega \in O[\omega]_\psi$ for short. For instance, the cell of maximal dimension is simply ω (as the corresponding sequence of face embeddings is empty), its $(n-1)$ -cells are $\{\mathfrak{s}_{[p]}\omega \mid [p] \in \omega^\bullet\} \cup \{\mathfrak{t}\omega\}$, and there is no cell of dimension $> n$.

- (2) The *boundary* $\partial O[\omega]$ of ω is the maximal subpresheaf of $O[\omega]$ not containing the cell ω . We write $\mathfrak{b}_\omega : \partial O[\omega] \hookrightarrow O[\omega]$ for the *boundary inclusion*. The set of boundary inclusions is denoted by \mathbf{B}

- (3) The *spine* $S[\omega]$ is the maximal subpresheaf of $\partial O[\omega]$ not containing the cell $\mathfrak{t}\omega$, and we write $\mathfrak{s}_\omega : S[\omega] \hookrightarrow O[\omega]$ for the *spine inclusion*. The set of spine inclusions is denoted by \mathfrak{S} .

Lemma 3.4.2. *For $\omega \in \mathbb{O}$, with $\dim \omega \geq 1$ the following square is a pushout and a pullback⁵, where all arrows are canonical inclusions:*

$$\begin{array}{ccc} \partial O[\mathfrak{t}\omega] & \longrightarrow & S[\omega] \\ \downarrow & & \downarrow \\ O[\mathfrak{t}\omega] & \longrightarrow & \partial O[\omega]. \end{array}$$

Lemma 3.4.3. *Let $n \geq 1$, $\nu \in \mathbb{O}_n$, $[l] \in \nu^l$, and $\psi \in \mathbb{O}_{n-1}$ be such that $\mathfrak{e}_{[l]}\nu = \mathfrak{t}\psi$, so that the grafting $\nu \circ_{[l]}\mathfrak{Y}_\psi$ is well-defined. Then the following square is a pushout:*

$$\begin{array}{ccc} O[\mathfrak{e}_{[l]}\nu] & \xrightarrow{\mathfrak{t}} & O[\psi] \\ \mathfrak{e}_{[l]}\downarrow & & \downarrow \mathfrak{s}_{[l]} \\ S[\nu] & \longrightarrow & S[\nu \circ_{[l]}\mathfrak{Y}_\psi]. \end{array}$$

Notation 3.4.4. Let $F : \mathbb{O} \rightarrow \text{hom } \mathcal{C}$ be a function that maps opetopes to morphisms in some category \mathcal{C} , and \mathfrak{M} the set of maps defined by $\mathfrak{M} := \{F(\omega) \mid \omega \in \mathbb{O}\}$. Then for $n \in \mathbb{N}$, we define $\mathfrak{M}_{\geq n} := \{F(\omega) \mid \omega \in \mathbb{O}_{\geq n}\}$, and similarly for $\mathfrak{M}_{>n}$, $\mathfrak{M}_{\leq n}$, $\mathfrak{M}_{<n}$, and $\mathfrak{M}_{=n}$. For convenience, the latter is abbreviated \mathfrak{M}_n . If $m \leq n$, we also let $\mathfrak{M}_{m,n} = \mathfrak{M}_{\geq m} \cap \mathfrak{M}_{\leq n}$. By convention, $\mathfrak{M}_{\leq n} = \emptyset$ if $n < 0$. For example, $\mathfrak{S}_{\geq 2} = \{\mathfrak{s}_\omega \mid \omega \in \mathbb{O}_{\geq 2}\}$, and $\mathfrak{S}_{n,n+1} = \mathfrak{S}_n \cup \mathfrak{S}_{n+1}$.

Definition 3.4.5. Write $\mathfrak{O} := \{\emptyset \hookrightarrow O[\omega] \mid \omega \in \mathbb{O}\}$ for the set of initial inclusions of the representables. Let

$$\mathfrak{A}_{k,n} := \mathfrak{O}_{<n-k} \cup \mathfrak{S}_{\geq n+1}. \quad (3.4.6)$$

Lemma 3.4.7. (1) *Let $X \in \mathcal{Psh}(\mathbb{O})$ such that $\mathfrak{S}_{n,n+1} \perp X$. Then $\mathfrak{B}_{n+1} \perp X$. In general, every morphism in $\mathfrak{B}_{\geq n+1}$ is an $\mathfrak{S}_{\geq n}$ -local isomorphism.*

- (2) *Let $X \in \mathcal{Psh}(\mathbb{O})$ such that $(\mathfrak{S}_{n,n+1} \cup \mathfrak{B}_{n+2}) \perp X$. Then $\mathfrak{S}_{n+2} \perp X$. In particular, if $(\mathfrak{S}_{n,n+1} \cup \mathfrak{B}_{\geq n+2}) \perp X$, then $\mathfrak{S}_{\geq n} \perp X$.*

Proof. (1) Let $\omega \in \mathbb{O}_{n+1}$. By 3-for-2, since the composite $\mathfrak{s}_\omega : S[\omega] \hookrightarrow \partial O[\omega] \hookrightarrow O[\omega]$ is in $\mathfrak{S}_{n,n+1}$, it suffices to show that $(S[\omega] \hookrightarrow \partial O[\omega]) \perp X$. Let $f : S[\omega] \rightarrow X$ be a morphism. The existence of a lift $\partial O[\omega] \rightarrow X$ follows from the existence of a lift $O[\omega] \rightarrow X$. Next, given two lifts $g, h : \partial O[\omega] \rightarrow X$ of f , by lemma 3.4.2, it suffices to show that they coincide on $O[\mathfrak{t}\omega]$ to show that they are equal. But since they coincide on $S[\omega]$, they coincide on $S[\mathfrak{t}\omega]$, and hence on $O[\mathfrak{t}\omega]$ since $\mathfrak{s}_{\mathfrak{t}\omega} \perp X$.

- (2) Let $\omega \in \mathbb{O}_{n+2}$ and $f : S[\omega] \rightarrow X$. Then the restriction $f|_{S[\mathfrak{t}\omega]}$ of f to $S[\mathfrak{t}\omega]$ extends to a unique $f|_{S[\mathfrak{t}\omega]}^{O[\mathfrak{t}\omega]}$. We now show that the following square commutes:

$$\begin{array}{ccc} \partial O[\mathfrak{t}\omega] & \longrightarrow & S[\omega] \\ \downarrow & & \downarrow f \\ O[\mathfrak{t}\omega] & \xrightarrow{f|_{S[\mathfrak{t}\omega]}^{O[\mathfrak{t}\omega]}} & X. \end{array} \quad (3.4.8)$$

⁵Recall that in a topos, the pushout of a monomorphism along any arrow is a monomorphism, and the pushout square is a pullback square. This property is sometimes called ‘‘adhesivity’’, and is a consequence of van Kampen-ness, or descent, for pushouts of monomorphisms.

Tautologically, we have $f|_{S[t\omega]} = f|_{S[t\omega]}^{O[t\omega]}|_{S[t\omega]}$, and in particular, $f|_{S[tt\omega]} = f|_{S[t\omega]}^{O[t\omega]}|_{S[tt\omega]}$. Since $S_{n-2} \perp X$, we have $f|_{O[tt\omega]} = f|_{S[t\omega]}^{O[t\omega]}|_{O[tt\omega]}$. Therefore, square (3.4.8) commutes, and by lemma 3.4.2, f extends to a unique $f|^{\partial O[\omega]} : \partial O[\omega] \rightarrow O[\omega]$ such that $f|_{O[t\omega]}^{\partial O[\omega]} = f|_{S[t\omega]}^{O[t\omega]}$. Since $B_n \perp X$, $f|_{\partial O[\omega]}$ extends to a unique morphism $O[\omega] \rightarrow X$. \square

Lemma 3.4.9. *Let $n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{n+2}$. Then the inclusion $S[t\omega] \hookrightarrow S[\omega]$ is a relative S_{n+1} -cell complex, i.e. a transfinite composition of pushouts of maps in S_{n+1} .*

Proof. We show that the morphism $S[t\omega] \hookrightarrow S[\omega]$ is a (finite) composite of pushouts of elements of S_{n+1} . Let $X_0 = S[t\omega]$. At stage $k \geq 0$, let J be the (necessarily finite) set of inclusions $j = (j_1, j_2)$ in $\text{Psh}(\mathbb{O})^{\rightarrow}$ as on the left

$$\begin{array}{ccc} S[\psi_j] & \xrightarrow{j_1} & X_k \\ \mathfrak{s}_{\psi_j} \downarrow & & \downarrow \\ O[\psi_j] & \xrightarrow{j_2} & S[\omega], \end{array} \quad \begin{array}{ccc} \coprod_{j \in J} S[\psi_j] & \xrightarrow{j_1} & X_k \\ \mathfrak{s}_{\psi_j} \downarrow & & \downarrow \\ \coprod_{j \in J} O[\psi_j] & \xrightarrow{j_2} & X_{k+1}. \end{array}$$

where $\psi_j \in \mathbb{O}_{n+1}$, such that j_1 does not factor through X_l for any $l < k$. Define X_{k+1} with the pushout on the right. There is a k , bounded by the height of the tree $\omega \in \text{tr } \mathfrak{Z}^n$, at which the sequence converges to $S[t\omega] = X_0 \hookrightarrow X_1 \hookrightarrow \dots \hookrightarrow X_k = S[\omega]$. \square

Corollary 3.4.10. (1) *Let $n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{n+2}$. Then the target embedding $t\omega \rightarrow \omega$ of ω is an $S_{n+1, n+2}$ -local isomorphism.*

(2) *Let $k, n \in \mathbb{N}$, and $\omega \in \mathbb{O}_{\geq n+2}$. Then the target embedding $t\omega \rightarrow \omega$ is an $A_{k, n}$ -local isomorphism.*

Proof. (1) In the square below

$$\begin{array}{ccc} S[t\omega] & \xrightarrow{\mathfrak{s}_{t\omega}} & O[t\omega] \\ r \downarrow & & \downarrow t \\ S[\omega] & \xrightarrow{\mathfrak{s}_{\omega}} & O[\omega] \end{array}$$

the map r is an S_{n+1} -local isomorphism by lemma 3.4.9, and the horizontal maps are in $S_{n+1, n+2}$. The result follows by 3-for-2.

(2) Let $\omega \in \mathbb{O}_m$. Since $S_{m-1, m+1} \subset A_{k, n}$ by definition, this follows from the previous point. \square

Corollary 3.4.11. *Let $\psi \in \mathbb{O}_n$.*

(1) *$tt = \mathfrak{s}_{\square} t : \psi \rightarrow I_{\psi}$ is in S_{n+2} .*

(2) *The morphisms $\mathfrak{s}_{\square}, t : \psi \rightarrow Y_{\psi}$ are $S_{n+1, n+2}$ -local isomorphisms.*

Proof. (1) The embedding $tt = \mathfrak{s}_{\square} t : \psi \rightarrow I_{\psi}$ is precisely the spine inclusion $\mathfrak{s}_{I_{\psi}}$ of the degenerate $(n+2)$ -opetope I_{ψ} .

(2) The source embedding $\mathfrak{s}_{\square} : \psi \rightarrow Y_{\psi}$ is precisely the spine inclusion $\mathfrak{s}_{Y_{\psi}}$ of the $(n+1)$ -opetope Y_{ψ} . The target embedding $t : \psi \rightarrow Y_{\psi}$ is the morphism $t : ttI_{\psi} \rightarrow tI_{\psi}$ and is the vertical arrow in the diagram below.

$$\begin{array}{ccc} \psi = S[I_{\psi}] & & \\ t \downarrow & \searrow \mathfrak{s}_{I_{\psi}} & \\ Y_{\psi} = tI_{\psi} & \xrightarrow{t} & I_{\psi}. \end{array}$$

The horizontal arrow is an $S_{n+1, n+2}$ -local isomorphism by corollary 3.4.10 and the diagonal arrow is in S_{n+2} by point (1). The result follows by 3-for-2. \square

3.5. Extensions.

Reminder 3.5.1. Recall that a functor between small categories $u : \mathcal{A} \rightarrow \mathcal{B}$ induces a restriction $u^* : \mathcal{Psh}(\mathcal{B}) \rightarrow \mathcal{Psh}(\mathcal{A})$ that admits both adjoints $u_! \dashv u^* \dashv u_*$, given by pointwise left and right Kan extensions.

Notation 3.5.2. Let $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{\infty\}$ be such that $m \leq n$, and let $\mathbb{O}_{m,n}$ be the full subcategory of \mathbb{O} spanned by opetopes ω such that $m \leq \dim \omega \leq n$. For instance, $\mathbb{O}_{m,\infty} = \mathbb{O}_{\geq m}$.

Definition 3.5.3 (Truncation). The inclusion $\iota_*^{\geq m} : \mathbb{O}_{m,n} \rightarrow \mathbb{O}_{\geq m}$ induces a restriction functor $(-)_m : \mathcal{Psh}(\mathbb{O}_{\geq m}) \rightarrow \mathcal{Psh}(\mathbb{O}_{m,n})$, called *truncation*, that have both a left adjoint $\iota_!^{\geq m}$ and a right adjoint $\iota_*^{\geq m}$. Explicitly, for $X \in \mathcal{Psh}(\mathbb{O}_{m,n})$, the presheaf $\iota_!^{\geq m} X$ is the “extension by 0”, i.e. $(\iota_!^{\geq m} X)_{m,n} = X$, and $(\iota_!^{\geq m} X)_\psi = \emptyset$ for all $\psi \in \mathbb{O}_{>n}$. On the other hand, $\iota_*^{\geq m} X$ is the “canonical extension” of X into a presheaf over $\mathbb{O}_{\geq m}$: we have $(\iota_*^{\geq m} X)_{m,n} = X$, and $\mathbb{B}_{>n} \perp \iota_*^{\geq m} X$, which uniquely determines $\iota_*^{\geq m} X$.

Likewise, the inclusion $\iota_*^{\leq n} : \mathbb{O}_{m,n} \rightarrow \mathbb{O}_{\leq n}$ induces a restriction functor $\mathcal{Psh}(\mathbb{O}_{\leq n}) \rightarrow \mathcal{Psh}(\mathbb{O}_{m,n})$, also denoted by $(-)_m$ and again called *truncation*, that have both a left adjoint $\iota_!^{\leq n}$ and a right adjoint $\iota_*^{\leq n}$. Explicitly, for $X \in \mathcal{Psh}(\mathbb{O}_{m,n})$, the presheaf $\iota_!^{\leq n} X$ is the “canonical extension” of X into a presheaf over $\mathbb{O}_{\leq n}$:

$$\iota_!^{\leq n} X = \operatorname{colim}_{O[\psi]_{m,n} \rightarrow X} O[\psi].$$

On the other hand, $\iota_*^{\leq n} X$ is the “terminal extension” of X in that $(\iota_*^{\leq n} X)_{m,n} = X$, and $(\iota_*^{\leq n} X)_\psi$ is a singleton, for all $\psi \in \mathbb{O}_{<n}$. Note that $\mathbb{O}_{<m} \perp \iota_*^{\leq n} X$, and that it is uniquely determined by this property.

For $n < \infty$, we write $(-)_{\leq n}$ for $(-)_{0,n} : \mathcal{Psh}(\mathbb{O}_{\geq 0}) = \mathcal{Psh}(\mathbb{O}) \rightarrow \mathcal{Psh}(\mathbb{O}_{0,n}) = \mathcal{Psh}(\mathbb{O}_{\leq n})$, and let $(-)_{<n} = (-)_{\leq n-1}$ if $n \geq 0$. Similarly, we note $(-)_{m,n} : \mathcal{Psh}(\mathbb{O}_{\leq \infty}) = \mathcal{Psh}(\mathbb{O}) \rightarrow \mathcal{Psh}(\mathbb{O}_{m,\infty}) = \mathcal{Psh}(\mathbb{O}_{\geq m})$ by $(-)_{\geq m}$, and let $(-)_{>m} = (-)_{\geq m+1}$.

Proposition 3.5.4. (1) *The functors $\iota_!^{\geq m}$, $\iota_*^{\geq m}$, $\iota_!^{\leq n}$, and $\iota_*^{\leq n}$ are fully faithful.*

(2) *A presheaf $X \in \mathcal{Psh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_!^{\geq m}$ if and only if $X_{>n} = \emptyset$.*

(3) *A presheaf $X \in \mathcal{Psh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_*^{\geq m}$ if and only if for all $\omega \in \mathbb{O}_{>n}$ we have $(\mathbf{b}_\omega)_{\geq m} \perp X$.*

(4) *A presheaf $X \in \mathcal{Psh}(\mathbb{O}_{\leq n})$ is in the essential image of $\iota_!^{\leq n}$ if and only if for all $\omega \in \mathbb{O}_{<m}$ we have $(\mathbf{o}_\omega)_{\leq n} \perp X$, i.e. X_ω is a singleton.*

Proof. The first point follows from the fact that $\iota_*^{\geq m}$ and $\iota_*^{\leq n}$ are fully faithful, and [SGA72, exposé I, proposition 5.6]. The rest is straightforward verifications. \square

Notation 3.5.5. To ease notations, we sometimes leave truncations implicit, e.g. point (3) of last proposition can be reworded as: a presheaf $X \in \mathcal{Psh}(\mathbb{O}_{\geq m})$ is in the essential image of $\iota_*^{\geq m}$ if and only if $\mathbb{B}_{>n} \perp X$.

4. THE OPETOPIC NERVE OF OPETOPIC ALGEBRAS

4.1. Opetopic algebras. Let $k \leq n \in \mathbb{N}$. Recall from notation 3.5.2 that $\mathbb{O}_{n-k,n} \hookrightarrow \mathbb{O}$ is the full subcategory of opetopes of dimension at least $n-k$ and at most n . A *k-coloured, n-dimensional opetopic algebra*, or (k,n) -opetopic algebra, will be an algebraic structure on a presheaf over $\mathbb{O}_{n-k,n}$. Specifically, we describe a monad on the category $\mathcal{Psh}(\mathbb{O}_{n-k,n})$, whose algebras are the (k,n) -opetopic algebras. Such an algebra X has “operations” (its cells of dimension n) that can be “composed” in ways encoded by $(n+1)$ -opetopes⁶. The operations of X will be “coloured” by its cells dimension $< n$, which determines which operations can be composed together.

As we will see, the fact that the operations and relations of an (k,n) -opetopic algebra are encoded by opetopes of dimension $> n$ results in the category $\mathcal{Alg}_{k,n}$ of (k,n) -opetopic algebras always having a canonical full and faithful *nerve functor* to the category $\mathcal{Psh}(\mathbb{O})$ of opetopic sets (theorem 4.5.11).

⁶Recall that an $(n+1)$ -opetope is precisely a pasting scheme of n -opetopes.

We now claim some examples. A classification of (k, n) -opetopic algebras is given by proposition 4.3.7.

Example 4.1.1 (Monoids and categories). Let $k = 0$ and $n = 1$. Then $\mathbb{O}_{n-k,n} = \mathbb{O}_1 = \{*\}$, and $\mathcal{Psh}(\mathbb{O}_{n-k,n}) = \text{Set}$. The category of $(0, 1)$ -opetopic algebras is precisely the category of associative monoids. If $k = 1$ instead, then $\mathbb{O}_{n-k,n} = \mathbb{O}_{0,1}$, and $\mathcal{Psh}(\mathbb{O}_{n-k,n}) = \text{Graph}$, the category of directed graphs. The category of $(1, 1)$ -opetopic algebras is precisely the category of small categories.

Example 4.1.2 (Coloured and uncoloured planar operads). Let $k = 0$ and $n = 2$. Then $\mathbb{O}_{n-k,n} = \mathbb{O}_2 \cong \mathbb{N}$, and $\mathcal{Psh}(\mathbb{O}_2) \simeq \text{Set}/\mathbb{N}$. The category of $(0, 2)$ -opetopic algebras is precisely the category of planar, uncoloured Set-operads. If $k = 1$ instead, then $\mathcal{Psh}(\mathbb{O}_{1,2})$ is the category of planar, coloured collections. The category of $(1, 2)$ -opetopic algebras is precisely the category of planar, coloured Set-operads.

Example 4.1.3 (Combinads). Let $k = 0$ and $n = 3$. Then $\mathbb{O}_{n-k,n} = \mathbb{O}_3$ is the set of planar finite trees, and $(0, 3)$ -opetopic algebras are exactly the *combinads* over the combinatorial pattern of non-symmetric trees, presented in [Lod12].

4.2. Parametric right adjoint monads. In preparation to the main results of this section, we survey elements of the theory of parametric right adjoint (p.r.a.) monads, which will be essential to the definition and description of (k, n) -opetopic algebras. A comprehensive treatment of this theory can be found in [Web07].

Definition 4.2.1 (Parametric right adjoint). Let $T : \mathcal{C} \rightarrow \mathcal{D}$ be a functor, and let \mathcal{C} have a terminal object 1. Then T factors as

$$\mathcal{C} = \mathcal{C}/1 \xrightarrow{T_1} \mathcal{D}/T1 \rightarrow \mathcal{D}.$$

We say that T is a *parametric right adjoint* (abbreviated p.r.a.) if T_1 has a left adjoint E .

We immediately restrict ourselves to the case $\mathcal{C} = \mathcal{D} = \mathcal{Psh}(\mathcal{A})$ for a small category \mathcal{A} . Then T_1 is the nerve of the restriction $E : \mathcal{A}/T1 \rightarrow \mathcal{Psh}(\mathcal{A})$, and the usual formula for nerve functors gives

$$TX_a = \sum_{x \in T1_a} \mathcal{Psh}(\mathcal{A})(Ex, X) \quad (4.2.2)$$

for $X \in \mathcal{Psh}(\mathcal{A})$ and $a \in \mathcal{A}$. In fact, it is clear that the data of the object $T1 \in \mathcal{Psh}(\mathcal{A})$ and of the functor E completely describe (via equation (4.2.2)) the functor T up to unique isomorphism.

Definition 4.2.3 (P.r.a monad). A *p.r.a. monad* is a monad whose endofunctor is a p.r.a. and whose unit and multiplication are cartesian natural transformations⁷.

Assume now that $T : \mathcal{Psh}(\mathcal{A}) \rightarrow \mathcal{Psh}(\mathcal{A})$ is a p.r.a. monad. Define Θ_0 to be the full subcategory of $\mathcal{Psh}(\mathcal{A})$ spanned by the image of $E : \mathcal{A}/T1 \rightarrow \mathcal{Psh}(\mathcal{A})$. Objects of Θ_0 are called *T-cardinals*. By [Web07, proposition 4.20], the Yoneda embedding $\mathcal{A} \hookrightarrow \mathcal{Psh}(\mathcal{A})$ factors as

$$\mathcal{A} \xrightarrow{i} \Theta_0 \xrightarrow{i_0} \mathcal{Psh}(\mathcal{A})$$

or in other words, representable presheaves are *T-cardinals*.

Every p.r.a. monad on a presheaf category is an example of a *monad with arities* [BMW12]. The theory of monads with arities provides a remarkable amount of information about the free-forgetful adjunction $\mathcal{Psh}(\mathcal{A}) \xleftrightarrow{\quad} \text{Alg}(T)$ and about the category of algebras $\text{Alg}(T)$. We summarise those results that we will use below.

Proposition 4.2.4. *The fully faithful functor $i_0 : \Theta_0 \hookrightarrow \mathcal{Psh}(\mathcal{A})$ is dense, or equivalently, its associated nerve functor $N_0 : \mathcal{Psh}(\mathcal{A}) \hookrightarrow \mathcal{Psh}(\Theta_0)$ is fully faithful.*

⁷P.r.a. monads on presheaf categories are a strict generalisation of polynomial monads, since an endofunctor $\text{Set}/I \rightarrow \text{Set}/I$ is a polynomial functor iff it is a p.r.a., see [Web07, example 2.4].

Proof. We denote the inclusion $\mathcal{A} \hookrightarrow \Theta_0$ by i , and note that N_0 is isomorphic to $i_* : \mathcal{Psh}(\mathcal{A}) \rightarrow \mathcal{Psh}(\Theta_0)$. Now, i is fully faithful, and by [SGA72, exposé I, proposition 5.6], this is equivalent to i_* being fully faithful. \square

Corollary 4.2.5. *Let $J_{\mathcal{A}} := \{\varepsilon_{\theta} : i_! i^* \theta \rightarrow \theta \mid \theta \in \Theta_0 - \text{im } i\}$, where ε_{θ} is the counit at θ . Then a presheaf $X \in \mathcal{Psh}(\Theta_0)$ is in the essential image of N_0 if and only if $J_{\mathcal{A}} \perp X$.*

Proof. By the formula $i_* Y = \mathcal{Psh}(\mathcal{A})(\Theta_0(i-, -), Y)$ we have the sequence of isomorphisms

$$\begin{aligned} (i_* Y)_{\theta} &= \mathcal{Psh}(\mathcal{A})(\Theta_0(i-, \theta), Y) \\ &\cong \mathcal{Psh}(\mathcal{A})(i^* \theta, i^* i_* Y) && \text{since } i_* \text{ is fully faithful} \\ &\cong \mathcal{Psh}(\Theta_0)(i_! i^* \theta, i_* Y) && \text{since } i_! \dashv i^*. \end{aligned}$$

It is easy to check that one direction of the previous isomorphism is pre-composition by ε_{θ} . Conversely, take $X \in \mathcal{Psh}(\Theta_0)$ such that $\varepsilon_{\theta} \perp X$ for all $\theta \in \Theta_0$. Then

$$\begin{aligned} i^* i_* X_{\theta} &\cong \mathcal{Psh}(\Theta_0)(\theta, i^* i_* X) \\ &\cong \mathcal{Psh}(\Theta_0)(i_! i^* \theta, X) \\ &\cong \mathcal{Psh}(\Theta_0)(\theta, X) && \text{since } \varepsilon_{\theta} \perp X \\ &\cong X_{\theta}, \end{aligned}$$

and thus $X \in \text{im } i^*$. Thus a presheaf X is isomorphic to one of the form $i_* Y$ if and only if we have $\varepsilon_{\theta} \perp X$ for all $\theta \in \Theta_0$. But if $\theta \in \text{im } i$, then the associated counit map is already an isomorphism, hence we can restrict to $J_{\mathcal{A}}$. \square

Notation 4.2.6. Let the “identity-on-objects / fully faithful” factorisation of the composite functor $F_T i_0 : \Theta_0 \hookrightarrow \mathcal{Psh}(\mathcal{A}) \rightarrow \text{Alg}(T)$ be denoted

$$\Theta_0 \xrightarrow{t} \Theta_T \xrightarrow{i_T} \text{Alg}(T) \tag{4.2.7}$$

Theorem 4.2.8. (1) *The fully faithful functor $i_T : \Theta_T \hookrightarrow \text{Alg}(T)$ is dense, or equivalently, its associated nerve functor $N_T : \text{Alg}(T) \hookrightarrow \mathcal{Psh}(\Theta_T)$ is fully faithful.*

(2) *The following diagram is an exact adjoint square⁸.*

$$\begin{array}{ccc} \mathcal{Psh}(\mathcal{A}) & \begin{array}{c} \xrightarrow{F_T} \\ \perp \\ \xleftarrow{U_T} \end{array} & \text{Alg}(T) \\ N_0 \downarrow & & \downarrow N_T \\ \mathcal{Psh}(\Theta_0) & \begin{array}{c} \xrightarrow{t_!} \\ \perp \\ \xleftarrow{t^*} \end{array} & \mathcal{Psh}(\Theta_T) \end{array}$$

In particular, both squares commute up to natural isomorphism.

(3) *(Nerve theorem) Any $X \in \mathcal{Psh}(\Theta_T)$ is in the essential image of N_T if and only if $t^* X$ is in the essential image of N_0 .*

Proof. See [Web07, theorem 4.10], and [BMW12, proposition 1.9]. \square

Corollary 4.2.9. *Let $J_T := t_! J_{\mathcal{A}} = \{t_! \varepsilon_{\theta} : t_! i_! i^* \theta \rightarrow t_! \theta \mid \theta \in \Theta_0 - \text{im } i\}$, where $\varepsilon_{\theta} : i_! i^* \theta \rightarrow \theta$ is the counit at θ . Then $X \in \mathcal{Psh}(\Theta_T)$ is in the essential image of N_T if and only if $J_T \perp X$.*

Proof. This follows from theorem 4.2.8 point (3), and from corollary 4.2.5. \square

⁸There exists a natural isomorphism $N_0 U_T \cong t^* N_T$ whose mate $t_! N_0 \rightarrow N_T F_T$ is invertible (satisfies the Beck-Chevalley condition).

4.3. Coloured \mathfrak{Z}^n -algebras. In this section, we extend the polynomial monad \mathfrak{Z}^n over $\text{Set}/\mathbb{O}_n = \mathcal{P}\text{sh}(\mathbb{O}_n)$ to a p.r.a. monad \mathfrak{Z}^n over $\mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$, for $k \leq n \in \mathbb{N}$.

This new setup will encompass more known examples (see proposition 4.3.7). For instance, recall that the polynomial monad \mathfrak{Z}^2 on $\text{Set}/\mathbb{O}_2 \cong \text{Set}/\mathbb{N}$ is exactly the monad of planar operads. The extension of \mathfrak{Z}^2 will retrieve *coloured* planar operads as algebras. Similarly, the polynomial monad \mathfrak{Z}^1 on Set is the free-monoid monad, which we would like to vary to obtain “coloured monoids”, i.e. small categories.

Let $k \leq n \in \mathbb{N}$. Let us define a p.r.a. endofunctor \mathfrak{Z}^n on the category $\mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$, that will restrict to the polynomial monad $\mathfrak{Z}^n : \mathcal{P}\text{sh}(\mathbb{O}_n) \rightarrow \mathcal{P}\text{sh}(\mathbb{O}_n)$ in the case $k = 0$. Following section 4.2, to define the p.r.a. endofunctor \mathfrak{Z}^n as the composite

$$\mathcal{P}\text{sh}(\mathbb{O}_{n-k,n}) \xrightarrow{\mathfrak{Z}_1^n} \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})/\mathfrak{Z}^n 1 \simeq \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1) \rightarrow \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n}),$$

up to unique isomorphism, it suffices to define its value $\mathfrak{Z}^n 1$ on the terminal presheaf, and to define a functor $E : \mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1 \rightarrow \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$.

Definition 4.3.1. Define $\mathfrak{Z}^n 1$ as:

$$(\mathfrak{Z}^n 1)_\psi := \{*\}, \quad (\mathfrak{Z}^n 1)_\omega := \{\nu \in \mathbb{O}_{n+1} \mid \mathfrak{t}\nu = \omega\}.$$

where $\psi \in \mathbb{O}_{n-k,n-1}$ and $\omega \in \mathbb{O}_n$. We define the functor $E : \mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1 \rightarrow \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$ as follows. On objects, for $* \in (\mathfrak{Z}^n 1)_\psi$ and $\nu \in (\mathfrak{Z}^n 1)_\omega$, let

$$E(*) := O[\psi], \quad E(\nu) := S[\nu],$$

and on morphisms as the canonical inclusions. The functor $\mathfrak{Z}_1^n : \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n}) \rightarrow \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n}/\mathfrak{Z}^n 1)$ is defined as the right adjoint to the left Kan extension of E along the Yoneda embedding. We now recover the endofunctor \mathfrak{Z}^n explicitly using equation (4.2.2): for $\psi \in \mathbb{O}_{n-k,n-1}$ we have $(\mathfrak{Z}^n X)_\psi \cong X_\psi$, and for $\omega \in \mathbb{O}_n$ we have

$$(\mathfrak{Z}^n X)_\omega \cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t}\nu = \omega}} \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})(S[\nu], X).$$

Note that $(\mathfrak{Z}^n X)_\omega$ matches with the “uncolored” version of \mathfrak{Z}_n of equation (3.1.2).

Recall from section 4.2 that a p.r.a. monad is a monad M whose unit $\text{id} \rightarrow M$ and multiplication $MM \rightarrow M$ are cartesian, and such that M is a p.r.a. endofunctor. We now endow the p.r.a. endofunctor \mathfrak{Z}^n with the structure of a p.r.a. monad over $\mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$. We first specify the unit and multiplication $\eta_1 : 1 \rightarrow \mathfrak{Z}^n 1$ and $\mu_1 : \mathfrak{Z}^n \mathfrak{Z}^n 1 \rightarrow \mathfrak{Z}^n 1$ on the terminal object 1, and extend them to cartesian natural transformations (lemma 4.3.3). Next, we check that the required monad identities hold for 1 (lemma 4.3.4), which automatically gives us the desired monad structure on \mathfrak{Z}^n .

Lemma 4.3.2. *The polynomial functor $\mathfrak{Z}^n \mathfrak{Z}^n : \text{Set}/\mathbb{O}_n \rightarrow \text{Set}/\mathbb{O}_n$ is given by*

$$\mathbb{O}_n \xleftarrow{\mathfrak{e}} E \longrightarrow \mathbb{O}_{n+2}^{(2)} \xrightarrow{\mathfrak{t}\mathfrak{t}} \mathbb{O}_n,$$

where

(1) $\mathbb{O}_{n+2}^{(2)}$ is the set of $(n+2)$ opetopes of height 2, i.e. of the form

$$Y_\nu \bigcirc_{[[p_i]]} Y_{\nu_i},$$

with $\nu, \nu_i \in \mathbb{O}_{n+1}$ and $[p_i]$ ranging over a (possibly empty) subset of ν^\bullet ,

(2) for $\xi \in \mathbb{O}_{n+2}^{(2)}$, $E(\xi) = \xi^|$,

(3) for $\xi \in \mathbb{O}_{n+2}^{(2)}$ and $[l] \in E(\xi) = \xi^|$, $\mathfrak{e}[l] = \mathfrak{e}_{[l]}\mathfrak{t}$.

We now define η_1 and μ_1 .

- (1) The presheaf $\mathfrak{Z}^n 1 \in \mathcal{Psh}(\mathbb{O}_{n-k,n})$ is pointed in the following way: the morphism $\eta_1 : 1 \rightarrow \mathfrak{Z}^n 1$ is the identity in dimension $< n$, and maps the unique element of 1_ω to Y_ω for $\omega \in \mathbb{O}_n$.
- (2) First, note that for every non-degenerate $\nu \in \mathbb{O}_{n+1}$, a morphism $x : S[\nu] \rightarrow \mathfrak{Z}^n 1$ in $\mathcal{Psh}(\mathbb{O}_{n-k,n})$ and an element of $\mathfrak{Z}^n \mathfrak{Z}^n 1$, this corresponds to an opetope $\bar{\nu} \in \mathbb{O}_{n+2}$ height 2 (see lemma 4.3.2) such that $s_{\square} \bar{\nu} = \nu$.

Next, the map of presheaves $\mu_1 : \mathfrak{Z}^n \mathfrak{Z}^n 1 \rightarrow \mathfrak{Z}^n 1$ is defined as the identity function on opetopes of dimension $< n$, and as the function

$$\mu_{1,\omega} : \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t} \nu = \omega}} \mathcal{Psh}(\mathbb{O}_{n-k,n})(S[\nu], \mathfrak{Z}^n 1) \rightarrow \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t} \nu = \omega}} \mathcal{Psh}(\mathbb{O}_{n-k,n})(S[\nu], 1)$$

$$\begin{aligned} (\nu, x) &\mapsto (\mathfrak{t} \bar{\nu}, *) && \text{if } \nu \text{ non-degen.} \\ (\nu, *) &\mapsto (\nu, *) && \text{if } \nu \text{ degen.} \end{aligned}$$

on opetopes $\omega \in \mathbb{O}_n$. This is well-defined thanks to **(Glob1)** (definition 3.3.2).

Lemma 4.3.3. *For $X \in \mathcal{Psh}(\mathbb{O}_{n-k,n})$, the unique morphism $! : X \rightarrow 1$ induces pullback squares*

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathfrak{Z}^n X \\ ! \downarrow & \lrcorner & \downarrow \mathfrak{Z}^n ! \\ 1 & \xrightarrow{\eta_1} & \mathfrak{Z}^n 1, \end{array} \quad \begin{array}{ccc} \mathfrak{Z}^n \mathfrak{Z}^n X & \xrightarrow{\mu_X} & \mathfrak{Z}^n X \\ \mathfrak{Z}^n \mathfrak{Z}^n ! \downarrow & \lrcorner & \downarrow \mathfrak{Z}^n ! \\ \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xrightarrow{\mu_1} & \mathfrak{Z}^n 1 \end{array}$$

Proof. Straightforward verifications, see appendix A for details. \square

We have suggestively named the topmost arrows η_X and μ_X , since this choice of pullback square for each X gives cartesian natural transformations $\eta : \text{id} \rightarrow \mathfrak{Z}^n$ and $\mu : \mathfrak{Z}^n \mathfrak{Z}^n \rightarrow \mathfrak{Z}^n$.

Lemma 4.3.4. *The following diagrams commute:*

$$\begin{array}{ccc} \mathfrak{Z}^n 1 & \xrightarrow{\eta_{\mathfrak{Z}^n 1}} & \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xleftarrow{\mathfrak{Z}^n \eta_1} & \mathfrak{Z}^n 1 \\ & \searrow & \downarrow \mu_1 & & \swarrow \\ & & \mathfrak{Z}^n 1, & & \end{array} \quad \begin{array}{ccc} \mathfrak{Z}^n \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xrightarrow{\mathfrak{Z}^n \mu_1} & \mathfrak{Z}^n \mathfrak{Z}^n 1 \\ \mu_{\mathfrak{Z}^n 1} \downarrow & & \downarrow \mu_1 \\ \mathfrak{Z}^n \mathfrak{Z}^n 1 & \xrightarrow{\mu_1} & \mathfrak{Z}^n 1. \end{array}$$

Proof. Straightforward computations. See appendix A for details. \square

Proposition 4.3.5. *The cartesian natural transformations μ and η give \mathfrak{Z}^n a structure of p.r.a. monad on $\mathcal{Psh}(\mathbb{O}_{n-k,n})$.*

Proof. This is a direct consequence of lemmas 4.3.3 and 4.3.4. \square

Clearly, when $k = 0$, we recover the usual polynomial monad on Set/\mathbb{O}_n .

Definition 4.3.6 (Opetopic algebra). Let $\text{Alg}_{k,n}$, the category of k -coloured n -dimensional opetopic algebras, be the Eilenberg–Moore category of \mathfrak{Z}^n considered as a monad on $\mathcal{Psh}(\mathbb{O}_{n-k,n})$.

Proposition 4.3.7. *Up to equivalence, and for small values of k and n , the category $\text{Alg}_{k,n}$ is given by the following table:*

| $k \backslash n$ | 0 | 1 | 2 | 3 |
|------------------|-----|-----|--------------------|-----------------------|
| 0 | Set | Mon | Op | Comb $_{\mathbb{PT}}$ |
| 1 | | Cat | Op $_{\text{col}}$ | Alg $_{1,3}$ |
| 2 | | | Alg $_{2,2}$ | Alg $_{2,3}$ |
| 3 | | | | Alg $_{3,3}$ |

where \mathbf{Set} is the category of sets, \mathbf{Mon} of monoids, \mathbf{Cat} of small categories, \mathbf{Op} of non coloured planar operads, \mathbf{Op}_{col} of coloured planar operads, and $\mathbf{Comb}_{\mathbb{P}\mathbb{T}}$ of combinads over the combinatorial pattern of planar trees [Lod12]. The lower half of the table is left empty since $\mathbf{Alg}_{k,n} = \mathbf{Alg}_{n,n}$ for $k \geq n$.

4.4. \mathfrak{Z}^n -cardinals and opetopic shapes.

Definition 4.4.1 (Opetopic shape). Following section 4.2, the category of \mathfrak{Z}^n -cardinals (definition 4.2.3) is the full subcategory $i_0 : \Theta_0 \hookrightarrow \mathcal{Psh}(\mathbb{O}_{n-k,n})$ whose objects are the representables $\omega \in \mathbb{O}_{n-k,n}$ and the spines $S[\nu]$, for $\nu \in \mathbb{O}_{n+1}$. Analogous to equation (4.2.7), we denote the (identity-on-objects, fully faithful) factorisation of $\mathfrak{Z}^n i_0 : \Theta_0 \hookrightarrow \mathcal{Psh}(\mathbb{O}_{n-k,n}) \rightarrow \mathbf{Alg}_{k,n}$ by

$$\Theta_0 \xrightarrow{z} \mathbb{A}_{k,n} \xrightarrow{u} \mathbf{Alg}_{k,n}.$$

We call the category $\mathbb{A}_{k,n}$ of free algebras on the \mathfrak{Z}^n -cardinals the category of (k, n) -opetopic shapes.

For the rest of this section, we fix parameters $k \leq n \in \mathbb{N}$ once and for all, and suppress them in notation whenever unambiguous, e.g. $\mathbb{A} := \mathbb{A}_{k,n}$, $\mathfrak{Z} := \mathfrak{Z}^n$, $\mathbf{Alg} := \mathbf{Alg}_{k,n}$.

Definition 4.4.2 (Spine). For $\omega \in \mathbb{O}_{n+1}$, let $\mathcal{S}_\omega := \mathbb{O}_{n-k,n}/S[\omega]$. We denote the colimit, in $\mathcal{Psh}(\mathbb{A})$, by

$$S[h\omega] := \text{colim} \left(\mathcal{S}_\omega \longrightarrow \mathbb{O}_{n-k,n} \xrightarrow{h} \mathbb{A} \longrightarrow \mathcal{Psh}(\mathbb{A}) \right),$$

and call it the *spine on $h\omega$* . Let $s_{h\omega} : S[h\omega] \hookrightarrow h\omega$ be the *spine inclusion of $h\omega$* , and with a slight abuse of notations,

$$\mathbf{S} := \left\{ s_{h\omega} : S[h\omega] \hookrightarrow h\omega \mid \omega \in \mathbb{O}_{n+1} \right\}.$$

The theory reviewed in section 4.2 has the following immediate consequences.

- Proposition 4.4.3.**
- (1) The inclusion $u : \mathbb{A} \hookrightarrow \mathbf{Alg}$ is dense, i.e., its associated nerve functor $N_u : \mathbf{Alg} \rightarrow \mathcal{Psh}(\mathbb{A})$ is fully faithful.
 - (2) Any $X \in \mathcal{Psh}(\mathbb{A})$ is in the essential image of N_u if and only if $\mathbf{S} \perp X$.
 - (3) The reflective adjunction $u : \mathcal{Psh}(\mathbb{A}) \xleftrightarrow{\quad} \mathbf{Alg} : N_u$ exhibits $\mathbf{Alg} \simeq \mathbf{S}^{-1} \mathcal{Psh}(\mathbb{A})$ as the localisation of $\mathcal{Psh}(\mathbb{A})$ at the set of morphisms \mathbf{S} .

Example 4.4.4. The category $\mathbb{A}_{1,1}$ is the category of simplices Δ , and $\mathbb{A}_{2,1}$ is the planar version of Moerdijk and Weiss’s category of dendrices Ω . Then proposition 4.4.3 is the well-known fact that \mathbf{Cat} and \mathbf{Op}_{col} have fully faithful nerve functors to $\mathcal{Psh}(\Delta)$ and to $\mathcal{Psh}(\Omega)$ respectively, exhibiting them as localisations of the respective presheaf categories at a set of *spine inclusions*.⁹

Our motivation stems from the following remarkable fact: there is a functor $\dot{h} : \mathbb{O}_{n-k,n+2} \rightarrow \mathbb{A}$ (that we will define below), that is neither full nor faithful (it is, however, *surjective* on objects and on morphisms), but is such that the composite functor $\mathbf{Alg} \hookrightarrow \mathcal{Psh}(\mathbb{A}) \xrightarrow{\dot{h}^*} \mathcal{Psh}(\mathbb{O}_{n-k,n+2})$ is a fully faithful right adjoint, and moreover exhibits \mathbf{Alg} as the category of models of a projective sketch on $\mathbb{O}_{n-k,n+2}^{\text{op}}$. In addition, the composite fully faithful functor $\mathbb{A} \hookrightarrow \mathcal{Psh}(\mathbb{O}_{n-k,n+2})$ is simply the nerve associated to \dot{h} (this says that \dot{h} is dense), and allows us to view \mathbb{A} as a full subcategory of $(n-k, n+2)$ -truncated opetopic sets, justifying the use of the term “opetopic shape”.

⁹Sometimes called “Grothendieck-Segal” colimits.

Definition 4.4.5. The functor $\dot{h} : \mathbb{O}_{n-k, n+2} \rightarrow \mathbb{A}$ is defined on objects as follows:

$$\begin{aligned} \dot{h} : \mathbb{O}_{n-k, n+2} &\longrightarrow \mathbb{A} \\ \psi \in \mathbb{O}_{\leq n} &\longmapsto \mathfrak{Z}O[\psi] \\ \omega \in \mathbb{O}_{n+1} &\longmapsto \mathfrak{Z}S[\omega] \\ \xi \in \mathbb{O}_{n+2} &\longmapsto \mathfrak{Z}S[\mathfrak{t}\xi]. \end{aligned}$$

On morphisms, \dot{h} is the same as the free functor \mathfrak{Z} on $\mathbb{O}_{n-k, n}$. Take $\omega \in \mathbb{O}_{n+1}$ and $\xi \in \mathbb{O}_{n+2}$.

- (1) Let $[p] \in \omega^\bullet$, and $\dot{h}\left(\mathfrak{s}_{[p]}\omega \xrightarrow{\mathfrak{s}_{[p]}} \omega\right) = \mathfrak{Z}\left(O[\mathfrak{s}_{[p]}\omega] \xrightarrow{\mathfrak{s}_{[p]}} S[\omega]\right)$.
- (2) Let $\dot{h}\left(\mathfrak{t}\omega \xrightarrow{\mathfrak{t}} \omega\right) = \left(\mathfrak{Z}O[\mathfrak{t}\omega] \xrightarrow{\dot{h}\mathfrak{t}} \mathfrak{Z}S[\omega]\right)$ correspond to the cell $\text{id}_{S[\omega]} \in \mathfrak{Z}S[\omega]_{\mathfrak{t}\omega}$ under the Yoneda embedding.
- (3) Let $\dot{h}\left(\mathfrak{t}\xi \xrightarrow{\mathfrak{t}} \xi\right) = \left(\mathfrak{Z}S[\mathfrak{t}\xi] \xrightarrow{\dot{h}\mathfrak{t}} \mathfrak{Z}S[\mathfrak{t}\xi]\right)$ be the identity map.
- (4) Let $[p] \in \xi^\bullet$. In order to define $\dot{h}\left(\mathfrak{s}_{[p]}\xi \xrightarrow{\mathfrak{s}_{[p]}} \xi\right) = \left(\mathfrak{Z}S[\mathfrak{s}_{[p]}\xi] \xrightarrow{\dot{h}\mathfrak{s}_{[p]}} \mathfrak{Z}S[\mathfrak{t}\xi]\right)$, it is enough to provide a morphism $\dot{h}\mathfrak{s}_{[p]} : S[\mathfrak{s}_{[p]}\xi] \rightarrow \mathfrak{Z}S[\mathfrak{t}\xi]$ in $\mathcal{P}\text{sh}(\mathbb{O}_{n-k, n})$, which we now construct.

Using equation (2.4.10), ξ decomposes as

$$\xi = \zeta \underset{[p]}{\circ} \mathbb{Y}_{\mathfrak{s}_{[p]}\xi} \underset{[[q_i]]}{\bigcirc} \zeta_i,$$

where $[q_i]$ ranges over $(\mathfrak{s}_{[p]}\xi)^\bullet$. The leaves of ζ_i are therefore a subset of the leaves of ξ . Precisely, a leaf address $[r] \in \zeta_i^\downarrow$ corresponds to the leaf address $[p[q_i]r] \in \xi^\downarrow$, defining an inclusion $f_i : S[\mathfrak{t}\zeta_i] \hookrightarrow S[\mathfrak{t}\xi]$ that maps the node $\wp_{\zeta_i}[r] \in (\mathfrak{t}\zeta_i)^\bullet$ to $\wp_\xi[p[q_i]r] \in (\mathfrak{t}\xi)^\bullet$.

Note that by definition, each f_i is an element of $\mathcal{P}\text{sh}(\mathbb{O}_{n-k, n})(S[\mathfrak{t}\zeta_i], S[\mathfrak{t}\xi]) \subseteq \mathfrak{Z}S[\mathfrak{t}\xi]_{\mathfrak{t}\mathfrak{t}\zeta_i}$, and since $\mathfrak{t}\mathfrak{t}\zeta_i = \mathfrak{t}\mathfrak{s}_{[p]}\zeta_i = \mathfrak{s}_{[q_i]}\mathfrak{s}_{[p]}\xi$ (by **(Glob1)** and **(Inner)**), we have $f_i \in \mathfrak{Z}S[\mathfrak{t}\xi]_{\mathfrak{s}_{[q_i]}\mathfrak{s}_{[p]}\xi}$.

Together, the f_i assemble into the required morphism $\dot{h}\mathfrak{s}_{[p]} : S[\mathfrak{s}_{[p]}\xi] \rightarrow \mathfrak{Z}S[\mathfrak{t}\xi]$, that maps the node $[q_i] \in (\mathfrak{s}_{[p]}\xi)^\bullet$ to f_i . So in conclusion, we have

$$\begin{aligned} \dot{h}\mathfrak{s}_{[p]} : S[\mathfrak{s}_{[p]}\xi] &\longrightarrow \mathfrak{Z}S[\mathfrak{t}\xi] \\ (\dot{h}\mathfrak{s}_{[p]})([q_i]) : S[\mathfrak{t}\zeta_i] &\longrightarrow S[\mathfrak{t}\xi] \\ \wp_{\zeta_i}[r] &\longmapsto \wp_\xi[p[q_i]r], \end{aligned}$$

for $[q_i] \in (\mathfrak{s}_{[p]}\xi)^\bullet$ and $[r] \in \zeta_i^\downarrow$.

This defines \dot{h} on object and morphisms, and functoriality is straightforward.

Example 4.4.6. Consider the case $(k, n) = (1, 1)$, so that $\dot{h} : \mathbb{O}_{0, 3} \rightarrow \mathbb{A}_{1, 1} \cong \mathbb{A}$. In low dimensions, we have $\dot{h}_{1, 1}\blacklozenge = [0]$, $\dot{h}_{1, 1}\blacksquare = [1]$, and $\dot{h}_{1, 1}\mathbf{n} = [n]$, for $n \in \mathbb{N}$. Consider now the following 3-opetope ξ :

$$\xi = \mathbb{Y}_3 \underset{[[*]]}{\circ} \mathbb{Y}_2 \underset{[[**]]}{\circ} \mathbb{Y}_1 = \left(\begin{array}{c} \begin{array}{c} \begin{array}{c} \cdot \\ \downarrow \\ \cdot \\ \downarrow \\ \cdot \\ \downarrow \\ \cdot \end{array} \\ \Downarrow \\ \begin{array}{c} \cdot \\ \downarrow \\ \cdot \\ \downarrow \\ \cdot \end{array} \\ \Downarrow \\ \begin{array}{c} \cdot \\ \downarrow \\ \cdot \\ \downarrow \\ \cdot \end{array} \\ \Downarrow \\ \cdot \end{array} \end{array} \Rightarrow \begin{array}{c} \cdot \\ \downarrow \\ \cdot \\ \downarrow \\ \cdot \end{array} \end{array} \right)$$

Then $\dot{h}_{1, 1}\xi = \mathfrak{Z}^1 S[\mathfrak{t}\xi] = [4]$. This result should be understood as the poset of points of ξ (represented as dots in the pasting diagram above) ordered by the topmost arrows of ξ .

Take the face embedding $s_{[\]} : \mathbf{3} \rightarrow \xi$. Then $\dot{h}_{1,1} s_{[\]}$ maps points 0, 1, 2, 3 of $\dot{h}_{1,1} \mathbf{3} = [3]$ to points 0, 1, 3, 4 of $\dot{h}_{1,1} \xi$, respectively. In other words, it “skips” point 2, which is exactly what the pasting diagram above depicts: the $[\]$ -source of ξ does not touch point 2. Likewise, the map $\dot{h}_{1,1} s_{[[**]]} : [1] = \dot{h}_{1,1} \mathbf{1} \rightarrow \dot{h}_{1,1} \xi$ maps 0, 1 to 3, 4, respectively.

Consider now the target embedding $t : \mathbf{4} \rightarrow \xi$. Since the target face touches all the points of ξ (this can be checked graphically, but more generally follows from **(Glob2)**), $\dot{h}_{1,1} t$ should be the identity map on $[4]$, which is precisely what the definition gives.

We spend the rest of this section proving various (rather technical) facts about the functor \dot{h} , which will allow us to construct the opetopic nerve functor (and to prove that it is fully faithful) in section 4.5.

Definition 4.4.7. Let $\omega, \omega' \in \mathbb{O}_{n+1}$. A morphism $f : \dot{h}\omega \rightarrow \dot{h}\omega'$ in \mathbb{A} is *diagrammatic* if there exists a $\xi \in \mathbb{O}_{n+2}$ and a $[p] \in \xi^\bullet$ such that $s_{[p]}\xi = \omega$, $t\xi = \omega'$, and $f = \dot{h}\left(\omega \xrightarrow{s_{[p]}} \xi\right)$. This situation is summarised by the following diagram, called a *diagram of f* .

$$\frac{\begin{array}{ccc} & & \xi \\ & \nearrow s_{[p]} & \uparrow t \\ \omega & & \omega' \end{array}}{\dot{h}\omega \xrightarrow{f} \dot{h}\omega'}$$

Example 4.4.8. Consider the case $(k, n) = (1, 1)$ again, and recall that $\mathbb{A}_{1,1} \cong \mathbb{A}$. Consider the map $f : [2] \rightarrow [3]$ mapping 0, 1, and 2 to 0, 2, and 3, respectively (in other words, f is the *1st simplicial coface map* [Jar06]). Taking ξ as on the left, we obtain a diagram of f on the right:

$$\xi = Y_2 \circ_{[[**]]} Y_2 = \left(\begin{array}{c} \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & & \cdot \end{array} \Rightarrow \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ & \downarrow & \\ \cdot & & \cdot \end{array} \end{array} \right), \quad \frac{\begin{array}{ccc} & & \xi \\ & \nearrow s_{[\]} & \uparrow t \\ \mathbf{2} & & \mathbf{2} \end{array}}{[2] \xrightarrow{g} [3]}$$

Consider now a non injective map $g : [2] \rightarrow [1]$ mapping 0, 1, and 2 to 0, 1, and 1, respectively (in other words, g is the *1st simplicial codegeneracy map* [Jar06]). Taking ζ as on the left, we obtain a diagram of g on the right:

$$\zeta = Y_2 \circ_{[[**]]} Y_0 = \left(\begin{array}{c} \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ \downarrow & \nearrow & \downarrow \\ \cdot & & \cdot \end{array} \Rightarrow \begin{array}{ccc} \cdot & \xrightarrow{\quad} & \cdot \\ & \downarrow & \\ \cdot & & \cdot \end{array} \end{array} \right), \quad \frac{\begin{array}{ccc} & & \zeta \\ & \nearrow s_{[[**]]} & \uparrow t \\ \mathbf{2} & & \mathbf{1} \end{array}}{[2] \xrightarrow{g} [1]}$$

On the one hand, lemma 4.4.9 below states that the composite of diagrammatic morphisms remains diagrammatic, and on the other hand, those two examples seem to indicate that all simplicial cofaces and codegeneracies are diagrammatic. One might thus expect all morphisms of \mathbb{A} to be in the image of $\dot{h}_{1,1} : \mathbb{O}_{0,3} \rightarrow \mathbb{A}_{1,1} \cong \mathbb{A}$. This is true, and a more general statement is proved in corollary 4.5.12.

Lemma 4.4.9. *If f_1 and f_2 are diagrammatic as on the left, the diagram on the right is well defined, and is a diagram of $f_2 f_1$.*

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & \xi_1 & & \xi_2 \\
 & \nearrow^{s_{[p_1]}} & \uparrow \mathfrak{t} & \nearrow^{s_{[p_2]}} & \uparrow \mathfrak{t} \\
 \omega_0 & & \omega_1 & & \omega_2 \\
 \hline
 \dot{h}\omega_0 & \xrightarrow{f_1} & \dot{h}\omega_1 & \xrightarrow{f_2} & \dot{h}\omega_2,
 \end{array} & &
 \begin{array}{ccc}
 & \xi_2 \square_{[p_2]} \xi_1 & \\
 & \nearrow^{s_{[p_2 p_1]}} & \uparrow \mathfrak{t} \\
 \omega_0 & & \omega_2 \\
 \hline
 \dot{h}\omega_0 & \xrightarrow{f_2 f_1} & \dot{h}\omega_2
 \end{array}
 \end{array}$$

Proof. It is a simple but lengthy matter of unfolding the definition of \dot{h} . See appendix A for details. \square

Lemma 4.4.10 (Contraction associativity formula). *Let $n \geq 2$, $\nu, \nu' \in \mathbb{O}_n$, and $[l] \in \nu^\bullet$ be such that $e_{[l]}\nu = e_{[l]}\nu'$. In particular, the grafting $\nu \circ_{[l]}\nu'$ is well-defined, and by **(Glob1)** and **(Glob2)**, $s_{\varrho_\nu[l]}\mathfrak{t}\nu = e_{[l]}\nu = e_{[l]}\nu' = \mathfrak{t}\nu'$. We have*

$$\mathfrak{t}(\nu \circ_{[l]}\nu') = (\mathfrak{t}\nu) \square_{\varrho_\nu[l]} (\mathfrak{t}\nu').$$

Proof. This is a direct consequence of the the fact that \mathfrak{Z}^{n-2} is a polynomial monad, i.e. a $(-)^*$ -algebra. \square

Lemma 4.4.11. (1) *Let $\omega \in \mathbb{O}_{n+1}$, and $\psi = \mathfrak{t}\omega$. Then the following is a diagram of $h\mathfrak{t} : \dot{h}\psi \rightarrow \dot{h}\omega$:*

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & & \xi \\
 & \nearrow^{s_{[l]}} & \uparrow \mathfrak{t} \\
 Y_\psi & & \omega \\
 \hline
 \dot{h}\psi & \xrightarrow{h\mathfrak{t}} & \dot{h}\omega,
 \end{array} & &
 \xi := Y_{Y_\psi} \circ_{[[l]]} Y_\omega.
 \end{array}$$

(Note that $\omega = \mathfrak{t}\xi$ by lemma 4.4.10)

(2) *Let $\beta, \omega \in \mathbb{O}_{n+1} = \text{tr } \mathfrak{Z}^{n-1}$, and $i : S[\beta] \rightarrow S[\omega]$ a morphism of presheaves. Then i corresponds to an inclusion of \mathfrak{Z}^{n-1} trees $\beta \hookrightarrow \omega$, mapping node at address $[q]$ to $[pq]$, where $[p] = i[] \in \omega^\bullet$.*

Write $\omega = \bar{\beta} \square_{[p]}\beta$, for an adequate $\bar{\beta} \in \mathbb{O}_{n+1}$. Then following is a diagram of hi :

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & & \xi \\
 & \nearrow^{s_{[p]}} & \uparrow \mathfrak{t} \\
 \beta & & \omega \\
 \hline
 \dot{h}\beta & \xrightarrow{hi} & \dot{h}\omega,
 \end{array} & &
 \xi := Y_{\bar{\beta}} \circ_{[[p]]} Y_\beta.
 \end{array}$$

(Note that $\omega = \mathfrak{t}\xi$ by lemma 4.4.10)

Proof. It is a simple matter of unfolding the definition of \dot{h} . See appendix A for details. \square

Lemma 4.4.12 (Diagrammatic lemma). *Let $\omega, \omega' \in \mathbb{O}_{n+1}$ with ω non degenerate, and $f : \dot{h}\omega \rightarrow \dot{h}\omega'$. Then f is diagrammatic.*

Proof (sketch, see appendix A for details). The idea is to proceed by induction on ω . The case $\omega = Y_\psi$ for some $\psi \in \mathbb{O}_n$ is fairly simple. In the inductive case ($\omega = \nu \circ_{[l]} Y_\psi$, for suitable ν, ψ , and $[l]$), we essentially show that f exhibits an inclusion of \mathfrak{Z}^n -trees $\omega \hookrightarrow \omega'$ by constructing a $(n+1)$ -opetope $\bar{\omega}$

such that $\omega' = \bar{\omega} \square_{[q]} \omega$. Thus by lemma 4.4.11, the following is a diagram of $\dot{h}f$:

$$\begin{array}{ccc} & & \xi \\ & \nearrow^{s[[q_1]]} & \uparrow t \\ \omega & & \omega' \\ \hline \dot{h}\omega & \xrightarrow{f} & \dot{h}\omega', \end{array} \quad \xi := Y_{\bar{\omega}} \circ_{[[q_1]]} Y_{\omega}.$$

□

4.5. The opetopic nerve functor. This section is entirely devoted to constructing the *opetopic nerve functor* $N : \mathcal{A}lg \longrightarrow \mathcal{P}sh(\mathbb{O})$, which is a fully faithful right adjoint and which exhibits $\mathcal{A}lg$ as the category of models of a projective sketch on \mathbb{O}^{op} (theorem 4.5.11).

Recall from corollary 4.2.9 that the reflective adjunction $\mathcal{P}sh(\mathbb{A}) \xleftrightarrow{\quad} \mathcal{A}lg$ exhibits $\mathcal{A}lg$ as the localisation of $\mathcal{P}sh(\mathbb{A})$ at the set \mathcal{S} of spine inclusions definition 4.4.2.

In corollary 4.5.9, we show an equivalence $\mathcal{S}_{n+1, n+2}^{-1} \mathcal{P}sh(\mathbb{O}_{n-k, n+2}) \simeq \mathcal{A}lg$, from which theorem 4.5.11 will follow directly.

Lemma 4.5.1. *The functor $\dot{h}_! : \mathcal{P}sh(\mathbb{O}_{n-k, n+2}) \longrightarrow \mathcal{P}sh(\mathbb{A})$ takes $\mathcal{S}_{n+1} \subseteq \mathcal{P}sh(\mathbb{O}_{n-k, n+2})^{\rightarrow}$ to $\mathcal{S} \subseteq \mathcal{P}sh(\mathbb{A})$ and takes morphisms in $\mathcal{S}_{n+2} \subseteq \mathcal{P}sh(\mathbb{O}_{n-k, n+2})^{\rightarrow}$ to \mathcal{S} -local isomorphisms.*

Proof. (1) We show that $\dot{h}_! \mathcal{S}_{n+1} = \mathcal{S}$. Take $\omega \in \mathbb{O}_{n+1}$. Then

$$\dot{h}_! S[\omega] = \dot{h}_! \operatorname{colim}_{\psi \in \mathcal{S}_{\omega}} O[\psi] \cong \operatorname{colim}_{\psi \in \mathcal{S}_{\omega}} \dot{h}_! O[\psi] = \operatorname{colim}_{\psi \in \mathcal{S}_{\omega}} A[\dot{h}\psi] = S[\omega].$$

(2) For $\omega \in \mathbb{O}_{n+2}$, the inclusion $S[t\omega] \longrightarrow S[\omega]$ is a relative \mathcal{S}_{n+1} -cell complex by lemma 3.4.9. Since $\dot{h}_!$ preserves colimits, and since $\dot{h}_! \mathcal{S}_{n+1} = \mathcal{S}$, we have that $\dot{h}_!(S[t\omega] \hookrightarrow S[\omega])$ is a relative \mathcal{S} -cell complex, and thus an \mathcal{S} -local isomorphism. In the square below

$$\begin{array}{ccc} \dot{h}_! S[t\omega] & \longrightarrow & \dot{h}_! S[\omega] \\ \dot{h}_! s_{t\omega} \downarrow & & \downarrow \dot{h}_! s_{\omega} \\ \dot{h}_! O[t\omega] & \xrightarrow{\dot{h}_! t} & \dot{h}_! O[\omega] \end{array}$$

we know that $\dot{h}_! t = O[\dot{h}t]$ is an isomorphism, and that $\dot{h}_! s_{t\omega} \in \mathcal{S}$ by the previous point. We have just shown that the top arrow is an \mathcal{S} -local isomorphism. By 3-for-2, we conclude that $\dot{h}_! s_{\omega}$ is too. □

We will prove two crucial lemmas that are key to the proof of theorem 4.5.11. Before doing so, let us make some preliminary remarks that will make the task easier.

Lemma 4.5.2. *Let $\omega \in \mathbb{O}_{n-k, n+2}$, and let $X \in \mathcal{S}_{n+1, n+2}^{\perp}$. Then the following are all spans of isomorphisms.*

- (1) $\mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_{\psi} \xleftarrow{\operatorname{id} \times X_{tt}} \mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_{I_{\psi}} \xrightarrow{\mathbb{A}(\dot{h}\omega, \dot{h}tt) \times \operatorname{id}} \mathbb{A}(\dot{h}\omega, \dot{h}I_{\psi}) \times X_{I_{\psi}}$, where $\psi \in \mathbb{O}_{n-1}$.
- (2) $\mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_{\psi} \xleftarrow{\operatorname{id} \times X_{s[\square]}} \mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_{Y_{\psi}} \xrightarrow{\mathbb{A}(\dot{h}\omega, \dot{h}s[\square]) \times \operatorname{id}} \mathbb{A}(\dot{h}\omega, \dot{h}Y_{\psi}) \times X_{Y_{\psi}}$, where $\psi \in \mathbb{O}_n$.
- (3) $\mathbb{A}(\dot{h}\omega, \dot{h}t\psi) \times X_{t\psi} \xleftarrow{\operatorname{id} \times X_t} \mathbb{A}(\dot{h}\omega, \dot{h}t\psi) \times X_{\psi} \xrightarrow{\mathbb{A}(\dot{h}\omega, \dot{h}t) \times \operatorname{id}} \mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_{\psi}$, where $\psi \in \mathbb{O}_{n+2}$.

Proof. (1) The first map is an isomorphism by corollary 3.4.11 point (1) and the second map is one by definition of \dot{h} .

(2) The first map is an isomorphism by corollary 3.4.11 point (2) and the second is one by definition of \dot{h} .

- (3) The first map is an isomorphism by corollary 3.4.10 point (1) and the second is one by definition of \dot{h} . \square

Lemma 4.5.3. *Let $\omega \in \mathbb{O}_{n-k,n+2}$. If $\psi \in \mathbb{O}_{n-k,n-2}$, then $\mathbb{A}(\dot{h}\omega, \dot{h}\psi) \cong \mathbb{O}_{n-k,n+2}(\omega, \psi)$.*

Proof. Easy verification. \square

The first of the two crucial propositions will provide us one half of an equivalence between the category $\mathcal{S}_{n+1,n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k,n+2})$ and Alg .

Proposition 4.5.4. *Let $X \in \mathcal{Psh}(\mathbb{O}_{n-k,n+2})$. If $\mathcal{S}_{n+1,n+2} \perp X$, then the unit $X \longrightarrow \dot{h}^* \dot{h}_! X$ is an isomorphism.*

Proof. It suffices to show that for each $\omega \in \mathbb{O}_{n-k,n+2}$, the map

$$X_\omega \longrightarrow h^* h_! X_\omega = \int^{\psi \in \mathbb{O}_{n-k,n+2}} \mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_\psi$$

is a bijection. We proceed to construct the required inverse via a cowedge $\mathbb{A}(\dot{h}\omega, \dot{h}-) \times X_{(-)} \longrightarrow X_\omega$, using a case analysis on $\omega \in \mathbb{O}_{n-k,n+2}$.

- (1) Assume $\omega \in \mathbb{O}_{n-k,n-1}$. We have $\mathbb{A}(\dot{h}\omega, h-) \cong \mathbb{O}_{n-k,n+2}(\omega, -)$ and this is just the change-of-variable formula.
- (2) Assume $\omega \in \mathbb{O}_n$. By lemma 4.5.2 and lemma 4.5.3, it suffices to consider the case $\psi \in \mathbb{O}_{n+1}$. We have the sequence of morphisms

$$\begin{aligned} \mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_\psi & \cong \left(\sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t}\nu = \omega}} \mathcal{Psh}(\mathbb{O}_{n-k,n+2})(S[\nu], S[\psi]) \right) \times \mathcal{Psh}(\mathbb{O}_{n-k,n+2})(S[\psi], X) \\ \longrightarrow \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t}\nu = \omega}} \mathcal{Psh}(\mathbb{O}_{n-k,n+2})(S[\nu], X) & \cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathfrak{t}\nu = \omega}} X_\nu \\ \xrightarrow{\mathfrak{t}} X_\omega. \end{aligned}$$

It is straightforward to verify that this defines a cowedge whose induced map is the required inverse.

- (3) Assume $\omega \in \mathbb{O}_{n+1}$. If ω is degenerate, say $\omega = \mathfrak{l}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$, then $\mathbb{A}(\dot{h}\omega, \dot{h}-) \cong \mathbb{A}(\dot{h}\phi, \dot{h}-)$ and we are in a case we have treated before. So let ω be non-degenerate. By lemmas 4.5.2 and 4.5.3, we may suppose $\psi \in \mathbb{O}_{n,n+1}$. Recall that for every $f \in \mathbb{A}(\dot{h}\omega, \dot{h}\psi)$, the diagrammatic lemma 4.4.12 computes a $\xi \in \mathbb{O}_{n+2}$ and $[p] \in \xi^\bullet$ such that $\mathfrak{s}_{[p]}\xi = \omega$, $\mathfrak{t}\xi = \psi$ and $\dot{h}\mathfrak{s}_{[p]} = f$. By corollary 3.4.10, $X_\xi \cong X_\psi$, and thus this gives a function

$$\begin{aligned} \mathbb{A}(\dot{h}\omega, \dot{h}\psi) \times X_\psi & \longrightarrow X_\omega \\ (f, x) & \longmapsto \mathfrak{s}_{[p]}(x). \end{aligned}$$

It is straightforward to verify that this assignment defines a cowedge, whose associated map is the required inverse.

- (4) Assume $\omega \in \mathbb{O}_{n+2}$. Then by definition of \dot{h} , $\mathbb{A}(\dot{h}\omega, \dot{h}-) \cong \mathbb{A}(\dot{h}\mathfrak{t}\omega, \dot{h}-)$, and this is the case we have just treated. \square

Corollary 4.5.5. *If $\mathcal{S}_{n+1,n+2} \perp X$, then $\mathcal{S} \perp \dot{h}_! X$.*

Proof. Recall from lemma 4.5.1 that $\mathbf{S} = \dot{h}_! \mathbf{S}_{n+1}$. Let $\omega \in \mathbb{O}_{n+1}$. We have

$$\begin{aligned}
 \mathcal{Psh}(\mathbb{A})(\dot{h}_! S[\omega], \dot{h}_! X) &\cong \mathcal{Psh}(\mathbb{O}_{n-k, n+2})(S[\omega], \dot{h}^* \dot{h}_! X) && \text{since } \dot{h}_! \dashv \dot{h}^* \\
 &\cong \mathcal{Psh}(\mathbb{O}_{n-k, n+2})(S[\omega], X) && \text{by proposition 4.5.4} \\
 &\cong \mathcal{Psh}(\mathbb{O}_{n-k, n+2})(\omega, X) && \text{since } \mathbf{S}_{n+1} \perp X \\
 &\cong \mathcal{Psh}(\mathbb{O}_{n-k, n+2})(\omega, \dot{h}^* \dot{h}_! X) && \text{by proposition 4.5.4} \\
 &\cong \mathcal{Psh}(\mathbb{A})(\dot{h}_! \omega, \dot{h}_! X) && \text{since } \dot{h}_! \dashv \dot{h}^*.
 \end{aligned}$$

□

This second crucial proposition will provide the other half of the equivalence between \mathcal{Alg} and the localisation $\mathbf{S}_{n+1, n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k, n+2})$.

Proposition 4.5.6. *Let $Y \in \mathcal{Psh}(\mathbb{A})$. If $\mathbf{S} \perp Y$, then the counit $\dot{h}_! \dot{h}^* Y \rightarrow Y$ is an isomorphism.*

Proof. It suffices to prove that for each $\lambda \in \mathbb{A}$, the map

$$(\dot{h}_! Y)_\lambda = \int^{\psi \in \mathbb{O}_{n-k, n+2}} \mathbb{A}(\lambda, \dot{h}\psi) \times Y_{\dot{h}\psi} \longrightarrow Y_\lambda \quad (4.5.7)$$

is a bijection. Since the map¹⁰

$$\begin{aligned}
 Y_\lambda &\longrightarrow \int^{\psi} \mathbb{A}(\lambda, \dot{h}\psi) \times Y_{\dot{h}\psi} \\
 y &\longmapsto \text{id} \otimes y
 \end{aligned}$$

is clearly a section, it is enough to prove that it is surjective. The map is well defined, as \dot{h} is surjective on objects and it is easy to verify that it is independent of the choice of an antecedent $\dot{h}\nu = \lambda$. We need to show that for every $\psi \in \mathbb{O}_{n-k, n+2}$, every $f \otimes y \in \mathbb{A}(\lambda, \dot{h}\psi) \times Y_{\dot{h}\psi}$ is equal, in the colimit (4.5.7), to $\text{id} \otimes y' \in \mathbb{A}(\lambda, \lambda) \times Y_\lambda$ for some $y' \in Y_\lambda$.

- (1) Assume $\lambda = \mathfrak{Z}\phi$, with $\phi \in \mathbb{O}_{n-k, n-1}$. Then $\mathbb{A}(\lambda, \dot{h}-) \cong \mathbb{O}_{n-k, n+2}(\phi, -)$ and any pair $f \otimes y \in \mathbb{O}_{n-k, n+2}(\phi, \psi) \times Y_\psi$ is related via a zig-zag relation

$$f \otimes y \xleftarrow{f} \text{id}_\phi \otimes (\dot{h}f)(y)$$

to an element of the required form.

- (2) Assume $\lambda = \mathfrak{Z}\omega$, with $\omega \in \mathbb{O}_n$. By lemma 4.5.2 and lemma 4.5.3, we may consider only the case where $\psi \in \mathbb{O}_{n+1}$. Note that $\mathbb{A}(\lambda, \dot{h}\psi) \cong \mathbb{A}(\dot{h}Y_\omega, \dot{h}\psi)$. Then $f \otimes y \in \mathbb{A}(\dot{h}Y_\omega, \dot{h}-)$ is related via a zig-zag relation

$$f \otimes y \xleftarrow{\mathfrak{s}_{[p]}} \text{id}_{\dot{h}Y_\omega} \otimes (\dot{h}\mathfrak{s}_{[p]})(y)$$

where we use lemma 4.4.12 to obtain $\mathfrak{s}_{[p]} : Y_\omega \rightarrow \xi$ such that $\dot{h}\mathfrak{s}_{[p]} = f$.

- (3) The case $\lambda = \mathfrak{Z}S[\omega]$, with $\omega \in \mathbb{O}_{n+1}$ identical to the previous one. □

Let the localisation of $\mathcal{Psh}(\mathbb{O}_{n-k, n+2})$ at the set of spine inclusions $\mathbf{S}_{n+1, n+2}$ be denoted

$$v : \mathcal{Psh}(\mathbb{O}_{n-k, n+2}) \xrightarrow{\leftarrow} \mathbf{S}_{n+1, n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k, n+2}) : N_v$$

On the other hand, recall from proposition 4.4.3 that we have a localisation $u : \mathcal{Psh}(\mathbb{A}) \xrightarrow{\leftarrow} \mathcal{Alg} : N_u$. We are now well-equipped to prove that \mathcal{Alg} is equivalent to the localised category $\mathbf{S}_{n+1, n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k, n+2})$.

¹⁰We use the notation $a \otimes b$ to refer to elements in a coend of sets of the form $\int^{c \in \mathcal{C}} A_c \times B_c$, with $A \in [\mathcal{C}^{\text{op}}, \text{Set}]$ and $B \in [\mathcal{C}, \text{Set}]$, our motivation being the tensor product of modules.

Proposition 4.5.8. *The adjunction $\dot{h}_! : \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{\perp} \mathcal{Psh}(\mathbb{A}) : \dot{h}^*$ restricts to an adjunction $\tilde{h}_! \dashv \tilde{h}^*$, as shown below.*

$$\begin{array}{ccc} S_{n+1,n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) & \xrightleftharpoons[\tilde{h}^*]{\tilde{h}_!} & \mathcal{Alg} \\ \downarrow v & & \downarrow N_u \\ \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) & \xrightleftharpoons[\dot{h}^*]{\dot{h}_!} & \mathcal{Psh}(\mathbb{A}). \end{array}$$

Proof. For all $Y \in \mathcal{Alg} \simeq S^{-1} \mathcal{Psh}(\mathbb{A})$, by lemma 4.5.1, we have that $h_! S_{n+1,n+2} \perp N_u Y$, or equivalently, $S_{n+1,n+2} \perp h^* N_u Y$. Thus $h^* N_u$ factors through $S_{n+1,n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k,n+2})$. Next, by corollary 4.5.5, $\dot{h}_! N_v$ factors through \mathcal{Alg} . \square

Corollary 4.5.9. *The adjunction $\tilde{h}_! : S_{n+1,n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{\perp} \mathcal{Alg} : \tilde{h}^*$ is an equivalence.*

Proof. This is a direct consequence of propositions 4.5.4 and 4.5.6. \square

Notation 4.5.10. With corollary 4.5.9 in hand, let

$$h : \mathcal{Psh}(\mathbb{O}) \xrightarrow{\perp} \mathcal{Alg} : N.$$

be the composite adjunction

$$\mathcal{Psh}(\mathbb{O}) \xrightarrow{\perp} \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{v} S_{n+1,n+2}^{-1} \mathcal{Psh}(\mathbb{O}_{n-k,n+2}) \xrightarrow{\tilde{h}_!} \mathcal{Alg}.$$

Theorem 4.5.11 (Nerve theorem for opetopic algebras). *The reflective adjunction $h : \mathcal{Psh}(\mathbb{O}) \rightarrow \mathcal{Alg} : N$ exhibits \mathcal{Alg} as the Gabriel–Ulmer localisation (section 1.6) of $\mathcal{Psh}(\mathbb{O})$ at the set of arrows \mathbb{A} . That is $\mathcal{Alg} \simeq \mathbb{A}^{-1} \mathcal{Psh}(\mathbb{O})$.*

Proof. By a general fact about composite localisations, the reflective adjunction $h : \mathcal{Psh}(\mathbb{O}) \rightarrow \mathcal{Alg} : N$ exhibits \mathcal{Alg} as the localisation of $\mathcal{Psh}(\mathbb{O})$ at the set $\mathbb{O}_{<n-k} \cup S_{n+1,n+2} \cup \mathbb{B}_{>n+2}$. The result then follows from lemma 3.4.7. \square

Corollary 4.5.12. *We have $\mathbb{A} = h \mathbb{O}_{n-k,n+2}$, i.e. $h = \dot{h} : \mathbb{O}_{n-k,n+2} \rightarrow \mathbb{A}$ is surjective on objects (by definition) and on morphisms.*

Proof. Let $\omega, \omega' \in \mathbb{O}_{n-k,n+2}$.

- (1) If $\dim \omega, \dim \omega' < n-1$, then $h\omega = \omega$ and $h\omega' = \omega'$ as presheaves over $\mathbb{O}_{n-k,n+2}$, and thus

$$\mathbb{A}(h\omega, h\omega') = h \mathcal{Psh}(\mathbb{O}_{n-k,n+2})(\omega, \omega') = h \mathbb{O}(\omega, \omega').$$

- (2) If $\dim \omega < n-1$ and $\dim \omega' \geq n-1$, then since $h\omega = \omega$ we have

$$\mathbb{A}(h\omega, h\omega') \cong \mathcal{Psh}(\mathbb{O}_{n-k,n+2})(\omega, h\omega') \cong h \mathcal{Psh}(\mathbb{O}_{n-k,n+2})(\omega, \omega') \cong h \mathbb{O}(\omega, \omega'),$$

where the second isomorphism comes from the fact that for $X \in \mathcal{Psh}(\mathbb{O}_{n-k,n})$, $\mathfrak{Z}^n X_{<n} = X_{<n}$.

- (3) If $\dim \omega \geq n-1$ and $\dim \omega' < n-1$, then $\mathbb{A}(h\omega, h\omega') = \emptyset$.

- (4) Lastly, assume $\dim \omega, \dim \omega' \geq n-1$. By corollary 3.4.11, if $\dim \omega = n-1$, then h maps $\mathbf{tt} : \omega \rightarrow \mathbb{I}_\omega$ to an isomorphism, and if $\dim \omega = n$, then h maps $\mathbf{s}_{\square} : \omega \rightarrow \mathbb{Y}_\omega$ to an isomorphism. By corollary 3.4.10, if $\dim \omega = n+2$, then h maps $\mathbf{t} : \mathbf{t}\omega \rightarrow \omega$ to an isomorphism as well. Hence, without loss of generality, we may assume that $\dim \omega = \dim \omega' = n+1$. If ω is non degenerate, then every morphism in $\mathbb{A}(h\omega, h\omega')$ is diagrammatic, thus in the image of h . Otherwise, if $\omega = \mathbb{I}_\phi$ for some $\phi \in \mathbb{O}_{n-1}$, then

$$\mathbb{A}(h\omega, h\omega') \cong \mathbb{A}(h\phi, h\omega') \cong h \mathbb{O}(\phi, \omega').$$

where the first isomorphism comes from corollary 3.4.11. \square

5. ALGEBRAIC TROMPE-L'ŒIL

As we saw in section 4, for each $k, n \geq 1$, we have a notion of k -coloured opetopic n -algebra. For such an algebra $B \in \mathcal{A}lg_{k,n}$, operations are n -cells (so that their shapes are n -opetopes), and colours are cells of dimension $n - k$ to $n - 1$, thus the “colour space” is stratified over k dimensions. Notable examples include

$$\mathcal{C}at \simeq \mathcal{A}lg_{1,1}, \quad \mathcal{O}p_{\text{col}} \simeq \mathcal{A}lg_{1,2}.$$

But are all $\mathcal{A}lg_{k,n}$ fundamentally different?

In this section, we answer this question negatively: in a sense that we make precise, the most “algebraically rich” notion of opetopic algebra is given in the case $(k, n) = (1, 3)$. Although opetopes can be arbitrarily complex, the algebraic data can be expressed by much simpler 3-opetopes, a.k.a. trees. We call this phenomenon *algebraic trompe-l'œil*, a french expression that literally translates as “fools-the-eye”. And indeed, the eye is fooled in two ways: by colour (proposition 5.1.4) and shape (proposition 5.2.11). In the former, we argue that the colours of an algebra $B \in \mathcal{A}lg_{k,n}$, expressing how operations may or may not be composed, only need 1 dimension, and thus that cells of dimension less than $n - 1$ do not bring new algebraic data, only geometrical one. For the latter, recall from section 3.1 that opetopes are trees of opetopes. In particular, 3-opetopes are just plain trees, and \mathbb{O}_3 already contains all the possible underlying trees of all opetopes. Consequently, operations of B , which are its n -cells, may be considered as 3-cells in a very similar 3-algebra B_{\dagger} . Finally, we combine those two results in theorem 5.2.12, which states that an algebra $B \in \mathcal{A}lg_{k,n}$ is exactly a presheaf $B \in \mathcal{P}sh(\mathbb{O}_{n-k,n})$ with a 1-coloured 3-algebra structure on $B_{n-1,n,\dagger}$.

5.1. Colour. For $B \in \mathcal{A}lg_{k,n}$, recall that the colours of B are its cells of dimension $n - k$ to $n - 1$. They express which operations (n -cells) of B may or may not be composed. However, since that criterion only depends on $(n - 1)$ -cells, constraints expressed by lower dimensional cells should be redundant. In proposition 5.1.4, we show that this is indeed the case, in that the algebra structure on B is completely determined by a 1-coloured n -algebra structure on $B_{n-1,n}$.

Lemma 5.1.1. *Let $k, n \geq 1$, and $\nu \in \mathbb{O}_{n+1}$. Then*

$$S[\nu]_{n-k,n} \cong \iota_!(S[\nu]_{n-1,n}),$$

where $\iota_!$ is the left adjoint to the truncation $\mathcal{P}sh(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}sh(\mathbb{O}_{n-1,n})$.

Proof. It follows from the fact that $S[\nu]$ is completely determined by the incidence relation of the n - and $(n - 1)$ -faces of ν . \square

Proposition 5.1.2. *For $X \in \mathcal{P}sh(\mathbb{O}_{n-k,n})$ we have $\mathfrak{Z}^n(X_{n-1,n}) \cong (\mathfrak{Z}^n X)_{n-1,n}$. Consequently, the truncation functor $(-)_n : \mathcal{P}sh(\mathbb{O}_{n-k,n}) \longrightarrow \mathcal{P}sh(\mathbb{O}_{n-1,n})$ lifts as*

$$\begin{array}{ccc} \mathcal{A}lg_{k,n} & \xrightarrow{(-)_n} & \mathcal{A}lg_{1,n} \\ \downarrow & & \downarrow \\ \mathcal{P}sh(\mathbb{O}_{n-k,n}) & \xrightarrow{(-)_n} & \mathcal{P}sh(\mathbb{O}_{n-1,n}). \end{array} \tag{5.1.3}$$

Proof. First, $\mathfrak{Z}^n(X_{n-1,n})_{n-1} = X_{n-1} = (\mathfrak{Z}^n X)_{n-1}$. Then, for $\psi \in \mathbb{O}_n$, we have

$$\begin{aligned} \mathfrak{Z}^n(X_{n-1,n})_\psi &= \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathbf{t}\nu = \psi}} \mathcal{P}\text{sh}(\mathbb{O}_{n-1,n})(S[\nu], X_{n-1,n}) \\ &\cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathbf{t}\nu = \psi}} \mathcal{P}\text{sh}(\mathbb{O}_{n-1,n})(\iota!(S[\nu]), X) \\ &\cong \sum_{\substack{\nu \in \mathbb{O}_{n+1} \\ \mathbf{t}\nu = \psi}} \mathcal{P}\text{sh}(\mathbb{O}_{n-1,n})(S[\nu], X) && \text{by lemma 5.1.1} \\ &= (\mathfrak{Z}^n X)_\psi. \end{aligned}$$

□

Proposition 5.1.4. *The square (5.1.3) is a pullback. That is, a \mathfrak{Z}^n -algebra structure on $X \in \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$ is completely determined by a \mathfrak{Z}^n -algebra structure on $X_{n-1,n}$.*

Proof. Let $X \in \mathcal{P}\text{sh}(\mathbb{O}_{n-k,n})$. By proposition 5.1.2, a \mathfrak{Z}^n -algebra structure on X restricts to one on $X_{n-1,n}$. Conversely, since $(\mathfrak{Z}^n X)_{<n} = X_{<n}$, a \mathfrak{Z}^n -algebra structure on $X_{n-1,n}$ extends to one on X . Since the truncation functor $(-)_n : \mathcal{A}\text{lg}_{k,n} \rightarrow \mathcal{A}\text{lg}_{1,n}$ is faithful, it establishes a bijective correspondence between the algebra structures on X and on $X_{n-1,n}$. □

5.2. Shape. We start by defining a functor $(-)_\dagger : \mathbb{O}_{n-1,n} \rightarrow \mathbb{O}_{2,3}$, for $n \geq 1$, mapping an n -opetope ω to the unique 3-opetope ω_\dagger having the same underlying polynomial tree, i.e. $\langle \omega_\dagger \rangle \cong \langle \omega \rangle$ (see notations of section 2.2).

Definition 5.2.1. (1) If $n = 1$, then $(-)_\dagger$ simply maps $\mathbb{O}_{0,1} = \left(\blacklozenge \xrightarrow{s^*,\mathbf{t}} \blacksquare \right)$ to the diagram $\left(\mathbf{0} \xrightarrow{s[\square],\mathbf{t}} \mathbf{Y}_0 \right)$.
(2) Assume now that $n \geq 2$. Recall that a 3-opetope is a \mathfrak{Z}^1 -tree, where \mathfrak{Z}^1 is given by

$$\{\blacksquare\} \xleftarrow{s} E_2 \xrightarrow{p} \mathbb{O}_2 \xrightarrow{t} \{\blacksquare\},$$

with $\mathbb{O}_2 = \{\mathbf{n} \mid n \in \mathbb{N}\}$ and $E_2(\mathbf{n}) = \mathbf{n}^\bullet$. Let $f : \mathfrak{Z}^{n-2} \rightarrow \mathfrak{Z}^1$ be given by

$$\begin{array}{ccccccc} \mathbb{O}_{n-2} & \xleftarrow{s} & E_{n-2} & \xrightarrow{p} & \mathbb{O}_{n-1} & \xrightarrow{t} & \mathbb{O}_{n-2} \\ f_0 \downarrow & & \downarrow f_2 & \lrcorner & \downarrow f_1 & & \downarrow f_0 \\ \{\blacksquare\} & \xleftarrow{s} & E_2 & \xrightarrow{p} & \mathbb{O}_2 & \xrightarrow{t} & \{\blacksquare\}, \end{array}$$

where $f_1(\psi) = \mathbf{m}$, for $m = \#\psi^\bullet$ the number of source faces of ψ , and where f_2 is fiberwise increasing. This morphism of polynomial functors induces a functor $f_* : \mathbb{O}_n = \text{tr } \mathfrak{Z}^{n-2} \rightarrow \text{tr } \mathfrak{Z}^1 = \mathbb{O}_3$ mapping an n -opetope to its underlying tree, seen as a 3-opetope. Explicitly,

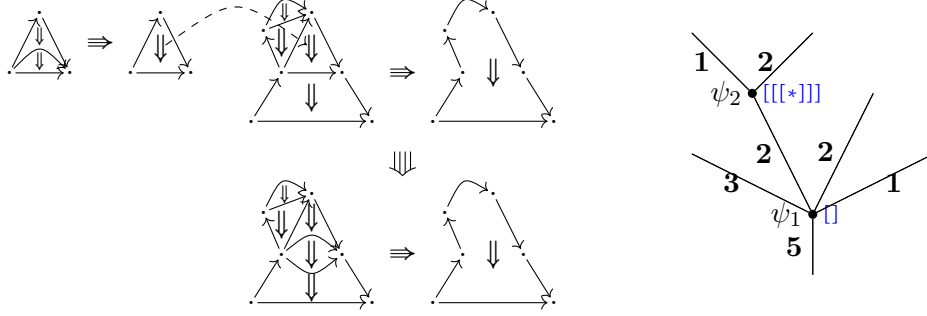
$$f_* \mathbf{l}_\phi = \mathbf{l}_*, \quad f_* \left(\mathbf{Y}_\psi \begin{array}{c} \bigcirc \\ \llbracket [p_i] \rrbracket \end{array} \nu_i \right) = \mathbf{m} \begin{array}{c} \bigcirc \\ \llbracket [*^i] \rrbracket \end{array} f_*(\nu_i),$$

where $\phi \in \mathbb{O}_{n-2}$, $\psi \in \mathbb{O}_{n-1}$, $\psi^\bullet = \{[p_0] < \dots < [p_{m-1}]\}$, and $\nu_0, \dots, \nu_{m-1} \in \mathbb{O}_n$. For $\omega \in \mathbb{O}_n$, since ω and ω_\dagger have the same underlying tree, they have the same number of source faces: $\#\omega^\bullet = \#\omega_\dagger^\bullet$, and we write a_ω for the unique increasing map $\omega^\bullet \rightarrow \omega_\dagger^\bullet$ with respect to the lexicographical order $<^{11}$.

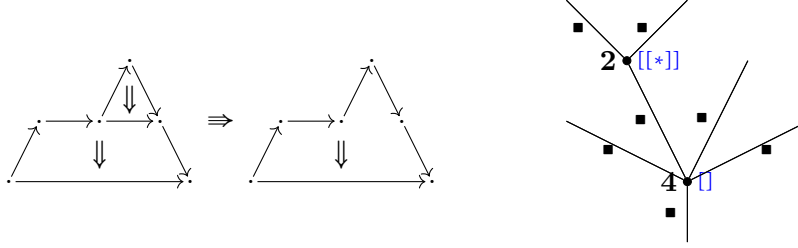
¹¹Intuitively, a_ω maps a node of the underlying tree $\langle \omega \rangle$ of ω to that same node in $\langle \omega_\dagger \rangle$, but using addresses. Since the source faces of ω and ω_\dagger are not the same, a_ω is not strictly speaking an identity, but rather a conversion of a “walking instruction in the tree ω ” (which is what an address is) to one in ω_\dagger . Explicitly, a node address $\llbracket [q_1] \cdots [q_k] \rrbracket \in \omega^\bullet$ (with $[q_{i+1}] \in s_{\llbracket [q_1] \cdots [q_i] \rrbracket} \omega$) is mapped to $\llbracket [f_{2,s[\square]} \omega [q_1] \cdots f_{2,s[\llbracket [q_1] \cdots [q_{k-1}]]} \omega [q_k] \rrbracket$.

- (3) Define now $(-)_\dagger : \mathbb{O}_{n-1,n} \longrightarrow \mathbb{O}_{2,3}$ as follows: for $\psi \in \mathbb{O}_{n-1}$ and $\omega \in \mathbb{O}_n$
- (a) $\psi_\dagger = f_1(\psi)$ as above;
 - (b) $\omega_\dagger = f_*(\omega)$ as above;
 - (c) we have $(t\omega)_\dagger = t\omega_\dagger$, so let $(t\omega \xrightarrow{t} \omega)_\dagger = ((t\omega)_\dagger \xrightarrow{t} \omega_\dagger)$;
 - (d) for $[p] \in \omega^\bullet$, we have $(s_{[p]}\omega)_\dagger = s_{a_\omega[p]}\omega_\dagger$, so let $(s_{[p]}\omega \xrightarrow{s_{[p]}} \omega)_\dagger = ((s_{[p]}\omega)_\dagger \xrightarrow{s_{a_\omega[p]}} \omega_\dagger)$.

Example 5.2.2. Consider the 4-opetope ω , represented graphically and in tree form below:



where ψ_1 and ψ_2 are the 3-opetopes on the top right and top left hand corner respectively. Then ω_\dagger is as follows:



Although the graphical representations of ω and ω_\dagger look nothing alike, note that their underlying undecorated trees are identical.

Notation 5.2.3. We abuse notations and let $(-)_\dagger : \mathcal{Psh}(\mathbb{O}_{n-1,n}) \longrightarrow \mathcal{Psh}(\mathbb{O}_{2,3})$ be the left Kan extension of $\mathbb{O}_{n-1,n} \xrightarrow{(-)_\dagger} \mathbb{O}_{2,3} \longrightarrow \mathcal{Psh}(\mathbb{O}_{2,3})$ along the Yoneda embedding. Explicitly, for $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$, we have

$$X_{\dagger, \mathbf{m}} = \sum_{\substack{\psi \in \mathbb{O}_{n-1} \\ \psi_\dagger = \mathbf{m}}} X_\psi, \quad X_{\dagger, \gamma} = \sum_{\substack{\omega \in \mathbb{O}_{n-1} \\ \omega_\dagger = \gamma}} X_\omega,$$

with $m \in \mathbb{N}$ and $\gamma \in \mathbb{O}_3$.

Remark 5.2.4. Clearly, $(-)_\dagger$ is faithful, and if $n \leq 3$, then $(-)_\dagger$ is also injective on object. Note that this is no longer the case if $n \geq 4$, as distinct n -opetopes may have the same underlying tree.

Notation 5.2.5. Let \mathcal{C} be a small category, and $X \in \mathcal{Psh}(\mathcal{C})$ be presheaf over \mathcal{C} . There is a canonical projection $(-)^{\natural} : \mathcal{C}/X \longrightarrow \mathcal{C}$, mapping $x \in X_c$ to its *shape* $c \in \mathcal{C}$. We may then see the category of elements \mathcal{C}/X of X as having objects elements of $\sum_{c \in \mathcal{C}} X_c$, and a morphism $f : x \longrightarrow y$ is a morphism $f : x^{\natural} \longrightarrow y^{\natural}$ in \mathcal{C} such that $f(y) = x$. A morphism $g : X \longrightarrow Y$ of presheaves over \mathcal{C} then amounts to a functor $g : \mathcal{C}/X \longrightarrow \mathcal{C}/Y$ that preserves shapes.

Remark 5.2.6. Take $n \geq 1$. For $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$, note that there is a canonical isomorphism $\mathcal{Psh}(\mathbb{O}_{n-1,n})/X \longrightarrow \mathcal{Psh}(\mathbb{O}_{2,3})/X_\dagger$, which is the identity on objects, maps $s_{[p]} : x \longrightarrow y$ to $s_{a_\omega[p]} x \longrightarrow y$, where $\omega = y^{\natural}$, and target embeddings to target embeddings.

- Lemma 5.2.7.** (1) For $\nu \in \mathbb{O}_{n+1}$, there exists a unique 4-opetope $\nu' \in \mathbb{O}_4$ such that $S[\nu]_{n-1,n,\dagger} \cong S[\nu']$.
- (2) Let $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$, $\nu \in \mathbb{O}_4$, and $f : S[\nu] \rightarrow X_{\dagger}$. Then there exists a unique $\nu' \in \mathbb{O}_{n+1}$ and $f' : S[\nu'] \rightarrow X$ such that $S[\nu']_{n-1,n,\dagger} = S[\Lambda]$, and $f'_{\dagger} = f$.

Proof. (1) If $\nu = \mathbf{l}_{\phi}$ for $\phi \in \mathbb{O}_{n-1}$, let $\nu' = \mathbf{l}_{\phi_{\dagger}}$. If $\nu = \mathbf{Y}_{\omega} \mathbb{O}_{[[p_i]]} \nu_i$, let

$$\nu' = \mathbf{Y}_{\omega_{\dagger}} \bigcirc_{[a_{\omega}[p_i]]} \nu'_i,$$

where the ν'_i are given by induction. The graftings are well defined since

$$\mathbf{t} \mathbf{s}_{[\square]} \nu'_i = \mathbf{t}(\mathbf{s}_{[\square]} \nu_i)_{\dagger} = (\mathbf{t} \mathbf{s}_{[\square]} \nu_i)_{\dagger} = (\mathbf{s}_{[p_i]} \omega)_{\dagger} = \mathbf{s}_{a_{\omega}[p_i]} \omega_{\dagger}.$$

The isomorphism $S[\nu]_{n-1,n,\dagger} \cong S[\nu']$ can easily be shown by induction on the structure of ν and using lemma 3.4.3.

- (2) For $\nu^{\bullet} = \{[p_1], \dots, [p_m]\}$, f maps $[p_i]$ to a cell $x_i \in X_{\dagger,2} = X_{n-1}$, and let $\psi_i \in \mathbb{O}_{n-1}$ be the shape of x_i as a cell of X . If $[p_i] = [p_j[q]]$ for some j and $[q]$, then $\mathbf{s}_{[q]} x_j = \mathbf{t} x_i$ in X_{\dagger} , so $\mathbf{s}_{a_{\psi_j}^{-1}[q]} x_j = \mathbf{t} x_i$ in X , and in particular, $\mathbf{s}_{a_{\psi_j}^{-1}[q]} \psi_j = \mathbf{t} \psi_i$. Consequently, the ψ_i s may be grafted together into a $(n+1)$ -opetope ν' such that $\nu'_{\dagger} = \nu$, and $\mathbf{s}_{a_{\nu'}^{-1}[p_i]} = \psi_i$. Define $f' : S[\nu'] \rightarrow X$ mapping $\mathbf{s}_{a_{\nu'}^{-1}[p_i]} \nu'$ to x_i , and observe that $f'_{\dagger} = f$. \square

Proposition 5.2.8. For $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$ we have $\mathfrak{Z}^3(X_{\dagger}) \cong (\mathfrak{Z}^n X)_{\dagger}$. Consequently, the functor $(-)_{\dagger}$ lifts as

$$\begin{array}{ccc} \mathcal{Alg}_{1,n} & \xrightarrow{(-)_{\dagger}} & \mathcal{Alg}_{1,3} \\ \downarrow & & \downarrow \\ \mathcal{Psh}(\mathbb{O}_{n-1,n}) & \xrightarrow{(-)_{\dagger}} & \mathcal{Psh}(\mathbb{O}_{2,3}). \end{array} \quad (5.2.9)$$

Proof. First, $\mathfrak{Z}^3(X_{\dagger})_2 = X_{\dagger,2} = X_{n-1} = (\mathfrak{Z}^n X)_{n-1} = (\mathfrak{Z}^n X)_{\dagger,2}$. Then,

$$\begin{aligned} \mathfrak{Z}^3(X_{\dagger})_3 &= \sum_{\nu \in \mathbb{O}_4} \mathcal{Psh}(\mathbb{O}_{2,3})(S[\nu], X_{\dagger}) \\ &\cong \sum_{\nu \in \mathbb{O}_{n+1}} \mathcal{Psh}(\mathbb{O}_{n-1,n})(S[\nu], X) && \text{by lemma 5.2.7} \\ &= (\mathfrak{Z}^n X)_n = (\mathfrak{Z}^n X)_{\dagger,3}. \end{aligned}$$

\square

Lemma 5.2.10. Let $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$ and $m : \mathfrak{Z}^n X \rightarrow X$. Then m is an algebra structure on X if and only if $m_{\dagger} : \mathfrak{Z}^3 X_{\dagger} \rightarrow X_{\dagger}$ is an algebra structure on X_{\dagger} .

Proof. Clearly, $(-)_{\dagger}$ maps the multiplication $\mu^n : \mathfrak{Z}^n \mathfrak{Z}^n \rightarrow \mathfrak{Z}^n$ to μ^3 , and the unit $\eta^n : \text{id} \rightarrow \mathfrak{Z}^n$ to η^3 . Since $(-)_{\dagger}$ is faithful, the square on the left commutes if and only if the square on the right commutes

$$\begin{array}{ccc} \mathfrak{Z}^n \mathfrak{Z}^n X & \xrightarrow{\mathfrak{Z}^n m} & \mathfrak{Z}^n X \\ \mu^n \downarrow & & \downarrow m \\ \mathfrak{Z}^n X & \xrightarrow{m} & X, \end{array} \quad \begin{array}{ccc} \mathfrak{Z}^3 \mathfrak{Z}^3 X_{\dagger} & \xrightarrow{\mathfrak{Z}^3 m_{\dagger}} & \mathfrak{Z}^3 X_{\dagger} \\ \mu^3 \downarrow & & \downarrow m_{\dagger} \\ \mathfrak{Z}^3 X_{\dagger} & \xrightarrow{m_{\dagger}} & X_{\dagger}, \end{array}$$

and likewise for the diagram involving η^n and η^3 . \square

Proposition 5.2.11. *The square (5.2.9) is a pullback. That is, a \mathfrak{Z}^n -algebra structure on $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$ is completely determined by a \mathfrak{Z}^3 -algebra structure on X_{\dagger} .*

Proof. Let $X \in \mathcal{Psh}(\mathbb{O}_{n-1,n})$. By proposition 5.2.8, a \mathfrak{Z}^n -algebra structure on X induces a \mathfrak{Z}^3 -algebra structure on X_{\dagger} .

Conversely, let $m : \mathfrak{Z}^3 X_{\dagger} \rightarrow X_{\dagger}$ be a \mathfrak{Z}^3 -algebra structure on X_{\dagger} , and define $m' : \mathfrak{Z}^n X \rightarrow X$ as the identity in dimension $n-1$, and mapping $f : S[\nu] \rightarrow X$ to $m(f_{\dagger}) \in X_{\dagger,2} = X_{n-1}$. Recall that f_{\dagger} is a map of the form $S[\nu'] \rightarrow X_{\dagger}$, for some ν' such that $t\nu' = (t\nu)_{\dagger}$, and thus m' is a map of opetopic sets. By lemma 5.2.10, it is a \mathfrak{Z}^n -algebra structure on X .

Since $(-)_{\dagger}$ is faithful, it establishes a bijective correspondence between the \mathfrak{Z}^n -algebra structures on X and the \mathfrak{Z}^3 -algebra structures on X_{\dagger} . \square

Theorem 5.2.12 (Algebraic trompe-l'œil). *The following square is a pullback:*

$$\begin{array}{ccc} \mathcal{Alg}_{k,n} & \xrightarrow{(-)_{n-1,n,\dagger}} & \mathcal{Alg}_{1,3} \\ \downarrow & & \downarrow \\ \mathcal{Psh}(\mathbb{O}_{n-k,n}) & \xrightarrow{(-)_{n-1,n,\dagger}} & \mathcal{Psh}(\mathbb{O}_{2,3}). \end{array} \quad (5.2.13)$$

In other words, a \mathfrak{Z}^n -algebra structure on $X \in \mathcal{Psh}(\mathbb{O}_{n-k,n})$ is completely determined by a \mathfrak{Z}^3 -algebra structure on $(X_{n-1,n})_{\dagger}$.

Proof. This is a direct consequence of proposition 5.1.4, proposition 5.2.11, and the pasting lemma for pullbacks. \square

APPENDIX A. OMITTED PROOFS

Proof of lemma 4.3.3. (1) Let P be the pullback of the cospan $1 \xrightarrow{\eta_1} \mathfrak{Z}^n 1 \xleftarrow{\mathfrak{Z}^n} \mathfrak{Z}^n X$. Since $(\mathfrak{Z}^n X)_{<n} = X_{<n}$, we trivially have $P_{<n} = X_{<n}$. Next, for $\omega \in \mathbb{O}_n$, we have

$$P_{\omega} = \{x \in \mathfrak{Z}^n X \mid \mathfrak{Z}^n!(x) = Y_{\omega}\} = \mathcal{Psh}(\mathbb{O}_{n-k,n})(S[Y_{\omega}], X) = X_{\omega}$$

as $S[Y_{\omega}] = O[\omega]$.

(2) Let P be the pullback of the cospan $\mathfrak{Z}^n \mathfrak{Z}^n 1 \xrightarrow{\mu_1} \mathfrak{Z}^n 1 \xleftarrow{\mathfrak{Z}^n} \mathfrak{Z}^n X$. As before, since $(\mathfrak{Z}^n X)_{<n} = X_{<n}$, we trivially have $P_{<n} = X_{<n} = (\mathfrak{Z}^n \mathfrak{Z}^n X)_{<n}$. Recall from lemma 4.3.2 that as a polynomial functor, $\mathfrak{Z}^n \mathfrak{Z}^n : \text{Set}/\mathbb{O}_n \rightarrow \text{Set}/\mathbb{O}_n$ is given by

$$\mathbb{O}_n \xleftarrow{e} E \longrightarrow \mathbb{O}_{n+2}^{(2)} \xrightarrow{tt} \mathbb{O}_n,$$

where $\mathbb{O}_{n+2}^{(2)}$ is the set of $(n+2)$ -opetopes of height at most 2. Then, for $\omega \in \mathbb{O}_n$, we have:

$$P_{\omega} = \left\{ (\xi, x) \mid x : \Lambda^t[\nu] \rightarrow X, \xi \in \mathbb{O}_{n+2}^{(2)}, t\xi = \nu \right\} = \left\{ x : S[t\xi] \mid \xi \in \mathbb{O}_{n+2}^{(2)} \right\} = \mathfrak{Z}^n \mathfrak{Z}^n X_{\omega}.$$

\square

Proof of lemma 4.3.4. For $X \in \mathcal{Psh}(\mathbb{O}_{n-k,n})$, $(\mathfrak{Z}^n X)_{<n} = X_{<n}$, thus all diagrams commute trivially in dimension $< n$.

(1) Let $\omega \in \mathbb{O}_n$ and $\nu \in \mathfrak{Z}^n 1_{\omega}$, i.e. $\nu \in \mathbb{O}_{n+1}$ such that $t\nu = \omega$. Then

$$\mu_1 \eta \mathfrak{Z}^n 1(\nu) = \mu_1 \left(Y_{Y_{t\nu}} \begin{array}{c} \circ \\ \square \end{array} Y_{\nu} \right) = t \left(Y_{Y_{t\nu}} \begin{array}{c} \circ \\ \square \end{array} Y_{\nu} \right) = \nu.$$

(2) Let $\omega \in \mathbb{O}_n$ and $\nu \in \mathfrak{Z}^n 1_\omega$, i.e. $\nu \in \mathbb{O}_{n+1}$ such that $t\nu = \omega$. Then

$$\mu_1(\mathfrak{Z}^n \eta_1)(\nu) = \mu_1 \left(Y_\nu \underset{[[p_i]]}{\circlearrowleft} Y_{Y_{s_{[p_i]}\nu}} \right) = t \left(Y_\nu \underset{[[p_i]]}{\circlearrowleft} Y_{Y_{s_{[p_i]}\nu}} \right) = \nu,$$

where $[p_i]$ ranges over ν^\bullet .

(3) Akin to lemma 4.3.2, one can show that elements of $\mathfrak{Z}^n \mathfrak{Z}^n \mathfrak{Z}^n 1_\omega$ are $(n+2)$ -opetopes ξ of height 3 such that $tt\xi = \omega$. Let ξ be such an opetope, and write it as

$$\xi = Y_\alpha \underset{[[p_i]]}{\circlearrowleft} \underbrace{\left(Y_{\beta_i} \underset{[[q_{i,j}]]}{\circlearrowleft} Y_{\gamma_{i,j}} \right)}_{X_i} = \underbrace{\left(Y_\alpha \underset{[[p_i]]}{\circlearrowleft} Y_{\beta_i} \right)}_Y \underset{[[p_i][q_{i,j}]]}{\circlearrowleft} Y_{\gamma_{i,j}}$$

where $\alpha, \beta_i, \gamma_{i,j} \in \mathbb{O}_n$, $[p_i]$ ranges over α^\bullet and $[q_{i,j}]$ over β_i^\bullet . Then

$$\begin{aligned} \mu_1(\mathfrak{Z}^n \mu_1)(\gamma) &= t \left(Y_\alpha \underset{[[p_i]]}{\circlearrowleft} Y_{tX_i} \right) \\ &= t \left(Y_{tY} \underset{[\emptyset_Y [[p_i][q_{i,j}]]]{\circlearrowleft} Y_{\gamma_{i,j}} \right) \quad \spadesuit \\ &= \mu_1 \mu \mathfrak{Z}^n 1(\gamma), \end{aligned}$$

where \spadesuit derives from the associativity axiom of the uncolored monad \mathfrak{Z}^n .

□

Proof of lemma 4.4.9. First, note that

$$\begin{aligned} t(\xi_2 \underset{[p_2]}{\square} \xi_1) &= tt(Y_{\xi_2} \underset{[[p_2]]}{\circlearrowleft} Y_{\xi_1}) && \text{by lemma 4.4.10} \\ &= ts[\](Y_{\xi_2} \underset{[[p_2]]}{\circlearrowleft} Y_{\xi_1}) && \text{(Glob2)} \\ &= t\xi_2 = \omega_2. \end{aligned}$$

Using (2.4.10), we write

$$\zeta = \xi_2 \underset{[p_2]}{\square} \xi_1 = \left(\alpha_2 \underset{[p_2]}{\circlearrowleft} \alpha_1 \right) \underset{[p_2 p_1]}{\circlearrowleft} Y_{\omega_0} \underset{[[q_i]]}{\circlearrowleft} \underbrace{\left(\beta_i \underset{[l_{i,j}]]{\circlearrowleft} \gamma_{i,j} \right)}_{\delta_i}$$

where $[q_i]$ ranges over ω_0^\bullet and $[l_{i,j}]$ over $\beta_i^!$, such that

$$\xi_1 = \alpha_1 \underset{[p_1]}{\circlearrowleft} Y_{\omega_0} \underset{[[q_i]]}{\circlearrowleft} \beta_i, \quad \xi_2 = \alpha_2 \underset{[p_2]}{\circlearrowleft} Y_{\omega_1} \underset{[\emptyset_{\xi_1} [p_1 [q_i] l_{i,j}]]}{\circlearrowleft} \gamma_{i,j}$$

Since leaf addresses of ξ_1 are of the form $[p_1 [q_i] l_{i,j}]$ for some $[q_i] \in \omega_0^\bullet$ and $[l_{i,j}] \in \beta_i^!$, and since $\omega_1 = t\xi_1$, the node addresses of ω_1 are of the form $\emptyset_{\xi_1} [p_1 [q_i] l_{i,j}]$, which justifies the decomposition of ξ_2 above.

Applying the definition of \dot{h} we have, for $[q_i] \in \omega_0^\bullet$, $[l_{i,j}] \in \beta_i^|$, and $[r] \in \gamma_{i,j}$,

$$\begin{aligned}
 \dot{h} s_{[p_2 p_1]} : S[\omega_0] &\longrightarrow \mathfrak{S}S[\omega_2] \\
 (\dot{h} s_{[p_2 p_1]})([q_i] : S[\mathfrak{t} \delta_i]) &\longrightarrow S[\omega_2] \\
 \wp_{\delta_i}[l_{i,j} r] &\longmapsto \wp_{\zeta}[p_2 p_1[q_i] l_{i,j} r]; & \spadesuit \\
 \dot{h} s_{[p_1]} : S[\omega_0] &\longrightarrow \mathfrak{S}S[\omega_1] \\
 (\dot{h} s_{[p_1]})([q_i] : S[\mathfrak{t} \beta_i]) &\longrightarrow S[\omega_1] \\
 \wp_{\beta_i}[l_{i,j}] &\longmapsto \wp_{\xi_1}[p_1[q_i] l_{i,j}]; & \diamond \\
 \dot{h} s_{[p_2]} : S[\omega_1] &\longrightarrow \mathfrak{S}S[\omega_2] \\
 (\dot{h} s_{[p_2]})(\wp_{\xi_1}[p_1[q_i] l_{i,j}]) : S[\mathfrak{t} \gamma_{i,j}] &\longrightarrow S[\omega_2] \\
 \wp_{\gamma_{i,j}}[r] &\longmapsto \wp_{\xi_2}[p_2 \wp_{\xi_1}[p_1[q_i] l_{i,j}] r]. & \clubsuit
 \end{aligned}$$

Thus,

$$\begin{aligned}
 (\dot{h} s_{[p_2 p_1]})([q_i])(\wp_{\delta_i}[l_{i,j} r]) &= \wp_{\zeta}[p_2 p_1[q_i] l_{i,j} r] && \text{by } \spadesuit \\
 &= \wp_{\xi_2}[p_2 \wp_{\xi_1}[p_1[q_i] l_{i,j}] r] && \heartsuit \\
 &= (\dot{h} s_{[p_2]})(\wp_{\xi_1}[p_1[q_i] l_{i,j}])(\wp_{\gamma_{i,j}}[r]) && \text{by } \clubsuit \\
 &= (\dot{h} s_{[p_2]})(\dot{h} s_{[p_1]})([q_i])(\wp_{\beta_i}[l_{i,j}])(\wp_{\gamma_{i,j}}[r]) && \text{by } \diamond \\
 &= (\dot{h} s_{[p_2]} \circ \dot{h} s_{[p_1]})([q_i])(\wp_{\delta_i}[l_{i,j} r]), && \blackstar
 \end{aligned}$$

where equality \heartsuit comes from the monad structure on \mathfrak{S} , and \blackstar from the definition of the composition in \mathfrak{A} when considered as the Kleisli category of \mathfrak{S} . \square

Proof of lemma 4.4.11. (1) Unfolding the definition of $\dot{h} s_{[]}$ we get, for $[q] \omega^\bullet$:

$$\begin{aligned}
 \dot{h} s_{[]} : S[\mathbf{Y}_\psi] &\longrightarrow \mathfrak{S}S[\omega] \\
 (\dot{h} s_{[]})([]) : S[\mathfrak{t} \mathbf{Y}_\omega] &\longmapsto S[\omega] \\
 \wp_{\mathbf{Y}_\omega}[[q]] &\longmapsto \wp_{\xi}[[[]][q]].
 \end{aligned}$$

Since $\wp_{\mathbf{Y}_\omega}[[q]] = [q] = \wp_{\xi}[[[]][q]]$, $\dot{h} s_{[]}$ corresponds to the cell $\text{id}_{S[\omega]} \in \mathfrak{S}S[\omega]_{\mathfrak{t}\omega}$, thus is equal to $\dot{h} \mathfrak{t}$ as claimed.

(2) First, graft trivial trees to ξ so it's in the form of equation (2.4.10):

$$\xi = \mathbf{Y}_{\bar{\beta}} \circ_{[[p]]} \mathbf{Y}_{\beta} \bigcirc_{[[q_i]]} \mathfrak{l}_{s_{[q_i]}\beta},$$

where $[q_i]$ ranges over β^\bullet . Unfolding the definition of $\dot{h} s_{[p]}$ we get, for $[q] \beta^\bullet$:

$$\begin{aligned}
 \dot{h} s_{[p]} : S[\beta] &\longrightarrow \mathfrak{S}S[\omega] \\
 (\dot{h} s_{[p]})([q]) : S[\mathfrak{t} \mathfrak{l}_{s_{[q]}\beta}] &\longrightarrow S[\omega] \\
 [] = \wp_{\mathfrak{l}_{s_{[q]}\beta}}[[]] &\longmapsto \wp_{\xi}[[p][q]] = [pq],
 \end{aligned}$$

thus $\dot{h} s_{[p]} = \dot{h} i$. \square

Proof of lemma 4.4.12. We proceed by induction on ω .

(1) Assume $\omega = \mathbf{Y}_\psi$ for some $\psi \in \mathbb{O}_n$. Then

$$\begin{aligned} \Delta(\dot{h}\mathbf{Y}_\psi, \dot{h}\omega') &= \Delta(\mathfrak{ZS}[\mathbf{Y}_\psi], \mathfrak{ZS}[\omega']) \\ &\cong (\mathfrak{ZS}[\omega])_\psi. \end{aligned}$$

Thus f corresponds to a unique morphism $\tilde{f} : S[\nu] \rightarrow S[\omega']$, for some $\nu \in \mathbb{O}_{n+1}$ such that $\mathbf{t}\nu = \psi$, and f is the composite

$$\dot{h}\mathbf{Y}_\psi = \dot{h}\nu \xrightarrow{\dot{h}\mathbf{t}} \dot{h}\nu \xrightarrow{\mathfrak{Z}\tilde{f}} \dot{h}\omega'.$$

Those two arrows are diagrammatic by lemma 4.4.11, and thus f is too by lemma 4.4.9.

(2) By induction, write $\omega = \nu_1 \circ_{[l]} \mathbf{Y}_{\psi_2}$ for some $\nu_1 \in \mathbb{O}_{n+1}$, $[l] \in \nu_1^\bullet$, and $\psi_2 \in \mathbb{O}_n$. Write $\psi_1 = \mathbf{t}\nu_1$, and $\nu_2 = \mathbf{Y}_{\psi_2}$. Then f restricts as f_i , $i = 1, 2$, given by the composite $\dot{h}\nu_i \rightarrow \dot{h}\omega \xrightarrow{f} \dot{h}\omega'$.

Let $[l']$ be the edge address of ω' such that $\mathbf{e}_{[l']}\omega' = f(\mathbf{e}_{[l]}\omega)$. Then ω' decomposes as $\omega' = \beta_1 \circ_{[l']}\beta_2$, for some $\beta_1, \beta_2 \in \mathbb{O}_{n+1}$ (in particular, β_1 and β_2 are sub \mathfrak{Z}^{n-1} -trees of ω'), and f_1 and f_2 factor as on the left

$$\begin{array}{ccc} \dot{h}\nu_i & \xrightarrow{\bar{f}_i} & \dot{h}\beta_i \\ & \searrow f_i & \downarrow u_i \\ & & \dot{h}\omega' \end{array} \qquad \begin{array}{ccc} & & \xi_i \\ & \nearrow \mathfrak{S}[\psi_i] & \uparrow \mathbf{t} \\ \nu_i & & \beta_i \\ \hline \dot{h}\nu_i & \xrightarrow{\bar{f}_i} & \dot{h}\beta_i. \end{array}$$

where u_i correspond to the subtree inclusion $\beta_i \hookrightarrow \omega'$. By induction, \bar{f}_i is diagrammatic, say with the diagram on the right above, and thus β_i can be written as $\beta_i = \bar{\nu}_i \circ_{[q_i]}\nu_i$, for some $\bar{\nu}_i \in \mathbb{O}_{n+1}$ and $[q_i] \in \bar{\nu}_i^\bullet$. In the case $i = 2$, note that $\beta_2 = \bar{\nu}_2 \circ_{[q_2]}\nu_2 = \bar{\nu}_2 \circ_{[q_2]}\mathbf{Y}_{\psi_2} = \bar{\nu}_2$.

On the one hand we have

$$\begin{aligned} \mathbf{e}_{[l']}\omega' &= f(\mathbf{e}_{[l]}\omega) && \text{by def. of } [l'] \\ &= f_1(\mathbf{e}_{[l]}\nu_1) && \text{since } \omega = \nu_1 \circ_{[l]}\mathbf{Y}_{\psi_2} \\ &= u_1 \bar{f}_1(\mathbf{e}_{[l]}\nu_1) && \text{since } f_1 = u_1 \bar{f}_1 \\ &= u_1(\mathbf{e}_{[q_1]}\beta_1) && \text{since } \beta_1 = \bar{\nu}_1 \circ_{[q_1]}\nu_1 \\ &= \mathbf{e}_{[q_1 l]}\omega, \end{aligned}$$

showing $[l'] = [q_1 l]$, and thus that $\bar{\nu}_1$ is of the form

$$\bar{\nu}_1 = \mu_1 \circ_{[q_1]}\mathbf{Y}_{\psi_1} \bigcirc_{[[r_{1,j}]]} \mu_{1,j}, \tag{A.0.1}$$

where $[r_{1,j}]$ ranges over $\psi_1^\bullet - \{\mathfrak{e}_{\nu_1}[l]\}$, and $\mu_1, \mu_{1,j} \in \mathbb{O}_{n+1}$. On the other hand,

$$\begin{aligned} \mathbf{e}_{[l']}\omega' &= f(\mathbf{e}_{[l]}\omega) && \text{by def. of } [l'] \\ &= f_2(\mathbf{e}_{[l]}\nu_2) && \text{since } \omega = \nu_1 \circ_{[l]}\mathbf{Y}_{\psi_2} \\ &= u_2 \bar{f}_2(\mathbf{e}_{[l]}\nu_2) && \text{since } f_2 = u_2 \bar{f}_2 \\ &= u_2(\mathbf{e}_{[q_2]}\beta_2) && \text{since } \beta_2 = \bar{\nu}_2 \circ_{[q_2]}\nu_2 \\ &= \mathbf{e}_{[l']}\omega', \end{aligned}$$

showing $[q_2] = []$, and so $s_{[]} \beta_2 = s_{[]} \bar{\nu}_2 = \psi_2$, and we can write β_2 as

$$\beta_2 = Y_{\psi_2} \bigcirc_{[[r_{2,j}]]} \mu_{2,j}, \quad (\text{A.0.2})$$

where $[r_{2,j}]$ ranges over ψ_2^\bullet , and $\mu_{2,j} \in \mathbb{O}_{n+1}$. Finally, we have

$$\begin{aligned} \omega' &= \beta_1 \circ_{[l']} \beta_2 = (\bar{\nu}_1 \square_{[q_1]} \nu_1) \circ_{[l']} \beta_2 \\ &= \left(\mu_1 \circ_{[q_1]} \nu_1 \bigcirc_{\wp_{\nu_1}^{-1}[r_{1,j}]} \mu_{1,j} \right) \circ_{[l']} \left(Y_{\psi_2} \bigcirc_{[[r_{2,j}]]} \mu_{2,j} \right) && \text{by (A.0.1) and (A.0.2)} \\ &= \left(\left(\mu_1 \circ_{[q_1]} \underbrace{\nu_1 \circ_{[l]} Y_{\psi_2}}_{=\omega} \right) \bigcirc_{[q_1] \cdot \wp_{\nu_1}^{-1}[r_{1,j}]} \mu_{1,j} \right) \bigcirc_{[l'[r_{2,j}]]} \mu_{2,j} && \text{rearranging terms} \\ &= \bar{\omega} \square_{[q_1]} \omega, \end{aligned}$$

for some $\bar{\omega}' \in \mathbb{O}_{n+1}$.¹² Finally, by lemma 4.4.11, the following is a diagram of $\hbar f$:

$$\begin{array}{ccc} & & \xi \\ & \nearrow s_{[[q_1]]} & \uparrow t \\ \omega & & \omega' \\ \hline \hbar\omega & \xrightarrow{f} & \hbar\omega', \end{array} \quad \xi := Y_{\bar{\omega}} \bigcirc_{[[q_1]]} Y_{\omega}.$$

□

¹²Specifically,

$$\omega' = \mu_1 \circ_{[q_1]} \left(\left(Y_{t\omega} \bigcirc_{[\wp_{\nu_1}^{-1}[r_{1,j}]]} \mu_{1,j} \right) \bigcirc_{[\wp_{\omega}[l'[r_{2,j}]]]} \mu_{2,j} \right) = \left(\left(\mu_1 \circ_{[q_1]} Y_{t\omega} \right) \bigcirc_{[q_1] \cdot [\wp_{\omega}[l'[r_{2,j}]]]} \mu_{1,j} \right) \bigcirc_{[q_1] \cdot [\wp_{\omega}[l'[r_{2,j}]]]} \mu_{2,j}.$$

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