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MORITA EQUIVALENCE FOR k -ALGEBRAS

ANNE-MARIE AUBERT, PAUL BAUM, ROGER PLYMEN,
AND MAARTEN SOLLEVELD

ABSTRACT. We review Morita equivalence for finite type k -algebras A and also a weakening of Morita equivalence which we call *stratified equivalence*. The spectrum of A is the set of equivalence classes of irreducible A -modules. For any finite type k -algebra A , the spectrum of A is in bijection with the set of primitive ideals of A . The stratified equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A . However, the stratified equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence. A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras associated to affine Weyl groups.

CONTENTS

1. Introduction	2
2. k -algebras	3
3. On the action of k	5
4. Finite type k -algebras	7
5. Morita equivalence for k -algebras	9
6. The exceptional set $\mathfrak{E}_A(X)$	11
7. Spectrum preserving morphisms	12
8. Algebraic Variation of k -structure	13
9. Stratified Equivalence	14
10. Permanence properties	18
11. Affine Hecke Algebras	20
11.1. Stratified equivalence for affine Hecke algebras	20
11.2. The affine Hecke algebras of $\mathrm{SL}_2(\mathbb{C})$	22
11.3. The affine Hecke algebras of $\mathrm{SL}_3(\mathbb{C})$	23
References	23

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1. INTRODUCTION

The subject of Morita equivalence begins with the classical paper of Morita [Mor]. Morita equivalence of rings is well-established and widely used [Lam, §18–19].

Now let k be the coordinate algebra of a complex affine variety. Equivalently, k is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. Morita equivalence for k -algebras, the starting-point of this article, is a more recent development, which introduces several nuances related to the k -module structure of the algebras.

A k -algebra is an algebra A over the complex numbers \mathbb{C} which is a k -module (with an evident compatibility between the algebra structure of A and the k -module structure of A). A is not required to have a unit. A is not required to be commutative. A k -algebra A is of finite type if as a k -module A is finitely generated. The spectrum of A is, by definition, the set of equivalence classes of irreducible A -modules. For any finite type k -algebra A , the spectrum of A is in bijection with the set of primitive ideals of A .

The set of primitive ideals of A will be denoted $\text{Prim}(A)$. When A is unital and commutative, let $\text{maxSpec}(A)$ denote the set of maximal ideals in A , and let $\text{Spec}(A)$ denote the set of prime ideals in A . In particular, let $A = k$. Then we have

$$\text{Prim}(A) \simeq \text{maxSpec}(A) \subset \text{Spec}(A).$$

The inclusion is strict, for the prime spectrum $\text{Spec}(A)$ contains a generic point, namely the zero ideal $\mathbf{0}$ in A . The set $\text{Prim}(A)$ consists of the \mathbb{C} -rational points of the affine variety underlying k .

So $\text{Prim}(A)$ is a generalisation, for k -algebras A of finite type, of the maximal ideal spectrum of a unital commutative ring, rather than the prime spectrum of a unital commutative ring. For a finite type k -algebra A , the zero ideal $\mathbf{0}$ is a primitive ideal if and only if $k = \mathbb{C}$ and $A = M_n(\mathbb{C})$ the algebra of all n by n matrices with entries in \mathbb{C} . Quite generally, any irreducible representation of a finite type k -algebra A is a surjection of A onto $M_n(\mathbb{C})$ for some n . This implies that any primitive ideal is maximal.

In some situations, Morita equivalence can be too strong and we are led to introduce a weakening of this concept, which we call *stratified equivalence*.

The stratified equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A . In addition, the stratified equivalence relation permits a tearing apart of *strata* in the primitive ideal space which is not allowed by Morita equivalence.

Denote by Γ a finite group. In §10, we show that stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.

A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras. Let (X, Y, R, R^\vee)

be a root datum in the standard sense [L]. This root datum delivers the following items:

- a finite Weyl group W_0
- an extended affine Weyl group $W_0X := W_0 \ltimes X$
- for each $q \in \mathbb{C}^\times$, an affine Hecke algebra \mathcal{H}_q
- a complex torus $T := \text{Hom}(X, \mathbb{C}^\times)$
- a complex variety $X := T/W_0$
- a canonical isomorphism $\mathcal{O}(X) \simeq Z(\mathcal{H}_q)$

Set $k := \mathcal{O}(X)$. Then, for all $q \in \mathbb{C}^\times$, \mathcal{H}_q is a unital finite type k -algebra. If $q = 1$, then \mathcal{H}_1 is the group algebra of the extended affine Weyl group:

$$\mathcal{H}_1 = \mathbb{C}[W_0X]$$

Theorem 1.1. *Except for q in a finite set of roots of unity, none of which is 1, \mathcal{H}_q and \mathcal{H}_1 are stratified equivalent as k -algebras. If $q \neq 1$ then \mathcal{H}_q and \mathcal{H}_1 are not Morita equivalent as k -algebras.*

A finite type k -algebra can be viewed as a noncommutative complex affine variety. So the setting of this article can be viewed as noncommutative algebraic geometry.

We conclude this article with a detailed account of the affine Hecke algebra \mathcal{H}_q attached to the Lie group $\text{SL}_2(\mathbb{C})$.

For an application of stratified equivalence to the representation theory of p -adic groups, see [ABPS].

2. k -ALGEBRAS

If X is an affine algebraic variety over the complex numbers \mathbb{C} , then $\mathcal{O}(X)$ will denote the coordinate algebra of X . Set $k = \mathcal{O}(X)$. Equivalently, k is a unital algebra over the complex numbers which is unital, commutative, finitely generated, and nilpotent-free. The Hilbert Nullstellensatz implies that there is an equivalence of categories

$$\left(\begin{array}{l} \text{unital commutative} \\ \text{finitely generated} \\ \text{nilpotent-free } \mathbb{C}\text{-algebras} \end{array} \right) \sim \left(\begin{array}{l} \text{affine complex al-} \\ \text{gebraic varieties} \end{array} \right)^{\text{op}}$$

$$\mathcal{O}(X) \mapsto X$$

Here “op” denotes the opposite category.

Definition 2.1. A k -algebra is a \mathbb{C} -algebra A such that A is a unital (left) k -module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda \omega) a \quad \forall (\lambda, \omega, a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1 a_2) = (\omega a_1) a_2 = a_1 (\omega a_2) \quad \forall (\omega, a_1, a_2) \in k \times A \times A.$$

Remark 2.2. A is not required to have a unit.

The centre of A will be denoted $Z(A)$:

$$Z(A) := \{c \in A \mid ca = ac \ \forall a \in A\}.$$

Remark 2.3. Let A be a unital k -algebra. Denote the unit of A by 1_A . The map $\omega \mapsto \omega.1_A$ is then a unital morphism of \mathbb{C} -algebras

$$k \longrightarrow Z(A)$$

i.e. unital k -algebra = unital \mathbb{C} -algebra A with a given unital morphism of \mathbb{C} -algebras

$$k \longrightarrow Z(A)$$

If A does not have a unit, a k -structure is equivalent to a unital morphism of \mathbb{C} -algebras

$$k \rightarrow Z(\mathcal{M}A)$$

where $\mathcal{M}A$ is the multiplier algebra of A .

Definition 2.4. Let A, B be two k -algebras. A *morphism of k -algebras* is a morphism of \mathbb{C} -algebras

$$f: A \rightarrow B$$

which is also a morphism of (left) k -modules,

$$f(\omega a) = \omega f(a) \quad \forall (\omega, a) \in k \times A.$$

Definition 2.5. Let A be a k -algebra. A *representation* of A [or a (left) A -module] is a \mathbb{C} -vector space V with given morphisms of \mathbb{C} -algebras

$$A \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

$$k \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

such that

- (1) $k \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$ is unital.
- (2) $(\omega a)v = \omega(av) = a(\omega v) \quad \forall (\omega, a, v) \in k \times A \times V$.

From now on in this article, A will denote a k -algebra.

A representation of A

$$A \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

$$k \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

will often be denoted

$$A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

it being understood that the action of k on V

$$k \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

is part of the given structure.

Definition 2.6. A representation $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ is *non-degenerate* if $AV = V$, i.e. for any $v \in V$, there are v_1, v_2, \dots, v_r in V and a_1, a_2, \dots, a_r in A with

$$v = a_1 v_1 + a_2 v_2 + \dots + a_r v_r.$$

Definition 2.7. A representation $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ is *irreducible* if $AV \neq \{0\}$ and there is no sub- \mathbb{C} -vector space W of V with:

$$\{0\} \neq W, \quad W \neq V$$

and

$$\omega w \in W \quad \forall (\omega, w) \in k \times W$$

and

$$aw \in W \quad \forall (a, w) \in A \times W$$

Definition 2.8. Two representations of the k -algebra A

$$\varphi_1: A \rightarrow \text{Hom}_{\mathbb{C}}(V_1, V_1)$$

$$\varphi_2: A \rightarrow \text{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of \mathbb{C} -vector spaces

$$T: V_1 \rightarrow V_2$$

with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V$$

and

$$T(\omega v) = \omega T(v) \quad \forall (\omega, v) \in k \times V$$

The *spectrum* of A , also denoted $\text{Irr}(A)$, is the set of equivalence classes of irreducible representations of A .

$$\text{Irr}(A) := \{\text{Irreducible representations of } A\} / \sim.$$

It can happen that the spectrum is empty. For example, let A comprise all 2 by 2 matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

with $x \in \mathbb{C}$. Then A is a commutative non-unital algebra of finite type (with $k = \mathbb{C}$) such that $\text{Irr}(A) = \emptyset$.

3. ON THE ACTION OF k

For a k -algebra A , $A_{\mathbb{C}}$ denotes the underlying \mathbb{C} -algebra of A . Then $A_{\mathbb{C}}$ is obtained from A by forgetting the action of k on A . For $A_{\mathbb{C}}$ there are the usual definitions : A *representation* of $A_{\mathbb{C}}$ [or a (left) $A_{\mathbb{C}}$ -*module*] is a \mathbb{C} -vector space V with a given morphism of \mathbb{C} -algebras

$$A_{\mathbb{C}} \longrightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

An $A_{\mathbb{C}}$ -module V is *irreducible* if $A_{\mathbb{C}}V \neq \{0\}$ and there is no sub- \mathbb{C} -vector space W of V with:

$$\{0\} \neq W, W \neq V$$

and

$$aw \in W \quad \forall (a, w) \in A_{\mathbb{C}} \times W$$

Two representations of $A_{\mathbb{C}}$

$$\varphi_1: A \rightarrow \text{Hom}_{\mathbb{C}}(V_1, V_1)$$

$$\varphi_2: A \rightarrow \text{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of \mathbb{C} -vector spaces

$$T: V_1 \rightarrow V_2$$

with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V$$

$\text{Irr}(A_{\mathbb{C}}) := \{\text{Irreducible representations of } A_{\mathbb{C}}\} / \sim$.

An $A_{\mathbb{C}}$ -module V for which the following two properties are valid is *strictly non-degenerate*

- $A_{\mathbb{C}}V = V$
- If $v \in V$ has $av = 0 \quad \forall a \in A_{\mathbb{C}}$, then $v = 0$.

Lemma 3.1. *Any irreducible $A_{\mathbb{C}}$ -module is strictly non-degenerate.*

Proof. Let V be an irreducible $A_{\mathbb{C}}$ -module. First, consider $A_{\mathbb{C}}V \subset V$. $A_{\mathbb{C}}V$ is preserved by the action of $A_{\mathbb{C}}$ on V . Cannot have $A_{\mathbb{C}}V = \{0\}$ since this would contradict the irreducibility of V . Therefore $A_{\mathbb{C}}V = V$.

Next, set

$$W = \{v \in V \mid av = 0 \quad \forall a \in A_{\mathbb{C}}\}$$

W is preserved by the action of $A_{\mathbb{C}}$ on V . W cannot be equal to V since this would imply $A_{\mathbb{C}}V = \{0\}$. Hence $W = \{0\}$. □

Lemma 3.2. *Let A be a k -algebra, and let V be a strictly non-degenerate $A_{\mathbb{C}}$ -module. Then there is a unique unital morphism of \mathbb{C} algebras*

$$k \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$$

which makes V an A -module.

Proof. Given $v \in V$, choose $v_1, v_2, \dots, v_r \in V$ and $a_1, a_2, \dots, a_r \in A$ with

$$v = a_1v_1 + a_2v_2 + \dots + a_rv_r$$

For $\omega \in k$, define ωv by :

$$\omega v = (\omega a_1)v_1 + (\omega a_2)v_2 + \dots + (\omega a_r)v_r$$

The second condition in the definition of strictly non-degenerate implies that ωv is well-defined. □

Lemma 3.2 will be referred to as the “ k -action for free lemma”.

If V is an A -module, $V_{\mathbb{C}}$ will denote the underlying $A_{\mathbb{C}}$ -module. $V_{\mathbb{C}}$ is obtained from V by forgetting the action of k on V .

Lemma 3.3. *If V is any irreducible A -module, then $V_{\mathbb{C}}$ is an irreducible $A_{\mathbb{C}}$ -module.*

Proof. Suppose that $V_{\mathbb{C}}$ is not an irreducible $A_{\mathbb{C}}$ -module. Then there is a sub- \mathbb{C} -vector space W of V with:

$$0 \neq W, \quad W \neq V$$

and

$$aw \in W \quad \forall (a, w) \in A \times W$$

Consider $AW \subset W$. AW is preserved by both the A -action on V and the k -action on V . Thus if $AW \neq \{0\}$, then V is not an irreducible A -module. Hence $AW = \{0\}$. Consider $kW \supset W$. kW is preserved by the k -action on V and is also preserved by the A -action on V because A annihilates kW . Since A annihilates kW , cannot have $kW = V$. Therefore $\{0\} \neq kW$, $kW \neq V$, which contradicts the irreducibility of the A -module V . \square

A corollary of Lemma 3.2 is :

Corollary 3.4. *For any k -algebra A , the map*

$$\begin{aligned} \text{Irr}(A) &\rightarrow \text{Irr}(A_{\mathbb{C}}) \\ V &\mapsto V_{\mathbb{C}} \end{aligned}$$

is a bijection.

Proof. Surjectivity follows from Lemmas 3.1 and 3.2. For injectivity, let V, W be two irreducible A -modules such that $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are equivalent $A_{\mathbb{C}}$ -modules. Let $T: V \rightarrow W$ be an isomorphism of \mathbb{C} vector spaces with

$$T(av) = aT(v) \quad \forall (a, v) \in A \times V$$

Given $v \in V$ and $\omega \in k$, choose $v_1, v_2, \dots, v_r \in V$ and $a_1, a_2, \dots, a_r \in A$ with

$$v = a_1v_1 + a_2v_2 + \dots + a_rv_r$$

Then

$$\begin{aligned} T(\omega v) &= T((\omega a_1)v_1 + (\omega a_2)v_2 + \dots + (\omega a_r)v_r) \\ &= (\omega a_1)Tv_1 + (\omega a_2)Tv_2 + \dots + (\omega a_r)Tv_r \\ &= \omega(a_1Tv_1 + a_2Tv_2 + \dots + a_rTv_r) \\ &= \omega(Tv). \end{aligned}$$

Hence $T: V \rightarrow W$ intertwines the k -actions on V, W and thus V, W are equivalent A -modules. \square

4. FINITE TYPE k -ALGEBRAS

An ideal I in a k -algebra A is *primitive* if I is the null-space of an irreducible representation of A , i.e. there is an irreducible representation of A $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ with

$$I = \{a \in A \mid \varphi(a) = 0\}$$

$\text{Prim}(A)$ denotes the set of all primitive ideals in A . The evident map

$$\text{Irr}(A) \rightarrow \text{Prim}(A)$$

sends an irreducible representation to its null-space. On $\text{Prim}(A)$ there is the Jacobson topology. If S is any subset of $\text{Prim}(A)$, $S \subset \text{Prim}(A)$, then the closure \overline{S} of S is :

$$\overline{S} := \{I \in \text{Prim}(A) \mid I \supset \bigcap_{L \in S} L\}$$

Definition 4.1. A k -algebra A is of *finite type* if, as a k -module, A is finitely generated.

For any finite type k -algebra A , the following three statements are valid :

- If $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ is any irreducible representation of A , then V is a finite dimensional \mathbb{C} vector space and $\varphi: A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ is surjective.
- The evident map $\text{Irr}(A) \rightarrow \text{Prim}(A)$ is a bijection.
- Any primitive ideal in A is a maximal ideal.

Since $\text{Irr}(A) \rightarrow \text{Prim}(A)$ is a bijection, the Jacobson topology on $\text{Prim}(A)$ can be transferred to $\text{Irr}(A)$ and thus $\text{Irr}(A)$ is topologized. Equivalently, $\text{Irr}(A)$ is topologized by requiring that $\text{Irr}(A) \rightarrow \text{Prim}(A)$ be a homeomorphism.

For a finite type k -algebra A ($k = \mathcal{O}(X)$), the central character is a map

$$\text{Irr}(A) \longrightarrow X$$

defined as follows. Let φ

$$\begin{aligned} A &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \\ k &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \end{aligned}$$

be an irreducible representation of A . I_V denotes the identity operator of V .

For $\omega \in k = \mathcal{O}(X)$, define $T_{\omega}: V \rightarrow V$ by

$$T_{\omega}(v) = \omega v$$

for all $v \in V$. Then T_{ω} is an intertwining operator for $A \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$. By Lemma 3.3 and Schur's Lemma, T_{ω} is a scalar multiple of I_V .

$$T_{\omega} = \lambda_{\omega} I_V \quad \lambda_{\omega} \in \mathbb{C}$$

The map

$$\omega \mapsto \lambda_{\omega}$$

is a unital morphism of \mathbb{C} algebras $\mathcal{O}(X) \rightarrow \mathbb{C}$ and thus is given by evaluation at a unique (\mathbb{C} rational) point p_{φ} of X :

$$\lambda_{\omega} = \omega(p_{\varphi}) \quad \forall \omega \in \mathcal{O}(X)$$

The *central character* $\text{Irr}(A) \longrightarrow X$ is

$$\varphi \mapsto p_{\varphi}$$

REMARK. Corollary 3.4 states that $\text{Irr}(A)$ depends only on the underlying \mathbb{C} algebra $A_{\mathbb{C}}$. The central character $\text{Irr}(A) \rightarrow X$, however, does depend on the structure of A as a k -module. A change in the action of k on $A_{\mathbb{C}}$ will

change the central character.

The central character $\text{Irr}(A) \rightarrow X$ is continuous where $\text{Irr}(A)$ is topologized as above and X has the Zariski topology. For a proof of this assertion see [KNS, Lemma 1, p.326]. From a somewhat heuristic non-commutative geometry point of view, $A_{\mathbb{C}}$ is a non-commutative complex affine variety, and a given action of k on $A_{\mathbb{C}}$, making $A_{\mathbb{C}}$ into a finite type k -algebra A , determines a morphism of algebraic varieties $A_{\mathbb{C}} \rightarrow X$.

5. MORITA EQUIVALENCE FOR k -ALGEBRAS

Definition 5.1. Let B be a k -algebra. A *right B -module* is a \mathbb{C} -vector space V with given morphisms of \mathbb{C} -algebras

$$\begin{aligned} B^{\text{op}} &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \\ k &\longrightarrow \text{Hom}_{\mathbb{C}}(V, V) \end{aligned}$$

such that:

- (1) $k \rightarrow \text{Hom}_{\mathbb{C}}(V, V)$ is unital
- (2) $v(\omega b) = (v\omega)b = (vb)\omega \quad \forall (v, \omega, b) \in V \times k \times B$.

B^{op} is the opposite algebra of B . V is non-degenerate if $VB = V$.

Note that a right B -module is the same as a left B^{op} -module.

With k fixed, let A, B be two k -algebras. An $A - B$ bimodule, denoted ${}_A V_B$, is a \mathbb{C} vector space V such that :

- (1) V is a left A -module.
- (2) V is a right B -module.
- (3) $a(vb) = (av)b \quad \forall (a, v, b) \in A \times V \times B$.
- (4) $\omega v = v\omega \quad \forall (\omega, v) \in k \times V$.

An $A - B$ bimodule ${}_A V_B$ is non-degenerate if $AV = V = VB$. I_V is the identity map of V . A is an $A - A$ bimodule in the evident way.

Definition 5.2. A k -algebra A has *local units* if given any finite set $\{a_1, a_2, \dots, a_r\}$ of elements of A , there is an idempotent $Q \in A$ ($Q^2 = Q$) with

$$Qa_j = a_jQ = a_j \quad j = 1, 2, \dots, r.$$

Definition 5.3. Let A, B be two k -algebras with local units. A *Morita equivalence* (between A and B) is given by a pair of non-degenerate bimodules

$${}_A V_B \quad {}_B W_A$$

together with isomorphisms of bimodules

$$\alpha: V \otimes_B W \rightarrow A$$

$$\beta: W \otimes_A V \rightarrow B$$

such that there is commutativity in the diagrams:

$$\begin{array}{ccc}
V \otimes_B W \otimes_A V & \xrightarrow{I_V \otimes \beta} & V \otimes_B B \\
\downarrow \alpha \otimes I_V & & \downarrow \cong \\
A \otimes_A V & \xrightarrow{\cong} & V
\end{array}$$

$$\begin{array}{ccc}
W \otimes_A V \otimes_B W & \xrightarrow{I_W \otimes \alpha} & W \otimes_A A \\
\downarrow \beta \otimes I_W & & \downarrow \cong \\
B \otimes_B V & \xrightarrow{\cong} & W
\end{array}$$

Let A, B two k -algebras with local units, and suppose given a Morita equivalence

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

The *linking algebra* is

$$L({}_A V_B, {}_B W_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

i.e. $L({}_A V_B, {}_B W_A)$ consists of all 2×2 matrices having (1, 1) entry in A , (2, 2) entry in B , (2, 1) entry in W , and (1, 2) entry in V . Addition and multiplication are matrix addition and matrix multiplication. Note that α and β are used in the matrix multiplication.

$L({}_A V_B, {}_B W_A)$ is a k -algebra. With $\omega \in k$, the action of k on $L({}_A V_B, {}_B W_A)$ is given by

$$\omega \begin{pmatrix} a & v \\ w & b \end{pmatrix} = \begin{pmatrix} \omega a & \omega v \\ \omega w & \omega b \end{pmatrix}$$

A Morita equivalence between A and B determines an equivalence of categories between the category of non-degenerate left A -modules and the category of non-degenerate left B -modules. Similarly for right modules. Also, a Morita equivalence determines isomorphisms (between A and B) of Hochschild homology, cyclic homology, and periodic cyclic homology.

EXAMPLE. For n a positive integer, let $M_n(A)$ be the k -algebra of all $n \times n$ matrices with entries in A . If A has local units, A and $M_n(A)$ are Morita equivalent as follows. For m, n positive integers, denote by $M_{m,n}(A)$ the set of all $m \times n$ (i.e. m rows and n columns) matrices with entries in A . Matrix multiplication then gives a map

$$M_{m,n}(A) \times M_{n,r}(A) \longrightarrow M_{m,r}(A)$$

With this notation, $M_{n,n}(A) = M_n(A)$ and $M_{1,1}(A) = M_1(A) = A$. Hence matrix multiplication gives maps

$$M_{1,n}(A) \times M_n(A) \longrightarrow M_{1,n}(A) \quad M_n(A) \times M_{n,1} \longrightarrow M_{n,1}(A)$$

Thus $M_{1,n}(A)$ is a right $M_n(A)$ -module and $M_{n,1}(A)$ is a left $M_n(A)$ -module.

Similarly, $M_{1,n}(A)$ is a left A -module and $M_{n,1}(A)$ is a right A -module.

With $A = A$ and $B = M_n(A)$, the bimodules of the Morita equivalence are $V = M_{1,n}(A)$ and $W = M_{n,1}(A)$.

Note that the required isomorphisms of bimodules

$$\alpha: V \otimes_B W \rightarrow A$$

$$\beta: W \otimes_A V \rightarrow B$$

are obtained by observing that the matrix multiplication maps

$$M_{1,n}(A) \times M_{n,1}(A) \rightarrow A$$

$$M_{n,1}(A) \times M_{1,n}(A) \rightarrow M_n(A)$$

factor through the quotients $M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A)$, $M_{n,1}(A) \otimes_A M_{1,n}(A)$ and so give bimodule isomorphisms

$$\alpha: M_{1,n}(A) \otimes_{M_n(A)} M_{n,1}(A) \rightarrow A$$

$$\beta: M_{n,1}(A) \otimes_A M_{1,n}(A) \rightarrow M_n(A)$$

If A has local units, then α and β are isomorphisms. Therefore A and $M_n(A)$ are Morita equivalent.

If A does not have local units, then α and β can fail to be isomorphisms, and there is no way to prove that A and $M_n(A)$ are Morita equivalent. In examples, this already happens with $n = 1$, and there is then no way to prove (when A does not have local units) that A is Morita equivalent to A . For more details on this issue see below where the proof is given that in the new equivalence relation A and $M_n(A)$ are equivalent even when A does not have local units.

A finite type k -algebra A has local units iff A is unital.

6. THE EXCEPTIONAL SET $\mathfrak{E}_A(X)$

A Morita equivalence between two finite type k -algebras A, B preserves the central character i.e. there is commutativity in the diagram

$$\begin{array}{ccc} \text{Irr}(A) & \longrightarrow & \text{Irr}(B) \\ cc \downarrow & & \downarrow cc \\ X & \xrightarrow{I_X} & X \end{array}$$

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters, and I_X is the identity map of X .

By the *exceptional set* $\mathfrak{E}(A)$ we shall mean the set of all $x \in X$ such that the fibre over x has cardinality greater than 1:

$$\mathfrak{E}(A) := \{x \in X : \# cc^{-1}(x) > 1\}.$$

If A and B are Morita equivalent as k -algebras, then we will have $\mathfrak{C}(A) = \mathfrak{C}(B)$. Contrapositively, we have the following useful

Corollary 6.1. *If $\mathfrak{C}(A) \neq \mathfrak{C}(B)$ then A and B are not Morita equivalent as k -algebras.*

7. SPECTRUM PRESERVING MORPHISMS

Let A, B two finite type k -algebras, and let $f: A \rightarrow B$ be a morphism of k -algebras.

Definition 7.1. f is *spectrum preserving* if

- (1) Given any primitive ideal $J \subset B$, there is a unique primitive ideal $I \subset A$ with $I \supset f^{-1}(J)$

and

- (2) The resulting map

$$\text{Prim}(B) \rightarrow \text{Prim}(A)$$

is a bijection.

For I and J as above, let V be the simple B -module with annihilator J . Then the simple A -module with annihilator I can be realized as a quotient of V and also as a subspace of V .

Example 7.2. *Let A, B two unital finite type k -algebras, and suppose given a Morita equivalence*

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

With the linking algebra $L({}_A V_B, {}_B W_A)$ as above, the inclusions

$$A \hookrightarrow L({}_A V_B, {}_B W_A) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mapsto b$$

are spectrum preserving morphisms of finite type k -algebras. The bijection

$$\text{Prim}(B) \longleftrightarrow \text{Prim}(A)$$

so obtained is the bijection determined by the given Morita equivalence.

Remark. If $f: A \rightarrow B$ is a spectrum preserving morphism of finite type k -algebras, then the resulting bijection

$$\text{Prim}(B) \longleftrightarrow \text{Prim}(A)$$

is a homeomorphism. For a proof of this assertion see [BN, Theorem 3, p.342]. Consequently, if A, B are two unital finite type k -algebras, and

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

is a Morita equivalence, then the resulting bijection

$$\text{Prim}(B) \longleftrightarrow \text{Prim}(A)$$

is a homeomorphism.

Definition 7.3. An ideal I in a k -algebra A is a k -ideal if $\omega a \in I \vee (\omega, a) \in k \times I$.

Remark. Any primitive ideal in a k -algebra A is a k -ideal.

Given A, B two finite type k -algebras, let $f: A \rightarrow B$ be a morphism of k -algebras.

Definition 7.4. f is *spectrum preserving with respect to filtrations* if there are k -ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A \quad \text{in } A$$

and k ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{r-1} \subset J_r = B \quad \text{in } B$$

with $f(I_j) \subset J_j$, ($j = 1, 2, \dots, r$) and $I_j/I_{j-1} \rightarrow J_j/J_{j-1}$, ($j = 1, 2, \dots, r$) is spectrum preserving.

The primitive ideal spaces of the subquotients I_j/I_{j-1} and J_j/J_{j-1} are the strata for stratifications of $\text{Prim}(A)$ and $\text{Prim}(B)$. Each stratum of $\text{Prim}(A)$ is mapped homeomorphically onto the corresponding stratum of $\text{Prim}(B)$. However, the map

$$\text{Prim}(A) \rightarrow \text{Prim}(B)$$

might not be a homeomorphism.

8. ALGEBRAIC VARIATION OF k -STRUCTURE

Let A be a unital \mathbb{C} -algebra, and let

$$\Psi: k \rightarrow Z(A[t, t^{-1}])$$

be a unital morphism of \mathbb{C} -algebras. Here t is an indeterminate, so $A[t, t^{-1}]$ is the algebra of Laurent polynomials with coefficients in A . As above Z denotes ‘‘center’’. For $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, $ev(\zeta)$ denotes the ‘‘evaluation at ζ ’’ map:

$$ev(\zeta): A[t, t^{-1}] \rightarrow A \\ \sum a_j t^j \mapsto \sum a_j \zeta^j$$

Consider the composition

$$k \xrightarrow{\Psi} Z(A[t, t^{-1}]) \xrightarrow{ev(\zeta)} Z(A).$$

Denote the unital k -algebra so obtained by A_ζ . $\forall \zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, the underlying \mathbb{C} -algebra of A_ζ is A .

$$(A_\zeta)_\mathbb{C} = A \quad \forall \zeta \in \mathbb{C}^\times$$

Such a family $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, of unital k -algebras, will be referred to as an *algebraic variation of k -structure with parameter space \mathbb{C}^\times* .

9. STRATIFIED EQUIVALENCE

With k fixed, consider the collection of all finite type k -algebras. On this collection, *stratified equivalence* is, by definition, the equivalence relation generated by the two elementary steps :

Elementary Step 1. If there is a morphism of k -algebras $f : A \rightarrow B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.

Elementary Step 2. If there is $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of k -structure with parameter space \mathbb{C}^\times , such that each A_ζ is a unital finite type k -algebra, then for any $\zeta, \eta \in \mathbb{C}^\times$, $A_\zeta \sim A_\eta$.

Thus, two finite type k -algebras A, B are equivalent if and only if there is a finite sequence $A_0, A_1, A_2, \dots, A_r$ of finite type k -algebras with $A_0 = A, A_r = B$, and for each $j = 0, 1, \dots, r-1$ one of the following three possibilities is valid :

- a morphism of k -algebras $A_j \rightarrow A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- a morphism of k -algebras $A_j \leftarrow A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of k -structure with parameter space \mathbb{C}^\times , is given such that each A_ζ is a unital finite type k -algebra, and $\eta, \tau \in \mathbb{C}^\times$ have been chosen with $A_j = A_\eta, A_{j+1} = A_\tau$.

To give a stratified equivalence relating A and B , the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology [BN] are determined:

$$\text{Prim}(A) \longleftrightarrow \text{Prim}(B) \quad \text{HP}_*(A) \cong \text{HP}_*(B)$$

Proposition 9.1. *If two unital finite type k -algebras A, B are Morita equivalent (as k -algebras) then they are stratified equivalent.*

$$A \underset{\text{Morita}}{\sim} B \implies A \sim B$$

Proof. Let A, B two unital finite type k -algebras, and suppose given a Morita equivalence

$${}_A V_B \quad {}_B W_A \quad \alpha: V \otimes_B W \rightarrow A \quad \beta: W \otimes_A V \rightarrow B$$

The linking algebra is

$$L({}_A V_B, {}_B W_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

The inclusions

$$A \hookrightarrow L({}_A V_B, {}_V W_A) \hookrightarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \leftarrow b$$

are spectrum preserving morphisms of finite type k -algebras. Hence A and B are stratified equivalent. \square

According to the above, a Morita equivalence of A and B gives a homeomorphism

$$\text{Prim}(A) \simeq \text{Prim}(B)$$

However, the bijection

$$\text{Prim}(A) \longleftrightarrow \text{Prim}(B)$$

obtained from a stratified equivalence might not be a homeomorphism, as in the following example.

Example 9.2. Let Y be a sub-variety of X . We will write \mathcal{I}_Y for the ideal in $\mathcal{O}(X)$ determined by Y , so that

$$\mathcal{I}_Y = \{\omega \in \mathcal{O}(X) \mid \omega(p) = 0 \quad \forall p \in Y\}.$$

Let A be the algebra of all 2×2 matrices whose diagonal entries are in $\mathcal{O}(X)$ and whose off-diagonal entries are in \mathcal{I}_Y . Addition and multiplication in A are matrix addition and matrix multiplication. As a k -module, A is the direct sum of $\mathcal{O}(X) \oplus \mathcal{O}(X)$ with $\mathcal{I}_Y \oplus \mathcal{I}_Y$.

$$A = \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{O}(X) \end{pmatrix}$$

Set $B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$, so that B is the coordinate algebra of the disjoint union $X \sqcup Y$. We have $\mathcal{O}(Y) = \mathcal{O}(X)/\mathcal{I}_Y$. As a $k = \mathcal{O}(X)$ -module, B is the direct sum $\mathcal{O}(X) \oplus (\mathcal{O}(X)/\mathcal{I}_Y)$.

Theorem 9.3. The algebras A and B are stratified equivalent but not Morita equivalent:

$$A \sim B \quad A \underset{\text{Morita}}{\not\sim} B$$

Proof. Let $M_2(\mathcal{O}(X))$ denote the algebra of all 2×2 matrices with entries in $\mathcal{O}(X)$. Consider the algebra morphisms

$$A \longrightarrow M_2(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \longleftarrow \mathcal{O}(X) \oplus \mathcal{O}(Y)$$

$$T \mapsto (T, t_{22}|Y) \quad (T_\omega, \theta) \leftarrow (\omega, \theta)$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \quad T_\omega = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}$$

The filtration of A is given by

$$\{0\} \subset \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{I}_Y \end{pmatrix} \subset A$$

and the filtration of $M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y)$ is given by

$$\{0\} \subset M_2(\mathcal{O}(X) \oplus \{0\}) \subset M_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y).$$

The rightward pointing arrow is spectrum preserving with respect to the indicated filtrations. The leftward pointing arrow is spectrum preserving (no filtrations needed). We infer that

$$A \sim B.$$

Note that

$$\text{Prim}(A) = X \text{ with each point of } Y \text{ replaced by two points}$$

and

$$\begin{aligned} \text{Prim}(B) &= \text{Prim}(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \\ &= X \sqcup Y \end{aligned}$$

The spaces $\text{Prim}(A)$ and $\text{Prim}(B)$ are not homeomorphic, and therefore A is not Morita equivalent to B . \square

Unlike Morita equivalence, stratified equivalence works well for finite type k -algebras whether or not the algebras are unital, e.g. A and $M_n(A)$ are stratified equivalent even when A is not unital. See Proposition 8.5 below.

For any k -algebra A there is the evident isomorphism of k -algebras $M_n(A) \cong A \otimes_{\mathbb{C}} M_n(\mathbb{C})$. Hence, using this isomorphism, if W is a representation of A and U is a representation of $M_n(\mathbb{C})$, then $W \otimes_{\mathbb{C}} U$ is a representation of $M_n(A)$.

As above, $M_{n,1}(\mathbb{C})$ denotes the $n \times 1$ matrices with entries in \mathbb{C} . Matrix multiplication gives the usual action of $M_n(\mathbb{C})$ on $M_{n,1}(\mathbb{C})$.

$$M_n(\mathbb{C}) \times M_{n,1}(\mathbb{C}) \longrightarrow M_{n,1}(\mathbb{C})$$

This is the unique irreducible representation of $M_n(\mathbb{C})$. For any k -algebra A , if W is a representation of A , then $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is a representation of $M_n(A)$.

Lemma 9.4. *Let A be a finite type k -algebra and let n be a positive integer. Then:*

- (i) *If W is an irreducible representation of A , $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is an irreducible representation of $M_n(A)$.*
- (ii) *The resulting map $\text{Irr}(A) \rightarrow \text{Irr}(M_n(A))$ is a bijection.*

Proof. For (i), suppose given an irreducible representation W of A . Let J be the primitive ideal in A which is the null space of W . Then the null space of $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is $J \otimes_{\mathbb{C}} M_n(\mathbb{C})$. Consider the quotient algebra $A \otimes_{\mathbb{C}} M_n(\mathbb{C}) / J \otimes_{\mathbb{C}} M_n(\mathbb{C}) = (A/J) \otimes_{\mathbb{C}} M_n(\mathbb{C})$. This is isomorphic to $M_r(\mathbb{C})$ where $A/J \cong M_r(\mathbb{C})$, and so $W \otimes_{\mathbb{C}} M_{n,1}(\mathbb{C})$ is irreducible.

Proposition 9.5. *Let A be a finite type k -algebra and let n be a positive integer, then A and $M_n(A)$ are stratified equivalent.*

Proof. Let $f: A \rightarrow M_n(A)$ be the morphism of k -algebras which maps $a \in A$ to the diagonal matrix

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix}$$

It will suffice to prove that $f: A \rightarrow M_n(A)$ is spectrum preserving.

Let J be an ideal in A . Denote by J^\diamond the ideal in $M_n(A)$ consisting of all $[a_{ij}] \in M_n(A)$ such that each a_{ij} is in J . Equivalently, $M_n(A)$ is $A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ and $J^\diamond = J \otimes_{\mathbb{C}} M_n(\mathbb{C})$. It will suffice to prove

- (1) If J is a primitive ideal in A , then J^\diamond is a primitive ideal in $M_n(A)$.
- (2) If L is any primitive ideal in $M_n(A)$, then there is a primitive ideal J in A with $L = J^\diamond$.

For (1), J primitive $\implies J^\diamond$ primitive, because the quotient algebra $M_n(A)/J^\diamond$ is $(A/J) \otimes_{\mathbb{C}} M_n(\mathbb{C})$ which is (isomorphic to) $M_r(\mathbb{C})$ where $A/J \cong M_r(\mathbb{C})$.

For (2), since \mathbb{C} is commutative, the action of \mathbb{C} on A can be viewed as both a left and right action. Matrix multiplication then gives a left and a right action of $M_n(\mathbb{C})$ on $M_n(A)$

$$M_n(\mathbb{C}) \times M_n(A) \rightarrow M_n(A)$$

$$M_n(A) \times M_n(\mathbb{C}) \rightarrow M_n(A)$$

for which the associativity rule

$$(\alpha\theta)\beta = \alpha(\theta\beta) \quad \alpha, \beta \in M_n(A) \quad \theta \in M_n(\mathbb{C})$$

is valid.

If V is any representation of $M_n(A)$, the associativity rule

$$(\alpha\theta)(\beta v) = \alpha[(\theta\beta)v] \quad \alpha, \beta \in M_n(A) \quad \theta \in M_n(\mathbb{C}) \quad v \in V$$

is valid.

Now let V be an irreducible representation of $M_n(A)$, with L as its null-space. Define a (left) action

$$M_n(\mathbb{C}) \times V \rightarrow V$$

of $M_n(\mathbb{C})$ on V by proceeding as in the proof of Lemma 4.2 (the “ k -action for free” lemma) i.e. given $v \in V$, choose $v_1, v_2, \dots, v_r \in V$ and $\alpha_1, \alpha_2, \dots, \alpha_r \in M_n(A)$ with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r$$

For $\theta \in M_n(\mathbb{C})$, define θv by :

$$\theta v = (\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \dots + (\theta\alpha_r)v_r$$

The strict non-degeneracy, Lemmas 4.1 and 4.3, of V implies that θv is well-defined as follows. Suppose that $u_1, u_2, \dots, u_s \in V$ and $\beta_1, \beta_2, \dots, \beta_r \in M_n(A)$ are chosen with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_s u_s$$

If α is any element of $M_n(A)$, then

$$\alpha[(\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \dots + (\theta\alpha_r)v_r - (\theta\beta_1)u_1 - (\theta\beta_2)u_2 - \dots - (\theta\beta_s)u_s] =$$

$$(\alpha\theta)[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r - \beta_1 u_1 - \beta_2 u_2 - \cdots - \beta_s u_s] = (\alpha\theta)[v - v] = 0$$

Use $f: A \rightarrow M_n(A)$ to make V into an A -module

$$av := f(a)v \quad a \in A \quad v \in V$$

The actions of A and $M_n(\mathbb{C})$ on V commute. Thus for each $\theta \in M_n(\mathbb{C})$, θV is a sub- A -module of V , where θV is the image of $v \mapsto \theta v$. Denote by E_{ij} the matrix in $M_n(\mathbb{C})$ which has 1 for its (i, j) entry and zero for all its other entries. Then, as an A -module, V is the direct sum

$$V = E_{11}V \oplus E_{22}V \oplus \cdots \oplus E_{nn}V$$

Moreover, the action of E_{ij} on V maps $E_{jj}V$ isomorphically (as an A -module) onto $E_{ii}V$. Hence as an $M_n(A) = A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ module, V is isomorphic to $(E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ — where \mathbb{C}^n is the standard representation of $M_n(\mathbb{C})$ i.e. is the unique irreducible representation of $M_n(\mathbb{C})$.

$$V \cong (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$$

$E_{11}V$ is an irreducible A -module since if not $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ would not be an irreducible $A \otimes_{\mathbb{C}} M_n(\mathbb{C})$ -module.

If J is the null space (in A) of $E_{11}V$, then $J^\diamond = J \otimes_{\mathbb{C}} M_n(\mathbb{C})$ is the null space of $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ and this completes the proof. \square

10. PERMANENCE PROPERTIES

Theorem 10.1. *Denote by Γ a finite group. Stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.*

Proposition 10.2. *Given k_1 -algebras $A \sim A'$ and k_2 -algebras $B \sim B'$ then we have*

$$A \otimes B \sim A' \otimes B'$$

as $k_1 \otimes k_2$ -algebras, all tensor products over \mathbb{C} .

Proof. We first prove that $A \otimes B \sim A' \otimes B$. Let $\{I_j\}$ be the filtration for A , and $\{I'_j\}$ be the filtration for A' . Then $\{I_j \otimes B\}$ (resp. $\{I'_j \otimes B\}$) will serve as filtrations for $A \otimes B$, (resp. $A' \otimes B$), in the sense that the maps

$$I_j/I_{j-1} \otimes B \rightarrow I'_j/I'_{j-1} \otimes B$$

are spectrum-preserving. A similar argument will prove that $A' \otimes B \sim A' \otimes B'$. Then we have

$$A \otimes B \sim A' \otimes B \sim A' \otimes B'$$

as required.

Let Ψ be a variation of k_1 -structure for A :

$$k_1 \rightarrow Z(A[t, t^{-1}])$$

and let

$$\Psi_B : k_2 \rightarrow Z(B)$$

be the k_2 -structure for B . Then

$$\Psi \otimes \Psi_B : k_1 \otimes k_2 \rightarrow Z(A[t, t^{-1}]) \otimes Z(B) \simeq Z(A \otimes B)[t, t^{-1}]$$

will be a variation of $k_1 \otimes k_2$ -structure for $A \otimes B$. \square

Lemma 10.3. *Assume that A and B both have Γ acting (as automorphisms of k -algebras), and we are given a Γ -equivariant morphism $\phi : A \rightarrow B$ which is spectrum preserving. Then $\phi \rtimes \Gamma : A \rtimes \Gamma \rightarrow B \rtimes \Gamma$ is a spectrum preserving morphism of k -algebras.*

Proof. According to Lemma A.1 in [ABPS], $\text{Irr}(A \rtimes \Gamma), \text{Irr}(B \rtimes \Gamma)$ are in bijection with the (twisted) extended quotients:

$$\begin{aligned} (\text{Irr}(A)//\Gamma)_{\mathfrak{h}} &\simeq \text{Irr}(A \rtimes \Gamma) \\ (\text{Irr}(B)//\Gamma)_{\mathfrak{h}} &\simeq \text{Irr}(B \rtimes \Gamma) \end{aligned}$$

Since $\phi : A \rightarrow B$ is spectrum preserving, the 2-cocycles \mathfrak{h} of Γ associated to $\text{Irr}(A)$ and $\text{Irr}(B)$ can be identified. Hence ϕ determines a bijection of the (twisted) extended quotients

$$(\text{Irr}(A)//\Gamma)_{\mathfrak{h}} \longleftarrow (\text{Irr}(B)//\Gamma)_{\mathfrak{h}}$$

Applying Lemma A.1 of [ABPS], this implies that $\phi \rtimes \Gamma$ is spectrum preserving. □

Proposition 10.4. *If A and B both have Γ acting (as automorphisms of k -algebras), and we are given a Γ -equivariant stratified equivalence $A \sim B$, then we have*

$$A \rtimes \Gamma \sim B \rtimes \Gamma$$

Proof. Suppose now that we are given a Γ -equivariant morphism of k -algebras $\phi : A \rightarrow B$ which is, in a Γ -equivariant way, spectrum preserving with respect to filtrations, i.e. the filtrations of A and B are preserved by the action of Γ . This gives a map of k -algebras $\phi \rtimes \Gamma : A \rtimes \Gamma$ to $B \rtimes \Gamma$. We claim that this map is spectrum-preserving with respect to filtrations.

Using the filtrations, consider the map $I_j/I_{j-1} \rightarrow J_j/J_{j-1}$. This is a Γ -equivariant spectrum preserving map. Hence by Lemma 10.3 above, the map

$$(I_j/I_{j-1}) \rtimes \Gamma \rightarrow (J_j/J_{j-1}) \rtimes \Gamma$$

is spectrum preserving. So the filtrations of $A \rtimes \Gamma$ and $B \rtimes \Gamma$ given by $I_j \rtimes \Gamma$ and $J_j \rtimes \Gamma$ are the required filtrations.

Suppose given a Γ -equivariant variation of k -structure for A , i.e. a Γ -equivariant unital map of \mathbb{C} -algebras

$$k \longrightarrow Z(A[t, t^{-1}])$$

where the Γ -action on k is trivial. Due to the triviality of this action, k is mapped to

$$Z(A[t, t^{-1}])^\Gamma = (Z(A))^\Gamma[t, t^{-1}]$$

By the standard inclusion

$$(Z(A))^\Gamma \subset Z(A \rtimes \Gamma)$$

k is mapped to $Z(A \rtimes \Gamma)[t, t^{-1}]$:

$$k \longrightarrow Z(A \rtimes \Gamma)[t, t^{-1}]$$

which is the required variation of k -structure for the crossed product $A \rtimes \Gamma$. □

11. AFFINE HECKE ALGEBRAS

Let (X, Y, R, R^\vee) be a root datum in the standard sense [L, p.73]. This root datum delivers the following items:

- a finite Weyl group W_0
- an extended affine Weyl group $W_0X := W_0 \times X$
- for each $q \in \mathbb{C}^\times$, an affine Hecke algebra \mathcal{H}_q
- a complex torus $T := \text{Hom}(X, \mathbb{C}^\times)$
- a complex variety $X := T/W_0$
- a canonical isomorphism $\mathcal{O}(X) \simeq Z(\mathcal{H}_q)$

The detailed construction of \mathcal{H}_q is described in the article of Lusztig [L, p.74]. Set $k := \mathcal{O}(X)$. Then, for all $q \in \mathbb{C}^\times$, \mathcal{H}_q is a unital finite type k -algebra. If $q = 1$, then \mathcal{H}_1 is the group algebra of the extended affine Weyl group:

$$\mathcal{H}_1 = \mathbb{C}[W_0X]$$

11.1. Stratified equivalence for affine Hecke algebras.

Theorem 11.1. *Except for q in a finite set of roots of unity, none of which is 1, \mathcal{H}_q is stratified equivalent to \mathcal{H}_1 :*

$$\mathcal{H}_q \sim \mathcal{H}_1.$$

Proof. Let J be Lusztig's asymptotic algebra [L, §8]. As a \mathbb{C} -vector space, J has a basis $\{t_v : v \in W_0X\}$, and there is a canonical structure of associative \mathbb{C} -algebra on J . The algebra \mathcal{H}_q is viewed as a k -algebra via the canonical isomorphism

$$\mathcal{O}(T/W_0) \cong Z(\mathcal{H}_q).$$

Lusztig's map ϕ_q maps $Z(\mathcal{H}_q)$ to $Z(J)$ by Proposition 1 in [ABP] and thus determines a unique k -structure for J such that the map ϕ_q is a morphism of k -algebras. J with this k -structure will be denoted J_q .

Let

$$a : W_0X \rightarrow \mathbb{N} \cup \{0\}$$

be the weight function defined by Lusztig.

The algebra J has a unit element of the form $1 = \sum_{d \in \mathcal{D}} d$ where \mathcal{D} is a certain set of involutions in W_a . For any $q \in \mathbb{C}^\times$, the \mathbb{C} -linear map $\psi_q : \mathcal{H}_q \rightarrow J$ defined by

$$(1) \quad \phi_q(C_w) = \sum h_{w,d,v} t_v$$

is a \mathbb{C} -linear homomorphism preserving 1. The summation is over the set

$$\{d \in \mathcal{D}, v \in W_0X : a(v) = a(d)\}.$$

The elements C_w are defined in [L, p.82] and form a \mathbb{C} -basis of \mathcal{H}_q . The coefficients $h_{w,d,v}$ originate in the multiplication rule for the C_w , see [L, p.82].

The map ϕ_q is injective. Thus all algebras \mathcal{H}_q with $q \in \mathbb{C}^\times$ appear as subalgebras of a single \mathbb{C} -algebra J .

Define

$$\begin{aligned} J_k &= \text{ideal generated by } \{t_v \in J : a(v) \leq k\} \\ I_k &= \phi_q^{-1}(J_k) \end{aligned}$$

We have the filtrations

$$(2) \quad \begin{aligned} 0 \subset I_1 \subset I_2 \subset \cdots \subset I_r &= \mathcal{H}_q \\ 0 \subset J_1 \subset J_2 \subset \cdots \subset J_r &= J \end{aligned}$$

We have

$$\begin{aligned} \text{Irr}(J) &= \bigsqcup \text{Irr}(J_k/J_{k-1}) \\ \text{Irr}(\mathcal{H}_q) &= \bigsqcup \text{Irr}(I_k/I_{k-1}) \end{aligned}$$

Let M be a simple \mathcal{H}_q -module (resp. simple J -module). The *weight* a_M of M is defined by Lusztig in [L, p.82]. Let $\text{Irr}(\mathcal{H}_q)_k$ (resp. $\text{Irr}(J)_k$) denote the set of simple \mathcal{H}_q -modules (resp. simple J -modules) of weight k . It follows from Lusztig's definition that we have

$$\begin{aligned} \text{Irr}(J_k/J_{k-1}) &= \text{Irr}(J)_k \\ \text{Irr}(I_k/I_{k-1}) &= \text{Irr}(\mathcal{H}_q)_k \end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccc} \mathcal{H}_q & \xrightarrow{\phi_q} & J \\ \uparrow & & \uparrow \\ I_k & \xrightarrow{\phi_q|_{I_k}} & J_k \\ \downarrow & & \downarrow \\ I_k/I_{k-1} & \xrightarrow{(\phi_q)_k} & J_k/J_{k-1} \end{array}$$

The middle horizontal map is the restriction of ϕ_q to I_k . This map is defined by equation (1) but with summation over the set

$$\{d \in \mathcal{D}, v \in W_0X : a(v) = a(d) \leq k\}.$$

The bottom horizontal map is middle horizontal map after descent to the quotient I_k/I_{k-1} . This map is defined by equation (1) but with summation over the set

$$\{d \in \mathcal{D}, v \in W_0X : a(v) = a(d) = k\}.$$

We now apply one of Lusztig's main theorems, Theorem 8.1 in [L]: Assume that $q \in \mathbb{C}^\times$ is either 1 or is not a root of 1. Let M' be a simple J -module of weight $a_{M'}$. The pre-image $\phi_q^{-1}(M')$ will contain a unique subquotient M of maximal weight $a_M = a_{M'}$. Then the map $M' \mapsto M$ is a bijection from the set of simple J -modules (up to isomorphism) to the set of simple \mathcal{H}_q -modules (up to isomorphism).

We reconcile our construction with Lusztig's bijection by observing that our map

$$(\phi_q)_k : I_k/I_{k-1} \rightarrow J_k/J_{k-1}$$

determines a map

$$(3) \quad \text{Irr}(J)_k \rightarrow \text{Irr}(\mathcal{H}_q)_k$$

This map is weight-preserving, by construction, and coincides with the Lusztig bijection. The map (3) is spectrum-preserving by Lusztig's theorem, and so the original map ϕ_q is spectrum preserving with respect to the filtrations (2).

\mathcal{H}_q is then stratified equivalent to \mathcal{H}_1 by the three elementary steps

$$\mathcal{H}_q \rightsquigarrow J_q \rightsquigarrow J_1 \rightsquigarrow \mathcal{H}_1.$$

The second elementary step (i.e. passing from J_q to J_1) is an algebraic variation of k -structure with parameter space \mathbb{C}^\times . The first elementary step uses Lusztig's map ϕ_q , and the third elementary step uses Lusztig's map ϕ_1 . Hence (provided q is not in the exceptional set of roots of unity—none of which is 1) \mathcal{H}_q is stratified equivalent to

$$\mathcal{H}_1 = \mathbb{C}[W_0X] = \mathcal{O}(T) \rtimes W_0$$

□

Remark 11.2. In Theorem 11.1, the condition on q can be replaced by the following more precise condition:

$$\sum_{w \in W_0} q^{\ell(w)} \neq 0$$

where $\ell : W_0X \rightarrow \mathbb{N}$ is the length function, see [Xi2, Theorem 3.2].

If $q \neq 1$ then \mathcal{H}_q and \mathcal{H}_1 are not Morita equivalent as k -algebras, for the exceptional sets are not equal:

$$q \neq 1 \implies \mathfrak{E}(\mathcal{H}_q) \neq \mathfrak{E}(\mathcal{H}_1)$$

11.2. The affine Hecke algebras of $\mathrm{SL}_2(\mathbb{C})$. If the root datum

$$(X, Y, R, R^\vee)$$

arises from a connected complex reductive Lie group G , then we will write $\mathcal{H}_q(G) = \mathcal{H}_q$.

An interesting situation arises for $\mathcal{H}_q(\mathrm{SL}_2(\mathbb{C}))$. Let s, t be the simple reflections in the (extended) affine Weyl group W_0X . When $q + 1 \neq 0$ there is an isomorphism of \mathbb{C} -algebras between \mathcal{H}_q and \mathcal{H}_1 :

$$\begin{aligned} \mathcal{H}_q &\simeq \mathcal{H}_1 \\ T_s &\mapsto \frac{q+1}{2}s + \frac{q-1}{2}, \quad T_t \mapsto \frac{q+1}{2}t + \frac{q-1}{2} \end{aligned}$$

On the other hand, when $q \neq 1$, \mathcal{H}_q and \mathcal{H}_1 cannot be isomorphic as k -algebras, for the exceptional sets are different. Let T denote the standard maximal torus in $\mathrm{SL}_2(\mathbb{C})$:

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} : z \neq 0 \right\}$$

We have $W_0 = \mathbb{Z}/2\mathbb{Z}$, $k = \mathcal{O}(T/W_0)$ the algebra of $\mathbb{Z}/2\mathbb{Z}$ -invariant Laurent polynomials in one variable. Note that the non-trivial element of W acts by

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mapsto \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}$$

The quotient variety T/W_0 comprises unordered pairs $\{z_1, z_2\}$ of nonzero complex numbers which satisfy the equation $z_1 z_2 = 1$.

The exceptional sets are

$$\mathfrak{E}(\mathcal{H}_1) = \{-1, -1\} \sqcup \{1, 1\}$$

$$\mathfrak{E}(\mathcal{H}_q) = \{-1, -1\} \sqcup \{q, 1/q\}$$

As q with $q \neq 1$ is deformed to 1, the point $\{-1, -1\}$ stays fixed, and the point $\{q, 1/q\}$ moves to the point $\{1, 1\}$.

11.3. The affine Hecke algebras of $\mathrm{SL}_3(\mathbb{C})$. The case of $\mathrm{SL}_3(\mathbb{C})$ is considered in §11.7 of Xi [Xi] where it is proved that \mathcal{H}_q and \mathcal{H}_1 are not isomorphic as \mathbb{C} -algebras whenever $q \neq 1$. As k -algebras, \mathcal{H}_q is not Morita equivalent to \mathcal{H}_1 whenever $q \neq 1$ because the exceptional set $\mathfrak{E}(\mathcal{H}_q)$ with $q \neq 1$ is not the same as $\mathfrak{E}(\mathcal{H}_1)$.

In contrast to this, except for a finite set of roots of unity (none of which is 1), according to Theorem 11.1, \mathcal{H}_q is stratified equivalent to \mathcal{H}_1 .

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