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| Arnaud Le Ny. Gibbs Measures for Long-Range Ising Models. 2019. hal-02343596

**HAL Id: hal-02343596**

**<https://hal.science/hal-02343596>**

Preprint submitted on 4 Nov 2019

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# Gibbs Measures for Long-Range Ising Models

Arnaud Le Ny \*

November 4, 2019

**Abstract:** This review-type paper is based on a talk given at the conference *États de la Recherche en Mécanique statistique*, which took place at IHP in Paris (December 10–14, 2018). We revisit old results from the 80’s about one dimensional long-range polynomially decaying Ising models (often called *Dyson models* in dimension one) and describe more recent results about interface fluctuations and interface states in dimensions one and two.

Based on a series of joint works with R. Bissacot, L. Coquille, E. Endo, A. van Enter, B. Kimura and W. Ruszel [42, 10, 25, 8, 39, 43].

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*AMS 2000 subject classification:* Primary- 60K35 ; secondary- 82B20.

*Keywords and phrases:* Long-range Ising models, phase coexistence, Gibbs *vs.* *g*-measures.

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# 1 Introduction

Gibbs measures for spin systems are probability measures defined on infinite product probability spaces of configurations of spins with values  $\pm 1$  attached, in our context, to each site of a lattices  $\mathbb{Z}^d$ , for  $d = 1, 2, 3$  in these notes. They are designed to represent equilibrium states in mathematical statistical mechanics, according to the  $2^d$  law of thermodynamics, in the aim of modelling phase transitions and extending Markov chains in a spatial context.

To avoid uniqueness of probability measures as usually got by the standard Kolmogorov construction in terms of consistent families of marginals at finite volumes, we consider them within the *DLR framework*, named after the independent constructions of Dobrushin on one hand [29], Lanford/Ruelle on the other hand [92], who introduced in the late 60's consistent systems of conditional probabilities w.r.t. the outside of finite volumes. With such use of conditional probabilities and boundary conditions, it appeared indeed possible to get different probability measures – thus different global behaviours – for the same local rules, provided by Gibbs specifications whose task is to specify the local conditional probabilities with boundary conditions prescribed outside finite sets.

The basic example of such spin systems is given by the standard Ising model, a famous Markov field with a specification given by the standard Boltzmann-Gibbs weights of the form  $e^{-\beta H}$  in order to get equilibrium states saturating a variational principle by solving an Entropy-Energy conflict. To get phase transitions, dimension is important, and phase transition in dimension 2 was presented in 1936 by Peierls, followed all over the 20th century by very rich studies on the structure of the convex set of Gibbs measures. In the early 70's, Dobrushin described an even richer structure in higher dimension, with the occurrence of rigid interface states in dimension 3, physically stable but non-translation invariant, called Dobrushin states.

In order to obtain phase transition in dimension 1, Kac/Thompson and Dyson have studied, also in the late 60's, infinite range versions of the Ising model, with long-range pair-potentials with polynomial decay leading to phase transition for very slow decays. These probability measures have recently been used to detect interesting phenomenon in dimension 1, and the extension of such models in dimension 2 for very slow decays had also been recently studied with the hope of interesting interface behaviours not detected in the past.

In these notes, we first describe in Section 2 the DLR framework and standard nearest-neighbours Ising models in dimensions  $d = 1, 2, 3$ . In Section 3 we focus on long-range Ising models in dimension 1 (*Dyson models*), and in Section 4 we describe recent results for long-range models in dimension 2.

## 2 DLR description of phase transitions – Ising models on $\mathbb{Z}^d$

We consider *Ising spins on  $d$ -regular lattices*, i.e. random variables  $\sigma_x, \omega_y$ , etc. attached at each sites  $x, y$ , etc. of  $S = \mathbb{Z}^d$  ( $d = 1, 2, 3$ ), and taking values in the single-spin state-space  $E = \{-1, +1\}$ . The latter is equipped with the discrete topology and with the discrete measurable structure, with an *a priori* probability counting measure  $\rho_0 = \frac{1}{2}(\delta_{-1} + \delta_{+1})$  and the power set  $\mathcal{E} = \mathcal{P}(E)$  as  $\sigma$ -algebra. We denote by  $\mathcal{S}$  the set of all the finite subsets of  $S$ , and sometimes write  $\Lambda \Subset \mathbb{Z}^d$  to denote such a set  $\Lambda \in \mathcal{S}$ .

Configurations  $\sigma = (\sigma_x)_{x \in S}, \omega = (\omega_x)_x$ , etc. belong to the *Configuration space*  $\Omega$ ,

$$(\Omega, \mathcal{F}, \rho) := (E^S, \mathcal{E}^{\otimes S}, \rho_0^{\otimes S})$$

equipped with the product topology and measurable structure. For either finite or infinite volumes  $\Delta \subset S$ , corresponding product spaces will be denoted  $(\Omega_\Delta, \mathcal{F}_\Delta, \rho_\Delta)$ . We also denote by  $\mathcal{M}_1(\Omega)$  the set of probability measures on them and use the subscript *inv* for the restriction to translation-invariant elements in an obvious sense (see [41]). The set of continuous functions, denoted by  $C(\Omega)$ , coincide with the set  $\mathcal{F}_{\text{qloc}}$  of quasilocal functions.

On  $(\Omega, \mathcal{F})$ , we consider *Ferromagnetic pair potentials*  $\Phi = \Phi^J$  with coupling functions

$$J = (J_{xy})_{x,y \in S}, \quad J_{xy} \geq 0$$

that are families of local functions  $(\Phi_A^J)_{A \in S}$  with  $\Phi_A^J = 0$  unless  $A = \{x, y\}$  for any pair  $\{x, y\} \subset S$ , in which case for any configuration  $\sigma \in \Omega$  on has:

$$\Phi_{\{x,y\}}^J(\sigma) = J_{xy} \sigma_x \sigma_y \quad (2.1)$$

We shall now forget the superscript  $J$ , and focus on two main types of coupling functions  $J$ :

- **Classical (homogeneous) *n.n.* Ising model:**

The interaction  $\Phi_A$  is non-null only for pairs  $A = \{x, y\}$  of nearest-neighbours (*n.n.*), also sometimes briefly written  $A = \langle xy \rangle$ , with couplings  $J = J^{n.n.}$  given by

$$J_{xy} = J^{n.n.} := J \mathbf{1}_{|x-y|=1}, \quad J > 0, \quad \text{for all } \{x, y\} \subset \mathbb{Z}^d$$

- **Long-range ferromagnetic Ising models with polynomial decay  $\alpha > d$ :**

$$J_{xy} = J_{xy}^\alpha := \frac{J}{|x-y|^\alpha}, \quad J > 0, \quad \text{for all } \{x, y\} \subset \mathbb{Z}^d \quad (2.2)$$

where  $|\cdot|$  denotes a canonical norm on  $\mathbb{Z}^d$ , with  $\alpha > d$  so that the couplings are summable:

$$\forall x \in \mathbb{Z}^d, \quad \sum_{y \in \mathbb{Z}^d} |J_{xy}^\alpha| < \infty$$

Given a finite volume  $\Lambda$  in  $\mathbb{Z}^d$ , for a prescribed boundary condition (*b.c.*)  $\omega_{\Lambda^c} \in \Omega_{\Lambda^c} = \{-1, 1\}^{\Lambda^c}$ , we define Hamiltonians on  $\Omega$  for any  $\sigma \in \Omega$  to be<sup>1</sup> the uniformly convergent series

$$H_\Lambda^\omega(\sigma_\Lambda) = - \sum_{\substack{x,y \in \Lambda \\ x \neq y}} J_{xy} \sigma_x \sigma_y - \sum_{\substack{x \in \Lambda \\ y \in \Lambda^c}} J_{xy} \sigma_x \omega_y \quad (2.3)$$

For a fixed inverse temperature  $\beta > 0$ , the *Gibbs specification* is determined by a family of probability kernels  $\gamma = (\gamma_\Lambda)_{\Lambda \in S}$  defined on  $\Omega_\Lambda \times \mathcal{F}_{\Lambda^c}$  by the Boltzmann-Gibbs weights

$$\gamma_\Lambda(\sigma|\omega) = \frac{1}{Z_\Lambda^\omega} e^{-\beta H_\Lambda^\omega(\sigma_\Lambda)} \quad (2.4)$$

where  $Z_\Lambda^\omega = \sum_{\sigma \in \Omega_\Lambda} e^{-\beta H_\Lambda^\omega(\sigma_\Lambda)}$  is the *partition function*, related to free energy.

---

<sup>1</sup>We also sometimes consider 'magnetic fields', either homogeneous ( $h \in \mathbb{R}$ ), inhomogeneous ( $h = (h_x)_{x \in S}$ ), or random ( $h = (h_x[\eta])_{x \in S}$  for random variables  $\eta$ 's playing the role of disorder in the *Random Field Ising Model*). The formal Hamiltonian reads then  $H_\Lambda[\eta] = - \sum_{x,y \in S} J_{xy} \sigma_x \sigma_y - \sum_{x \in S} h_x[\eta] \sigma_x$ .

**Remark 1** Due to this Boltzmann-Gibbs form (2.4), finite-volume Gibbs measures at temperature  $T = \beta^{-1} > 0$  are designed to maximize Entropy minus Energy, satisfying a variational principle in concordance with the 2<sup>d</sup> principle of thermodynamics. After some work, infinite-volume Gibbs measures are also shown to represent equilibrium states at infinite volume : they are the one(s) who minimize(s) free energy " $F = U - TS$ ", or equivalently the one(s) that, at a fixed 'energy', maximize(s) 'entropy'. We do not describe this variational approach in these notes, although it justifies the heuristics behind Entropy-Energy arguments used in the low temperature proofs of phase transitions within Peierls or Pirogov-Sinai strategies [110, 13, 41, 113, 97].

Within this DLR Framework, a Gibbs measure  $\mu$  is then defined to be a probability measure on  $\mathcal{M}_1(\Omega)$  whose conditional probabilities with boundary condition  $\omega$  outside  $\Lambda$ , are of the form of the kernels  $\gamma_\Lambda(\cdot|\omega)$  and thus satisfy the DLR equations:

$$\mu\gamma_\Lambda = \mu, \text{ for all } \Lambda \in \mathbb{Z} \quad (2.5)$$

Alternatively, DLR equations (2.5) satisfied by Gibbs measures  $\mu$  for a specification  $\gamma$  read

$$\mu(\cdot) = \int \frac{1}{Z_\Lambda^\omega} e^{-\beta H_\Lambda(\cdot_\Lambda \omega_{\Lambda^c})} d\mu(\omega) \quad (2.6)$$

Equation (2.6) is the starting point of the extremal decompositions of Gibbs measures leading to the Choquet simplex structure of sets of Gibbs measures (see below).

DLR equations (2.6) and (2.5) also mean that, for a subset  $\Lambda$  finite, regular versions of conditional probabilities of  $\mu$  w.r.t.  $\mathcal{F}_{\Lambda^c}$  should satisfy

$$\mu[\cdot | \mathcal{F}_{\Lambda^c}][\omega] = \frac{1}{Z_\Lambda^\omega} e^{-\beta H_\Lambda^\omega(\cdot_\Lambda)}, \mu\text{-a.s.}(\omega)$$

We denote by  $\mathcal{G}(\gamma)$  the set of Gibbs measures, and  $\mathcal{G}_{\text{inv}}(\gamma)$  for translation-invariant ones.

Existence of Gibbs measures ( $\mathcal{G}(\gamma) \neq \emptyset$ ) is insured by our compact finite-state space framework, and more generally from the existence of continuous versions of conditional probabilities (equivalent to *Quasilocality*, see [41, 60, 97] or Section 3.2).

In Equilibrium statistical mechanics, one is more often interested in multiplicity of Gibbs measures, called *phase transition* when  $|\mathcal{G}| > 1$ . In next subsection, we describe the fundamental case of classical Ising models where two different *phases* exist at low temperature, because then entropic effects cannot perturbate enough energetic minimizers.

In such cases, a general result<sup>2</sup> on DLR measures is the following:

**Theorem 1** *The set  $\mathcal{G}(\gamma)$  of DLR measures for a given specification  $\gamma$  is a convex subset of  $\mathcal{M}_1^+(\Omega)$  whose extreme boundary is denoted  $\text{ex}\mathcal{G}(\gamma)$ , and satisfies the following properties:*

1. *The extreme elements of  $\mathcal{G}(\gamma)$  are the probability measures  $\mu \in \mathcal{G}(\gamma)$  that are trivial on the tail  $\sigma$ -field  $\mathcal{F}_\infty := \bigcap_{\Lambda \in \mathcal{S}} \mathcal{F}_{\Lambda^c}$ :*

$$\text{ex}\mathcal{G}(\gamma) = \left\{ \mu \in \mathcal{G}(\gamma) : \mu(B) = 0 \text{ or } 1, \forall B \in \mathcal{F}_\infty \right\} \quad (2.7)$$

---

<sup>2</sup>There is a proof of this result avoiding Krein-Milman Theorem abstract theorem. It can be made within a similar scheme as the ergodic decomposition theorem or de Finetti's description of exchangeable measures, following a general demonstration of Dynkin, see [34, 60, 97].

Moreover, distinct extreme elements  $\mu, \nu \in \text{ex}\mathcal{G}(\gamma)$  are mutually singular:  $\exists B \in \mathcal{F}_\infty$ ,  $\mu(B) = 1$  and  $\nu(B) = 0$ , and more generally, each  $\mu \in \mathcal{G}(\gamma)$  is uniquely determined within  $\mathcal{G}(\gamma)$  by its restriction to  $\mathcal{F}_\infty$

2.  $\mathcal{G}(\gamma)$  is a Choquet simplex: Any  $\mu \in \mathcal{G}(\gamma)$  can be written in a unique way as

$$\mu = \int_{\text{ex}\mathcal{G}(\gamma)} \nu \cdot \alpha_\nu(d\nu) \quad (2.8)$$

where  $\alpha_\mu \in \mathcal{M}_1^+(\text{ex}\mathcal{G}(\gamma), e(\text{ex}\mathcal{G}(\gamma)))$  is defined for all  $M \in e(\text{ex}\mathcal{G}(\gamma))$  by (2.9) below.

The weights  $\alpha_\mu(M)$  are associated with any measurable subset of measures  $M \in e(\text{ex}\mathcal{G}(\gamma))$ , the  $\sigma$ -algebra of evaluation maps on spaces of measures [54, 60, 97]. They represent the relative weights of typical configurations of the extremal Gibbs measures in the mixture (2.8),

$$\alpha_\mu(M) = \mu \left[ \left\{ \omega \in \Omega : \exists \nu \in M, \lim_n \gamma_{\Lambda_n}(C|\omega) = \nu(C), \forall C \in \mathcal{C} \right\} \right] \quad (2.9)$$

Extremal Gibbs measures are sometimes called *States* or *Phases*, while *Pure states* concern translation-invariant extremal Gibbs measures, such as the +- or --states got by weak limits with homogeneous all +- or all --boundary conditions in our ferromagnetic spin systems.

We emphasize that even concerning *n.n.* homogeneous Ising models, there can be infinitely many's non-transition-invariant extremal Gibbs measures entering in the extremal decomposition (2.8), for *e.g.*  $d = 3$  or on Cayley trees [60]. We describe the case of anisotropic long-range models in dimension 2, for which this holds for slow decays of the interaction (Section 4.2).

Except in one occasion (for some decays  $\alpha \in (3, 4)$ , see Section 3), we shall consider mostly **ferromagnetic** couplings *i.e.*  $J \geq 0$ . In particular, we enjoy *FKG and monotonicity preserving properties*, among other reasons because they yield the existence of two extremal infinite-volume Gibbs measures as weak limits of the all -- or all +-boundary conditions:

$$\mu^-(\cdot) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \gamma_\Lambda^{\beta\Phi}(\cdot|+) \text{ and } \mu^+(\cdot) := \lim_{\Lambda \uparrow \mathbb{Z}^d} \gamma_\Lambda^{\beta\Phi}(\cdot|+) \quad (2.10)$$

In this framework, uniqueness is insured by  $\mu^- = \mu^+$ , while phase transition is got by proving that  $\mu^- \neq \mu^+$ . Moreover, for any other Gibbs measure  $\mu \in \mathcal{G}(\gamma)$ , the following stochastic domination inequalities hold:

$$\mu^- \leq \mu \leq \mu^+$$

Here we use the *FKG order* ' $\leq$ ', meaning that the bounds are valid for expectations of increasing functions. We shall sometimes write such expectations  $\langle \cdot \rangle^+$ ,  $\langle \cdot \rangle^-$ ,  $\langle \cdot \rangle^0$ , and  $\langle \cdot \rangle^\pm$ , for respectively the all +-, all --, free, and  $\pm$ -Dobrushin boundary condition", or  $\langle \cdot \rangle^\omega$  for general b.c.  $\omega$ . We add a subscript  $\Lambda$ , or sometimes  $L$ , and write  $\langle \cdot \rangle_L$  for the finite-volume versions on square boxes  $\Lambda = \Lambda_L = ([-L, +L] \cap \mathbb{Z})^d$ , and also  $\mathbb{P}_L$  for the corresponding probabilities.

In the particular case of *n.n.* Ising models, for which the precise results that  $\mathcal{G}(\gamma) = [\mu^-, \mu^+]$  in  $2d$  but not in  $3d$  have been established in the seventies at low temperature for *n.n.* Ising models, culminating with the independent results of Aizenman or Higuchi around 1980 [2, 70]. In Section 4, we provide hints to prove that the absence of translation-invariant extremal states other than  $\mu^-$  and  $\mu^+$  is also true for long-range polynomially decaying potentials in  $2d$  (at least for fast decays  $\alpha > 3$ ) and provide partial results from [25] for very long range models with slow decays  $\alpha \in (2, 3)$ .

As another general result for  $n.n$  models, let us quote the explicit values of the *magnetization*  $\mu[\sigma_0]$  in  $2d$  by Onsager ([102], 1944) and the result of uniqueness in homogeneous fields by Lee and Yang ([96], 1952). For a complete, rigorous and didactic presentation of this classical Ising model, one should really read the book of Friedli/Velenik ([54], Chap. 3).

## 2.1 Phase transition *vs.* uniqueness results in the classical $n.n.$ case

- **Uniqueness in dimension  $d = 1$**

For  $n.n.$  Ising models and more generally *finite-range random fields in one dimension*, uniqueness is well known due to existence and uniqueness results of the invariant measure of irreducible Markov chains, see *e.g.* Chapter 3 of [60]. Called in generality *Markov random fields*, they are indeed also *reversible Markov chains* and there is a one to one correspondance : one says that *Global and local Markov properties are equivalent*. This is not always the case, as seen in *e.g.* [52, 65, 119] or to some extend for long range models, see Section 2.3. and [10].

Heuristically, writing the free energy under the form " $F = U - TS$ ", one has two so-called *Ground states* (minimizers of the Hamiltonian), the all  $+$ - and the all  $-$ -*configurations*. Inserting a droplet of defects in one of this phase would have a constant, volume-independent, energetic cost. It is thus always beaten by entropy in the thermodynamic limit, at any positive temperature. Thus, at any temperature, only a unique disorder phase appears.

For a more rigorous presentation of such Entropy *vs.* Energy arguments, see [110, 13, 113].

- **Phase transition at low temperature for  $d = 2$**

We shall briefly sketch the standard argument of Peierls to prove phase transition for the  $2d$  Ising model at low temperature, but state first a more general result. For the full convex picture at any temperature, with only two translation-invariant extremal Gibbs measures see [2, 70, 60] or the discussion in [25].

**Theorem 2** *Consider  $\gamma$  to be the specification (2.4) of the  $n.n.$  Ising model in dimension 2. Then there exists a critical inverse temperature  $0 < \beta_c < +\infty$  such that*

- $\mathcal{G}(\gamma) = \{\mu\}$  for all  $\beta < \beta_c$ .
- $\mathcal{G}(\gamma) = [\mu^-, \mu^+]$  for all  $\beta > \beta_c$  where the extremal phases  $\mu^- \neq \mu^+$  can be selected via " $-$ "-or " $+$ "-boundary conditions: for all  $f \in \mathcal{F}_{\text{loc}}$ ,

$$\mu^- [f] := \lim_{\Lambda \uparrow \mathcal{S}} \gamma_\Lambda [f \mid -] \text{ and } \mu^+ [f] := \lim_{\Lambda \uparrow \mathcal{S}} \gamma_\Lambda [f \mid +]. \quad (2.11)$$

Moreover, for any  $\mu \in \mathcal{G}(\gamma)$ , for any bounded increasing  $f$ ,  $\mu^- [f] \leq \mu [f] \leq \mu^+ [f]$ , and the extremal phases have opposite magnetizations<sup>3</sup>  $m^*(\beta) := \mu^+ [\sigma_0] = -\mu^- [\sigma_0] > 0$ .

Let us sketch now the *Peierls argument* ([103], 1936), also derived/discovered by Griffiths ([67] (1964)) or Dobrushin ([28], 1965). This geometrical approach uses the interface between different spin values  $+$ 's and  $-$ 's to define the Hamiltonian in terms of closed circuits called

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<sup>3</sup>We denote  $\mu[f]$  for the expectation  $\mathbb{E}_\mu[f]$ .

There are closed boundaries and open ones, the latter starting and ending on the edges of the array. We show that at sufficiently low temperatures the area enclosed by closed boundaries and cut off by open ones is only a small fraction of the total area. Hence the majority of the magnets must be of equal sign and the model corresponds to our definition of a ferromagnet. We give an upper limit for the number of boundary lines passing through any one point, assuming statistical equilibrium. Each boundary that starts at a given point may be described as a succession of steps of equal length, of which each may have the direction  $+x$ ,  $-x$ ,  $+y$  or  $-y$ . Since each element of the

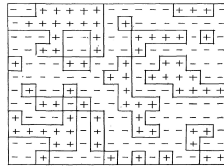


Fig. 1. Example of boundary lines.

boundary separates two magnets of opposite sign, the boundary has an energy  $2\epsilon l$ , if  $l$  is its length, in units of the distance between neighbouring magnets. However, not every succession of steps represents a possible boundary. Boundaries are restricted by the conditions that

- (1) A boundary must either return to its starting point, or end on an edge.
- (2) It passes through none of its points more than once.
- (3) Not more than two boundaries can pass through any one point and, if there are two, the two steps adjoining the point on one boundary must be different from those on the other.
- (4) No part of a boundary can lie outside the array of magnets.

Figure 1: Original Peierls contours

*Contours*, to get temperature-dependent bounds on the energy of configurations, that eventually leads to phase transition at low temperature by rigorous entropy *vs.* energy arguments. For a complete presentation of the argument, one could *e.g.* consult [100].

Consider a finite volume  $\Lambda \in \mathcal{S}$ , start with the boundary condition  $+$  and take the probability measure  $\gamma_\Lambda(\cdot | +)$ . A *path* in  $\mathbb{Z}^2$  is a finite sequence  $\pi = \{i_1, \dots, i_n\}$  of sites such that  $i_j$  and  $i_k$  are *n.n.* ( $|j - k| = 1$ ). We call dual of  $\mathbb{Z}^2$  the set  $\mathbb{Z}^2 + (\frac{1}{2}, \frac{1}{2})$  and define a *contour*  $\gamma = (r_1, \dots, r_n)$ , of length  $|\gamma| = n \in \mathbb{N}^*$ , to be a sequence of points in the dual such that  $(r_j, \dots, r_n, r_1, \dots, r_{j-1})$  is a path for all  $j = 1, \dots, n$ . A *contour*  $\gamma$  of the dual of  $\mathbb{Z}^2$  is said to *occur* in the configuration  $\sigma$ , or simply to be a *contour of*  $\sigma$ , if it separates clusters of  $+$ 's or  $-$ 's in  $\omega$ , i.e. if  $\gamma \subset \{b + (\frac{1}{2}, \frac{1}{2}) : b = \{i, j\}, \|i - j\|_1 = 1, \omega_i \neq \omega_j\}$ . To relate Hamiltonians with contours, one observe that the event  $\gamma$  that a contour occurs requires an energy proportional to its length  $|\gamma|$  (*i.e.* the perimeter of the droplet), so that if  $\beta$  is large long contours will be very improbable w.r.t. the probability  $\gamma_\Lambda^{\beta\Phi}(\cdot | +)$ . Thus, one relates the energy of a contour with its length to get the following *Peierls's estimate*:

$$\gamma_\Lambda^{\beta\Phi}(\gamma | +) \leq e^{-2\beta|\gamma|}$$

From this, thanks to an entropic bound counting the number of contours of a given length, it is possible to estimate the probability that the spin at the origin takes value  $-1$ , an event which implies the occurrence of contours, by

$$\gamma_\Lambda^{\beta\Phi}(\sigma_0 = - | +) \leq \sum_{l \geq 1} l 3^l e^{-2\beta l} \tag{2.12}$$

Using (2.10), this yields the weak convergence as  $\beta$  goes to infinity of  $\mu_\beta^+$  to the Dirac measure  $\delta_+$ , while the  $-$ -phase can be similarly proved to converge to the Dirac measure  $\delta_-$ .

• **Phase transition and Dobrushin states in  $d = 3$**



In our ferromagnetic models, phase transitions at higher dimensions are implied by those of lower dimensions, by stochastic domination. In particular, for such models, the critical temperature in dimension 3 is at least the one in dimension 2 :  $T_c(d = 2) < T_c(d = 3)$ .

Nevertheless, there could be intermediate ranges of temperature where the phase diagram could coincide or not with the  $2d$ -picture : either there are *only (2) t.i. extremal Gibbs measures and no non-translation-invariant extremal Gibbs measures*, either there are *(at least countably) many's non-t.i. extremal Gibbs measures*<sup>4</sup>.

The  $3d$ -picture, where there are indeed countably many's non-translation-invariant extremal Gibbs measures, have been first described by Dobrushin in 1972 [30]. The original idea is to use the mixed so-called  $\pm$ - *Dobrushin b.c.* (located at the origin), defined such that

$$\forall x = (x_1, x_2, x_3) \in \mathbb{Z}^3, \quad \begin{aligned} \pm_x &= +1 && \text{if } x_1 \geq 0 \\ \pm_x &= -1 && \text{otherwise} \end{aligned}$$

and to prove that the corresponding limiting Gibbs states  $\mu^\pm$  cannot be translation-invariant as soon as there is phase transition in  $2d$ , so for temperatures  $T \leq T_c(2) < T_c(3)$ . Such a temperature, where some extremal states cease to be translation-invariant, is called the *Roughening temperature* (see e.g. [13, 12, 54]). The infinite-volume limit  $\mu^\pm$  would exhibit more coexistence near this plane, and more +’s or –’s, further up or down from it. This yields a *non-translation-invariant extremal states*  $\mu^\pm$ , which thus cannot be a convex mixture of the other extremal states  $\mu^+$  and  $\mu^-$ , so in particular

$$\mu^\pm \neq \frac{1}{2}\mu^- + \frac{1}{2}\mu^+$$

As a consequence, the microscopic interface separating the +’s and –’s would not fluctuate much when the volume increases, and stay located near the original plane : one says that this interface is *rigid*. This construction could be done for any horizontal plane  $\pi : x = h$ , or even more any plane in  $\mathbb{Z}^3$ , and thus a countable family of different ‘Dobrushin’ b.c., so that one gets at least countably many’s non-translation-invariant extremal Gibbs measures  $\mu_\pi^\pm$ . In Section 4, we detail a bit more the proof van Beijeren provided afterwards ([6], 1976) in the case of some *anisotropic long-range Ising models on  $\mathbb{Z}^2$* .

As we shall see now, this rigidity does not hold for  $d \leq 2$  for *n.n.* Ising models [60, 2, 70], nor for long-range models in  $d = 1$  [47, 60], neither for anisotropic long-range Ising models in  $d = 2$ , where Gibbs measures got by Dobrushin b.c. are not Dobrushin *states* : they are either non-extremal, either non-translation-invariant as shown in [25], see Section 4. For more general results on translation-invariant extremal Gibbs measures for finite-range Ising model, see [11, 108].

Note that in case of rigidity, this mixed  $\mu^\pm$ -states give rise to many peculiar measures, such as some local but non global Markov measure [52, 65, 75, 119] or some Gibbs measure which is not the limit of any finite measures with b.c. [24].

## 2.2 Fluctuations and rigidity of interfaces in the *n.n.* cases ( $d = 2, 3$ )

The absence of non-translation-invariant extremal Gibbs measures for the Ising models has been a long-standing case of studies in the seventies. While Dobrushin was formalizing the  $3d$ -picture, Gallavotti studied the asymptotic behavior of the microscopic interface separating the

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<sup>4</sup>Dobrushin states, got by weak limit with mixed Dobrushin b.c. centered at any plane  $\pi$  are known to exist in  $3d$ , but it is open whether there do exist other non t.i. extremal Gibbs measures.

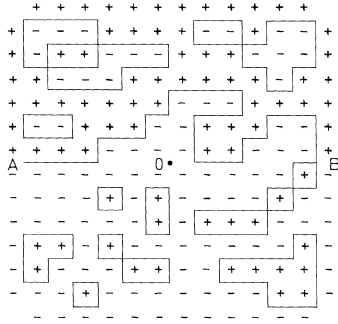


Fig. 2.

Figure 2: Microscopic interface : Gallavotti line

two phases in the  $2d$  case ([59], 1972). In particular, starting from a square of basis  $L$  growing to the whole space  $\mathbb{Z}^2$ , he proved that with high probability, the interface will fluctuate at distance  $\sqrt{L}$ , either up, or either down with equiprobability. This has also been formalized in [101] who combined these results with correlation inequalities [93, 94, 69, 95] to eventually get a non-extremal but translation-invariant Gibbs measure

$$\mu^\pm = \frac{1}{2}\mu^- + \frac{1}{2}\mu^+ \quad (2.13)$$

On the contrary to dimension 3 where the fluctuations of the interface remain bounded [30], these fluctuations have been afterwards shown to have a *Gaussian profile* by Abraham/Reed [1, 1976), fluctuating indeed as  $\sqrt{L}$  for a box  $\Lambda$  of basis  $L$ . With such boundary conditions, this interface will eventually fluctuate up or down, with probability  $\frac{1}{2}$  each, at a ballistic speed. These properties have been extended to many other mixed non translation-invariant boundary conditions one can imagine (See [101] and references therein). The difficulty afterwards was to be able to prove even for boundary conditions one could not imagine, and even for Gibbs measures that could arise without any boundary condition.

The studies eventually culminate by the works of Aizenman [2] or Higuchi [70], excluding translation-invariant extremal states other than  $\{\mu^-, \mu^+\}$  by percolation methods based on a previous work of Russo [112]. This eventually leads to the full convex picture  $\mathcal{G}(\gamma) = [\mu^-, \mu^+]$  so that the convex decomposition (2.8) reduces to :

$$\forall \mu \in \mathcal{G}(\gamma) = \alpha_\mu^+ \mu^+ + (1 - \alpha_\mu^+) \mu^-$$

where the weights  $\alpha_\mu^+$  are given by (2.9).

The behavior of this interface in  $2d$  has been refined up to the critical point, see the more precise results by Higuchi [71, 72], Greenberg/Ioffe [66] or other investigations of Bricmont *et al.* [12, 14, 105]. See also results got by percolation approach by Gielis/Grimmett [63] or more recently by Cerf/Zhou [22].

### 3 Long-range Ising models in dimension one (*Dyson models*)

We briefly describe the history of these long-range models since its modern introduction in 1969 by Kac and Thompson, where phase transition for decays  $1 < \alpha \leq 2$  was conjectured

[80]. The ( $1 < \alpha < 2$ )-cases were solved by Dyson at the same time using a bound on the magnetization with this of a hierarchical model [35]. He extends its results and partially solved the borderline case  $\alpha = 2$  in 1971 [36], while the complete proof of this rich hybrid case was provided in 1982 by Fröhlich/Spencer [57], with afterwards many peculiar properties that we will not describe here, see *e.g.* Aizenman *et al.* [5, 3] or Imbrie *et al.* [73, 74].

**Proposition 1** *The Dyson model with specification  $\gamma$  and potential (2.1) exhibits a phase transition at low temperature for slow decays  $1 < \alpha \leq 2$ :*

$$\exists \beta_c > 0, \text{ such that } \beta > \beta_c \implies \mu^- \neq \mu^+ \text{ and } \mathcal{G}(\gamma) = [\mu^-, \mu^+]$$

where the extremal phases  $\mu^+$  and  $\mu^-$  are translation-invariant. They have in particular opposite magnetisations  $\mu^+[\sigma_0] = -\mu^-[\sigma_0] = M_0(\beta, \alpha) > 0$  at low temperature.

It is known that all Gibbs measures for Dyson models are translation-invariant [60, 47].

Phase transition in these long-range models takes its origin in the possibility, due to the infinite range of the interaction, for the entropy to lose against energy at low temperature, for slow decays  $\alpha \leq 2$ , thanks to a **volume-dependent energy cost** needed to create a droplet of the opposite phase in a ground-state configuration, as for *n.n.* Ising models in dimension  $d \geq 2$ . In these estimates, the dimension  $d$  is replaced as a parameter by the decay  $\alpha$ , so that the latter can be used to tune the dimension, in a continuous manner. See *e.g.* [40].

The original estimate was already observed by Landau/Lifschitz [91], and is sometimes called *Landau estimate* [106]. In our situation, start with the +phase, got by monotone weak limit with homogeneous +-boundary condition as defined in (2.10), for our pair-potential  $\Phi$  long-range couplings  $J = J^\alpha$  as in (2.2), for  $d = 1$  and  $\alpha > 1$ .

Write the excess energy  $h_L := H_\Lambda(-|+) - H_\Lambda(+|+)$  at volume  $\Lambda = \Lambda_L$  to be the cost of inserting of droplet of the opposite phase, for finite-volume intervals  $\Lambda$  of length  $2L$ . *Landau estimate* tells that the *finite-volume excess energy*  $h_\Lambda$  is has indeed a volume-dependent order:

$$h_L^+ \approx \sum_{j=-L}^L \sum_{k \geq L}^{\infty} \frac{1}{k^\alpha} \approx C \cdot L^{2-\alpha}. \quad (3.14)$$

While it had been already been used to get uniqueness for fast decays  $\alpha > 2$  [110, 115, 113], it tells us in particular that the energetic cost to insert droplet/interval  $\Lambda$  of length  $L$  of the opposite phase, is volume-dependent for  $\alpha \in (1, 2)$ . Thus, for very long ranges (also called slow decays), the probability of occurrence of a droplet of the opposite phase is depressed at least by

$$c \exp -\beta \zeta L^{2-\alpha}, c, \zeta > 0, (\alpha < 2) \quad (3.15)$$

The analogy with  $d > 1$  where the bounds goes as  $c \exp -\beta L^{(d-1)/d}$  is evident, but we warn the reader that other analogies exist (for *e.g.* critical exponents in [4]). The results described here can be completed by the concise introduction of Littin/Picco ([100], 2017), or any of the introduction in the series of papers of Cassandro *et al.* [17, 20, 21, 18, 19].

A crucial step to formalize these ideas has been the 2005 paper of Cassandro/Ferrari/Merola/Pressuti which provided an explicit and rigorous geometric description of Gibbs measures in the phase transition region. We describe it in next section.

### 3.1 Triangle-contour construction – Peierls-like argument

In this subsection, we sketch the *triangle-contour construction* in one-dimension, for long-range Ising models with slow decays. Premises of these notions were originally coined to treat the borderline case  $\alpha = 2$  by Fröhlich/Spencer [57], with the introduction of *spin-flip* or *interface* points, *pre-contours* and *contours* inspired by the dipole described by the same authors for the two-dimensional Coulomb gas in [56]. The geometric description of configurations in terms of these contours lead to a bijection as soon as one leave some possible ambiguities. This was done later by the *triangle-contour* description of Cassandro *et al.* by randomizing the lengths of the droplets, in order to be able to generate uniquely one geometric construction by configuration. The explicit construction together to the required quasi-additivity needed to get Peierls estimate have been afterwards developed in a series of paper of Cassandro *et al.*, starting from [17] with some technical restrictions<sup>5</sup> partially reduced afterwards, either by Littin/Picco [100] or by Bissacot *et al.* [9].

A *contour* associated with a configuration  $\sigma$  will be formed by droplet(s) of the opposite phase, well separated enough so that one recovers some weak subadditivity on their Hamiltonians of the form of (3.20). To avoid ambiguities and get a bijection between configurations and contours, the main idea of [17] has been to randomize the length of the droplets, to be able to call them one-by-one in a procedure inspired by coarse-graining in one dimension [27, 16]. We refer to the Cassandro *et al.* series of papers and to the thesis of Littin [99] for the proofs of the bijection configuration-triangle first, and triangle-contours afterwards.

We first introduce the necessary notions to get such relevant contours in  $1d$ . Then we describe the Peierls estimate they obtain in this one-dimensional long-range context. In addition, this triangle construction also allows an unambiguous notion of microscopic interface (with mesoscopic fluctuations) in the phase transition region, as we shall see in next subsection.

#### Step 1. Bijection configuration-triangles

For +-b.c., there is a unique ground state, the + configuration s.t.  $+_x = +1, \forall x \in \mathbb{Z}$ .

In this one dimensional model, impurities from this ground states are characterised by the existence of *spin-flip points*  $x \in \mathbb{Z}$  on the dual lattice, yielding an interface at  $(x, x + 1)$  when  $\sigma_x \sigma_{x+1} = -1$ . Start from a configuration  $\sigma$  and enumerate the defects ('-') from, say, the left boundary; the first spin-flip point separates then a row of consecutive plusses to a (maybe singled) row of the opposite phase, which flips again at the next spin-flip point, and so on. One would like to group the rows of defects into classes separated enough to be considered as almost independent, depending on the decay  $\alpha$ . Triangles are then built on rows of identical spins between two spin-flip points.

To complete the construction, Cassandro *et al.* provided an algorithm to get uniquely, from a configuration  $\sigma$ , a family of triangles

$$\bar{T} = (T_1, \dots, T_k, \dots, T_n)$$

ordered by lengths

$$|T_k| \leq |T_{k-1}|$$

Triangles  $T = T_k$ 's are subsets of the dual lattice, whose length  $|T|$  is the number of sites imbetween two spin-flip points. The algorithm provided by Cassandro *et al.* (see also Picco *et al.* [99, 100, 118]) is such that the triangles of the family  $\bar{T}$  satisfy the following properties:

<sup>5</sup>Large *n.n.* couplings  $J(1) = J \gg 1$  and a restricted range of decays  $1 < \alpha_* < \alpha < 2$  for some  $\alpha_* \approx 1, 41..$

1. Triangles are well separated one to the other:

$$\text{dist}(T, T') > \min(|T|, |T'|)$$

so that they indeed represent droplets of the opposite phase, with  $+$ 's imbetween.

2. The associated Hamiltonian  $H_\Lambda^+(\bar{T}) = H_\Lambda^+(\sigma)$  is additive :

$$H_\Lambda^+(\bar{T}) = \sum_{k=1}^n H_\Lambda^+(T_k).$$

3. The energetic cost needed to remove the smaller triangle,

$$H_k(\bar{T}) := H(T_k, T_{k+1}, \dots, T_n) - H(T_{k+1}, \dots, T_n)$$

satisfies

$$H_k(\bar{T}) \geq \kappa_\alpha |T|^{2-\alpha} \tag{3.16}$$

with  $\kappa_\alpha := 2(3 - 2^{3-\alpha})$ .

Note that  $\kappa_\alpha > 0$  only for  $1 < \alpha_* < \alpha < 2$ , which is the reason of the original restriction on decays. It had been avoided by Litten/Picco by providing a similar bound for contours (and not triangles, see [100]). For the sake of simplicity, we describe here the version of the construction with these technical constraints ( $J(1) \gg 1$  and  $\ln 3 / \ln 2 = \alpha_* < \alpha < 2$ ). Note also that in their construction they could be triangles inside triangles.

In such geometric construction, one key points are to avoid ambiguities in the choice of the geometric objects, and second to insure that the process described indeed leads to something. This was done in [17], pursued and upgrades in the series of papers [20, 21, 19, 18], also described in a didactic way in [99, 100, 118].

## Step 2 : "Contours" as bands of nearby "triangles"

The second ingredient needed for Peierls estimate machinery is a subadditivity, of the form:

$$H(T_1, T_2) \geq \zeta H(T_1) + H(T_2) \tag{3.17}$$

when  $T_1$  and  $T_2$  are two different non-overlapping droplets/triangles. As shown in *e.g.* [17, 118], this cannot be always the case for any pair of triangles. The idea in the definition of contours is the following : if (3.17) does not hold, then group the triangles  $(T_1, T_2)$  in the same *contour*. This is in particular the case when one has

$$\text{dist}(T, T') > C(\delta) \min(|T|, |T'|)^\delta, \quad \delta \geq 1 \tag{3.18}$$

so one could group together the triangles that are too close, in order to form a *contour*. The original choice in [17] was made with  $\delta = 3$ , and they indeed describe an algorithm producing a family of contours  $\bar{\Gamma} = \bar{\Gamma}(\bar{T}) = \bar{\Gamma}(\sigma)$  such that

1. To a configuration  $\sigma$  there corresponds a unique family of contours

$$\bar{\Gamma} = (\Gamma_1, \dots, \Gamma_2)$$

where  $\Gamma_i = \{T_{i,j}, 1 \leq j \leq k_i\}$  is formed by triangles well separated from each other, *i.e.* satisfying (3.18).

2. The length of the contours is the sums of the lengths of the triangles belonging to it :

$$|\Gamma| = \sum_{T \in \Gamma} |T|.$$

3. Contours generated by the triangles generated by a configuration  $\sigma$  are themselves also well-separated :

$$\text{dist}(\Gamma, \Gamma') > C \min(\Gamma, \Gamma')^3 \quad (3.19)$$

where

$$\text{dist}(\Gamma, \Gamma') = \min_{T \in \Gamma, T' \in \Gamma'} \text{dist}(T, T')$$

Other technical conditions are needed to insure the convergence and uniqueness of the algorithm, see [17]. When two triangles  $T$  and  $T'$  belonging to different contours have disjoint support, one says that they are mutually external, but this is not always the case, see [19, 100].

### Step 3 : Quasi-additive bounds of the Hamiltonians

To get its estimate, and avoid too strong dependencies between contours, Peierls used

$$H(\Gamma_0, \Gamma_1, \dots, \Gamma_n) = H(\Gamma_0) + H(\Gamma_1, \dots, \Gamma_n)$$

but in fact the following weak form of subadditivity is enough :

$$H(\Gamma_0, \Gamma_1, \dots, \Gamma_n) \geq \zeta H(\Gamma_0) + H(\Gamma_1, \dots, \Gamma_n), 0 < \zeta < 1. \quad (3.20)$$

This was proved in [17] for slow decays and extended to the whole range of decays  $1 < \alpha \leq 2$  by Littin *et al.*, with an extension of Landau estimate to contours, with a removing cost estimated as:

$$H(\Gamma) \geq \zeta_\alpha \sum_{\tilde{T} \in \Gamma} |\tilde{T}|^{2-\alpha} \quad (3.21)$$

with  $\zeta_\alpha > 0$  for  $\alpha_* < \alpha < 2$  (for other decays, a mixed energy-entropy argument is needed [17, 100, 9]).

### Step 4 : Peierls argument

A necessary condition to have  $\sigma_0 = -1$  is that the origin 0 is contained in the support of some contour  $\Gamma$ , so that :

$$\mu_\Lambda^+(\sigma_0 = -1) \leq \mu_\Lambda^+(\{\exists \Gamma : 0 \in \Gamma\}) \leq \sum_{\Gamma \ni 0} \mu_\Lambda(\Gamma)$$

and, using (3.21), relate it to the lengths of the triangles to get:

$$\mu_\Lambda^+(\sigma_0 = -1) \leq \sum_m \sum_{\Gamma: |\Gamma|=m, 0 \in \Gamma} e^{-\frac{\beta \zeta}{2} \sum_{T \in \Gamma} |T|^{2-\alpha}}$$

To conclude, one uses an entropy estimate counting the number of such triangles [17] to get for  $m \geq 1$  and some  $b$  large enough,

$$\sum_{\Gamma: |\Gamma|=m, 0 \in \Gamma} e^{-\frac{\beta \zeta}{2} \sum_{T \in \Gamma} |T|^{2-\alpha}} \leq 2m e^{-bm^{2-\alpha}}$$

and eventually phase transition for  $\beta$  large enough.

### 3.2 Non-Gibbsianness in 1d : decimation of Dyson models

A particular consequence of this phase transition is that it provides an example of a non-Gibbsian measure in dimension one, briefly described here (see also [42]).

In this subsection, we use the well-known characterization of Gibbs measures as being *quasilocal* and *non-null*. Quasilocality is a Feller-type property equivalent to the existence of continuous versions of conditional probabilities, in the product topology of the discrete one on  $E$ , providing an interpretation of Gibbs measures as natural extensions Markov fields. The rigorous proof of the equivalence was coined by Kozlov ([90], 1974) and Sullivan ([117], 1976). Note that in one implication (from quasilocality to Gibbsianness), some non-trivial issues about translation-invariance arise, see discussion in *e.g.* [41, 48, 97]

When  $\mu \in \mathcal{G}(\gamma)$  is quasilocal, then for any  $f$  local and  $\Lambda \in \mathcal{S}$ , the conditional expectations of  $f$  w.r.t. the outside of  $\Lambda$  are  $\mu$ -a.s. given by  $\gamma_\Lambda f$ , by (2.6), and this is itself a continuous function of the boundary condition by (2.4). Thus, one gets for any  $\omega$

$$\lim_{\Delta \uparrow \mathbb{Z}} \sup_{\omega^1, \omega^2 \in \Omega} \left| \mu[f|\mathcal{F}_{\Lambda^c}](\omega_\Delta \omega_{\Delta^c}^1) - \mu[f|\mathcal{F}_{\Lambda^c}](\omega_\Delta \omega_{\Delta^c}^2) \right| = 0 \quad (3.22)$$

As described in whole generality by van Enter *at al.* [41], this does not always hold for *renormalized* Gibbs measures; Let us describe now the simple such transformation leading to essential discontinuity when applied to, so-called *decimation*.

**Decimation Transformation:** It is defined on the configuration space as

$$T: (\Omega, \mathcal{F}) \longrightarrow (\Omega', \mathcal{F}') = (\Omega, \mathcal{F}); \quad \omega \longmapsto \omega' = (\omega'_i)_{i \in \mathbb{Z}}, \quad \text{with } \omega'_i = \omega_{2i} \quad (3.23)$$

It acts on measures in a canonical way: denote  $\nu^+ := T\mu^+$  the decimation of the  $+$ -phase

$$\forall A' \in \mathcal{F}', \quad \nu^+(A') = \mu^+(T^{-1}A') = \mu^+(A) \quad \text{where } A = T^{-1}A' = \{\omega : \omega' = T(\omega) \in A'\}.$$

In the seminal work of van Enter/Fernández/Sokal ([41], 1993), non-quasilocality of the decimated measure  $\nu^+$  is proved in dimension 2 at low enough temperature, as soon as a phase transition is possible for an Ising model on the *decorated lattice*, which consists of a version of  $\mathbb{Z}^2$  where the "even" sites have been removed. Here, the role of the image 'decorated' lattice is played by the set of odd sites,  $2\mathbb{Z} + 1$ , which can be identified with  $\mathbb{Z}$  itself, and when a phase transition holds for the Dyson specification – thus at low enough temperature for  $1 < \alpha \leq 2$  – the same is true for a constrained specification *with alternating constraint* due to the alternating configuration, yielding non-Gibbsianness of  $\nu^+$ .

**Theorem 3** [42] *For any  $1 < \alpha \leq 2$ , at low enough temperature, the decimation  $\nu = T\mu$  of any Gibbs measure  $\mu$  of the Dyson model is non-quasilocal, hence non-Gibbs.*

For the full proof, see [42]. Here, we only pick-up a sketch of the proof.

The point of essential discontinuity we exhibit, called the *bad configuration* for the image measure  $\nu^+$  is the *alternating configuration*  $\omega'_{\text{alt}}$  defined for any  $i \in \mathbb{Z}$  as  $(\omega'_{\text{alt}})_i = (-1)^i$ . To get the essential discontinuity, the choice of  $f(\sigma') = \sigma'_0$  and conditioning outside  $\{0\}$  will be enough. Due to cancelations and symmetries, conditioning by this alternating configuration yields a constrained model that is again a model of Dyson-type which has a low-temperature transition in our range of decays  $1 < \alpha \leq 2$ . The proof essentially goes along the lines sketched

in [41, 98], with the role the “annulus” played by two large intervals  $[-N, -L-1]$  and  $[L+1, N]$  to the left and to the right of the central interval  $[-L, +L]$ . If we constrain the spins in these two intervals to be either plus or minus, within these two intervals the measures on the unfixed spins are close to those of the Dyson-type model in a positive, or negative, magnetic field. As those measures are unique ([96, 86]) no influence from the boundary can be transmitted by via the “annulus”.. However, due to the long range of the Dyson interaction, there may be also a direct influence from the boundary to the central interval. To overcome this difficulty, we choose  $N(L)$  large enough as  $N = L^{\frac{1}{\alpha-1}}$ , in order to make this direct influence as small as he wants.

The main tool to justify this rigorously is to consider the “Equivalence of boundary conditions” concept coined by Bricmont/Lebowitz/Pfister in the beautiful paper [13], by considering b.c.  $\omega^\pm$  either in the +- or in the --neighbourhoods of the alternated configuration. Write  $\Lambda' = \Lambda'(L) = [-L, +L]$  and  $\Delta' = \Delta'(N) = [-N, +N]$ , with  $N > L$  and denote formally by  $H$  the Hamiltonian of the constrained specifications for  $\omega_1^+$  and  $\omega_2^+$  as prescribed. One can bound uniformly in  $L$  the relative Hamiltonians as

$$\left| H_{\Lambda, \omega_1^+}(\sigma_\Lambda) - H_{\Lambda, \omega_2^+}(\sigma_\Lambda) \right| \leq C < \infty. \quad (3.24)$$

as soon as one takes  $N = N(L) = O(L^{\frac{1}{\alpha-1}})$ . Then one gets by [13] (see also [54]) that all of the limiting Gibbs states obtained by these boundary conditions have the same measure zero sets, an equivalent decomposition into extremal Gibbs states (presumably trivial here, as the Gibbs measure will be unique, as we shall see), and thus yield the same magnetisation :  $M^+ = M^+(\omega, N, L) = M^+(\omega_1^+, N, L) = M^+(\omega_2^+, N, L)$  is indeed independent of  $\omega$  as soon as it belongs to the pre-image of the +-neighbourhood of the alternating configuration. To get (3.24), we use the long-range structure of the interaction to get a uniform bound

$$\left| H_{\Lambda, \omega_1^+}(\sigma_\Lambda) - H_{\Lambda, \omega_2^+}(\sigma_\Lambda) \right| \leq 2 \sum_{x=-L}^L \sum_{k>N} \frac{1}{k^\alpha} < 2L \frac{N^{1-\alpha}}{1-\alpha}$$

so that choosing  $N = N(L)$  such that  $2L \frac{N^{1-\alpha}}{1-\alpha} = 1$  will yields the seeked essential discontinuity, so one can choose

$$N(L) = L^{\frac{1}{\alpha-1}}. \quad (3.25)$$

Once we got rid of any possible direct asymptotic effects due to the long range by choosing a large enough annulus as above, the main point is now that freezing the primed spins to be minus can overcome the +-boundary condition when the frozen annulus  $\Delta' \setminus \Lambda'$  is in a --state, for  $L$  and  $N(L)$  large enough. The corresponding magnetization can then be made as close as possible to the magnetisation of the Dyson model with an homogeneous external field  $h_x = -$  everywhere, which at low enough temperature is smaller than and close to the magnetisation of the Dyson model under the --phase, i.e to  $-M_0(\beta, \alpha) < 0$  (and this --phase is also unique). The magnetisation with the constraint  $\omega^+$  will thus be close to or bigger than  $+M_0(\beta, \alpha)$  so that a non-zero difference is created at low enough temperature.

Note that this non-Gibbsianness might be of some importance in the use of renormalization group in Neurosciences simulations, see [23] and references therein.

### 3.3 Mesoscopic interfaces and (non-) $g$ -measure property

An another important consequence of the arising phase transition in one-dimension for long-range model with slow decays is the occurrence of mesoscopic fluctuations of the interface



(point) got with mixed  $+$ -Dobrushin b.c. ( $-$  on the left side of the integrer line,  $+$  on the other side). As we show in [10], these fluctuations implies a wetting phenomena (propagation of a droplet of the opposite phase), which have itself an important consequence on the continuity properties of one-sided conditional probabilities, providing a seemingly first example of Gibbs measure which is not a  $g$ -measures in [10]. We describe this result here; it requires to describe the interface fluctuations results of [19], and the intermediate wetting consequences also derived in [10].

### Dobrushin boundary conditions and *Interface point*:

For homogeneous boundary conditions, since the number of spin-flip points is even, every spin-flip point was an extremity of some droplet/triangle. If we consider now a Dobrushin-type boundary condition, then the number of spin-flip points becomes odd, and so there exists a *unique* spin-flip point which is not the vertex of any triangle.

This point is called the *interface point*. To describe where it can be located, let discretise the interval  $[-1, +1]$  as

$$T_L = \left\{ -1 - \frac{1}{2L}, -1 + \frac{1}{2L}, \dots, -\frac{1}{2L}, \frac{1}{2L}, \dots, 1 + \frac{1}{2L} \right\},$$

and consider the  $'-+'$ -Dobrushin boundary condition. Given a configuration  $\omega$  in  $\Lambda = \Lambda_L$ , let  $I^* \equiv I^*(\omega) \in \Lambda^*$  be the interface point of the configuration  $\omega$ , and for  $\theta \in T_L$ , denote by  $\mathcal{S}_{\Lambda, \theta} = \{\omega : I^* = \theta L\}$  the set of configurations in  $\Lambda$  for which the interface point lies at  $\theta L$ .

Define now for each  $\theta \in T_L$  the probability to have an interface in  $\theta L$  by

$$\mu_{\Lambda}^{-+}[I^* = \theta L] = \frac{Z_{\theta, \Lambda}^{-+}}{Z_{\Lambda}^{-+}},$$

where the partition functions  $Z_{\theta, \Lambda}^{-+} = \sum_{\omega \in \mathcal{S}_{\Lambda, \theta}} e^{-\beta H_{\Lambda}^{-+}(\omega)}$  and  $Z_{\Lambda}^{-+} = \sum_{\theta \in T_L} Z_{\theta, \Lambda}^{-+}$  are defined via the Hamiltonian  $H_{\Lambda}^{-+}$  in volume  $\Lambda$  with Dobrushin boundary conditions. For  $i \in \Lambda$ , the conditional expectation of  $\omega_i$ , given  $I^* = \theta L$ , is then

$$\mu_{\theta, \Lambda}^{-+}[\omega_i] := \mu_{\Lambda}^{-+}[\omega_i | I^* = \theta L] = \frac{1}{Z_{\theta, \Lambda}^{-+}} \sum_{\omega \in \mathcal{S}_{\Lambda, \theta}} \omega_i e^{-\beta H_{\Lambda}^{-+}(\omega)}.$$

The expectation of  $\omega_i$  can then be written in terms of  $\mu_{\theta, \Lambda}^{-+}[\omega_i]$  as

$$\mu_{\Lambda_L}^{-+}[\omega_i] = \sum_{\theta \in T_L} \mu_{\theta, \Lambda_L}^{-+}[\omega_i] \mu_{\Lambda_L}^{-+}(I^* = \theta L). \quad (3.26)$$

Most of the results of this section are based on a convergent cluster expansion for partition functions from [17, 19], where one in particular learns:

**Proposition 2** *For all  $\alpha \in (\alpha_*, 2)$ , there exists  $\beta_0(\alpha) > 0$  s.t. for all  $\beta > \beta_0$  and  $\theta \in T_L$ ,*

$$\log Z_{\theta, \Lambda}^{-+} - \log Z_{\Lambda}^{-} = -c_L(\alpha) L^{2-\alpha} + e^{-2\beta(\zeta(\alpha)+J)} \frac{L^{2-\alpha}}{(2-\alpha)(\alpha-1)} f_{\alpha}(\theta) (1 \pm e^{-c_1(\alpha)\beta}) (1 + o(L))$$

where  $\zeta(\alpha) = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$  is the Riemann zeta function and  $f_{\alpha}(\theta) = (1+\theta)^{2-\alpha} + (1-\theta)^{2-\alpha}$ ,  $c_L = c_L(\alpha) > 0$ ,  $c_1 = c_1(\alpha)$ , and  $J = J(1) \gg 1$ .

The estimation of expectation under the  $+$ -phase has also been estimated in [19]:

**Theorem 4** *For all  $\alpha \in (\alpha_*, 2)$ ,  $\exists \beta_0(\alpha)$ ,  $c_1 > 0$  s.t.  $\forall \beta \geq \beta_0$ , uniformly in  $\Lambda \Subset \mathbb{Z}$ ,*

$$\mu_\Lambda^+[\omega_i] = 1 - \left[ 2e^{-2\beta(\zeta(\alpha)+J)} \left( 1 \pm e^{-c_1(\alpha)\beta} \right) (1 + o(1)) \right], \text{ for all } i \in \Lambda \quad (3.27)$$

Thus, after taking the infinite-volume limit, at low temperatures, the magnetisation satisfies:

$$1 - \left[ 2e^{-2\beta(\zeta(\alpha)+J)} \left( 1 + e^{-c_1(\alpha)\beta} \right) \right] \leq \mu^+[\omega_i] \leq 1 - \left[ 2e^{-2\beta(\zeta(\alpha)+J)} \left( 1 - e^{-c_1(\alpha)\beta} \right) \right]. \quad (3.28)$$

### Consequence of interface fluctuations : Wetting transition

For a fixed  $N > 1$ , consider the  $+$ -phase  $\mu^+$ , conditioned on the event  $-\omega_{-N,-1}$  of the occurrence of a droplet of  $-$ 's in an interval  $[-N, -1]$ . Then we claim in [10] that there are two intervals of length of order  $L$ , left and right of the fixed interval and of the form  $\left[ -N - \frac{(1-sL^{\frac{\alpha}{2}-1})}{2}L, -N - 1 \right]$ , and  $[0, \frac{(1-sL^{\frac{\alpha}{2}-1})}{2}L]$ , such that for  $N \gg L$  both large enough, satisfying  $LN^{1-\alpha} = o(1)$ , the magnetisation of the spins in one of these intervals conditioned on the event  $\{\omega_{-N,-1} = -\omega_{-N,-1}\}$  is negative. These intervals play the role of a “completely wet region” in a wetting transition.

**Proposition 3** *Let  $\alpha \in (\alpha_*, 2)$  and  $\beta_0 \equiv \beta_0(\alpha)$  as above. Then, there exists  $\beta_1 > \beta_0$  such that, for any  $\beta > \beta_1$ , there exist  $s = s(\beta, \alpha)$ ,  $\lambda = \lambda(\beta, \alpha, s) > 0$  and  $L_0 \equiv L_0(\alpha, \beta) \geq 1$  such that, for any  $L > L_0$ , there exists  $N_0(L) > L$  such that, for any  $N \geq N_0(L)$ ,*

$$\mu^+(\omega_i | -\omega_{-N,-1}) \leq -\lambda m, \quad (3.29)$$

for every  $i \in \left[ -N - \frac{(1-sL^{\frac{\alpha}{2}-1})}{2}L, -N - 1 \right] \cup \left[ 0, \frac{(1-sL^{\frac{\alpha}{2}-1})}{2}L \right]$ , where  $m = \langle \omega_0 \rangle^+ > 0$ .

The main idea of our proof is to choose  $N$  large enough for the total influence of all spins left of the interval to be bounded by a (small) constant, so that one can neglect boundary effects beyond  $-N$  by equivalence of boundary conditions as in [13, 42]. Then inside the interval of length  $L$ , the interface separating the  $+$ - and  $-$  phases is w.h.p. within the same window as with the Dobrushin boundary conditions. If afterwards we move the  $+$ -boundary to the right, the location of the interface, by monotonicity, can also move only to the right, that is away from the frozen interface.

### Consequence of wetting : discontinuity of 1-sided conditional probabilities

We deduce from the wetting transition the discontinuity of any  $g$ -function associated with  $\mu^+$ , which in turn cannot be a  $g$ -measure. Let us first introduce a bit more  $g$ -functions and  $g$ -measures in our context.

In Dynamical systems, similarly to Gibbs measures in mathematical statistical mechanics,  $g$ -measures are defined by combining topological and measurable notions, with the introduction of transition functions (the ‘ $g$ ’-functions) having to be continuous functions of the *past only*. One requires continuity of single-site *one-sided* conditional probabilities and says that  $\mu$  is a  $g$ -measure if there exists a (past-measurable) *continuous* and non-null function  $g_0$  which gives “one-sided” conditional probabilities, that is non-null conditional probabilities for events

localised on the right half line (the “future”), given a boundary condition fixed only to the left (the “past”). To formalize it, define  $T : \{-1, +1\}^{(-\infty, 0]} \rightarrow \{-1, +1\}^{(-\infty, 0]}$  be the shift  $(Tx)_n = x_{n-1}$ . Denote by  $\mathcal{P}$  the class of positive  $g$ -functions  $g : \{-1, +1\}^{(-\infty, 0]} \rightarrow (0, 1]$  such that  $\sum_{y \in T^{-1}x} g(y) = 1$ , for all  $x \in \{-1, +1\}^{(-\infty, 0]}$ . We shall use the *past* and *future*  $\sigma$ -algebras  $\mathcal{F}_{<0}$  and  $\mathcal{F}_{>0}$  generated by the projections indexed by negative and positive integers.

**Definition 1** *A probability measure is a  $g$ -measure, if there is a non null continuous  $g$ -function  $g_0$ , defined on the left (“past”) half-line configuration space, such that, for each  $\omega_0 \in \{-1, +1\}$  and  $\mu$  a.e.  $\tau = (\tau_j)_{j < 0} \in \{-1, +1\}^{(-\infty, 0)}$ ,*

$$\mu[\omega_0 | \mathcal{F}_{<0}](\tau) := \mathbb{E}_\mu[\mathbf{1}_{\sigma_0 = \omega_0} | \mathcal{F}_{<0}](\tau) = g_0(\tau \omega_0). \quad (3.30)$$

For translation-invariant measures, it is extended to any site  $i$  with conditional probabilities w.r.t. to the past at site  $i$  given by  $g_i = g$ .

Discontinuity of any candidate  $g^+$  to represent a  $g$ -function for  $\mu^+$  will be a consequence of the entropic repulsion phenomenon describe above. In the following lemma from [10],  $\mu_{\mathbb{Z}_+}^{+, \omega}[\cdot]$  denotes expectations under a measure  $\mu_{\mathbb{Z}_+}^{+, \omega}$  constrained to be  $\omega$  on  $\mathbb{Z}^-$ , with +-b.c. otherwise. The neighborhoods  $\mathcal{N}_{N,L}^{+, \text{left}}(\omega_{\text{alt}})$  (resp.  $\mathcal{N}_{N,L}^{-, \text{left}}(\omega_{\text{alt}})$ ) are the configurations which coincide with the alternate configuration with +-b.c. (resp. -b.c.) beyond  $N > L$ .

**Lemma 1** *Consider the alternating configuration  $\omega_{\text{alt}} = ((\omega_{\text{alt}})_i)_{i \in \mathbb{Z}}$  defined by  $(\omega_{\text{alt}})_i = (-1)^i$ , and take a Dyson model with polynomial decay  $\alpha_* < \alpha < 2$  at sufficiently low temperature. Then, there exist  $L_0 \geq 1$  and  $\delta > 0$  such that for any  $L > L_0$  there is an  $N > L$ , with  $LN^{1-\alpha} = o(1)$ , such that for every  $\omega^+ \in \mathcal{N}_{N,L}^{+, \text{left}}(\omega_{\text{alt}})$  and  $\omega^- \in \mathcal{N}_{N,L}^{-, \text{left}}(\omega_{\text{alt}})$ ,*

$$\left| \mu_{\mathbb{Z}_+}^{+, \omega^+}[\sigma_0] - \mu_{\mathbb{Z}_+}^{+, \omega^-}[\sigma_0] \right| > \delta. \quad (3.31)$$

As a corollary, we obtain the main result of [10]:

**Theorem 5** *For  $\mu$  being either the +- or the -- phase of a Dyson model with decay  $\alpha_* < \alpha < 2$  at sufficiently low temperature, the one-sided conditional probability  $\mu[\omega_0 | \mathcal{F}_{<0}](\cdot)$  is essentially discontinuous at  $\omega_{\text{alt}}$ . Therefore, none of the Gibbs measures  $\mu$  for the Dyson model in this phase transition region is a  $g$ -measure.*

To describe the  $g$ -functions, we need regular versions of conditional probabilities given the outside of infinite sets, because so is the past (it is the complement of  $\mathbb{Z}^-$ , whose conditional probabilities are not provided by the DLR equations). Various constructions of such *Global specifications* [51, 98, 42, 10] to represent these regular versions eventually allow us to consider, for given pasts, the expression of the  $g$ -functions as the magnetisations of Dyson models under various conditionings, see Equation (3.32) below. Studying continuity reduces in fact to studying the stability of interfaces when changing the boundary conditions arbitrary far away in the past.

Starting from  $\mu^+$ , we introduce  $g^+$  to be the candidate to be the  $g$ -function representing (a version of) the single-site conditional probabilities (3.30) as a function of the past. Just as in [51, 42], we introduce thus for any “past” configuration  $\omega \in \Omega$ :

$$g^+(\omega) := \mu^+[\omega_0 | \mathcal{F}_{<0}](\omega)$$

Using the expression in terms of global specifications (see [51, 98, 42]) and constrained measures, one gets,  $\mu^+$ -a.s. ( $\omega$ ), the following candidate:

$$g^+(\omega) = \mu_{\mathbb{Z}^+}^{+,\omega} \otimes \delta_{\omega_{(\mathbb{Z}^+)^c}}[\omega_0] \quad (3.32)$$

where  $\mu_S^{+,\omega}$  is the constrained measure on  $(\Omega_S, \mathcal{F}_S)$  for  $S = \mathbb{Z}^+$  here.

Previous works, using monotony and right-continuity [51, 42], insure that  $\mu^+$  is then indeed “specified” by  $g^+$ , in the sense that it is invariant by its left action:  $\mu^+ g^+ = \mu^+$ .

To prove that  $\mu^+$  is not a  $g$ -measures, we prove that  $g^+$  can take significantly different values on sub-neighborhoods  $\mathcal{N}_{N,L}^{\pm,\text{left}}(\omega_{\text{alt}}) \subset \mathcal{N}_L(\omega_{\text{alt}})$ , for  $L$  large and  $N$  larger. To do so, we introduce the particular alternating configuration  $\omega_{\text{alt}}$ . To prove that it is a bad configuration, one should find two sub-neighborhoods on which the value of  $g^+$  differs.

Consider the sub-neighborhoods  $\mathcal{N}_{N,L}^{\pm,\text{left}}(\omega_{\text{alt}})$  for  $L < N$ , whose size is adjusted later. All together, this leads us to consider a partially frozen Dyson model, either frozen into  $+$  outside  $I_n$ , or into  $-$  in the “annulus”  $[-N, -L]$ , and the alternating one  $\omega_{\text{alt}}$  in  $[-L, -1]$ .

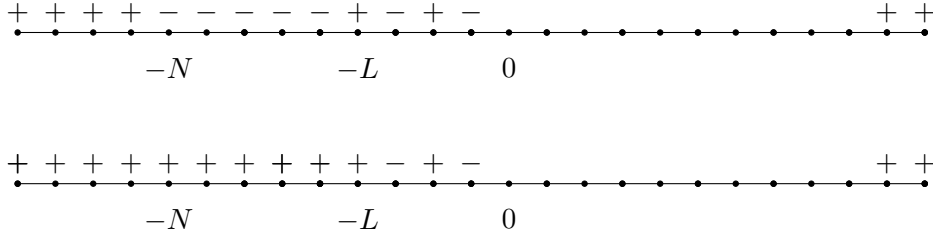


Figure 1 : Left  $\pm$  Neighborhoods of  $\omega_{\text{alt}}$

By (3.32), for a  $\mu^+$ -a.s. given  $\omega$ , the value taken by  $g^+$  will be the infinite-volume limit of the magnetisation of the finite-volume Gibbs measure of a Dyson-model on  $\Lambda = [0, n]$ , with the same decay  $\alpha < 2$  and  $\omega$ -dependent inhomogeneous external fields  $h_x[\omega], x \geq 0$ . For configurations  $\omega := \omega^-$  on the sub-neighborhood  $\mathcal{N}_{N,L}^{-,\text{left}}(\omega_{\text{alt}})$ , one gets external fields

$$\forall x \geq 0, h_x[\omega] = \sum_{k=1}^L \frac{(-1)^k}{(k+x)^\alpha} - \sum_{k=L+1}^N \frac{1}{(k+x)^\alpha} + \sum_{k \geq N} \frac{\omega_{-k}}{(k+x)^\alpha} + \sum_{k \geq n} \frac{1}{(k+x)^\alpha}$$

while for  $\omega := \omega^+ \in \mathcal{N}_{N,L}^{+,\text{left}}(\omega_{\text{alt}})$ , we get:

$$\forall x \geq 0, h_x[\omega] = \sum_{k=1}^L \frac{(-1)^k}{(k+x)^\alpha} + \sum_{k=L+1}^N \frac{1}{(k+x)^\alpha} + \sum_{k \geq N} \frac{\omega_{-k}}{(k+x)^\alpha} + \sum_{k \geq n} \frac{1}{(k+x)^\alpha}$$

We recognize a long-range RFIM with dependent biased, disordered external field, whose distribution is linked to the original measure  $\mu$  itself via the distribution of the past. When the fields are homogeneous one can use correlation inequalities and uniqueness via Lee-Yang [96] type arguments – as were e.g. used to prove essential discontinuities for the decimation of Dyson model in Section 3.2. – but here this external field will change signs, depending on  $x \in [0, n]$ . For  $n, L, N(L)$  large enough, it starts by being negative at 0 and, due to the  $+$ -boundary procedure far away, it becomes positive for  $x$  large.

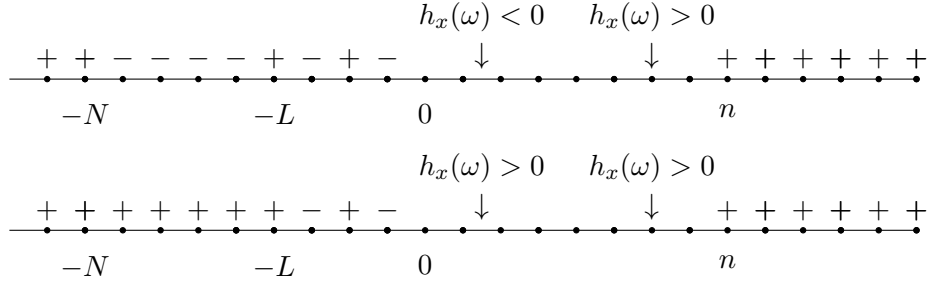


Figure 2: Inhomogeneous  $\omega$ -dependent external fields

On the contrary, on the neighborhood  $\mathcal{N}_{N,L}^{-,\text{left}}$ , the inhomogeneous magnetic field  $h_x(\omega)$  will stay negative far enough to the past so that a  $--$ -phase is still felt at the origin in the limits, while on the neighborhood  $\mathcal{N}_{N,L}^{+,\text{left}}$ , a  $++$ -phase is always selected for  $N$  and  $L$  of adjusted size. In the last case, we need to evaluate the effect of large interval of minuses on its outside, faraway through an intermediate neutral interval, and eventually the lack of  $g$ -measure property is a consequence of the *entropic repulsion in wetting phenomena* described above. The precise and rigorous proof is more involved and delicate, so we omit it in these notes and refer to [10].

### 3.4 Other results – external fields; random b.c. and metastates

#### External fields :

A general study for inhomogeneous external field or alternated ones is still to be done. Uniqueness has been proved in various situations, in a series of papers of Kerimov (see *e.g.* [85, 86]), while Bissacot *et al.* have considered both uniqueness and phase transition issues in the case of decaying fields. Correlated external fields are currently studied by Littin in a work in progress.

#### Disordered fields :

As for higher dimensionnal  $n.n.$  Ising model where randomness yeild a dimension reduction, adding a random *i.i.d.* magnetic field reduces the phase transition to ranges  $\alpha \leq \frac{3}{2}$ . Uniqueness was known by Aizenman-Wher type arguments, while a contour proof of phase transition has been provided by [20]. For  $\alpha \geq \frac{3}{2}$ , the peculiarities of the unique phase according to realizations of the external fields have been described in [21].

#### Random b.c. and metastates:

As is higher dimensional standard  $n.n.$  Ising model, the behaviour under random *incoherent* b.c., in the sense that they are drawn from untypical b.c. (say *i.i.d.* when phase transition holds) also leads to a difficult toy model for spin-glasses [44, 45, 46]. In the corks in progress [39, 43], we consider the Dyson model with b.c. drawn from *i.i.d.* sequences and describe a non-trivial metastate behaviour, with again a critical decay value  $\alpha = \frac{3}{2}$  discriminating between two different global behaviours.

## 4 Long-range Ising model in dimension two

In dimension two, let us focus on two different type of models, an *isotropic* one where everybody interacts with everybody with a strength decaying with the distance (for decays  $\alpha > 2$ ), or *anisotropic* models, where only sites on the same horizontal or vertical axis interact (but possibly for longer decays  $\alpha > 1$ ). We investigate the translation-invariance of extremal states, in the direction of the validity of AH theorem in the most common isotropic case, and on the other hand we describe the existence of rigid (extremal and non translation-invariant) Dobrushin states in the anisotropic case with slow decays.

In this section, we describe the results of [25], and add a detailed proof of van Beijeren's techniques, already known for long-range models but whose proof was well hidden in the appendix of a (not obviously) related paper of Bricmont *et al* [15].

### 4.1 Absence of Dobrushin states in the isotropic cases

Consider classical  $2d$  extensions of long-range Dyson models, with an isotropic pair potential, (*i.e.* a uniform polynomial decay  $\alpha > 2$ ) of the form

$$J_{x,y}^\alpha = J_{x,y}^{n.n.} + \frac{1}{|x-y|^\alpha}, \quad \forall x, y \in \mathbb{Z}^2 \quad (4.33)$$

In our ferromagnetic framework, phase transition at any decay  $\alpha > 2$  holds at low temperature by stochastic domination of the corresponding *n.n.*-model. Nevertheless, different critical values, as in  $d = 1$  with  $\alpha = 2$ , have been identified although they do not manifest in phase transition phenomenon. In Fourier analysis techniques or mean-field/lace expansion questions,  $\alpha = 4$  appear to be an important threshold, while at  $\alpha \leq 3$ , some peculiarities appear for non-Ferromagnetic or disordered models, and the (Gertzik)-Pirogov-Sinai picture is 'probably' not valid anymore [61, 62, 107].

In [25], we mainly consider decays  $2 < \alpha < 4$ , distinguishing between a 'medium-range' picture  $3 < \alpha \leq 4$ , and a 'very long-range one'  $2 < \alpha \leq 3$ . By stochastic domination of the corresponding *n.n.*-case, phase transition holds at low temperature  $T \leq T_c(\alpha, d = 2)$  and the pure phases  $\mu^-$  and  $\mu^+$  are built by the standard monotone weak limit procedure.

We write  $\omega = (\pm, h)$  for the so-called Dobrushin b.c. centered at height  $h \in \mathbb{Z}$ :

$$\omega_x = \begin{cases} +1, & \text{on } \{(x_1, x_2) \in \mathbb{Z}^2 : x_2 \geq h\} \\ -1, & \text{on } \{(x_1, x_2) \in \mathbb{Z}^2 : x_2 < h\} \end{cases} \quad (4.34)$$

For a given height  $h$ , write  $\mu^{(\pm, h)}$  for any (sub-sequential) weak limit of sequences  $(\mu_\Lambda^{(\pm, h)})_\Lambda$ . As in previous sections, the Gibbs measure  $\mu^{(\pm, h)}$  is called a Dobrushin state if it is extremal and is *not* translation-invariant. In this work, we exclude their existence in both cases; we either use an energy estimate and, as in Section 3.2 and 3.3, 'Equivalence of b.c.' from [13] in the shorter-range case  $0 < \alpha < 3$ , or a strategy of Fröhlich/Pfister [58] using relative entropy estimates to exclude cohabitation of translation-invariance and extremality in the longer range cases  $2 < \alpha \leq 3$ .

#### Medium ranges $3 < \alpha \leq 4$ :

In this case, ferromagnetism is not needed and the results got are more general. By comparing Hamiltonians of different Dobrushin b.c. located at two consecutive planes, we

see that this energy difference is already uniformly bounded for decays  $\alpha > 3$ , allowing us to avoid entropic considerations, while for longer-range decays we shall see that relative entropy estimates and ferromagnetism are needed to incorporate entropic effects.

The energetic observation we use is that the difference between two Dobrushin conditions is obtained by flipping all spins in two half-lines, so if the maximal energy between a half-line left of the origin and a half-plane right of the origin is uniformly bounded, the arguments of equivalence of boundary conditions of [13] apply and we can conclude that there is no 'pure' interface Gibbs state, or said differently no interface state. What we show in [25] is that it holds for decays  $\alpha > 3$ .

Denote  $H_\Lambda^\pm(\sigma)$  the Hamiltonian with Dobrushin b.c. in  $\Lambda = \Lambda_L = ([-L, +L] \cap \mathbb{Z})^2$  :

$$-H_\Lambda^\pm(\sigma) = \sum_{x,y \in \Lambda} \sigma_x \sigma_y J_{x,y} + \sum_{x \in \Lambda, y \in \Lambda^u} \sigma_x J_{x,y} - \sum_{x \in \Lambda, y \in \Lambda^d} \sigma_x J_{x,y}$$

where  $\Lambda^u = \{(x, y) : y \geq 0\} \cap \Lambda^c$  and  $\Lambda^d = \{(x, y) : y < 0\} \cap \Lambda^c$ . Let  $\overline{H}_\Lambda^\pm(\sigma)$  be defined as the shifted Hamiltonian, with upward-shifted Dobrushin b.c.:

$$-\overline{H}_\Lambda^\pm(\sigma) = \sum_{x,y \in \Lambda} \sigma_x \sigma_y J_{x,y} + \sum_{x \in \Lambda, y \in \Lambda^{u+1}} \sigma_x J_{x,y} - \sum_{x \in \Lambda, y \in \Lambda^{d-1}} \sigma_x J_{x,y}$$

where  $\Lambda^{u+1} = \{(x, y) : j \geq 1\} \cap \Lambda^c$  and  $\Lambda^{d-1} = \{(x, y) : j \leq 0\} \cap \Lambda^c$ . Then we can estimate

$$\begin{aligned} & |H_\Lambda^\pm(\sigma) - \overline{H}_\Lambda^\pm(\sigma)| \\ &= \left| \sum_{x \in \Lambda, y \in \Lambda^u} \sigma_x J_{x,y} - \sum_{x \in \Lambda, y \in \Lambda^{u+1}} \sigma_x J_{x,y} + \sum_{x \in \Lambda, y \in \Lambda^{d-1}} \sigma_x J_{x,y} - \sum_{x \in \Lambda, y \in \Lambda^d} \sigma_x J_{x,y} \right| \\ &\leq \sum_{(y_1, 0) \in \Lambda^c} \sum_{(x_1, y_1) \in \Lambda} O(|(x_1 - y_1)^2 + y_1^2|)^{-\alpha/2} \\ &\leq \sum_{y_1=L+1}^{\infty} \sum_{x_1=0}^L O((y_1 - x_1)^{1-\alpha} + (x_1 + y_1)^{1-\alpha}) \end{aligned}$$

so that  $|H_\Lambda^\pm(\sigma) - \overline{H}_\Lambda^\pm(\sigma)| \leq C(\alpha) = O(L^{3-\alpha})$  which is uniformly bounded for  $\alpha > 3$ .

Now, one proceed as [13] by using 'Equivalence of boundary conditions' : Finite energy difference implies that the states obtained as weak limits are absolutely continuous w.r.t. each other and should have the same components in their extremal decomposition. When the limit state is an extremal Gibbs measure, the state and its translate would thus be equal, and thus the state would be translation-invariant.

As described briefly in [25], the case of fast decays  $\alpha > 3$  falls in fact within the framework of the Gertzik-Pirogov-Sinai theory [107]. These models satisfy a Peierls condition at low enough temperature as shown in [61, 62]. In such a framework, all the Gibbs measures should be translation-invariant, as described in the review [32]. From this, coupled with the fact recently extended to more general contexts by Raoufi [108] that the  $\mu^+$  and  $\mu^-$  states are the only translation-invariant extremal states, one gets also the convex decompositions in terms of these pure states. For the standard Dobrushin b.c. located at the origin, one recovers

$$\mu^{(\pm, 0)} = \lim_{\Lambda \uparrow \mathbb{Z}^2} \mu_\Lambda^{(\pm, 0)} = \frac{1}{2}(\mu^- + \mu^+)$$

**Longer ranges  $2 < \alpha < 3$ :**

In this case, we first need to consider the zero-temperature case and investigate the asymptotic behavior of the *Energy difference for Dobrushin ground states and shifted ground state*, obtained by shifting the spin on a half-line only. Indeed, although the maximal interaction energy between a half-line left of the origin and a half-plane right of it is infinite, we show in [25] that the expected interaction energy in a state with Dobrushin boundary conditions still remains finite. We use there both the ‘‘antisymmetry’’ between up and down and the ferromagnetic character of the interaction. The argument uses the fact that the interaction of the negative half-line  $\{i < 0, j = 0\}$  and the positive half-line  $\{i \geq 0, j = 0\}$  is finite, while the interaction of the half-line with any plus spin above the line is canceled by the interaction with the reflected minus spin below the line.

To see this, split the lattice  $\mathbb{Z}^2$  into  $A^+ = \{(i, j) : j \geq 1\} \cup \{(i, 0) : i > 0\}$ ,  $A^- = \{(i, j) : j \leq -1\}$  and  $A^0 = \{(i, 0) : i \leq 0\}$ . Consider the Dobrushin ground states  $\sigma_{GS}$  in the sense that we put all +1 in  $A^+ \cup A^0$  and -1 in  $A^-$  with energy  $H^\pm(\sigma_{GS})$ . We call after  $\sigma_{GS,step}$  the configuration  $\sigma_{GS}$  which is flipped on the half line  $A^0$ , consisting thus in +1 in  $A^+$  and -1 in  $A^0 \cup A^-$ , and estimate the energy difference. Then

$$-H(\sigma_{GS}) = \frac{1}{2} \sum_{x,y \in A^+} J_{xy} + \frac{1}{2} \sum_{x,y \in A^-} J_{xy} - \sum_{x \in A^+, y \in A^-} J_{xy} + \sum_{x \in A^+, y \in A^0} J_{xy} - \sum_{x \in A^0, y \in A^-} J_{xy}$$

and

$$-H(\sigma_{GS,step}) = \frac{1}{2} \sum_{x,y \in A^+} J_{xy} + \frac{1}{2} \sum_{x,y \in A^-} J_{xy} - \sum_{x \in A^+, y \in A^-} J_{xy} - \sum_{x \in A^+, y \in A^0} J_{xy} + \sum_{x \in A^0, y \in A^-} J_{xy}$$

writing as before  $x = (x_1, x_2), y = (y_1, y_2)$ , the energy difference is equal to

$$|H(\sigma_{GS}) - H(\sigma_{GS,step})| = 2 \left| \sum_{x \in A^+, y \in A^0} J_{xy} - \sum_{y \in A^0, x \in A^-} J_{xy} \right|$$

which by symmetry of the couplings  $J_{xy}$  is uniformly bounded for  $\alpha > 2$ .

A similar argument will still hold at low but positive temperatures, and we sketch it now. For a complete rigorous proof, consult [25]. The main observation is that the interaction energy of a spin interacting with a half-plane at distance  $l$  is maximally of order  $O(l^{2-\alpha})$ , but its expectation in the Gibbs state with Dobrushin b.c.  $(\pm, h)$  at more or less the same height is  $O(l^{1-\alpha})$ . Summing over the line just above the interface gives then that the total expected energy cost of shifting is uniformly bounded, thus the relative entropy, which between two Gibbs measures corresponds to the expectation of Hamiltonians difference, computed in one of them between the two putative Dobrushin states is finite. This implies that, once they are extremal, these states are the same, using again the same relative entropy arguments as *e.g.* in [13, 58]. This implies the translation invariance of the measures got by weak limits of Dobrushin b.c. and the absence of Dobrushin states.

## 4.2 Rigidity in anisotropic cases

We consider two different but similar anisotropic long-range models on  $\mathbb{Z}^2$  here : First, a mixed long and *n.n.* translation-invariant interaction whose interaction are *n.n.* vertically and ‘Dyson-like’ horizontally, *i.e.* of the form



$$J_{x,y} = 1 \quad \text{if } x_1 = y_1 \quad \text{and} \quad y_2 = x_2 \pm 1; \quad J_{x,y} = 0 \quad \text{if} \quad |y_2 - x_2| > 1$$

and

$$J_{x,y} = |x_1 - y_1|^{-\alpha_1} \quad \text{if} \quad x_2 = y_2$$

and secondly a 'Dyson-like' long-range interactions in both horizontal and vertical directions, with not necessarily the same powers,  $\alpha_1, \alpha_2$ , where at least one of the two is in  $(1, 2)$ :

$$J_{x,y} = |x_2 - y_2|^{-\alpha_2} \quad \text{if} \quad x_1 = y_1, \quad J_{x,y} = |x_1 - y_1|^{-\alpha_1} \quad \text{if} \quad x_2 = y_2$$

Let us see, as proposed in [6] and described in the appendix of [15], that in dimension 2 it is possible to get rigidity for anisotropic, but still rather symmetric long-range models.

In fact, this is the case as soon as one keeps :

- Some monotonicity properties of the couplings  $J$  as a function of the graph distance.
- Symmetry w.r.t. the horizontal axis  $x_2 = 0$  (the one of the Dobrushin b.c.).
- Spontaneous magnetization of the one-dimensional system with the same coupling (decoupled from the rest of the lattice).

We provide a description for the more general model. Recall that it has interactions along the horizontal and vertical axis only, with polynomial decays  $\alpha_1$  and  $\alpha_2$  :

$$\begin{aligned} J_{xy} &= |x_2 - y_2|^{-\alpha_2} & \text{if } x_1 = y_1 \\ J_{xy} &= |x_1 - y_1|^{-\alpha_1} & \text{if } x_2 = y_2 \\ J_{xy} &= 0 & \text{otherwise} \end{aligned} \tag{4.35}$$

**Theorem 6** *For the anisotropic models described above, at low enough temperature and slow horizontal decay  $1 < \alpha_1 \leq 2$ , there exists a Dobrushin state  $\mu^\pm$ , non-translation-invariant and extremal, selected as weak limit with horizontal boundary conditions  $(\pm, 0)$ . It is such that on the horizontal line  $\Delta_1 : x_2 = 0$ ,*

$$\forall x = (x_1, 0) \in \Delta_1, \langle \sigma_x \rangle^\pm > 0$$

**Remark :** The vertical decay does not play any role with this horizontal Dobrushin b.c. but might enter in fluctuations in the case when the interface is not rigid, but rough.

Our proof is a detailed adaptation to the case of infinite-range models performed in the Appendix B of [15]. The particular form of the interactions, with pair-potentials along two lines only, allows us to present completely the proof and to take some shortcuts. Mathematically speaking, it also comes as a generalization already noticed by van Beijeren [6] in its proof of Dobrushin's rigidity result in dimension 3 [30]. To get strict positivity of the magnetization under the putative Dobrushin state, and thus non-translation invariance of any weak limit, one compares it with the spontaneous magnetization of a one-dimensional 'Dyson-like' auxiliary system using a duplicate trick as follows :

1. **Step 1: Duplicate** a configuration  $\sigma \in \{-1, +1\}^{\mathbb{Z}^2}$  from the original  $2d$ -system with Hamiltonian  $H^\pm$  of a  $1d$  long-range Ising model with the same polynomial decay  $\alpha_1$ , restricted to the horizontal line, decoupled from the rest of the plane, with +-b.c. instead

of  $\pm$ . Call it  $H'^+$ , write its ferromagnetic coupling  $J'$  and pick a configuration  $\sigma'$  according to it. By independent duplication, write formally the joint Hamiltonian

$$\mathbb{H}^{(\pm,+)}(\sigma, \sigma') := H^\pm(\sigma) + H'^+(\sigma') \quad (4.36)$$

Consider a volume  $\Lambda = \Lambda_L = ([-L, L] \cap \mathbb{Z})^2$  and partition it naturally as  $\Lambda = \Lambda^+ \cup \Lambda^0 \cup \Lambda^-$ , where  $\Lambda^0$  is the line  $\{x_2 = 0\}$ ,  $\Lambda^+$  the (strict) upper half-plane  $\{x_2 > 0\}$  and  $\Lambda^-$  the lower one  $\{x_2 < 0\}$ . For these volumes, (4.36) reads

$$\begin{aligned} -\mathbb{H}_{\Lambda, \Lambda^0}^{(\pm,+)}(\sigma, \sigma') = & \sum_{x,y \in \Lambda} J_{xy} \sigma_x \sigma_y + \sum_{x \in \Lambda} \left( \sum_{y \in \Lambda^c, y_1 \geq 0} J_{xy} \sigma_x - \sum_{y \in \Lambda^c, y_1 < 0} J_{xy} \sigma_x \right) \\ & + \sum_{x,y \in \Lambda_0} J'_{xy} \sigma'_x \sigma'_y + \sum_{x \in \Lambda_0, y \in \Lambda^c, y_1 = 0} J'_{xy} \sigma'_x \end{aligned} \quad (4.37)$$

To make use of the symmetry w.r.t the horizontal axis, we define the symmetric of  $x \in \mathbb{Z}^2$  as  $\bar{x} = (x_1, -x_2)$  for any  $x = (x_1, x_2) \in \mathbb{Z}^2$  and remark that

$$J_{\bar{x}\bar{y}} = J_{xy} \geq 0, \quad J_{x,\bar{y}} = J_{\bar{x},y} \geq 0, \quad J_{xy} \geq J_{x\bar{y}} \quad (4.38)$$

Rewrite the joint Hamiltonian  $-\mathbb{H}_{\Lambda, \Lambda^0}^{(\pm,+)}(\sigma, \sigma')$  as

$$\sum_{x \in \Lambda^+} \left( \sum_{y \in \Lambda} J_{xy} \sigma_x \sigma_y + \sum_{y: y_1 = x_1, |y_2| > L} \text{sgn}(y_2) J_{xy} \sigma_x + \sum_{y: |y_1| > L, y_2 = x_2} J_{xy} \sigma_x \right) \quad (4.39)$$

$$+ \sum_{x \in \Lambda^-} \left( \sum_{y \in \Lambda} J_{xy} \sigma_x \sigma_y + \sum_{y: y_1 = x_1, |y_2| > L} \text{sgn}(y_2) J_{xy} \sigma_x - \sum_{y: |y_1| > L, y_2 = x_2} J_{xy} \sigma_x \right) \quad (4.40)$$

$$+ \sum_{x \in \Lambda^0} \left( \sum_{y \in \Lambda^0} J_{xy} \sigma_x \sigma_y + \sum_{y: y_1 = x_1, |y_2| < L} J_{xy} \sigma_x \sigma_y + \sum_{y: y_1 = x_1, |y_2| > L} \text{sgn}(y_2) J_{xy} \sigma_x \right) \quad (4.41)$$

$$+ \sum_{y: |y_1| > L, y_2 = 0} J_{xy} \sigma_x + \sum_{y \in \Lambda_0} J'_{xy} \sigma'_x \sigma'_y + \sum_{y: |y_1| > L, y_2 = 0} J'_{xy} \sigma'_x \quad (4.42)$$

Any  $x \in \Lambda^+$  can be mapped one-to-one into  $\bar{x} \in \Lambda^-$  so that (4.40) becomes

$$\sum_{x \in \Lambda^+} \left( \sum_{y \in \Lambda} J_{\bar{x}\bar{y}} \sigma_{\bar{x}} \sigma_y + \sum_{y: y_1 = \bar{x}_1, |y_2| > L} \text{sgn}(y_2) J_{xy} \sigma_{\bar{x}} - \sum_{y: |y_1| > L, y_2 = \bar{x}_2} J_{\bar{x}\bar{y}} \sigma_{\bar{x}} \right)$$

while the restriction to the horizontal line (4.41) can be written

$$\sum_{x \in \Lambda^0} \left( \sum_{y \in \Lambda^0} J_{xy} \sigma_x \sigma_y + \sum_{y_1 = x_1: 0 < y_2 < L} (J_{xy} \sigma_x \sigma_y + J_{x\bar{y}} \sigma_x \sigma_{\bar{y}}) + \sum_{y: y_1 = x_1, y_2 > L} (J_{xy} - J_{x\bar{y}}) \sigma_x \right) \quad (4.43)$$

**2. Step 2. Use the symmetries.** By considering also sites  $y$  in the upper half-plane  $\Lambda^+$

and using symmetric sites, (4.39) and (4.40) merge into a term

$$\begin{aligned}
\mathbb{H}_{\Lambda^+, \partial\Lambda_0} &:= \sum_{x \in \Lambda^+} \left( \sum_{y \in \Lambda^+: y_1 = x_1} (J_{xy} \sigma_x \sigma_y + J_{x\bar{y}} \sigma_x \sigma_{\bar{y}}) + \sum_{y: y_1 = x_1, y_2 > L} (J_{xy} \sigma_x - J_{x\bar{y}} \sigma_x) \right) \\
&+ \sum_{y \in \Lambda^+, y_2 = x_2} J_{xy} \sigma_x \sigma_y + \sum_{y \in \Lambda^c, y_2 = x_2} J_{xy} \sigma_x \\
&+ \sum_{y \in \Lambda^+: y_1 = x_1} (J_{x\bar{y}} \sigma_{\bar{x}} \sigma_y + J_{\bar{x}\bar{y}} \sigma_{\bar{x}} \sigma_{\bar{y}}) + \sum_{y: y_1 = x_1, y_2 > L} (J_{x\bar{y}} \sigma_{\bar{x}} - J_{\bar{x}\bar{y}} \sigma_{\bar{x}}) \\
&+ \sum_{y \in \Lambda^+, y_2 = x_2} J_{x\bar{y}} \sigma_{\bar{x}} \sigma_{\bar{y}} - \sum_{y \in \Lambda^c, y_2 = x_2} J_{x\bar{y}} \sigma_{\bar{x}}
\end{aligned}$$

Use first the symmetries  $J_{x\bar{y}} = J_{xy}$  and  $J_{\bar{x}\bar{y}} = J_{x\bar{y}}$  for all  $x, y \in \Lambda$  :

$$\begin{aligned}
\mathbb{H}_{\Lambda^+, \partial\Lambda_0} &= \sum_{x \in \Lambda^+} \left( \sum_{y \in \Lambda^+: y_1 = x_1} J_{xy} (\sigma_x \sigma_y + \sigma_{\bar{x}} \sigma_{\bar{y}}) + \sum_{y: y_1 = x_1, y_2 > L} (J_{xy} - J_{x\bar{y}}) \sigma_x \right) \\
&+ \sum_{y \in \Lambda^+, y_2 = x_2} J_{xy} \sigma_x \sigma_y + \sum_{y \in \Lambda^c, y_2 = x_2} J_{xy} \sigma_x \\
&+ \sum_{y \in \Lambda^+: y_1 = x_1} (J_{x\bar{y}} \sigma_{\bar{x}} \sigma_y + J_{xy} \sigma_{\bar{x}} \sigma_{\bar{y}}) - \sum_{y: y_1 = x_1, y_2 > L} (J_{xy} - J_{x\bar{y}}) \sigma_{\bar{x}} \\
&+ \sum_{y \in \Lambda^+, y_2 = x_2} J_{xy} \sigma_{\bar{x}} \sigma_{\bar{y}} - \sum_{y \in \Lambda^c, y_2 = x_2} J_{xy} \sigma_{\bar{x}}
\end{aligned}$$

Rearrange terms to eventually get

$$\begin{aligned}
\mathbb{H}_{\Lambda^+, \partial\Lambda_0} &= \sum_{x \in \Lambda^+} \left( \sum_{y \in \Lambda^+: y_1 = x_1} (J_{xy} (\sigma_x \sigma_y + \sigma_{\bar{x}} \sigma_{\bar{y}}) + J_{x\bar{y}} (\sigma_x \sigma_{\bar{y}} + \sigma_{\bar{x}} \sigma_y)) \right) \\
&+ \sum_{y: y_1 = x_1, y_2 > L} (J_{xy} - J_{x\bar{y}}) (\sigma_x - \sigma_{\bar{x}}) \\
&+ \sum_{y \in \Lambda^+, y_2 = x_2} J_{xy} (\sigma_x \sigma_y + \sigma_{\bar{x}} \sigma_{\bar{y}}) + \sum_{y \in \Lambda^c, y_2 = x_2} J_{xy} (\sigma_x - \sigma_{\bar{x}})
\end{aligned}$$

Similar arrangements hold for the horizontal part, using also with that  $J' = J$  on  $\Lambda^0$ , and the fact that on  $\Delta^0$  our symmetry reduces to identity :

$$\begin{aligned}
\mathbb{H}_{\Lambda^0, \partial\Lambda^+} &= \sum_{x \in \Lambda^0} \left( \sum_{y \in \Lambda^0} J_{xy} (\sigma_x \sigma_y + \sigma'_x \sigma'_y) + \sum_{y: y_1 = x_1, y_2 > L} J_{xy} \sigma_x (\sigma_y + \sigma_{\bar{y}}) \right) \\
&+ \sum_{y: |y_1| > L, y_2 = 0} J_{xy} (\sigma_x + \sigma'_x) + \sum_{y: y_1 = x_1, 0 < y_2 < L} (J_{xy} - J_{x\bar{y}}) \sigma_x
\end{aligned}$$

Thus, we need to get information on  $\mathbb{H}_{\Lambda}^{\pm,+}(\sigma, \sigma') = \mathbb{H}_{\Lambda^+, \partial\Lambda_0} + \mathbb{H}_{\Lambda^0, \partial\Lambda^+}$ . We remark that it looks like a ferromagnetic system with the variable  $\sigma_x \sigma_y + \sigma_{\bar{x}} \sigma_{\bar{y}}$  and  $\sigma_x - \sigma_{\bar{x}}$  instead of pair-potential (quadratic) part and a (linear) self-interaction. This is exactly the case in the trick of Percus [104], as follows.

3. **Step 3. Change the variables**  $(\sigma, \sigma')$  into an adaptation of the duplicate set  $\{s, t\}$  of Percus [94, 104]. It uses the symmetric  $\bar{x}$  of any site  $x \in \mathbb{Z}^2$  w.r.t  $\{x_2 = 0\}$  (Compare [15] page 19):

$$\begin{aligned}\forall x \in \Lambda^+, s_x &= \sigma_x + \sigma_{\bar{x}} \quad , \quad t_x = \sigma_x - \sigma_{\bar{x}} \\ \forall x \in \Lambda^0, s_x &= \sigma_x + \sigma'_x \quad , \quad t_x = \sigma_x - \sigma'_x\end{aligned}$$

The new variables take value in  $\{-1, 0, +1\}$  with some trivial constraints, but they have nice extra properties to deal with. In particular one has

$$\begin{aligned}\forall x, y \in \Lambda^+, \sigma_x \sigma_y + \sigma_{\bar{x}} \sigma_{\bar{y}} &= \frac{1}{2}(s_x s_y + t_x t_y) \quad , \quad \sigma_x \sigma_{\bar{y}} + \sigma_{\bar{x}} \sigma_y = \frac{1}{2}(s_x s_y - t_x t_y) \\ \forall x, y \in \Lambda^0, \sigma_x \sigma_y + \sigma'_x \sigma'_y &= \frac{1}{2}(s_x s_y + t_x t_y) \quad , \quad \sigma_x \sigma'_y - \sigma'_x \sigma_y = \frac{1}{2}(s_x s_y - t_x t_y)\end{aligned}$$

and also among other useful relations valid for any  $x$  and  $y$ ,

$$\sigma_x \sigma_y + \sigma_x \sigma_{\bar{y}} = s_y \frac{s_x + t_x}{2}$$

so that we eventually get the joint Hamiltonian

$$\begin{aligned}-\mathbb{H}_{\Lambda, \Lambda^0}^{\pm, +}(s, t) &= \sum_{x \in \Lambda^+} \left( \sum_{y \in \Lambda^+, y_1 = x_1} \left( \frac{J_{xy} + J_{x\bar{y}}}{2} s_x s_y + \frac{J_{xy} - J_{x\bar{y}}}{2} t_x t_y \right) \right. \\ &+ \sum_{y_1 = x_1, y_2 > L} (J_{xy} - J_{x\bar{y}}) t_x + \sum_{y \in \Lambda^+, y_2 = x_2} \frac{J_{xy}}{2} (s_x s_y + t_x t_y) + \sum_{y \in \Lambda^c, y_2 = x_2} J_{xy} t_x \\ &+ \sum_{y \in \Lambda^+, y_1 = x_1} (J_{xy} + J_{x\bar{y}}) \frac{s_x + t_x}{2} s_y + \sum_{y: y_1 = x_1, y_2 > L} (J_{xy} - J_{x\bar{y}}) s_y \Big) \\ &+ \sum_{x \in \Lambda^0} \left( \sum_{y \in \Lambda^0} J_{xy} (s_x s_y + t_x t_y) + \sum_{y_1 = x_1, y_2 > L} J_{xy} s_y \frac{s_x + t_x}{2} + \sum_{y: |y_1| > L, y_2 = 0} J_{xy} s_y \right. \\ &+ \left. \sum_{y: y_1 = x_1, 0 < y_2 < L} (J_{xy} - J_{x\bar{y}}) \frac{s_x + t_x}{2} \right)\end{aligned}$$

#### 4. Step 4 : Correlation inequalities and symmetries

For both  $x, y \in \Lambda^+$ , one has  $J_{xy} \geq J_{x\bar{y}}$  so that from the form (4.44) given above, it is now obvious that the joint Hamiltonian  $\mathbb{H}$  has ferromagnetic pair interactions, or single-site interactions. Then use a generalisation of the GHS inequalities as given in the original Griffiths, Hurst and Sherman or Kelly and Sherman papers [68, 81], or the extension of them by Lebowitz [93, 94]<sup>6</sup> to conclude as in [6, 15] that the coordinates are positively magnetized; in particular

$$\langle t_x \rangle_{\Lambda^+, \Lambda^0}^{\pm, +} \geq 0.$$

Restriction to the horizontal layer gives that

$$\forall x \in \Lambda^0, \langle \sigma_x \rangle_{\Lambda}^{\pm} \geq \langle \sigma'_x \rangle_{\Lambda^0}^+ > 0 \text{ at low T.}$$

<sup>6</sup>See also Georgii [60] p 447 for a precise genealogy, pair-interaction is essential.

The second expectation is performed for the one-dimensional Gibbs states with + b.c. at the same temperature. Thus, as soon as spontaneous magnetization occurs for the latter, this implies the existence of a non-translation-invariant Gibbs states in dimension two. notice that this lower-dimensional phase-transition condition is not fulfilled in the isotropic long-range models treated above, because their well-definedness requires  $\alpha > 2$ , for which there is no phase transition in dimension one. To get such a phase transition and positive magnetization, one has to consider very long-ranges with decays  $1 < \alpha_1 \leq 2$ , acting on a horizontal line only.

**Acknowledgments:** I particularly thank Aernout van Enter, Pierre Picco and Yvan Velenik for particularly interesting discussions and encouragements to work on these long-range Ising models. My research during this period have been partly funded by *Labex Bézout*, funded by ANR, reference ANR-10-LABX-58. A major part of these notes have been prepared during the academic year 2017-2018 at Eurandom (TU/e, Eindhoven), supported by CNRS, Eurandom and the dutch consortium *NETWORKS* (Eurandom-Lorentz Center-CWI).

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