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Adaptive Domain Decomposition method for Saddle Point problem in Matrix Form

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Abstract

We introduce an adaptive domain decomposition (DD) method for solving saddle point problems defined as a block two by two matrix. The algorithm does not require any knowledge of the constrained space. We assume that all sub matrices are sparse and that the diagonal blocks are the sum of positive semi definite matrices. The latter assumption enables the design of adaptive coarse space for DD methods, see [5].

1 Introduction

Solving saddle point problems with parallel algorithms is very important for many branches of scientific computing: fluid (see e.g. [13]) and solid mechanics, computational electromagnetism, inverse problem and optimization.

We are interested in domain decomposition (DD) methods since they are naturally well-fitted to modern parallel architectures. For specific systems of partial differential equations with a saddle point formulation, efficient DD methods have been designed, see e.g. [16, 12, 17] and [20] references therein.

Here as in [14, 2, 4], we consider the problem in the form of a two by two block matrix. Let m and n be two integers with $m < n$. Let A $n \times n$ SPD matrix and B be a sparse $m \times n$ full rank matrix of constraints and C a $m \times m$ non negative matrix (in particular, $C = 0$ is allowed), we consider the following saddle point matrix:

$$\mathcal{A} := \begin{pmatrix} A & B^T \\ B & -C \end{pmatrix}. \quad (1)$$

When the kernel of matrix B is known, very efficient multigrid methods have been designed in the context of finite element methods, see e.g. [10, 9, 1, 18, 6].

Here we do not assume any knowledge on the kernel of matrix B . But in order to build a scalable method, we assume that all three matrices are sparse and that A and C are the sum of positive semi definite matrices. The latter assumption enables the design of adaptive coarse space for DD methods, see [5].

2 Schur complement preconditioning

The sparse $n \times n$ SPD matrix A is preconditioned by a two-level Schwarz type DD method :

$$M_A^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^N R_i^T (R_i A R_i^T)^{-1} R_i, \quad (2)$$

where R_0 is full rank $\dim(V_0) \times n$ where V_0 denotes the space spanned by the columns of R_0^T . The following assumptions are crucial to ensure the final method is scalable:

Assumption 2.1 (dimension and structure of the coarse space)

- *The coarse space dimension, $\dim(V_0)$, is $O(N)$ typically 10-20 times N .*
- *The coarse space is made of extensions by zero of local vectors.*

Using the GenEO method [19], it is possible to fix in advance two constants $0 < \lambda_m < 1 < \lambda_M$ and then build a coarse space V_0 such that A^{-1} is spectrally equivalent to M_A^{-1} :

$$\frac{1}{\lambda_M} M_A^{-1} \leq A^{-1} \leq \frac{1}{\lambda_m} M_A^{-1},$$

The coarse space V_0 is made extensions by zero of local generalized eigenvalue problems and its dimension is typically proportional to the number of subdomains. This corresponds to Assumption 2.1.

Our aim is first to precondition the Schur complement $-S$ of matrix \mathcal{A} (eq. (1)) where

$$S := C + B A^{-1} B^T, \quad (3)$$

via the spectrally equivalent preconditioning of the spectrally equivalent Schur complement M_S ,

$$M_S := C + B M_A^{-1} B^T. \quad (4)$$

This is done in § 3. Then in § 4.3 we will use M_S to precondition the saddle point matrix \mathcal{A} .

Note that M_S is by definition a sum of $N + 2$ positive semi definite matrices

$$M_S := B R_0^T (R_0 A R_0^T)^{-1} R_0 B^T + C + \sum_{i=1}^N B R_i^T (R_i A R_i^T)^{-1} R_i B^T. \quad (5)$$

Since B is a sparse matrix, it is interesting to introduce, for all $0 \leq i \leq N$, \tilde{R}_i the restriction operator on the support of $\mathfrak{S}(B R_i^T)$ so that $\tilde{R}_i^T \tilde{R}_i B R_i^T = B R_i^T$. Then by defining for $0 \leq i \leq N$,

$$\tilde{B}_i := \tilde{R}_i B R_i^T,$$

the operator M_S is rewritten as

$$M_S := \tilde{R}_0^T \tilde{B}_0 (R_0 A R_0^T)^{-1} \tilde{B}_0^T \tilde{R}_0 + C + \sum_{i=1}^N \tilde{R}_i^T \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T \tilde{R}_i. \quad (6)$$

We consider a partition of unity on $H := \mathbb{R}^m$ defined with local diagonal matrices $(\tilde{D}_i)_{1 \leq i \leq N} \in \mathbb{R}^{\dim(\text{Im}(B R_i^T)) \times \dim(\text{Im}(B R_i^T))}$:

$$\sum_{i=1}^N \tilde{R}_i^T \tilde{D}_i \tilde{R}_i = I_H.$$

Remark 2.1 *This partition of unity exists since*

$$B = \sum_{i=1}^N B R_i^T \tilde{D}_i R_i = \sum_{i=1}^N \tilde{R}_i^T \tilde{R}_i (B R_i^T \tilde{D}_i R_i)$$

is full rank.

3 Preconditioning of M_S

We make the following assumption

Assumption 3.1 *There exist symmetric positive semidefinite matrices $(\tilde{C}_i)_{1 \leq i \leq N}$ such that*

$$C = \sum_{i=1}^N \tilde{R}_i^T \tilde{C}_i \tilde{R}_i.$$

This assumption is not so restrictive. Indeed, if $C = 0$, it is automatically satisfied, this corresponds to a minimization problem with constraints enforced exactly without penalization nor relaxation. Moreover, we have:

Lemma 3.1 *If C is a diagonal matrix, Assumption 3.1 is satisfied.*

Proof If C is a diagonal matrix, it suffices to take

$$\tilde{C}_i := \tilde{R}_i C \tilde{R}_i^T \tilde{D}_i,$$

which is a diagonal non negative matrix. Indeed, we have then:

$$C \mathbf{P} = \sum_{i=1}^N C \tilde{R}_i^T \tilde{D}_i \tilde{R}_i \mathbf{P} = \sum_{i=1}^N \tilde{R}_i^T (\tilde{R}_i C \tilde{R}_i^T \tilde{D}_i) \tilde{R}_i \mathbf{P}.$$

■

Then, the operator M_S is the sum of a non local but low rank matrix S_0 :

$$S_0 := \tilde{R}_0^T \tilde{B}_0 (R_0 A R_0^T)^{-1} \tilde{B}_0^T \tilde{R}_0,$$

and of S_1 which is a sum of N local positive semi definite matrices:

$$S_1 := \sum_{i=1}^N \tilde{R}_i^T (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \tilde{R}_i,$$

that is

$$M_S = S_0 + S_1.$$

Note that we may assume that S_1 is invertible whereas it does not make sense for S_0 . By factorizing $R_0 A R_0^T = L_0 L_0^T \in \mathbb{R}^{\dim(V_0) \times \dim(V_0)}$, we have a Cholevsky factorization of S_0 :

$$S_0 = \tilde{R}_0^T \tilde{B}_0 L_0^{T-1} L_0^{-1} \tilde{B}_0^T \tilde{R}_0.$$

By Sherman-Morrison's technique, solving $M_S \mathbf{P} = \mathbf{G}$ for some $\mathbf{G} \in \mathbb{R}^m$, amounts to solving

$$\begin{pmatrix} S_1 & \tilde{R}_0^T \tilde{B}_0 L_0^{T-1} \\ L_0^{-1} \tilde{B}_0^T \tilde{R}_0 & -I \end{pmatrix} \begin{pmatrix} \mathbf{P} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{G} \\ \mathbf{0} \end{pmatrix},$$

with $\mathbf{y} := L_0^{-1} \tilde{B}_0^T \tilde{R}_0 \mathbf{P} \in \mathbb{R}^{\dim(V_0)}$. This is equivalent to solving its Schur complement:

$$A_0 \mathbf{y} := (I + L_0^{-1} \tilde{B}_0^T \tilde{R}_0 S_1^{-1} \tilde{R}_0^T \tilde{B}_0 L_0^{T-1}) \mathbf{y} = L_0^{-1} \tilde{B}_0^T \tilde{R}_0 S_1^{-1} \mathbf{G} \in \mathbb{R}^{\dim(V_0)}. \quad (7)$$

In order to solve this equation, we consider two options: a direct solver or a Krylov method.

As for the first method, it requires the computation of the entries of the matrix $A_0 \in \mathbb{R}^{\dim(V_0) \times \dim(V_0)}$ whose dimension is small. But computing its entries amounts to solve $\dim(V_0)$ linear systems with S_1 . Even if each solve is made scalable by a suitable parallel preconditioner M_{S_1} defined below in § 3.1, the whole thing has a $O(N)$ complexity and is thus not scalable.

The second option, using a Krylov method, is interesting since the matrix-vector product with A_0 is scalable using the preconditioner M_{S_1} when solving a linear system with operator S_1 . Now when solving (7) with a unpreconditioned Krylov method, convergence is in at most $\dim(V_0)$ iterations which is still not scalable. In order to build a spectrally equivalent preconditioner $M_{A_0}^{-1}$ to A_0 , we use again the preconditioner $M_{S_1}^{-1}$ of S_1 . and then use it in a preconditioned conjugate gradient algorithm to solve eq. (7), see § 3.2.

We consider next the construction of a preconditioner $M_{S_1}^{-1}$ to S_1 .

3.1 Equivalent preconditioner of S_1

3.1.1 One-level DD

As in [5] chapter 7, we begin with a one-level Neumann-Neumann type DD method defined in terms of the Fictitious Space Lemma (FSL) [15, 7]. This study will be the basis for constructing the two-level preconditioner. Recall

$$H := \mathbb{R}^m$$

and let a be the following bilinear form:

$$a : H \times H \rightarrow \mathbb{R} \quad a(\mathbf{P}, \mathbf{Q}) := (S_1 \mathbf{P}, \mathbf{Q}).$$

Let

$$H_D := \prod_{i=1}^N \mathbb{R}^{\text{rank}(\tilde{B}_i)},$$

and b be the following bilinear form:

$$b : H_D \times H_D \rightarrow \mathbb{R} \quad b(\mathcal{P}, \mathcal{Q}) := \sum_{i=1}^N ((\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \mathbf{P}_i, \mathbf{Q}_i).$$

We define \mathcal{R} :

$$\mathcal{R} : \quad H_D \rightarrow \quad H \\ (\mathbf{P}_i)_{1 \leq i \leq N} \mapsto \sum_{i=1}^N \tilde{R}_i^T \tilde{D}_i \mathbf{P}_i,$$

We now check the three assumptions of the FSL.

Surjectivity of \mathcal{R} For any $\mathbf{P} \in H$, we have:

$$\mathbf{P} = \sum_{i=1}^N \tilde{R}_i^T \tilde{D}_i \tilde{R}_i \mathbf{P},$$

so that

$$\mathbf{P} = \mathcal{R}((\tilde{R}_i \mathbf{P})_{1 \leq i \leq N}). \quad (8)$$

Continuity of \mathcal{R} On one hand, we have using k_0 the number of neighbours of a subdomain plus one, $k_0 := \max_{1 \leq i \leq N} \#\mathcal{O}(i)$ where $\mathcal{O}(i) := \{1 \leq j \leq N \mid \tilde{R}_i \tilde{D}_i S_1 \tilde{D}_j \tilde{R}_j^T \neq 0\}$:

$$\begin{aligned} a(\mathcal{R}(\mathcal{P}), \mathcal{R}(\mathcal{P})) &= \|(\sum_{i=1}^N \tilde{R}_i^T \tilde{D}_i \mathbf{P}_i)\|_a^2 \leq k_0 \sum_{i=1}^N \|\tilde{R}_i^T \tilde{D}_i \mathbf{P}_i\|_a^2 \\ &= k_0 \left(\left(\sum_{j \in \mathcal{O}(i)} \tilde{R}_j^T (\tilde{C}_j + \tilde{B}_j (R_j A R_j^T)^{-1} \tilde{B}_j^T) \tilde{R}_j \right) \tilde{R}_i^T \tilde{D}_i \mathbf{P}_i, \tilde{R}_i^T \tilde{D}_i \mathbf{P}_i \right). \end{aligned}$$

On the other hand, we have by definition:

$$b(\mathcal{P}, \mathcal{P}) = \sum_{i=1}^N \left((\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \mathbf{P}_i, \mathbf{P}_i \right).$$

We can take:

$$c_R := \max_{1 \leq i \leq N} \max_{P_i \in \mathbb{R}^{\text{rank}(\tilde{B}_i)}} \frac{(\sum_{j \in \mathcal{O}(i)} \tilde{R}_j \tilde{R}_j^T (\tilde{C}_j + \tilde{B}_j (R_j A R_j^T)^{-1} \tilde{B}_j^T) \tilde{R}_j \tilde{R}_i^T \tilde{D}_i \mathbf{P}_i, \tilde{D}_i \mathbf{P}_i)}{((\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \mathbf{P}_i, \mathbf{P}_i)}.$$

Stable decomposition Let $\mathbf{P} \in H$, we start from its decomposition (8) and estimate its b -norm

$$b(\mathcal{P}, \mathcal{P}) = \sum_{i=1}^N \left((\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \tilde{R}_i \mathbf{P}, \tilde{R}_i \mathbf{P} \right) = a(\mathbf{P}, \mathbf{P}),$$

so that we can take $c_T = 1$.

3.1.2 Two-level DD

In order to control the value of c_R defined above, we introduce a two-level preconditioner exactly as in [5], § 7.8.3, page 197. In our case, the generalized eigenvalue value problem in each subdomain $1 \leq i \leq N$ to be solved to build the coarse space reads:

$$\begin{aligned} \tilde{D}_i \left(\sum_{j \in \mathcal{O}(i)} \tilde{R}_j \tilde{R}_j^T (\tilde{C}_j + \tilde{B}_j (R_j A R_j^T)^{-1} \tilde{B}_j^T) \tilde{R}_j \tilde{R}_i^T \right) \tilde{D}_i \mathbf{P}_{ik} & \\ = \lambda_{ik} (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \mathbf{P}_{ik}. & \end{aligned} \quad (9)$$

It can be solved in $O(1)$ communications. The coarse space is defined as follows. Let τ_{S_1} be a user-defined threshold; for each subdomain $1 \leq i \leq N$, we introduce a subspace $W_i \subset \mathbb{R}^{\text{rank}(\tilde{B}_i)}$:

$$\tilde{W}_i := \text{Span}\{\mathbf{P}_{ik} \mid \lambda_{ik} > \tau_{S_1}\}. \quad (10)$$

Then the coarse space \tilde{W}_0 is defined by

$$\tilde{W}_0 := \bigoplus_{1 \leq i \leq N} \tilde{R}_i^T \tilde{D}_i \tilde{W}_i.$$

Let Z_{S_1} be a rectangular matrix whose columns span the coarse space \tilde{W}_0 . Let \tilde{P}_0 be the S_1 orthogonal projection from \mathbb{R}^m on \tilde{W}_0 whose formula is

$$\tilde{P}_0 = Z_{S_1} (Z_{S_1}^T S_1 Z_{S_1})^{-1} Z_{S_1}^T S_1. \quad (11)$$

Finally, the preconditioner for S_1 reads

$$\begin{aligned} M_{S_1}^{-1} &:= Z_{S_1} (Z_{S_1}^T S_1 Z_{S_1})^{-1} Z_{S_1}^T + (I - \tilde{P}_0) \\ &\times \left(\sum_{i=1}^N \tilde{R}_i^T \tilde{D}_i (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T)^\dagger \tilde{D}_i \tilde{R}_i \right) (I - \tilde{P}_0^T). \end{aligned} \quad (12)$$

Recall that we have for $\alpha := \max(1, \frac{k_0}{\tau_{S_1}})$:

$$\frac{1}{\alpha} M_{S_1}^{-1} \leq S^{-1} \leq M_{S_1}^{-1}.$$

Remark 3.1 *Solving a linear system with the Schur complement*

$$(\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T) \mathbf{P}_i = \mathbf{G}_i,$$

amounts to solving an augmented sparse system of the form

$$- \begin{pmatrix} R_i A R_i^T & \tilde{B}_i^T \\ \tilde{B}_i & -\tilde{C}_i \end{pmatrix} \begin{pmatrix} \mathbf{U}_i \\ \mathbf{P}_i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{G}_i \end{pmatrix}.$$

3.2 Preconditioner for A_0

Let us define

$$M_{A_0} := (I + L_0^{-1} \tilde{B}_0^T \tilde{R}_0 M_{S_1}^{-1} \tilde{R}_0^T \tilde{B}_0 L_0^{T-1}), \quad (13)$$

which is spectrally equivalent to matrix A_0 :

$$\frac{1}{\alpha} M_{A_0} \leq A_0 \leq M_{A_0}.$$

3.2.1 Scalability of the computation of M_{A_0}

Compared to A_0 , the computation of the entries of M_{A_0} is scalable since $\tilde{R}_0^T \tilde{B}_0 = B^T R_0$ has a sparse structure, see Assumption 2.1. Consider for instance the term corresponding to the coarse space in the formula for $M_{S_1}^{-1}$ (see eq. (12)):

$$L_0^{-1} \tilde{B}_0^T \tilde{R}_0 Z_{S_1} (Z_{S_1}^T S_1 Z_{S_1})^{-1} Z_{S_1}^T \tilde{R}_0^T \tilde{B}_0 L_0^{T-1}.$$

Matrix $Z_{S_1}^T \tilde{R}_0^T \in \mathbb{R}^{\dim(\tilde{W}_0) \times \dim(V_0)}$ is the product of two DD coarse spaces so that its computation is $O(1)$.

For the other computations, we need extra assumptions 3.2 and 3.3. Indeed, let's look at the sum over the subdomains in (12) and we isolate one subdomain $1 \leq i \leq N$:

$$L_0^{-1} \tilde{B}_0^T \tilde{R}_0 (I - \tilde{P}_0) \tilde{R}_i^T \tilde{D}_i (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T)^\dagger \tilde{D}_i \tilde{R}_i (I - \tilde{P}_0^T) \tilde{R}_0^T \tilde{B}_0 L_0^{T-1}.$$

This can be decomposed into a four term sum, the first term being:

$$L_0^{-1} \tilde{B}_0^T \tilde{R}_0 \tilde{R}_i^T \tilde{D}_i (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T)^\dagger \tilde{D}_i \tilde{R}_i \tilde{R}_0^T \tilde{B}_0 L_0^{T-1},$$

Assumption 3.2 We assume that, for all $1 \leq i \leq N$, the local matrix

$$\tilde{D}_i \tilde{R}_i \tilde{R}_0^T \tilde{B}_0 = \tilde{D}_i \tilde{R}_i B^T R_0$$

has $O(1)$ non zero columns.

The second term being:

$$-L_0^{-1} \tilde{B}_0^T \tilde{R}_0 \tilde{R}_i^T \tilde{D}_i (\tilde{C}_i + \tilde{B}_i (R_i A R_i^T)^{-1} \tilde{B}_i^T)^\dagger \tilde{D}_i \tilde{R}_i \tilde{P}_0^T \tilde{R}_0^T \tilde{B}_0 L_0^{T-1}.$$

We focus on the term

$$\tilde{D}_i \tilde{R}_i \tilde{P}_0^T \tilde{R}_0^T \tilde{B}_0 = \tilde{D}_i \tilde{R}_i S_1 Z_{S_1} (Z_{S_1}^T S_1 Z_{S_1})^{-1} Z_{S_1}^T B^T R_0.$$

Assumption 3.3 We assume that, for all $1 \leq i \leq N$, the local matrix $\tilde{D}_i \tilde{R}_i S_1 Z_{S_1}$ has $O(1)$ non zero columns

Using this assumption, the computation of second term is scalable. The two other terms can also be computed in a scalable way.

Note that, the resulting matrix M_{A_0} being of small size, it can be factorized by a direct method.

4 Recap

4.1 Setup for the Schur complement preconditioner M_S

We have a setup phase which is composed of three stages, the first two ones can be done concurrently:

1. Build the two-level preconditioner M_A^{-1} for A , see eq. (2),
2. Build the two-level preconditioner $M_{S_1}^{-1}$ for S_1 , see eq. (12),
3. Compute the entries of matrix M_{A_0} , see eq. (13) and factorize it.

4.2 Preconditioning of the Schur complement S by M_S^{-1}

Applying preconditioner M_S^{-1} is performed by solving $M_S \mathbf{P} = \mathbf{G} \in \mathbb{R}^m$ with several Krylov solves with spectrally equivalent preconditioners :

Algorithm 1 M_S^{-1} matvec product

INPUT: $\mathbf{G} \in \mathbb{R}^m$

1. Solve $S_1 \mathbf{G}' = \mathbf{G}$ by using the two-level preconditioner $M_{S_1}^{-1}$.

Compute the right handside $L_0^{-1} \tilde{B}_0^T \tilde{R}_0 \mathbf{G}'$ of eq. (7).

2. Solve eq. (7) for \mathbf{y} using the inverse of M_{A_0} as preconditioner and the matrix vector product with A_0 is made using a Krylov solver for S_1^{-1} preconditioned by $M_{S_1}^{-1}$.

3. Solve $S_1 \mathbf{P} = \mathbf{G} - \tilde{R}_0^T \tilde{B}_0 L_0^{T-1} \mathbf{y}$ for \mathbf{P} using the two-level preconditioner $M_{S_1}^{-1}$

OUTPUT: $\mathbf{P} = M_S^{-1} \mathbf{G}$

4.3 DD solver for the saddle point system

We now consider the solving of the saddle point problem:

$$\begin{pmatrix} A & B^T \\ B & -C \end{pmatrix} \begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix} = \begin{pmatrix} \mathbf{F}_U \\ \mathbf{F}_P \end{pmatrix}. \quad (14)$$

The following three factor factorization, see e.g. [3]:

$$\begin{pmatrix} A & B^T \\ B & 0 \end{pmatrix} = \begin{pmatrix} I & 0 \\ BA^{-1} & I \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & -(C + BA^{-1}B^T) \end{pmatrix} \begin{pmatrix} I & A^{-1}B^T \\ 0 & I \end{pmatrix},$$

shows that solving eq. (14) can be performed by solving sequentially three linear systems, two with A and one with $C + BA^{-1}B^T$. This leads to the following algorithm:

Algorithm 2 DD saddle point solver

INPUT: $\begin{pmatrix} \mathbf{F}_U \\ \mathbf{F}_P \end{pmatrix} \in \mathbb{R}^{n+m}$

1. Solve $A \mathbf{G}_U = \mathbf{F}_U$ by a PCG with M_A^{-1} as a preconditioner

Compute $\mathbf{G}_P := \mathbf{F}_P - B \mathbf{G}_U$

2. Solve $(C + BA^{-1}B^T) \mathbf{P} = -\mathbf{G}_P$ by a PCG with M_S^{-1} as a preconditioner, see Algorithm 1.

Compute $\mathbf{G}_U := \mathbf{F}_U - B^T \mathbf{P}$

3. Solve $A \mathbf{U} = \mathbf{G}_U$ by a PCG with M_A^{-1} as a preconditioner

OUTPUT: $\begin{pmatrix} \mathbf{U} \\ \mathbf{P} \end{pmatrix}$ the solution to (14).

5 Variants

In § (2), we start with a two level additive Schwarz method (ASM) eq. (2) as a preconditioner for matrix A in (1). Another possibility is to start from a balancing Neumann-Neumann (BNN) or more generally a SORAS [8] or BDD-H [11] type method:

$$M_{ASORAS}^{-1} := R_0^T (R_0 A R_0^T)^{-1} R_0 + \sum_{i=1}^N R_i^T A_i^R ob^{-1} R_i, \quad (15)$$

where for each subdomain $1 \leq i \leq N$, $A_i^R ob$ is a local Neumann matrix (BNN algorithm) or an arbitrary invertible matrix. In order to build the preconditioners for matrix S and then \mathcal{A} , it is sufficient to replace matrices $R_i A R_i^T$ with $A_i^R ob$ in the above sections.

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