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MORITA EQUIVALENCE FOR $k$-ALGEBRAS

ANNE-MARIE AUBERT, PAUL BAUM, ROGER PLYMEN,
AND MAARTEN SOLLEVELD

Abstract. We review Morita equivalence for finite type $k$-algebras $A$
and also a weakening of Morita equivalence which we call stratified equiva-
rence. The spectrum of $A$ is the set of equivalence classes of irreducible
$A$-modules. For any finite type $k$-algebra $A$, the spectrum of $A$ is in
bijection with the set of primitive ideals of $A$. The stratified equivalence
relation preserves the spectrum of $A$ and also preserves the periodic
cyclic homology of $A$. However, the stratified equivalence relation per-
mits a tearing apart of strata in the primitive ideal space which is not
allowed by Morita equivalence. A key example illustrating the distinc-
tion between Morita equivalence and stratified equivalence is provided
by affine Hecke algebras associated to affine Weyl groups.

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1. Introduction

The subject of Morita equivalence begins with the classical paper of Morita [Mor]. Morita equivalence of rings is well-established and widely used [Lam, §18–19].

Now let $k$ be the coordinate algebra of a complex affine variety. Equivalently, $k$ is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. Morita equivalence for $k$-algebras, the starting-point of this article, is a more recent development, which introduces several nuances related to the $k$-module structure of the algebras.

A $k$-algebra is an algebra $A$ over the complex numbers $\mathbb{C}$ which is a $k$-module (with an evident compatibility between the algebra structure of $A$ and the $k$-module structure of $A$). $A$ is not required to have a unit. $A$ is not required to be commutative. A $k$-algebra $A$ is of finite type if as a $k$-module $A$ is finitely generated. The spectrum of $A$ is, by definition, the set of equivalence classes of irreducible $A$-modules. For any finite type $k$-algebra $A$, the spectrum of $A$ is in bijection with the set of primitive ideals of $A$.

The set of primitive ideals of $A$ will be denoted $\text{Prim}(A)$. When $A$ is unital and commutative, let $\text{maxSpec}(A)$ denote the set of maximal ideals in $A$, and let $\text{Spec}(A)$ denote the set of prime ideals in $A$. In particular, let $A = k$. Then we have

$$\text{Prim}(A) \cong \text{maxSpec}(A) \subset \text{Spec}(A).$$

The inclusion is strict, for the prime spectrum $\text{Spec}(A)$ contains a generic point, namely the zero ideal $\mathbf{0}$ in $A$. The set $\text{Prim}(A)$ consists of the $\mathbb{C}$-rational points of the affine variety underlying $k$.

So $\text{Prim}(A)$ is a generalisation, for $k$-algebras $A$ of finite type, of the maximal ideal spectrum of a unital commutative ring, rather than the prime spectrum of a unital commutative ring. For a finite type $k$-algebra $A$, the zero ideal $\mathbf{0}$ is a primitive ideal if and only if $k = \mathbb{C}$ and $A = M_n(\mathbb{C})$ the algebra of all $n$ by $n$ matrices with entries in $\mathbb{C}$. Quite generally, any irreducible representation of a finite type $k$-algebra $A$ is a surjection of $A$ onto $M_n(\mathbb{C})$ for some $n$. This implies that any primitive ideal is maximal.

In some situations, Morita equivalence can be too strong and we are led to introduce a weakening of this concept, which we call stratified equivalence.

The stratified equivalence relation preserves the spectrum of $A$ and also preserves the periodic cyclic homology of $A$. In addition, the stratified equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence.

Denote by $\Gamma$ a finite group. In §10, we show that stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.

A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras. Let $(X, Y, R, R^\vee)$
be a root datum in the standard sense [4]. This root datum delivers the following items:

- a finite Weyl group \( W_0 \)
- an extended affine Weyl group \( W_0 X := W_0 \ltimes X \)
- for each \( q \in \mathbb{C}^\times \), an affine Hecke algebra \( \mathcal{H}_q \)
- a complex torus \( T := \text{Hom}(X, \mathbb{C}^\times) \)
- a complex variety \( X := T / W_0 \)
- a canonical isomorphism \( \mathcal{O}(X) \cong Z(\mathcal{H}_q) \)

Set \( k := \mathcal{O}(X) \). Then, for all \( q \in \mathbb{C}^\times \), \( \mathcal{H}_q \) is a unital finite type \( k \)-algebra. If \( q = 1 \), then \( \mathcal{H}_1 \) is the group algebra of the extended affine Weyl group:

\[
\mathcal{H}_1 = \mathbb{C}[W_0 X]
\]

**Theorem 1.1.** Except for \( q \) in a finite set of roots of unity, none of which is 1, \( \mathcal{H}_q \) and \( \mathcal{H}_1 \) are stratified equivalent as \( k \)-algebras. If \( q \neq 1 \) then \( \mathcal{H}_q \) and \( \mathcal{H}_1 \) are not Morita equivalent as \( k \)-algebras.

A finite type \( k \)-algebra can be viewed as a noncommutative complex affine variety. So the setting of this article can be viewed as noncommutative algebraic geometry.

We conclude this article with a detailed account of the affine Hecke algebra \( \mathcal{H}_q \) attached to the Lie group \( \text{SL}_2(\mathbb{C}) \).

For an application of stratified equivalence to the representation theory of \( p \)-adic groups, see [ABPS].

2. \( k \)-algebras

If \( X \) is an affine algebraic variety over the complex numbers \( \mathbb{C} \), then \( \mathcal{O}(X) \) will denote the coordinate algebra of \( X \). Set \( k = \mathcal{O}(X) \). Equivalently, \( k \) is a unital algebra over the complex numbers which is unital, commutative, finitely generated, and nilpotent-free. The Hilbert Nullstellensatz implies that there is an equivalence of categories

\[
\left( \text{unital commutative} \atop \text{finitely generated} \atop \text{nilpotent-free } \mathbb{C}\text{-algebras} \right) \sim \left( \text{affine complex algebraic varieties} \right)^{\text{op}}
\]

\( \mathcal{O}(X) \mapsto X \)

Here “op” denotes the opposite category.

**Definition 2.1.** A \( k \)-algebra is a \( \mathbb{C} \)-algebra \( A \) such that \( A \) is a unital (left) \( k \)-module with:

\[
\lambda(\omega a) = \omega(\lambda a) = (\lambda \omega)a \quad \forall (\lambda, \omega, a) \in \mathbb{C} \times k \times A
\]

and

\[
\omega(a_1 a_2) = (\omega a_1) a_2 = a_1 (\omega a_2) \quad \forall (\omega, a_1, a_2) \in k \times A \times A.
\]
Remark 2.2. $A$ is not required to have a unit.

The centre of $A$ will be denoted $Z(A)$:

$$Z(A) := \{ c \in A \mid ca = ac \ \forall a \in A \}.$$ 

Remark 2.3. Let $A$ be a unital $k$-algebra. Denote the unit of $A$ by $1_A$. The map $\omega \mapsto \omega.1_A$ is then a unital morphism of $C$-algebras

$$k \rightarrow Z(A)$$

i.e. unital $k$-algebra $=$ unital $C$-algebra $A$ with a given unital morphism of $C$-algebras

$$k \rightarrow Z(A)$$

If $A$ does not have a unit, a $k$-structure is equivalent to a unital morphism of $C$-algebras

$$k \rightarrow Z(MA)$$

where $MA$ is the multiplier algebra of $A$.

Definition 2.4. Let $A,B$ be two $k$-algebras. A morphism of $k$-algebras is a morphism of $C$-algebras

$$f : A \rightarrow B$$

which is also a morphism of (left) $k$-modules,

$$f(\omega a) = \omega f(a) \quad \forall (\omega, a) \in k \times A.$$ 

Definition 2.5. Let $A$ be a $k$-algebra. A representation of $A$ [or a (left) $A$-module] is a $C$-vector space $V$ with given morphisms of $C$-algebras

$$A \rightarrow \text{Hom}_C(V,V)$$

$$k \rightarrow \text{Hom}_C(V,V)$$

such that

1. $k \rightarrow \text{Hom}_C(V,V)$ is unital.
2. $(\omega a)v = \omega (av) = a(\omega v) \ \forall (\omega, a, v) \in k \times A \times V.$

From now on in this article, $A$ will denote a $k$-algebra.

A representation of $A$

$$A \rightarrow \text{Hom}_C(V,V)$$

$$k \rightarrow \text{Hom}_C(V,V)$$

will often be denoted

$$A \rightarrow \text{Hom}_C(V,V)$$

it being understood that the action of $k$ on $V$

$$k \rightarrow \text{Hom}_C(V,V)$$

is part of the given structure.
Definition 2.6. A representation $\varphi: A \to \text{Hom}_C(V,V)$ is non-degenerate if $AV = V$, i.e. for any $v \in V$, there are $v_1, v_2, \ldots, v_r$ in $V$ and $a_1, a_2, \ldots, a_r$ in $A$ with

$$v = a_1v_1 + a_2v_2 + \cdots + a_rv_r.$$ 

Definition 2.7. A representation $\varphi: A \to \text{Hom}_C(V,V)$ is irreducible if $AV \neq \{0\}$ and there is no sub-$C$-vector space $W$ of $V$ with:

$$\{0\} \neq W, \ W \neq V$$

and

$$\omega w \in W \ \forall (\omega, w) \in k \times W$$

and

$$aw \in W \ \forall (a, w) \in A \times W$$

Definition 2.8. Two representations of the $k$-algebra $A$

$$\varphi_1: A \to \text{Hom}_C(V_1, V_1)$$

$$\varphi_2: A \to \text{Hom}_C(V_2, V_2)$$

are equivalent if there is an isomorphism of $C$-vector spaces

$$T: V_1 \to V_2$$

with

$$T(av) = aT(v) \ \forall (a, v) \in A \times V$$

and

$$T(\omega v) = \omega T(v) \ \forall (\omega, v) \in k \times V$$

The spectrum of $A$, also denoted $\text{Irr}(A)$, is the set of equivalence classes of irreducible representations of $A$.

$$\text{Irr}(A) := \{\text{Irreducible representations of } A\}/\sim.$$ 

It can happen that the spectrum is empty. For example, let $A$ comprise all 2 by 2 matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

with $x \in \mathbb{C}$. Then $A$ is a commutative non-unital algebra of finite type (with $k = \mathbb{C}$) such that $\text{Irr}(A) = \emptyset$.

3. On the action of $k$

For a $k$-algebra $A$, $A_C$ denotes the underlying $C$-algebra of $A$. Then $A_C$ is obtained from $A$ by forgetting the action of $k$ on $A$. For $A_C$ there are the usual definitions: A representation of $A_C$ [or a (left) $A_C$-module] is a $C$-vector space $V$ with a given morphism of $C$-algebras

$$A_C \to \text{Hom}_C(V, V)$$
An $A_C$-module $V$ is irreducible if $A_C V \neq \{0\}$ and there is no sub-$C$-vector space $W$ of $V$ with:
\[ \{0\} \neq W , \ W \neq V \]
and
\[ aw \in W \ \forall (a,w) \in A_C \times W \]

Two representations of $A_C$
\[ \varphi_1 : A \to \text{Hom}_C(V_1,V_1) \]
\[ \varphi_2 : A \to \text{Hom}_C(V_2,V_2) \]
are equivalent if there is an isomorphism of $C$-vector spaces $T : V_1 \to V_2$ with
\[ T(av) = aT(v) \ \forall (a,v) \in A \times V \]

$Irr(A_C) := \{\text{Irreducible representations of } A_C\}/\sim$.

An $A_C$-module $V$ for which the following two properties are valid is strictly non-degenerate

- $A_C V = V$
- If $v \in V$ has $av = 0 \ \forall a \in A_C$, then $v = 0$.

**Lemma 3.1.** Any irreducible $A_C$-module is strictly non-degenerate.

**Proof.** Let $V$ be an irreducible $A_C$-module. First, consider $A_C V \subset V$. $A_C V$ is preserved by the action of $A_C$ on $V$. Cannot have $A_C V = \{0\}$ since this would contradict the irreducibility of $V$. Therefore $A_C V = V$.

Next, set
\[ W = \{v \in V | av = 0 \ \forall a \in A_C\} \]

$W$ is preserved by the action of $A_C$ on $V$. $W$ cannot be equal to $V$ since this would imply $A_C V = \{0\}$. Hence $W = \{0\}$. \qed

**Lemma 3.2.** Let $A$ be a $k$-algebra, and let $V$ be a strictly non-degenerate $A_C$-module. Then there is a unique unital morphism of $C$ algebras
\[ k \to \text{Hom}_C(V,V) \]
which makes $V$ an $A$-module.

**Proof.** Given $v \in V$, choose $v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with
\[ v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r \]

For $\omega \in k$, define $\omega v$ by:
\[ \omega v = (\omega a_1) v_1 + (\omega a_2) v_2 + \cdots + (\omega a_r) v_r \]
The second condition in the definition of strictly non-degenerate implies that $\omega v$ is well-defined. \qed
Lemma 3.2 will be referred to as the "$k$-action for free lemma".

If $V$ is an $A$-module, $V_C$ will denote the underlying $A_C$-module. $V_C$ is obtained from $V$ by forgetting the action of $k$ on $V$.

**Lemma 3.3.** If $V$ is any irreducible $A$-module, then $V_C$ is an irreducible $A_C$-module.

**Proof.** Suppose that $V_C$ is not an irreducible $A_C$-module. Then there is a sub-$C$-vector space $W$ of $V$ with:

$$0 \neq W, \ W \neq V$$

and

$$aw \in W \ \forall (a, w) \in A \times W$$

Consider $AW \subset W$. $AW$ is preserved by both the $A$-action on $V$ and the $k$-action on $V$. Thus if $AW \neq \{0\}$, then $V$ is not an irreducible $A$-module. Hence $AW = \{0\}$. Consider $kW \supset W$. $kW$ is preserved by the $k$-action on $V$ and is also preserved by the $A$-action on $V$ because $A$ annihilates $kW$. Since $A$ annihilates $kW$, cannot have $kW = V$. Therefore $\{0\} \neq kW$, $kW \neq V$, which contradicts the irreducibility of the $A$-module $V$. \hfill \square

A corollary of Lemma 3.2 is:

**Corollary 3.4.** For any $k$-algebra $A$, the map

$$\text{Irr}(A) \to \text{Irr}(A_C)$$

$$V \mapsto V_C$$

is a bijection.

**Proof.** Surjectivity follows from Lemmas 3.1 and 3.2. For injectivity, let $V,W$ be two irreducible $A$-modules such that $V_C$ and $W_C$ are equivalent $A_C$-modules. Let $T: V \to W$ be an isomorphism of $C$ vector spaces with

$$T(av) = aT(v) \ \forall (a, v) \in A \times V$$

Given $v \in V$ and $\omega \in k$, choose $v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with

$$v = a_1v_1 + a_2v_2 + \cdots + a_rv_r$$

Then

$$T(\omega v) = T((\omega a_1)v_1 + (\omega a_2)v_2 + \cdots + (\omega a_r)v_r)$$

$$= (\omega a_1)T v_1 + (\omega a_2)T v_2 + \cdots + (\omega a_r)T v_r$$

$$= \omega(a_1Tv_1 + a_2Tv_2 + \cdots + a_rTv_r)$$

$$= \omega(Tv).$$

Hence $T: V \to W$ intertwines the $k$-actions on $V,W$ and thus $V,W$ are equivalent $A$-modules. \hfill \square

4. **Finite type $k$-algebras**

An ideal $I$ in a $k$-algebra $A$ is *primitive* if $I$ is the null-space of an irreducible representation of $A$, i.e. there is an irreducible representation of $A$ $\varphi : A \to \text{Hom}_C(V,V)$ with

$$I = \{a \in A \mid \varphi(a) = 0\}$$
Prim($A$) denotes the set of all primitive ideals in $A$. The evident map $\text{Irr}(A) \to \text{Prim}(A)$ sends an irreducible representation to its null-space. On Prim($A$) there is the Jacobson topology. If $S$ is any subset of Prim($A$), $S \subset \text{Prim}(A)$, then the closure $\overline{S}$ of $S$ is:

$$\overline{S} := \{I \in \text{Prim}(A) | I \supset \cap_{L \in S} L\}$$

**Definition 4.1.** A $k$-algebra $A$ is of **finite type** if, as a $k$-module, $A$ is finitely generated.

For any finite type $k$-algebra $A$, the following three statements are valid:

- If $\varphi: A \to \text{Hom}_C(V,V)$ is any irreducible representation of $A$, then $V$ is a finite dimensional $C$ vector space and $\varphi: A \to \text{Hom}_C(V,V)$ is surjective.
- The evident map $\text{Irr}(A) \to \text{Prim}(A)$ is a bijection.
- Any primitive ideal in $A$ is a maximal ideal.

Since $\text{Irr}(A) \to \text{Prim}(A)$ is a bijection, the Jacobson topology on Prim($A$) can be transferred to Irr($A$) and thus Irr($A$) is topologized. Equivalently, Irr($A$) is topologized by requiring that $\text{Irr}(A) \to \text{Prim}(A)$ be a homeomorphism.

For a finite type $k$-algebra $A$ ($k = \mathcal{O}(X)$), the central character is a map $\text{Irr}(A) \to X$ defined as follows. Let $\varphi$

$$A \to \text{Hom}_C(V,V)$$

$$k \to \text{Hom}_C(V,V)$$

be an irreducible representation of $A$. $I_V$ denotes the identity operator of $V$.

For $\omega \in k = \mathcal{O}(X)$, define $T_\omega: V \to V$ by

$$T_\omega(v) = \omega v$$

for all $v \in V$. Then $T_\omega$ is an intertwining operator for $A \to \text{Hom}_C(V,V)$. By Lemma 3.3 and Schur’s Lemma, $T_\omega$ is a scalar multiple of $I_V$.

$$T_\omega = \lambda_\omega I_V$$

$$\lambda_\omega \in \mathbb{C}$$

The map

$$\omega \mapsto \lambda_\omega$$

is a unital morphism of $\mathbb{C}$ algebras $\mathcal{O}(X) \to \mathbb{C}$ and thus is given by evaluation at a unique ($\mathbb{C}$ rational) point $p_\varphi$ of $X$:

$$\lambda_\omega = \omega(p_\varphi) \quad \forall \omega \in \mathcal{O}(X)$$

The **central character** $\text{Irr}(A) \to X$ is

$$\varphi \mapsto p_\varphi$$

**Remark.** Corollary 3.4 states that $\text{Irr}(A)$ depends only on the underlying $\mathbb{C}$ algebra $A_\mathbb{C}$. The central character $\text{Irr}(A) \to X$, however, does depend on the structure of $A$ as a $k$-module. A change in the action of $k$ on $A_\mathbb{C}$ will
change the central character.

The central character \( \text{Irr}(A) \rightarrow X \) is continuous where \( \text{Irr}(A) \) is topologized as above and \( X \) has the Zariski topology. For a proof of this assertion see [KNS, Lemma 1, p.326]. From a somewhat heuristic non-commutative geometry point of view, \( A_C \) is a non-commutative complex affine variety, and a given action of \( k \) on \( A_C \), making \( A_C \) into a finite type \( k \)-algebra \( A \), determines a morphism of algebraic varieties \( A_C \rightarrow X \).

5. Morita equivalence for \( k \)-algebras

Definition 5.1. Let \( B \) be a \( k \)-algebra. A right \( B \)-module is a \( \mathbb{C} \)-vector space \( V \) with given morphisms of \( \mathbb{C} \)-algebras

\[
B^{\text{op}} \rightarrow \text{Hom}_\mathbb{C}(V,V) \\
k \rightarrow \text{Hom}_\mathbb{C}(V,V)
\]

such that:

1. \( k \rightarrow \text{Hom}_\mathbb{C}(V,V) \) is unital
2. \( v(\omega b) = (v\omega)b = (vb)\omega \quad \forall (v,\omega,b) \in V \times k \times B \).

\( B^{\text{op}} \) is the opposite algebra of \( B \). \( V \) is non-degenerate if \( VB = V \).

Note that a right \( B \)-module is the same as a left \( B^{\text{op}} \)-module.

With \( k \) fixed, let \( A, B \) be two \( k \)-algebras. An \( A-B \) bimodule, denoted \( AV_B \), is a \( \mathbb{C} \) vector space \( V \) such that :

1. \( V \) is a left \( A \)-module.
2. \( V \) is a right \( B \)-module.
3. \( a(vb) = (av)b \quad \forall (a,v,b) \in A \times V \times B \).
4. \( \omega v = v\omega \quad \forall (\omega,v) \in k \times V \).

An \( A-B \) bimodule \( AV_B \) is non-degenerate if \( AV = V = VB \). \( I_V \) is the identity map of \( V \). \( A \) is an \( A-A \) bimodule in the evident way.

Definition 5.2. A \( k \)-algebra \( A \) has local units if given any finite set \( \{a_1,a_2,\ldots,a_r\} \) of elements of \( A \), there is an idempotent \( Q \in A \) (\( Q^2 = Q \)) with

\[ Qa_j = a_jQ = a_j \quad j = 1,2,\ldots,r. \]

Definition 5.3. Let \( A, B \) be two \( k \)-algebras with local units. A Morita equivalence (between \( A \) and \( B \)) is given by a pair of non-degenerate bimodules

\[
AV_B \quad BW_A
\]

together with isomorphisms of bimodules

\[
\alpha: V \otimes_B W \rightarrow A \\
\beta: W \otimes_A V \rightarrow B
\]

such that there is commutativity in the diagrams:
Let $A$, $B$ two $k$-algebras with local units, and suppose given a Morita equivalence

\[ AV_B \quad BW_A \quad \alpha : V \otimes_B W \to A \quad \beta : W \otimes_A V \to B \]

The linking algebra is

\[ L(AV_B, BW_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix} \]

i.e. $L(AV_B, BW_A)$ consists of all $2 \times 2$ matrices having $(1, 1)$ entry in $A$, $(2, 2)$ entry in $B$, $(2, 1)$ entry in $W$, and $(1, 2)$ entry in $V$. Addition and multiplication are matrix addition and matrix multiplication. Note that $\alpha$ and $\beta$ are used in the matrix multiplication.

$L(AV_B, BW_A)$ is a $k$-algebra. With $\omega \in k$, the action of $k$ on $L(AV_B, BW_A)$ is given by

\[ \omega \begin{pmatrix} a & v \\ w & b \end{pmatrix} = \begin{pmatrix} \omega a & \omega v \\ \omega w & \omega b \end{pmatrix} \]

A Morita equivalence between $A$ and $B$ determines an equivalence of categories between the category of non-degenerate left $A$-modules and the category of non-degenerate left $B$-modules. Similarly for right modules. Also, a Morita equivalence determines isomorphisms (between $A$ and $B$) of Hochschild homology, cyclic homology, and periodic cyclic homology.

**Example.** For $n$ a positive integer, let $M_n(A)$ be the $k$-algebra of all $n \times n$ matrices with entries in $A$. If $A$ has local units, $A$ and $M_n(A)$ are Morita equivalent as follows. For $m, n$ positive integers, denote by $M_{m,n}(A)$ the set of all $m \times n$ (i.e. $m$ rows and $n$ columns) matrices with entries in $A$. Matrix multiplication then gives a map

\[ M_{m,n}(A) \times M_{n,r}(A) \to M_{m,r}(A) \]

With this notation, $M_{n,n}(A) = M_n(A)$ and $M_{1,1}(A) = M_1(A) = A$. Hence matrix multiplication gives maps
M\textsubscript{1,n}(A) × M\textsubscript{n}(A) \longrightarrow M\textsubscript{1,n}(A)\quad M\textsubscript{n}(A) × M\textsubscript{n,1} \longrightarrow M\textsubscript{n,1}(A)

Thus $M_{1,n}(A)$ is a right $M_n(A)$-module and $M_{n,1}(A)$ is a left $M_n(A)$-module.

Similarly, $M_{1,n}(A)$ is a left $A$-module and $M_{n,1}(A)$ is a right $A$-module.

With $A = A$ and $B = M_n(A)$, the bimodules of the Morita equivalence are $V = M_{1,n}(A)$ and $W = M_{n,1}(A)$.

Note that the required isomorphisms of bimodules

$\alpha: V \otimes_B W \to A$

$\beta: W \otimes_A V \to B$

are obtained by observing that the matrix multiplication maps

$M_{1,n}(A) \times M_{n,1}(A) \to A$

$M_{n,1}(A) \times M_{1,n}(A) \to M_n(A)$

factor through the quotients $M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A)$, $M_{n,1}(A) \otimes_A M_{1,n}(A)$

and so give bimodule isomorphisms

$\alpha: M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A) \to A$

$\beta: M_{n,1}(A) \otimes_A M_{1,n}(A) \to M_n(A)$

If $A$ has local units, then $\alpha$ and $\beta$ are isomorphisms. Therefore $A$ and $M_n(A)$ are Morita equivalent.

If $A$ does not have local units, then $\alpha$ and $\beta$ can fail to be isomorphisms, and there is no way to prove that $A$ and $M_n(A)$ are Morita equivalent. In examples, this already happens with $n = 1$, and there is then no way to prove (when $A$ does not have local units) that $A$ is Morita equivalent to $A$.

For more details on this issue see below where the proof is given that in the new equivalence relation $A$ and $M_n(A)$ are equivalent even when $A$ does not have local units.

A finite type $k$-algebra $A$ has local units iff $A$ is unital.

6. The Exceptional Set $\mathcal{E}_A(X)$

A Morita equivalence between two finite type $k$-algebras $A, B$ preserves the central character i.e. there is commutativity in the diagram

\[
\begin{array}{ccc}
\text{Irr}(A) & \longrightarrow & \text{Irr}(B) \\
\downarrow_{cc} & & \downarrow_{cc} \\
X & \xrightarrow{I_X} & X \\
\end{array}
\]

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters, and $I_X$ is the identity map of $X$.

By the exceptional set $\mathcal{E}(A)$ we shall mean the set of all $x \in X$ such that the fibre over $x$ has cardinality greater than 1:

$\mathcal{E}(A) := \{ x \in X : \# cc^{-1}(x) > 1 \}$. 
If $A$ and $B$ are Morita equivalent as $k$-algebras, then we will have $\mathcal{E}(A) = \mathcal{E}(B)$. Contrapositively, we have the following useful

**Corollary 6.1.** If $\mathcal{E}(A) \neq \mathcal{E}(B)$ then $A$ and $B$ are not Morita equivalent as $k$-algebras.

7. **Spectrum preserving morphisms**

Let $A$, $B$ two finite type $k$-algebras, and let $f: A \rightarrow B$ be a morphism of $k$-algebras.

**Definition 7.1.** $f$ is spectrum preserving if

1. Given any primitive ideal $J \subset B$, there is a unique primitive ideal $I \subset A$ with $I \supset f^{-1}(J)$

and

2. The resulting map $\text{Prim}(B) \rightarrow \text{Prim}(A)$

is a bijection.

For $I$ and $J$ as above, let $V$ be the simple $B$-module with annihilator $J$. Then the simple $A$-module with annihilator $I$ can be realized as a quotient of $V$ and also as a subspace of $V$.

**Example 7.2.** Let $A$, $B$ two unital finite type $k$-algebras, and suppose given a Morita equivalence

$A \otimes_B W \rightarrow A \otimes_B V \rightarrow B$

With the linking algebra $L(A \otimes_B V, B \otimes_A W)$ as above, the inclusions

$$A \hookrightarrow L(A \otimes_B V, B \otimes_A W) \hookrightarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mapsto b$$

are spectrum preserving morphisms of finite type $k$-algebras. The bijection $\text{Prim}(B) \leftrightarrow \text{Prim}(A)$

so obtained is the bijection determined by the given Morita equivalence.

**Remark.** If $f:A \rightarrow B$ is a spectrum preserving morphism of finite type $k$-algebras, then the resulting bijection $\text{Prim}(B) \leftrightarrow \text{Prim}(A)$

is a homeomorphism. For a proof of this assertion see [BN] Theorem 3, p.342]. Consequently, if $A$, $B$ are two unital finite type $k$-algebras, and

$A \otimes_B W \rightarrow A \otimes_B V \rightarrow B$

is a Morita equivalence, then the resulting bijection $\text{Prim}(B) \leftrightarrow \text{Prim}(A)$

is a homeomorphism.
Definition 7.3. An ideal $I$ in a $k$-algebra $A$ is a $k$-ideal if $\omega a \in I \ \forall \ (\omega, a) \in k \times I$.

Remark. Any primitive ideal in a $k$-algebra $A$ is a $k$-ideal.

Given $A, B$ two finite type $k$-algebras, let $f: A \to B$ be a morphism of $k$-algebras.

Definition 7.4. $f$ is spectrum preserving with respect to filtrations if there are $k$-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A \ \text{in} \ A$$

and $k$ ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{r-1} \subset J_r = B \ \text{in} \ B$$

with $f(I_j) \subset J_j$, $(j = 1, 2, \ldots, r)$ and $I_j/I_{j-1} \to J_j/J_{j-1}$, $(j = 1, 2, \ldots, r)$ is spectrum preserving.

The primitive ideal spaces of the subquotients $I_j/I_{j-1}$ and $J_j/J_{j-1}$ are the strata for stratifications of Prim$(A)$ and Prim$(B)$. Each stratum of Prim$(A)$ is mapped homeomorphically onto the corresponding stratum of Prim$(B)$. However, the map

$$\text{Prim}(A) \to \text{Prim}(B)$$

might not be a homeomorphism.

8. Algebraic Variation of $k$-structure

Let $A$ be a unital $\mathbb{C}$-algebra, and let

$$\Psi: k \to Z \left(A[t, t^{-1}]\right)$$

be a unital morphism of $\mathbb{C}$-algebras. Here $t$ is an indeterminate, so $A[t, t^{-1}]$ is the algebra of Laurent polynomials with coefficients in $A$. As above $Z$ denotes “center”. For $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, $ev(\zeta)$ denotes the “evaluation at $\zeta$" map:

$$ev(\zeta): A[t, t^{-1}] \to A$$

$$\sum a_j t^j \mapsto \sum a_j \zeta^j$$

Consider the composition

$$k \xrightarrow{\Psi} Z \left(A[t, t^{-1}]\right) \xrightarrow{ev(\zeta)} Z(A).$$

Denote the unital $k$-algebra so obtained by $A_\zeta$. $\forall \zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, the underlying $\mathbb{C}$-algebra of $A_\zeta$ is $A$.

$$(A_\zeta)_\mathbb{C} = A \quad \forall \zeta \in \mathbb{C}^\times$$

Such a family $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, of unital $k$-algebras, will be referred to as an algebraic variation of $k$-structure with parameter space $\mathbb{C}^\times$. 
9. Stratified Equivalence

With \( k \) fixed, consider the collection of all finite type \( k \)-algebras. On this collection, \textit{stratified equivalence} is, by definition, the equivalence relation generated by the two elementary steps:

\textbf{Elementary Step 1.} If there is a morphism of \( k \)-algebras \( f: A \to B \) which is spectrum preserving with respect to filtrations, then \( A \sim B \).

\textbf{Elementary Step 2.} If there is \( \{A_\zeta\}, \zeta \in \mathbb{C}^x \), an algebraic variation of \( k \)-structure with parameter space \( \mathbb{C}^x \), such that each \( A_\zeta \) is a unital finite type \( k \)-algebra, then for any \( \zeta, \eta \in \mathbb{C}^x \), \( A_\zeta \sim A_\eta \).

Thus, two finite type \( k \)-algebras \( A, B \) are equivalent if and only if there is a finite sequence \( A_0, A_1, A_2, \ldots, A_r \) of finite type \( k \)-algebras with \( A_0 = A, A_r = B \), and for each \( j = 0, 1, \ldots, r - 1 \) one of the following three possibilities is valid:

\begin{itemize}
  \item a morphism of \( k \)-algebras \( A_j \to A_{j+1} \) is given which is spectrum preserving with respect to filtrations.
  \item a morphism of \( k \)-algebras \( A_j \leftarrow A_{j+1} \) is given which is spectrum preserving with respect to filtrations.
  \item \( \{A_\zeta\}, \zeta \in \mathbb{C}^x \), an algebraic variation of \( k \)-structure with parameter space \( \mathbb{C}^x \), is given such that each \( A_\zeta \) is a unital finite type \( k \)-algebra, and \( \eta, \tau \in \mathbb{C}^x \) have been chosen with \( A_j = A_\eta, A_{j+1} = A_\tau \).
\end{itemize}

To give a stratified equivalence relating \( A \) and \( B \), the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology \([BN]\) are determined:

\[ \text{Prim}(A) \leftrightarrow \text{Prim}(B) \quad \text{HP}_*(A) \cong \text{HP}_*(B) \]

\textbf{Proposition 9.1.} If two unital finite type \( k \)-algebras \( A, B \) are Morita equivalent (as \( k \)-algebras) then they are stratified equivalent.

\[ A \sim_{\text{Morita}} B \implies A \sim B \]

\textbf{Proof.} Let \( A, B \) two unital finite type \( k \)-algebras, and suppose given a Morita equivalence

\[ \alpha: V \otimes_B W \to A \quad \beta: W \otimes_A V \to B \]

The linking algebra is

\[ L(A \otimes_B W) := \begin{pmatrix} A & V \\ W & B \end{pmatrix} \]

The inclusions

\[ A \to L(A \otimes_B W) \leftarrow B \]
are spectrum preserving morphisms of finite type \( k \)-algebras. Hence \( A \) and \( B \) are stratified equivalent.

According to the above, a Morita equivalence of \( A \) and \( B \) gives a homeomorphism

\[
\text{Prim}(A) \simeq \text{Prim}(B)
\]

However, the bijection

\[
\text{Prim}(A) \leftrightarrow \text{Prim}(B)
\]

obtained from a stratified equivalence might not be a homeomorphism, as in the following example.

**Example 9.2.** Let \( Y \) be a sub-variety of \( X \). We will write \( \mathcal{I}_Y \) for the ideal in \( \mathcal{O}(X) \) determined by \( Y \), so that

\[
\mathcal{I}_Y = \{ \omega \in \mathcal{O}(X) \mid \omega(p) = 0 \quad \forall p \in Y \}.
\]

Let \( A \) be the algebra of all \( 2 \times 2 \) matrices whose diagonal entries are in \( \mathcal{O}(X) \) and whose off-diagonal entries are in \( \mathcal{I}_Y \). Addition and multiplication in \( A \) are matrix addition and matrix multiplication. As a \( k \)-module, \( A \) is the direct sum of \( \mathcal{O}(X) \oplus \mathcal{O}(X) \) with \( \mathcal{I}_Y \oplus \mathcal{I}_Y \).

\[
A = \begin{pmatrix}
\mathcal{O}(X) & \mathcal{I}_Y \\
\mathcal{I}_Y & \mathcal{O}(X)
\end{pmatrix}
\]

Set \( B = \mathcal{O}(X) \oplus \mathcal{O}(Y) \), so that \( B \) is the coordinate algebra of the disjoint union \( X \cup Y \). We have \( \mathcal{O}(Y) = \mathcal{O}(X)/\mathcal{I}_Y \). As a \( k = \mathcal{O}(X) \)-module, \( B \) is the direct sum \( \mathcal{O}(X) \oplus (\mathcal{O}(X)/\mathcal{I}_Y) \).

**Theorem 9.3.** The algebras \( A \) and \( B \) are stratified equivalent but not Morita equivalent:

\[
A \sim B \quad A \not\sim_{\text{Morita}} B
\]

**Proof.** Let \( M_2(\mathcal{O}(X)) \) denote the algebra of all \( 2 \times 2 \) matrices with entries in \( \mathcal{O}(X) \). Consider the algebra morphisms

\[
A \rightarrow M_2(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \leftarrow \mathcal{O}(X) \oplus \mathcal{O}(Y)
\]

\[
T \mapsto (T, t_{22}|Y) \quad (T_{\omega}, \theta) \leftrightarrow (\omega, \theta)
\]

where

\[
T = \begin{pmatrix}
t_{11} & t_{12} \\
t_{21} & t_{22}
\end{pmatrix}
\]

\[
T_{\omega} = \begin{pmatrix}
\omega & 0 \\
0 & 0
\end{pmatrix}
\]

The filtration of \( A \) is given by

\[
\{0\} \subset \begin{pmatrix}
\mathcal{O}(X) & \mathcal{I}_Y \\
\mathcal{I}_Y & \mathcal{I}_Y
\end{pmatrix} \subset A
\]

and the filtration of \( M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y) \) is given by
\{0\} \subset M_2(\mathcal{O}(X) \oplus \{0\}) \subset M_2(\mathcal{O}(X) \oplus \mathcal{O}(Y)).

The rightward pointing arrow is spectrum preserving with respect to the indicated filtrations. The leftward pointing arrow is spectrum preserving (no filtrations needed). We infer that \( A \sim B. \)

Note that \( \mathrm{Prim}(A) = X \) with each point of \( Y \) replaced by two points and \( \mathrm{Prim}(B) = \mathrm{Prim}(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \)

The spaces \( \mathrm{Prim}(A) \) and \( \mathrm{Prim}(B) \) are not homeomorphic, and therefore \( A \) is not Morita equivalent to \( B. \)

Unlike Morita equivalence, stratified equivalence works well for finite type \( k \)-algebras whether or not the algebras are unital, e.g. \( A \) and \( M_n(A) \) are stratified equivalent even when \( A \) is not unital. See Proposition 8.5 below.

For any \( k \)-algebra \( A \) there is the evident isomorphism of \( k \)-algebras \( M_n(A) \cong A \otimes \mathbb{C} M_n(\mathbb{C}). \) Hence, using this isomorphism, if \( W \) is a representation of \( A \) and \( U \) is a representation of \( M_n(\mathbb{C}) \), then \( W \otimes U \) is a representation of \( M_n(A) \).

As above, \( M_{n,1}(\mathbb{C}) \) denotes the \( n \times 1 \) matrices with entries in \( \mathbb{C} \). Matrix multiplication gives the usual action of \( M_n(\mathbb{C}) \) on \( M_{n,1}(\mathbb{C}). \)

\[ M_n(\mathbb{C}) \times M_{n,1}(\mathbb{C}) \longrightarrow M_{n,1}(\mathbb{C}) \]

This is the unique irreducible representation of \( M_n(\mathbb{C}). \) For any \( k \)-algebra \( A \), if \( W \) is a representation of \( A \), then \( W \otimes \mathbb{C} M_{n,1}(\mathbb{C}) \) is a representation of \( M_n(A) \).

**Lemma 9.4.** Let \( A \) be a finite type \( k \)-algebra and let \( n \) be a positive integer. Then:

(i) If \( W \) is an irreducible representation of \( A \), \( W \otimes \mathbb{C} M_{n,1}(\mathbb{C}) \) is an irreducible representation of \( M_n(A) \).

(ii) The resulting map \( \mathrm{Irr}(A) \rightarrow \mathrm{Irr}(M_n(A)) \) is a bijection.

**Proof.** For (i), suppose given an irreducible representation \( W \) of \( A \). Let \( J \) be the primitive ideal in \( A \) which is the null space of \( W \). Then the null space of \( W \otimes \mathbb{C} M_{n,1}(\mathbb{C}) \) is \( J \otimes \mathbb{C} M_n(\mathbb{C}). \) Consider the quotient algebra \( A \otimes \mathbb{C} M_n(\mathbb{C})/J \otimes \mathbb{C} M_n(\mathbb{C}) = (A/J) \otimes \mathbb{C} M_n(\mathbb{C}). \) This is isomorphic to \( M_{rn}(\mathbb{C}) \) where \( A/J \cong M_r(\mathbb{C}) \), and so \( W \otimes \mathbb{C} M_{n,1}(\mathbb{C}) \) is irreducible.

**Proposition 9.5.** Let \( A \) be a finite type \( k \)-algebra and let \( n \) be a positive integer, then \( A \) and \( M_n(A) \) are stratified equivalent.
Proof. Let \( f: A \rightarrow M_n(A) \) be the morphism of \( k \)-algebras which maps \( a \in A \) to the diagonal matrix
\[
\begin{bmatrix}
a & 0 & \ldots & 0 \\
0 & a & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & a
\end{bmatrix}
\]
It will suffice to prove that \( f: A \rightarrow M_n(A) \) is spectrum preserving.

Let \( J \) be an ideal in \( A \). Denote by \( J^\circ \) the ideal in \( M_n(A) \) consisting of all \( [a_{ij}] \in M_n(A) \) such that each \( a_{ij} \) is in \( J \). Equivalently, \( M_n(A) \) is \( A \otimes C \otimes M_n(\mathbb{C}) \) and \( J^\circ = J \otimes C \otimes M_n(\mathbb{C}) \). It will suffice to prove

1. If \( J \) is a primitive ideal in \( A \), then \( J^\circ \) is a primitive ideal in \( M_n(A) \).
2. If \( L \) is any primitive ideal in \( M_n(A) \), then there is a primitive ideal \( J \) in \( A \) with \( L = J^\circ \).

For (1), \( J \) primitive \( \implies \) \( J^\circ \) primitive, because the quotient algebra \( M_n(A)/J^\circ \) is \( (A/J) \otimes \mathbb{C} \otimes M_n(\mathbb{C}) \) which is (isomorphic to) \( M_{rn}(\mathbb{C}) \) where \( A/J \cong M_r(\mathbb{C}) \).

For (2), since \( C \) is commutative, the action of \( C \) on \( A \) can be viewed as both a left and right action. Matrix multiplication then gives a left and a right action of \( M_n(C) \) on \( M_n(A) \)

\[
M_n(C) \times M_n(A) \rightarrow M_n(A) \\
M_n(A) \times M_n(C) \rightarrow M_n(A)
\]

for which the associativity rule
\[
(\alpha \theta \beta) = \alpha(\theta \beta) \quad \alpha, \beta \in M_n(A) \quad \theta \in M_n(\mathbb{C})
\]
is valid.

If \( V \) is any representation of \( M_n(A) \), the associativity rule
\[
(\alpha \theta)(\beta v) = \alpha[(\theta \beta)v] \quad \alpha, \beta \in M_n(A) \quad \theta \in M_n(\mathbb{C}) \quad v \in V
\]
is valid.

Now let \( V \) be an irreducible representation of \( M_n(A) \), with \( L \) as its null-space. Define a (left) action
\[
M_n(C) \times V \rightarrow V
\]
of \( M_n(C) \) on \( V \) by proceeding as in the proof of Lemma 4.2 (the “\( k \)-action for free” lemma) i.e. given \( v \in V \), choose \( v_1, v_2, \ldots, v_r \in V \) and \( \alpha_1, \alpha_2, \ldots, \alpha_r \in M_n(A) \) with
\[
v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r
\]
For \( \theta \in M_n(\mathbb{C}) \), define \( \theta v \) by :
\[
\theta v = (\theta \alpha_1) v_1 + (\theta \alpha_2) v_2 + \cdots + (\theta \alpha_r) v_r
\]
The strict non-degeneracy, Lemmas 4.1 and 4.3, of \( V \) implies that \( \theta v \) is well-defined as follows. Suppose that \( u_1, u_2, \ldots, u_s \in V \) and \( \beta_1, \beta_2, \ldots, \beta_r \in M_n(A) \) are chosen with
\[
v = \beta_1 u_1 + \beta_2 u_2 + \cdots + \beta_s u_s
\]
If \( \alpha \) is any element of \( M_n(A) \), then
\[
\alpha[(\theta \alpha_1) v_1 + (\theta \alpha_2) v_2 + \cdots + (\theta \alpha_r) v_r - (\theta \beta_1) u_1 - (\theta \beta_2) u_2 - \cdots - (\theta \beta_s) u_s] =
\]
Moreover, the action of \( E \) Denote by Theorem 10.1. \( \alpha \)
entries. Then, as an
\( J \)
If \( A \)
be an irreducible
module) onto \( E \) is a sub-\( A \)-module of \( V \), where \( \theta V \) is
the matrix in \( M_n(\mathbb{C}) \) which has 1 for its \((i, j)\) entry and zero for all its other entries. Then, as an \( A \)-module, \( V \) is the direct sum
\[ V = E_{11}V \oplus E_{22}V \oplus \cdots \oplus E_{nn}V \]
Moreover, the action of \( E_{ij} \) on \( V \) maps \( E_{jj}V \) isomorphically (as an \( A \)-module) onto \( E_{ii}V \). Hence as an \( M_n(\mathbb{A}) = A \otimes \mathbb{C} M_n(\mathbb{C}) \) module, \( V \) is isomorphic to \( (E_{11}V) \otimes \mathbb{C} \mathbb{C}^n \) — where \( \mathbb{C}^n \) is the standard representation of \( M_n(\mathbb{C}) \) i.e. is the unique irreducible representation of \( M_n(\mathbb{C}) \).
\[ V \cong (E_{11}V) \otimes \mathbb{C} \mathbb{C}^n \]
\( E_{11}V \) is an irreducible \( A \)-module since if not \( V = (E_{11}V) \otimes \mathbb{C} \mathbb{C}^n \) would not be an irreducible \( A \otimes \mathbb{C} M_n(\mathbb{C}) \)-module.
If \( J \) is the null space (in \( A \)) of \( E_{11}V \), then \( J^\circ = J \otimes \mathbb{C} M_n(\mathbb{C}) \) is the null space of \( V = (E_{11}V) \otimes \mathbb{C} \mathbb{C}^n \) and this completes the proof.

10. Permanence properties

**Theorem 10.1.** Denote by \( \Gamma \) a finite group. Stratified equivalence persists under the formation of tensor products \( A \otimes B \), and the formation of crossed products \( A \rtimes \Gamma \).

**Proposition 10.2.** Given \( k_1 \)-algebras \( A \sim A' \) and \( k_2 \)-algebras \( B \sim B' \) then we have
\[ A \otimes B \sim A' \otimes B' \]
as \( k_1 \otimes k_2 \)-algebras, all tensor products over \( \mathbb{C} \).

**Proof.** We first prove that \( A \otimes B \sim A' \otimes B \). Let \( \{I_j\} \) be the filtration for \( A \),
and \( \{I'_j\} \) be the filtration for \( A' \). Then \( \{I_j \otimes B\} \) (resp. \( \{I'_j \otimes B\} \))
will serve as filtrations for \( A \otimes B \), (resp. \( A' \otimes B \)), in the sense that the maps
\[ I_j/I_{j-1} \otimes B \rightarrow I'_j/I'_{j-1} \otimes B \]
are spectrum-preserving. A similar argument will prove that \( A' \otimes B \sim A' \otimes B' \).
Then we have
\[ A \otimes B \sim A' \otimes B \sim A' \otimes B' \]
as required.

Let \( \Psi \) be a variation of \( k_1 \)-structure for \( A \):
\[ k_1 \rightarrow Z(A[t, t^{-1}]) \]
and let
\[ \Psi_B : k_2 \rightarrow Z(B) \]
be the \( k_2 \)-structure for \( B \). Then
\[ \Psi \otimes \Psi_B : k_1 \otimes k_2 \rightarrow Z(A[t, t^{-1}]) \otimes Z(B) \cong Z(A \otimes B)[t, t^{-1}] \]
will be a variation of \( k_1 \otimes k_2 \)-structure for \( A \otimes B \). \qed
Lemma 10.3. Assume that $A$ and $B$ both have $\Gamma$ acting (as automorphisms of $k$-algebras), and we are given a $\Gamma$-equivariant morphism $\phi : A \to B$ which is spectrum preserving. Then $\phi \rtimes \Gamma : A \rtimes \Gamma \to B \rtimes \Gamma$ is a spectrum preserving morphism of $k$-algebras.

Proof. According to Lemma A.1 in [ABPS], $\text{Irr}(A \rtimes \Gamma), \text{Irr}(B \rtimes \Gamma)$ are in bijection with the (twisted) extended quotients:

\[(\text{Irr}(A)/\text{sl} \subset \text{sh.left})_{\Gamma} \cong \text{Irr}(A \rtimes \Gamma)\]
\[(\text{Irr}(B)/\text{sl} \subset \text{sh.left})_{\Gamma} \cong \text{Irr}(B \rtimes \Gamma)\]

Since $\phi : A \to B$ is spectrum preserving, the 2-cocycles $\underline{\epsilon}$ of $\Gamma$ associated to $\text{Irr}(A)$ and $\text{Irr}(B)$ can be identified. Hence $\phi$ determines a bijection of the (twisted) extended quotients

\[(\text{Irr}(A)/\text{sl} \subset \text{sh.left})_{\Gamma} \leftarrow (\text{Irr}(B)/\text{sl} \subset \text{sh.left})_{\Gamma}\]

Applying Lemma A.1 of [ABPS], this implies that $\phi \rtimes \Gamma$ is spectrum preserving. □

Proposition 10.4. If $A$ and $B$ both have $\Gamma$ acting (as automorphisms of $k$-algebras), and we are given a $\Gamma$-equivariant stratified equivalence $A \sim B$, then we have

$A \rtimes \Gamma \sim B \rtimes \Gamma$

Proof. Suppose now that we are given a $\Gamma$-equivariant morphism of $k$-algebras $\phi : A \to B$ which is, in a $\Gamma$-equivariant way, spectrum preserving with respect to filtrations, i.e. the filtrations of $A$ and $B$ are preserved by the action of $\Gamma$. This gives a map of $k$-algebras $\phi \rtimes \Gamma : A \rtimes \Gamma \to B \rtimes \Gamma$. We claim that this map is spectrum-preserving with respect to filtrations.

Using the filtrations, consider the map $I_j/I_{j-1} \to J_j/J_{j-1}$. This is a $\Gamma$-equivariant spectrum preserving map. Hence by Lemma [10.3] above, the map

\[(I_j/I_{j-1}) \rtimes \Gamma \to (J_j/J_{j-1}) \rtimes \Gamma\]

is spectrum preserving. So the filtrations of $A \rtimes \Gamma$ and $B \rtimes \Gamma$ given by $I_j \rtimes \Gamma$ and $J_j \rtimes \Gamma$ are the required filtrations.

Suppose given a $\Gamma$-equivariant variation of $k$-structure for $A$, i.e. a $\Gamma$-equivariant unital map of $C$-algebras

$k \to Z(A[t,t^{-1}])$

where the $\Gamma$-action on $k$ is trivial. Due to the triviality of this action, $k$ is mapped to

$Z(A[t,t^{-1}])^\Gamma = (Z(A))^\Gamma[t,t^{-1}]$

By the standard inclusion

\[(Z(A))^\Gamma \subset Z(A \rtimes \Gamma)\]

$k$ is mapped to $Z(A \rtimes \Gamma)[t,t^{-1}]$:

$k \to Z(A \rtimes \Gamma)[t,t^{-1}]$

which is the required variation of $k$-structure for the crossed product $A \rtimes \Gamma$. □
11. Affine Hecke Algebras

Let \((X,Y,R,R')\) be a root datum in the standard sense [L, p.73]. This root datum delivers the following items:

- a finite Weyl group \(W_0\)
- an extended affine Weyl group \(W_0 X := W_0 \ltimes X\)
- for each \(q \in \mathbb{C}^\times\), an affine Hecke algebra \(H_q\)
- a complex torus \(T := \text{Hom}(X, \mathbb{C}^\times)\)
- a complex variety \(X := T/W_0\)
- a canonical isomorphism \(O(X) \simeq Z(H_q)\)

The detailed construction of \(H_q\) is described in the article of Lusztig [L, p.74]. Set \(k := O(X)\). Then, for all \(q \in \mathbb{C}^\times\), \(H_q\) is a unital finite type \(k\)-algebra. If \(q = 1\), then \(H_1\) is the group algebra of the extended affine Weyl group:

\[ H_1 = \mathbb{C}[W_0 X] \]


**Theorem 11.1.** Except for \(q\) in a finite set of roots of unity, none of which is 1, \(H_q\) is stratified equivalent to \(H_1\):

\[ H_q \sim H_1. \]

**Proof.** Let \(J\) be Lusztig’s asymptotic algebra [L, §8]. As a \(\mathbb{C}\)-vector space, \(J\) has a basis \(\{t_v : v \in W_0 X\}\), and there is a canonical structure of associative \(\mathbb{C}\)-algebra on \(J\). The algebra \(H_q\) is viewed as a \(k\)-algebra via the canonical isomorphism

\[ O(T/W_0) \simeq Z(H_q). \]

Lusztig’s map \(\phi_q\) maps \(Z(H_q)\) to \(Z(J)\) by Proposition 1 in [ABP] and thus determines a unique \(k\)-structure for \(J\) such that the map \(\phi_q\) is a morphism of \(k\)-algebras. \(J\) with this \(k\)-structure will be denoted \(J_q\).

Let

\[ a : W_0 X \to \mathbb{N} \cup \{0\} \]

be the weight function defined by Lusztig.

The algebra \(J\) has a unit element of the form \(1 = \sum_{d \in D} t_v^d\) where \(D\) is a certain set of involutions in \(W_a\). For any \(q \in \mathbb{C}^\times\), the \(\mathbb{C}\)-linear map \(\psi_q : H_q \to J\) defined by

\[ \phi_q(C_w) = \sum h_{w,d,v} t_v \]

is a \(\mathbb{C}\)-linear homomorphism preserving 1. The summation is over the set \(\{d \in D, v \in W_0 X : a(v) = a(d)\}\).

The elements \(C_w\) are defined in [L, p.82] and form a \(\mathbb{C}\)-basis of \(H_q\). The coefficients \(h_{w,d,v}\) originate in the multiplication rule for the \(C_w\), see [L, p.82].

The map \(\phi_q\) is injective. Thus all algebras \(H_q\) with \(q \in \mathbb{C}^\times\) appear as subalgebras of a single \(\mathbb{C}\)-algebra \(J\).
Define
\[ J_k = \text{ideal generated by } \{ t_v \in J : a(v) \leq k \} \]
\[ I_k = \phi_q^{-1}(J_k) \]

We have the filtrations
\[ 0 \subset I_1 \subset I_2 \subset \ldots \subset I_r = \mathcal{H}_q \]
\[ 0 \subset J_1 \subset J_2 \subset \ldots \subset J_r = J \]

We have
\[ \text{Irr}(J_k) = \bigcup \text{Irr}(J_k/J_{k-1}) \]
\[ \text{Irr}(\mathcal{H}_q)_k = \bigcup \text{Irr}(I_k/I_{k-1}) \]

Let \( M \) be a simple \( \mathcal{H}_q \)-module (resp. simple \( J \)-module). The weight \( a_M \) of \( M \) is defined by Lusztig in \([L, p.82]\). Let \( \text{Irr}(\mathcal{H}_q)_k \) (resp. \( \text{Irr}(J)_k \)) denote the set of simple \( \mathcal{H}_q \)-modules (resp. simple \( J \)-modules) of weight \( k \). It follows from Lusztig’s definition that we have
\[ \text{Irr}(J_k/J_{k-1}) = \text{Irr}(J)_k \]
\[ \text{Irr}(I_k/I_{k-1}) = \text{Irr}(\mathcal{H}_q)_k \]

Consider the commutative diagram
\[
\begin{array}{ccc}
\mathcal{H}_q & \xrightarrow{\phi_q} & J \\
\uparrow & & \uparrow \\
I_k & \xrightarrow{\phi_q|_{I_k}} & J_k \\
\downarrow & & \downarrow \\
I_k/I_{k-1} & \xrightarrow{(\phi_q)_k} & J_k/J_{k-1} \\
\end{array}
\]

The middle horizontal map is the restriction of \( \phi_q \) to \( I_k \). This map is defined by equation (1) but with summation over the set \( \{ d \in D, v \in W_0X : a(v) = a(d) \leq k \} \).

The bottom horizontal map is middle horizontal map after descent to the quotient \( I_k/I_{k-1} \). This map is defined by equation (1) but with summation over the set \( \{ d \in D, v \in W_0X : a(v) = a(d) = k \} \).

We now apply one of Lusztig’s main theorems, Theorem 8.1 in \([L]\): Assume that \( q \in \mathbb{C}^\times \) is either 1 or is not a root of 1. Let \( M' \) be a simple \( J \)-module of weight \( a_{M'} \). The pre-image \( \phi_q^{-1}(M') \) will contain a unique subquotient \( M \) of maximal weight \( a_M = a_{M'} \). Then the map \( M' \mapsto M \) is a bijection from the set of simple \( J \)-modules (up to isomorphism) to the set of simple \( \mathcal{H}_q \)-modules (up to isomorphism).

We reconcile our construction with Lusztig’s bijection by observing that our map
\[ (\phi_q)_k : I_k/I_{k-1} \to J_k/J_{k-1} \]
determines a map
\[ \text{Irr}(J)_k \to \text{Irr}(\mathcal{H}_q)_k \]
This map is weight-preserving, by construction, and coincides with the Lusztig bijection. The map \( \phi_q \) is spectrum preserving with respect to the filtrations \( \mathcal{H}_q \).

\( \mathcal{H}_q \) is then stratified equivalent to \( \mathcal{H}_1 \) by the three elementary steps

\[ H_q \sim J_q \sim J_1 \sim H_1. \]

The second elementary step (i.e. passing from \( J_q \) to \( J_1 \)) is an algebraic variation of \( k \)-structure with parameter space \( \mathbb{C} \times \). The first elementary step uses Lusztig’s map \( \phi_q \), and the third elementary step uses Lusztig’s map \( \phi_1 \).

Hence (provided \( q \) is not in the exceptional set of roots of unity—none of which is 1) \( \mathcal{H}_q \) is stratified equivalent to

\[ H_1 = \mathbb{C}[W_0 X] = \mathcal{O}(T) \rtimes W_0. \]

\( \square \)

**Remark 11.2.** In Theorem 11.1 the condition on \( q \) can be replaced by the following more precise condition:

\[ \sum_{w \in W_0} q^{\ell(w)} \neq 0 \]

where \( \ell : W_0 X \rightarrow \mathbb{N} \) is the length function, see [Xi2, Theorem 3.2].

If \( q \neq 1 \) then \( \mathcal{H}_q \) and \( \mathcal{H}_1 \) are not Morita equivalent as \( k \)-algebras, for the exceptional sets are not equal:

\[ q \neq 1 \implies \mathcal{E}(\mathcal{H}_q) \neq \mathcal{E}(\mathcal{H}_1). \]

### 11.2. The affine Hecke algebras of \( \text{SL}_2(\mathbb{C}) \)

If the root datum \((X, Y, R, R^\vee)\) arises from a connected complex reductive Lie group \( G \), then we will write \( \mathcal{H}_q(G) = \mathcal{H}_q \).

An interesting situation arises for \( \mathcal{H}_q(\text{SL}_2(\mathbb{C})) \). Let \( s, t \) be the simple reflections in the (extended) affine Weyl group \( W_0 X \). When \( q + 1 \neq 0 \) there is an isomorphism of \( \mathbb{C} \)-algebras between \( \mathcal{H}_q \) and \( \mathcal{H}_1 \):

\[ \mathcal{H}_q \cong \mathcal{H}_1 \]

\[ T_s \mapsto q \frac{1 + s}{2} + \frac{q - 1}{2}, \quad T_t \mapsto q \frac{1 + t}{2} + \frac{q - 1}{2} \]

On the other hand, when \( q \neq 1 \), \( \mathcal{H}_q \) and \( \mathcal{H}_1 \) cannot be isomorphic as \( k \)-algebras, for the exceptional sets are different. Let \( T \) denote the standard maximal torus in \( \text{SL}_2(\mathbb{C}) \):

\[ \left\{ \left( \begin{array}{cc} z & 0 \\ 0 & 1/z \end{array} \right) : z \neq 0 \right\} \]

We have \( W_0 = \mathbb{Z}/2\mathbb{Z}, k = \mathcal{O}(T/W_0) \) the algebra of \( \mathbb{Z}/2\mathbb{Z} \)-invariant Laurent polynomials in one variable. Note that the non-trivial element of \( W \) acts by

\[ \left( \begin{array}{cc} z & 0 \\ 0 & 1/z \end{array} \right) \mapsto \left( \begin{array}{cc} 1/z & 0 \\ 0 & z \end{array} \right) \]

The quotient variety \( T/W_0 \) comprises unordered pairs \( \{z_1, z_2\} \) of nonzero complex numbers which satisfy the equation \( z_1 z_2 = 1 \).
The exceptional sets are
\[ E(\mathcal{H}_1) = \{-1,-1\} \sqcup \{1,1\} \]
\[ E(\mathcal{H}_q) = \{-1,-1\} \sqcup \{q,1/q\} \]

As \( q \) with \( q \neq 1 \) is deformed to 1, the point \( \{-1,-1\} \) stays fixed, and the point \( \{q,1/q\} \) moves to the point \( \{1,1\} \).

11.3. The affine Hecke algebras of \( SL_3(\mathbb{C}) \).

The case of \( SL_3(\mathbb{C}) \) is considered in §11.7 of Xi [Xi] where it is proved that \( \mathcal{H}_q \) and \( \mathcal{H}_1 \) are not isomorphic as \( \mathbb{C} \)-algebras whenever \( q \neq 1 \). As \( k \)-algebras, \( \mathcal{H}_q \) is not Morita equivalent to \( \mathcal{H}_1 \) whenever \( q \neq 1 \) because the exceptional set \( E(\mathcal{H}_q) \) with \( q \neq 1 \) is not the same as \( E(\mathcal{H}_1) \).

In contrast to this, except for a finite set of roots of unity (none of which is 1), according to Theorem 11.1, \( \mathcal{H}_q \) is stratified equivalent to \( \mathcal{H}_1 \).

References


