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COLLABORATION WITH PAUL BAUM – CONJECTURES, CONVERSATIONS, CAFÉS, AND CONFERENCES

ANNE-MARIE AUBERT, ROGER PLYMEN, AND MAARTEN SOLLEVELD

1. Paul Baum (by Roger Plymen)

1.1. **The beginning.** My story, in relation to Paul, begins in Fall 1983, lunch time in IHES, France. In those halcyon days before email, smartphones and all the rest, conversation and discussion were much more important. Alain Connes had just had a conversation with Pierre Deligne, a conversation which roamed around the representation theory of p-adic groups, on the one hand, and on "the conjecture" on the other hand. As a result, Alain earnestly encouraged me to work on the conjecture for p-adic groups. I liked the sound of this idea, but there were two issues:

- I did not know what a p-adic number was
- I did not know (neither did anyone else) how to formulate the conjecture for p-adic groups.

I assumed that the right-hand-side of the conjecture (which came to be known as the Baum-Connes conjecture) would not change under reformulation (this has turned out broadly speaking to be true) and started on the complex topological K-theory of the reduced C^*-algebra of reductive p-adic groups such as SL_2. This was, to put it mildly, a steep learning curve.

1.2. **Early days.** Sometime in Spring 1989, shortly after the publication of his seminal paper with Alain Connes [BC], Paul called me with his idea for formulating the BC-conjecture for all second countable locally compact groups.

In his seminal paper with Alain Connes on the Chern character for discrete groups [BC], there may have been intimations of the universal example, but it did not have a name, and was certainly not centre stage. However, while on his exercise bike, Paul had the crucial idea of defining the universal example $\underline{E}G$ as a contractible space on which G acts properly, and bringing this concept centre stage. It became the cornerstone of the left-hand-side of what came to be know as the Baum-Connes conjecture. Paul was then able to bring his vast experience of K-homology into the subject.

In the context of the equivariant Chern character, the *extended quotient* $X//\Gamma$ appeared for the first time in [BC]. The notation $X//\Gamma$ was influenced by an article of Cartier [Car].

I had already (in some kind of anticipation) done some reading on affine buildings, and realized immediately that the affine building of G qualified as

universal example. And so it came about that the BC-conjecture for p-adic groups was, for the first time, well-formulated. With Paul and Nigel Higson, we eventually published a proof of BC for the p-adic general linear group, see [BHP2].

1.3. **LLC for the general linear group** GL_n**.** I learnt the formulation of the LLC (Local Langlands Conjecture) from the excellent expository article by Steve Kudla [K]. At the time, I was trying to relate the BC conjecture for GL_n to the LLC for GL_n. The right-hand-side of BC is formulated in terms of topological K-theory. I wondered whether, thanks to the equivariant Chern character, one could replace the K-theory by the cohomology of an extended quotient, and then strip away the cohomology altogether. Would a valuable geometric statement remain?

The answer is Yes, and after much experimentation, I eventually published with Jacek Brodzki, a CR Note [BrP1] which, for the first time, related the BC-conjecture with the LLC (Local Langlands Conjecture) for GL_n. This was formulated entirely in terms of Langlands parameters, a point of view which has turned out to be very fruitful, see the works of Moussaoui et al [M1, M2, AMS1, AMS2].

1.4. **ABPS.** I was pleased with the CR Note with Jacek, but had no real thoughts on how it might be developed.

It was Paul who realised that [BrP2] should be regarded as a refinement of the Bernstein program, and therefore applicable to all reductive p-adic groups. Anne-Marie Aubert joined the team, and then, some time later, Maarten Solleveld also joined us.

In this way, the ABPS (Aubert-Baum-Plymen-Solleveld) conjecture was born: see Section 3. The ABPS collaboration created a stream of publications, all of them around the ABPS conjecture [ABPS1] – [ABPS10].

1.5. **What have I learnt from Paul?** To seek and find the essential simplicities lurking within an apparently very technical and obscure subject. Paul insists on understanding each mathematical topic very thoroughly before moving on – this may involve asking apparently naive questions. Such questions often uncover gaps in understanding. Working with Paul, I have absorbed, by osmosis, a great deal of algebraic topology.

Paul was one of the first topologists to adapt thoroughly to the noncommutative vision of Alain Connes.

One thing I have found working with Paul is his adaptability. On one occasion, I was working with Steve Millington, trying to prove the BC-conjecture for reductive adelic groups. We were thoroughly stuck, had no real idea how to proceed. Paul arrived, and within a day or two had sufficiently adapted to the issues to make some good suggestions. This, together with a crucial remark of Vincent Lafforgue, eventually led to a complete proof, see [BMP1], [BMP2].

On another occasion I was working remotely with Jamila Jawdat on a rather delicate K-theory issue in connection with the p-adic group SL_n. Paul generously concentrated on these issues for several days, during one of his regular visits to me in England, and we solved the problem, see [JP].

In general, Paul and I like to get away from maths departments, and prefer to work in quiet cafés, preferably with a good view, and a reliable source of food and drink. This routine has served us well over a period of three decades.

2. Working with Paul Baum (by Anne-Marie Aubert)

In 2003, in the middle of the summer holidays, I received an email from Roger Plymen, with whom I was already working, suggesting that Paul should join our project. Roger's email was beginning as follows "I have had long discussions this week with Paul Baum. He has been supplying many new ideas". Soon after, our collaboration started, mainly via exchanging emails and files.

We met sometime later at the Institut Henri Poincaré in Paris that Paul was visiting. I remember that when my mother and I arrived, Paul was at the café working with Nigel Higson. Then we went to his office and had very interesting discussions on our work, on the LLC and on the construction of supercuspidal representations of p-adic groups, a question that has known many developments over the years but that is not yet solved in its full generality. I always learn a lot from talking and exchanging emails with Paul! At this occasion, Paul and my mother managed also to talk together (in French!), and I found out that Paul was bringing his mother too in some of his travels and that she liked meeting his colleagues.

The first talk by Paul that I attended was at the conference in honor of Alain Connes, "Géométrie Non Commutative", again at IHP, in April 2007. A large part of the talk, entitled "Geometric Structure in the Representation Theory of p-adic Groups", was on our joint works [ABP1], [ABP2], on the very early stages of what is now known as the "ABPS conjecture" (see Section 3 below). That year, Paul gave talks on the conjecture in several places, including Oxford, Moscow, Göttingen, Melbourne and Warsaw, and numerous talks on other parts of his work. I am amazed at the number of talks that Paul is able to give on many different subjects and in many different places around the world.

In July 2008, I attended another talk by Paul (with exactly the same title) and again at a birthday conference, with almost the same title "New Directions in Non-commutative Geometry", but this times in Manchester, and in honor of Roger. Paul was speaking right before me and he presented, with a lot of kindness, his talk as an introduction of mine. At that occasion I talked about my painting with Paul, and I discovered with interest that his father, Mark Baum, was a painter.

My most vivid memory of mathematical discussions with Paul and Roger is our day at the reservoir in Manchester. Paul and I were visiting Roger for a few days, and one morning Roger took the two of us (and my mother) to a very pleasant café, and we were sitting there all day talking of math and, at the same time, looking at the beautiful view and enjoying nice food. I remember that we were working on the last section of our G_2 paper [ABP4], and, at the end of the day, essentially all the missing technical points were achieved. We were looking at the representations in the principal series of the p-adic group G_2 which correspond, thanks to Alan Roche's results, to Iwahori-spherical representations of the endoscopic group SO_4. We had partitioned the extended quotient $T^\vee//W$, where $T^\vee$ and $W \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ are the standard maximal torus and the Weyl group of $SO_4(\mathbb{C})$, respectively, into four subsets $(T^\vee//W)_i$ corresponding to the 4 unipotent classes C_1, C_2, C_3, C_4 of $SO_4(\mathbb{C})$, and the issue was to prove that the coordinate algebra of $(T^\vee//W)_i$ is "equivalent" to the part $J_{\mathbf{c}_i}$ of the Lusztig asymptotic algebra J that was attached by him to the two-sided cell $\mathbf{c}_i$ corresponding to the unipotent class C_i. Results on this type for an arbitrary split p-reductive are still far to reach, even in the case of Iwahori-spherical representations. This equivalence is not a Morita equivalence, nor is it spectrum-preserving in general. For instance, for the trivial unipotent class, it is spectrum-preserving with respect to a filtration of length 2. An equivalence of this kind appeared previously in joint work of Paul with Victor Nistor [BN] and in [BrP2]. It is one of the ingredients of the strong form of the ABPS conjecture as stated for split reductive groups in [ABPS1] (based on Paul's Takagi Lectures, at the University of Tokyo in 2012), and is the main subject of [ABPS10].

One of the other mysterious aspects of the conjecture ABPS is its K-theoretical facet, which provides a bridge between two different mathematical worlds: that of BC, and that of the LLC. Paul, Maarten, Roger and I explored this aspect in [ABPS5]. In particular, by using the Bernstein decomposition of the reduced C^*-algebra of a reductive p-adic group, we obtained a very precise (still conjectural in general) description of the K-theory of the latter. I was very glad and honored to have the opportunity to present our work on the subject at the Fields Institute in Toronto, for the "Conference on Geometry, Representation Theory and the Baum-Connes Conjecture", on July the 20th 2016. A very special day: Paul's 80th birthday! This great day saw also talks by Roger, Maarten, Nanhua Xi and Paul himself. That conference was the first time we met all four together as co-workers. It was also a wonderful occasion for me to meet several other co-authors of Paul, working in many different domains.

3. The ABPS conjecture (by Maarten Solleveld)

The first (still rudimentary) version of their conjectures appeared in [ABP1]. I became aware of this paper when I was a PhD-student under the guidance

of Eric Opdam in Amsterdam. This work was strikingly similar to what Eric and I had been trying to do in the context of affine Hecke algebras, so far with limited success. Basically, the common theme was the interplay of noncommutative geometry and representation theory, with the goal to derive results about the latter. The obvious difference between the attempts of Eric and me and [ABP1] was that Anne-Marie, Roger and Paul had managed to formulate and write down their conjectures in a clear way, and to already prove a few instances. This paper, together with the closely related work [BHP3], became a major source of inspiration for my research.

Nevertheless, for some years the ABP-trio and I continued to work independently on their related topics. It was not until Paul's visit to Göttingen in 2011 that this changed. Paul brought with him some very interesting questions, which set me thinking. And then, with his usual relentless enthusiasm, Paul convinced me to join the team. I remember well how Paul and I spent several days discussing our conjectures. Paul, Roger and Anne-Marie had thought about this much more than I had, but I came in with new ideas from affine Hecke algebras. Paul wanted to know everything! He always strove to make things clear from a broad perspective, and could persistently ask questions for hours. Shortly afterwards, these findings were polished in extensive email conversations between the four of us. This led to the definitive formulation of the ABPS conjectures for split groups, published in [ABPS1].

What amazed at that time was Paul's activity. I thought I visited a reasonable number of conferences and seminars, but it was very little compared to Paul – at the tender age of 75. He was travelling the world like an explorer, and just in holidays from teaching, seemingly non-stop! And everywhere he gave talks, in his own unmistakable style. For example, at some point Paul addressed an audience in Kopenhagen. He started with a story about his father, and drew some comparisons between Europe and the United States. The lengthy introduction came to a climax with the words "Europe was in ruins, again. As usual, hahaha!" Needless to say, this was completely irrelevant and unrelated to the mathematical content. But Paul was clearly entertaining everybody in the hall, himself included.

In the years 2011–2017, the ABPS conjecture provided ample inspiration for new papers. We managed to verify it for principal series representations of split groups [ABPS6, ABPS7] and for inner forms of general linear groups. Anne-Marie's PhD student Ahmed Moussaoui found a proof for symplectic and orthogonal groups [M2].

To test our conjecture in more complicated cases, like non-split groups, we embarked on a deep study of inner forms of special linear groups [ABPS4, ABPS8]. To our surprise, this unearthed a counterexample! See [ABPS8, Example 5.5]. Of course this was distressing as well as mighty interesting.

Here Paul's vast experience with conjectures came in handy. Namely, not long before it had become clear that there were serious problems with the Baum–Connes conjecture for non-exact groups. Paul's solution: modify the conjecture, so that all previous supporting examples remain valid, and the problematic cases become confirming examples for the refined conjecture. For BC this has been done by modifying the notion of crossed products [BGW].

For p-adic groups, we did something similar. We generalized the notion of extended quotients to twisted extended quotients, which are relevant for non-split groups. In those terms we finally formulated the ABPS conjectures for all reductive p-adic groups [ABPS9, ABPS5], at the same time making the link with the local Langlands correspondence more precise. These conjectures remain a hard nut to crack. Although there currently is good progress towards a proof of the ABPS conjectures for depth zero representations of arbitrary p-adic groups, a general proof is not yet in sight.

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