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MORITA EQUIVALENCE FOR k-ALGEBRAS

ANNE-MARIE AUBERT, PAUL BAUM, ROGER PLYMEN,
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Abstract. We review Morita equivalence for finite type k-algebras A and also a weakening of Morita equivalence which we call *stratified equivalence*. The spectrum of A is the set of equivalence classes of irreducible A-modules. For any finite type k-algebra A, the spectrum of A is in bijection with the set of primitive ideals of A. The stratified equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A. However, the stratified equivalence relation permits a tearing apart of strata in the primitive ideal space which is not allowed by Morita equivalence. A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras associated to affine Weyl groups.

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1. Introduction

The subject of Morita equivalence begins with the classical paper of Morita [Mor]. Morita equivalence of rings is well-established and widely used [Lam, §18–19].

Now let k be the coordinate algebra of a complex affine variety. Equivalently, k is a unital algebra over the complex numbers which is commutative, finitely generated, and nilpotent-free. Morita equivalence for k-algebras, the starting-point of this article, is a more recent development, which introduces several nuances related to the k-module structure of the algebras.

A k-algebra is an algebra A over the complex numbers $\mathbb{C}$ which is a k-module (with an evident compatibility between the algebra structure of A and the k-module structure of A). A is not required to have a unit. A is not required to be commutative. A k-algebra A is of finite type if as a k-module A is finitely generated. The spectrum of A is, by definition, the set of equivalence classes of irreducible A-modules. For any finite type k-algebra A, the spectrum of A is in bijection with the set of primitive ideals of A.

The set of primitive ideals of A will be denoted $\mathrm{Prim}(A)$. When A is unital and commutative, let $\mathrm{maxSpec}(A)$ denote the set of maximal ideals in A, and let $\mathrm{Spec}(A)$ denote the set of prime ideals in A. In particular, let $A = k$. Then we have

$$\mathrm{Prim}(A) \simeq \mathrm{maxSpec}(A) \subset \mathrm{Spec}(A).$$

The inclusion is strict, for the prime spectrum $\mathrm{Spec}(A)$ contains a generic point, namely the zero ideal $\mathbf{0}$ in A. The set $\mathrm{Prim}(A)$ consists of the $\mathbb{C}$-rational points of the affine variety underlying k.

So $\mathrm{Prim}(A)$ is a generalisation, for k-algebras A of finite type, of the maximal ideal spectrum of a unital commutative ring, rather than the prime spectrum of a unital commutative ring. For a finite type k-algebra A, the zero ideal $\mathbf{0}$ is a primitive ideal if and only if $k = \mathbb{C}$ and $A = M_n(\mathbb{C})$ the algebra of all n by n matrices with entries in $\mathbb{C}$. Quite generally, any irreducible representation of a finite type k-algebra A is a surjection of A onto $M_n(\mathbb{C})$ for some n. This implies that any primitive ideal is maximal.

In some situations, Morita equivalence can be too strong and we are led to introduce a weakening of this concept, which we call *stratified equivalence*.

The stratified equivalence relation preserves the spectrum of A and also preserves the periodic cyclic homology of A. In addition, the stratified equivalence relation permits a tearing apart of *strata* in the primitive ideal space which is not allowed by Morita equivalence.

Denote by Γ a finite group. In §10, we show that stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.

A key example illustrating the distinction between Morita equivalence and stratified equivalence is provided by affine Hecke algebras. Let $(X, Y, R, R^\vee)$

be a root datum in the standard sense [L]. This root datum delivers the following items:

– a finite Weyl group W_0
– an extended affine Weyl group $W_0X := W_0 \ltimes X$
– for each $q \in \mathbb{C}^\times$, an affine Hecke algebra $\mathcal{H}_q$
– a complex torus $T := \mathrm{Hom}(X, \mathbb{C}^\times)$
– a complex variety $X := T/W_0$
– a canonical isomorphism $\mathcal{O}(X) \simeq Z(\mathcal{H}_q)$

Set $k := \mathcal{O}(X)$. Then, for all $q \in \mathbb{C}^\times$, $\mathcal{H}_q$ is a unital finite type k-algebra. If $q = 1$, then $\mathcal{H}_1$ is the group algebra of the extended affine Weyl group:

$$\mathcal{H}_1 = \mathbb{C}[W_0X]$$

Theorem 1.1. *Except for q in a finite set of roots of unity, none of which is 1, $\mathcal{H}_q$ and $\mathcal{H}_1$ are stratified equivalent as k-algebras. If $q \neq 1$ then $\mathcal{H}_q$ and $\mathcal{H}_1$ are not Morita equivalent as k-algebras.*

A finite type k-algebra can be viewed as a noncommutative complex affine variety. So the setting of this article can be viewed as noncommutative algebraic geometry.

We conclude this article with a detailed account of the affine Hecke algebra $\mathcal{H}_q$ attached to the Lie group $\mathrm{SL}_2(\mathbb{C})$.

For an application of stratified equivalence to the representation theory of p-adic groups, see [ABPS].

2. k-ALGEBRAS

If X is an affine algebraic variety over the complex numbers $\mathbb{C}$, then $\mathcal{O}(X)$ will denote the coordinate algebra of X. Set $k = \mathcal{O}(X)$. Equivalently, k is a unital algebra over the complex numbers which is unital, commutative, finitely generated, and nilpotent-free. The Hilbert Nullstellensatz implies that there is an equivalence of categories

$$\left(\begin{array}{l}\text{unital commutative}\\ \text{finitely generated}\\ \text{nilpotent-free } \mathbb{C}\text{-algebras}\end{array}\right) \sim \left(\begin{array}{l}\text{affine complex al-}\\ \text{gebraic varieties}\end{array}\right)^{\mathrm{op}}$$
$$\mathcal{O}(X) \mapsto X$$

Here "op" denotes the opposite category.

Definition 2.1. A *k-algebra* is a $\mathbb{C}$-algebra A such that A is a unital (left) k-module with:

$$\lambda(\omega a) = \omega(\lambda a) = (\lambda\omega)a \quad \forall(\lambda, \omega, a) \in \mathbb{C} \times k \times A$$

and

$$\omega(a_1a_2) = (\omega a_1)a_2 = a_1(\omega a_2) \quad \forall(\omega, a_1, a_2) \in k \times A \times A.$$

Remark 2.2. A is not required to have a unit.

The centre of A will be denoted $Z(A)$:

$$Z(A) := \{c \in A \mid ca = ac \ \forall a \in A\}.$$

Remark 2.3. Let A be a unital k-algebra. Denote the unit of A by 1_A. The map $\omega \mapsto \omega.1_A$ is then a unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A)$$

i.e. unital k-algebra = unital $\mathbb{C}$-algebra A with a given unital morphism of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A)$$

If A does not have a unit, a k-structure is equivalent to a unital morphism of $\mathbb{C}$-algebras

$$k \to Z(\mathcal{M}A)$$

where $\mathcal{M}A$ is the multiplier algebra of A.

Definition 2.4. Let A, B be two k-algebras. A *morphism of k-algebras* is a morphism of $\mathbb{C}$-algebras

$$f\colon A \to B$$

which is also a morphism of (left) k-modules,

$$f(\omega a) = \omega f(a) \quad \forall (\omega, a) \in k \times A.$$

Definition 2.5. Let A be a k-algebra. A *representation* of A [or a (left) *A-module*] is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

such that

(1) $k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$ is unital.
(2) $(\omega a)v = \omega(av) = a(\omega v) \quad \forall (\omega, a, v) \in k \times A \times V$.

From now on in this article, A will denote a k-algebra.

A representation of A

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

will often be denoted

$$A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$

it being understood that the action of k on V

$$k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$

is part of the given structure.

Definition 2.6. A representation $\varphi: A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is *non-degenerate* if $AV = V$, i.e. for any $v \in V$, there are v_1, v_2, …, v_r in V and a_1, a_2, …, a_r in A with

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r.$$

Definition 2.7. A representation $\varphi: A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is *irreducible* if $AV \neq \{0\}$ and there is no sub-$\mathbb{C}$-vector space W of V with:

$$\{0\} \neq W \ , \ W \neq V$$

and

$$\omega w \in W \quad \forall (\omega, w) \in k \times W$$

and

$$aw \in W \quad \forall (a, w) \in A \times W$$

Definition 2.8. Two representations of the k-algebra A

$$\varphi_1: A \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_1)$$
$$\varphi_2: A \to \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of $\mathbb{C}$-vector spaces

$$T: V_1 \to V_2$$

with

$$T(av) = aT(v) \quad \forall\, (a, v) \in A \times V$$

and

$$T(\omega v) = \omega T(v) \quad \forall\, (\omega, v) \in k \times V$$

The *spectrum* of A, also denoted $\mathrm{Irr}(A)$, is the set of equivalence classes of irreducible representations of A.

$$\mathrm{Irr}(A) := \{\text{Irreducible representations of A}\}/\sim .$$

It can happen that the spectrum is empty. For example, let A comprise all 2 by 2 matrices of the form

$$\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$$

with $x \in \mathbb{C}$. Then A is a commutative non-unital algebra of finite type (with $k = \mathbb{C}$) such that $\mathrm{Irr}(A) = \varnothing$.

3. On the action of k

For a k-algebra A, $A_{\mathbb{C}}$ denotes the underlying $\mathbb{C}$-algebra of A. Then $A_{\mathbb{C}}$ is obtained from A by forgetting the action of k on A. For $A_{\mathbb{C}}$ there are the usual definitions : A *representation* of $A_{\mathbb{C}}$ [or a (left) $A_{\mathbb{C}}$-*module*] is a $\mathbb{C}$-vector space V with a given morphism of $\mathbb{C}$-algebras

$$A_{\mathbb{C}} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

An $A_{\mathbb{C}}$-module V is *irreducible* if $A_{\mathbb{C}}V \neq \{0\}$ and there is no sub-$\mathbb{C}$-vector space W of V with:

$$\{0\} \neq W \ , \ W \neq V$$

and

$$aw \in W \quad \forall (a,w) \in A_{\mathbb{C}} \times W$$

Two representations of $A_{\mathbb{C}}$

$$\varphi_1 \colon A \to \mathrm{Hom}_{\mathbb{C}}(V_1, V_1)$$
$$\varphi_2 \colon A \to \mathrm{Hom}_{\mathbb{C}}(V_2, V_2)$$

are *equivalent* if there is an isomorphism of $\mathbb{C}$-vector spaces

$$T \colon V_1 \to V_2$$

with

$$T(av) = aT(v) \quad \forall\, (a,v) \in A \times V$$

$\mathrm{Irr}(A_{\mathbb{C}})$:=$\{$Irreducible representations of $A_{\mathbb{C}}\}/\sim$.

An $A_{\mathbb{C}}$-module V for which the following two properties are valid is *strictly non-degenerate*

- $A_{\mathbb{C}}V = V$
- If $v \in V$ has $av = 0 \ \forall a \in A_{\mathbb{C}}$, then $v = 0$.

Lemma 3.1. *Any irreducible $A_{\mathbb{C}}$-module is strictly non-degenerate.*

Proof. Let V be an irreducible $A_{\mathbb{C}}$-module. First, consider $A_{\mathbb{C}}V \subset V$. $A_{\mathbb{C}}V$ is preserved by the action of $A_{\mathbb{C}}$ on V. Cannot have $A_{\mathbb{C}}V = \{0\}$ since this would contradict the irreducibility of V. Therefore $A_{\mathbb{C}}V = V$.
Next, set

$$W = \{v \in V | av = 0 \ \forall a \in A_{\mathbb{C}}\}$$

W is preserved by the action of $A_{\mathbb{C}}$ on V. W cannot be equal to V since this would imply $A_{\mathbb{C}}V = \{0\}$. Hence $W = \{0\}$.

□

Lemma 3.2. *Let A be a k-algebra, and let V be a strictly non-degenerate $A_{\mathbb{C}}$-module. Then there is a unique unital morphism of $\mathbb{C}$ algebras*

$$k \to \mathrm{Hom}_{\mathbb{C}}(V, V)$$

which makes V an A-module.

Proof. Given $v \in V$, choose $v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r$$

For $\omega \in k$, define ωv by :

$$\omega v = (\omega a_1) v_1 + (\omega a_2) v_2 + \cdots + (\omega a_r) v_r$$

The second condition in the definition of strictly non-degenerate implies that ωv is well-defined. □

Lemma 3.2 will be referred to as the "k-action for free lemma".

If V is an A-module, $V_{\mathbb{C}}$ will denote the underlying $A_{\mathbb{C}}$-module. $V_{\mathbb{C}}$ is obtained from V by forgetting the action of k on V.

Lemma 3.3. *If V is any irreducible A-module, then $V_{\mathbb{C}}$ is an irreducible $A_{\mathbb{C}}$-module.*

Proof. Suppose that $V_{\mathbb{C}}$ is not an irreducible $A_{\mathbb{C}}$-module. Then there is a sub-$\mathbb{C}$-vector space W of V with:

$$0 \neq W, \quad W \neq V$$

and

$$aw \in W \qquad \forall (a, w) \in A \times W$$

Consider $AW \subset W$. AW is preserved by both the A-action on V and the k-action on V. Thus if $AW \neq \{0\}$, then V is not an irreducible A-module. Hence $AW = \{0\}$. Consider $kW \supset W$. kW is preserved by the k-action on V and is also preserved by the A-action on V because A annihilates kW. Since A annihilates kW, cannot have $kW = V$. Therefore $\{0\} \neq kW$, $kW \neq V$, which contradicts the irreducibility of the A-module V. □

A corollary of Lemma 3.2 is :

Corollary 3.4. *For any k-algebra A, the map*

$$\begin{aligned} \mathrm{Irr}(A) &\to \mathrm{Irr}(A_{\mathbb{C}}) \\ V &\mapsto V_{\mathbb{C}} \end{aligned}$$

is a bijection.

Proof. Surjectivity follows from Lemmas 3.1 and 3.2. For injectivity, let V, W be two irreducible A-modules such that $V_{\mathbb{C}}$ and $W_{\mathbb{C}}$ are equivalent $A_{\mathbb{C}}$-modules. Let $T: V \to W$ be an isomorphism of $\mathbb{C}$ vector spaces with

$$T(av) = aT(v) \quad \forall\, (a, v) \in A \times V$$

Given $v \in V$ and $\omega \in k$, choose $v_1, v_2, \ldots, v_r \in V$ and $a_1, a_2, \ldots, a_r \in A$ with

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_r v_r$$

Then

$$\begin{aligned} T(\omega v) &= T((\omega a_1)v_1 + (\omega a_2)v_2 + \cdots + (\omega a_r)v_r) \\ &= (\omega a_1)Tv_1 + (\omega a_2)Tv_2 + \cdots + (\omega a_r)Tv_r \\ &= \omega(a_1 Tv_1 + a_2 Tv_2 + \cdots + a_r Tv_r) \\ &= \omega(Tv). \end{aligned}$$

Hence $T: V \to W$ intertwines the k-actions on V, W and thus V, W are equivalent A-modules. □

4. Finite type k-algebras

An ideal I in a k-algebra A is *primitive* if I is the null-space of an irreducible representation of A, i.e. there is an irreducible representation of A $\varphi: A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ with

$$I = \{a \in A \mid \varphi(a) = 0\}$$

$\mathrm{Prim}(A)$ denotes the set of all primitive ideals in A. The evident map

$$\mathrm{Irr}(A) \to \mathrm{Prim}(A)$$

sends an irreducible representation to its null-space. On $\mathrm{Prim}(A)$ there is the Jacobson topology. If S is any subset of $\mathrm{Prim}(A)$, $S \subset \mathrm{Prim}(A)$, then the closure $\overline{S}$ of S is :

$$\overline{S} := \{I \in \mathrm{Prim}(A) \mid I \supset \cap_{L \in S} L\}$$

Definition 4.1. A k-algebra A is of *finite type* if, as a k-module, A is finitely generated.

For any finite type k-algebra A, the following three statements are valid :

- If $\varphi: A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is any irreducible representation of A, then V is a finite dimensional $\mathbb{C}$ vector space and $\varphi: A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$ is surjective.
- The evident map $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ is a bijection.
- Any primitive ideal in A is a maximal ideal.

Since $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ is a bijection, the Jacobson topology on $\mathrm{Prim}(A)$ can be transferred to $\mathrm{Irr}(A)$ and thus $\mathrm{Irr}(A)$ is topologized. Equivalently, $\mathrm{Irr}(A)$ is topologized by requiring that $\mathrm{Irr}(A) \to \mathrm{Prim}(A)$ be a homeomorphism.

For a finite type k-algebra A ($k = \mathcal{O}(X)$), the central character is a map

$$\mathrm{Irr}(A) \longrightarrow X$$

defined as follows. Let φ

$$A \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V, V)$$

be an irreducible representation of A. I_V denotes the identity operator of V.

For $\omega \in k = \mathcal{O}(X)$, define $T_\omega: V \to V$ by

$$T_\omega(v) = \omega v$$

for all $v \in V$. Then T_ω is an intertwining operator for $A \to \mathrm{Hom}_{\mathbb{C}}(V, V)$. By Lemma 3.3 and Schur's Lemma, T_ω is a scalar multiple of I_V.

$$T_\omega = \lambda_\omega I_V \qquad \lambda_\omega \in \mathbb{C}$$

The map

$$\omega \mapsto \lambda_\omega$$

is a unital morphism of $\mathbb{C}$ algebras $\mathcal{O}(X) \to \mathbb{C}$ and thus is given by evaluation at a unique ($\mathbb{C}$ rational) point p_φ of X:

$$\lambda_\omega = \omega(p_\varphi) \qquad \forall \omega \in \mathcal{O}(X)$$

The *central character* $\mathrm{Irr}(A) \longrightarrow X$ is

$$\varphi \mapsto p_\varphi$$

Remark. Corollary 3.4 states that $\mathrm{Irr}(A)$ depends only on the underlying $\mathbb{C}$ algebra $A_{\mathbb{C}}$. The central character $\mathrm{Irr}(A) \to X$, however, does depend on the structure of A as a k-module. A change in the action of k on $A_{\mathbb{C}}$ will

change the central character.

The central character $\mathrm{Irr}(A) \to X$ is continuous where $\mathrm{Irr}(A)$ is topologized as above and X has the Zariski topology. For a proof of this assertion see [KNS, Lemma 1, p.326]. From a somewhat heuristic non-commutative geometry point of view, $A_{\mathbb{C}}$ is a non-commutative complex affine variety, and a given action of k on $A_{\mathbb{C}}$, making $A_{\mathbb{C}}$ into a finite type k-algebra A, determines a morphism of algebraic varieties $A_{\mathbb{C}} \to X$.

5. Morita equivalence for k-algebras

Definition 5.1. Let B be a k-algebra. A *right B-module* is a $\mathbb{C}$-vector space V with given morphisms of $\mathbb{C}$-algebras

$$B^{\mathrm{op}} \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$
$$k \longrightarrow \mathrm{Hom}_{\mathbb{C}}(V,V)$$

such that:

(1) $k \to \mathrm{Hom}_{\mathbb{C}}(V,V)$ is unital

(2) $v(\omega b) = (v\omega)b = (vb)\omega \quad \forall (v,\omega,b) \in V \times k \times B.$

B^{op} is the opposite algebra of B. V is non-degenerate if $VB = V$.

Note that a right B-module is the same as a left B^{op}-module.

With k fixed, let A, B be two k-algebras. An $A-B$ bimodule, denoted ${}_AV_B$, is a $\mathbb{C}$ vector space V such that :

(1) V is a left A-module.
(2) V is a right B-module.
(3) $a(vb) = (av)b \qquad \forall\, (a,v,b) \in A \times V \times B.$
(4) $\omega v = v\omega \qquad \forall\, (\omega, v) \in k \times V.$

An $A-B$ bimodule ${}_AV_B$ is non-degenerate if $AV = V = VB$. I_V is the identity map of V. A is an $A-A$ bimodule in the evident way.

Definition 5.2. A k-algebra A has *local units* if given any finite set $\{a_1, a_2, \ldots, a_r\}$ of elements of A, there is an idempotent $Q \in A$ ($Q^2 = Q$) with

$$Qa_j = a_jQ = a_j \qquad j = 1, 2, \ldots, r.$$

Definition 5.3. Let A, B be two k-algebras with local units. A *Morita equivalence* (between A and B) is given by a pair of non-degenerate bimodules

$${}_AV_B \quad {}_BW_A$$

together with isomorphisms of bimodules

$$\alpha\colon V \otimes_B W \to A$$
$$\beta\colon W \otimes_A V \to B$$

such that there is commutativity in the diagrams:

$$\begin{tikzcd} V \otimes_B W \otimes_A V \arrow[r, "I_V \otimes \beta"] \arrow[d, "\alpha \otimes I_V"] & V \otimes_B B \arrow[d, "\cong"] \\ A \otimes_A V \arrow[r, "\cong"'] & V \end{tikzcd}$$

$$\begin{tikzcd} W \otimes_A V \otimes_B W \arrow[r, "I_W \otimes \alpha"] \arrow[d, "\beta \otimes I_W"] & W \otimes_A A \arrow[d, "\cong"] \\ B \otimes_B V \arrow[r, "\cong"'] & W \end{tikzcd}$$

Let A, B two k-algebras with local units, and suppose given a Morita equivalence

$$_AV_B \qquad _BW_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B$$

The *linking algebra* is

$$L(_AV_B, _BW_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

i.e. $L(_AV_B, _BW_A)$ consists of all 2×2 matrices having (1, 1) entry in A, (2, 2) entry in B, (2, 1) entry in W, and (1, 2) entry in V. Addition and multiplication are matrix addition and matrix multiplication. Note that α and β are used in the matrix multiplication.

$L(_AV_B, _BW_A)$ is a k-algebra. With $\omega \in k$, the action of k on $L(_AV_B, _BW_A)$ is given by

$$\omega \begin{pmatrix} a & v \\ w & b \end{pmatrix} = \begin{pmatrix} \omega a & \omega v \\ \omega w & \omega b \end{pmatrix}$$

A Morita equivalence between A and B determines an equivalence of categories between the category of non-degenerate left A-modules and the category of non-degenerate left B-modules. Similarly for right modules. Also, a Morita equivalence determines isomorphisms (between A and B) of Hochschild homology, cyclic homology, and periodic cyclic homology.

Example. For n a positive integer, let $\mathrm{M}_n(A)$ be the k-algebra of all $n \times n$ matrices with entries in A. If A has local units, A and $\mathrm{M}_n(A)$ are Morita equivalent as follows. For m, n positive integers, denote by $\mathrm{M}_{m,n}(A)$ the set of all $m \times n$ (i.e. m rows and n columns) matrices with entries in A. Matrix multiplication then gives a map

$$\mathrm{M}_{m,n}(A) \times \mathrm{M}_{n,r}(A) \longrightarrow \mathrm{M}_{m,r}(A)$$

With this notation, $\mathrm{M}_{n,n}(A) = \mathrm{M}_n(A)$ and $\mathrm{M}_{1,1}(A) = \mathrm{M}_1(A) = A$. Hence matrix multiplication gives maps

$$\mathrm{M}_{1,n}(A) \times \mathrm{M}_n(A) \longrightarrow \mathrm{M}_{1,n}(A) \qquad \mathrm{M}_n(A) \times \mathrm{M}_{n,1} \longrightarrow \mathrm{M}_{n,1}(A)$$

Thus $\mathrm{M}_{1,n}(A)$ is a right $\mathrm{M}_n(A)$-module and $\mathrm{M}_{n,1}(A)$ is a left $\mathrm{M}_n(A)$-module.
Similarly, $\mathrm{M}_{1,n}(A)$ is a left A-module and $\mathrm{M}_{n,1}(A)$ is a right A-module.
With $A = A$ and $B = \mathrm{M}_n(A)$, the bimodules of the Morita equivalence are $V = \mathrm{M}_{1,n}(A)$ and $W = \mathrm{M}_{n,1}(A)$.
Note that the required isomorphisms of bimodules

$$\alpha: V \otimes_B W \to A$$
$$\beta: W \otimes_A V \to B$$

are obtained by observing that the matrix multiplication maps

$$\mathrm{M}_{1,n}(A) \times \mathrm{M}_{n,1}(A) \to A$$
$$\mathrm{M}_{n,1}(A) \times \mathrm{M}_{1,n}(A) \to \mathrm{M}_n(A)$$

factor through the quotients $\mathrm{M}_{1,n}(A) \otimes_{\mathrm{M}_n(A)} \mathrm{M}_{n,1}(A)$, $\quad \mathrm{M}_{n,1}(A) \otimes_A \mathrm{M}_{1,n}(A)$ and so give bimodule isomorphisms

$$\alpha: \mathrm{M}_{1,n}(A) \otimes_{\mathrm{M}_n(A)} \mathrm{M}_{n,1}(A) \to A$$
$$\beta: \mathrm{M}_{n,1}(A) \otimes_A \mathrm{M}_{1,n}(A) \to \mathrm{M}_n(A)$$

If A has local units, then α and β are isomorphisms. Therefore A and $\mathrm{M}_n(A)$ are Morita equivalent.

If A does not have local units, then α and β can fail to be isomorphisms, and there is no way to prove that A and $\mathrm{M}_n(A)$ are Morita equivalent. In examples, this already happens with $n = 1$, and there is then no way to prove (when A does not have local units) that A is Morita equivalent to A. For more details on this issue see below where the proof is given that in the new equivalence relation A and $\mathrm{M}_n(A)$ are equivalent even when A does not have local units.

A finite type k-algebra A has local units iff A is unital.

6. The exceptional set $\mathfrak{E}_A(X)$

A Morita equivalence between two finite type k-algebras A, B preserves the central character i.e. there is commutativity in the diagram

$$\begin{tikzcd} \mathrm{Irr}(A) \arrow[r] \arrow[d, "cc"'] & \mathrm{Irr}(B) \arrow[d, "cc"] \\ X \arrow[r, "I_X"'] & X \end{tikzcd}$$

where the upper horizontal arrow is the bijection determined by the given Morita equivalence, the two vertical arrows are the two central characters, and I_X is the identity map of X.

By the *exceptional set* $\mathfrak{E}(A)$ we shall mean the set of all $x \in X$ such that the fibre over x has cardinality greater than 1:

$$\mathfrak{E}(A) := \{x \in X : \sharp\, cc^{-1}(x) > 1\}.$$

If A and B are Morita equivalent as k-algebras, then we will have $\mathfrak{E}(A) = \mathfrak{E}(B)$. Contrapositively, we have the following useful

Corollary 6.1. *If $\mathfrak{E}(A) \neq \mathfrak{E}(B)$ then A and B are not Morita equivalent as k-algebras.*

7. Spectrum preserving morphisms

Let A, B two finite type k-algebras, and let $f\colon A \to B$ be a morphism of k-algebras.

Definition 7.1. f is *spectrum preserving* if

(1) Given any primitive ideal $J \subset B$, there is a unique primitive ideal $I \subset A$ with $I \supset f^{-1}(J)$

and

(2) The resulting map
$$\mathrm{Prim}(B) \to \mathrm{Prim}(A)$$
is a bijection.

For I and J as above, let V be the simple B-module with annihilator J. Then the simple A-module with annihilator I can be realized as a quotient of V and also as a subspace of V.

Example 7.2. *Let A, B two unital finite type k-algebras, and suppose given a Morita equivalence*
$$_AV_B \qquad _BW_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B$$
With the linking algebra $L(_AV_B, {}_BW_A)$ as above, the inclusions
$$A \hookrightarrow L(_AV_B, {}_VW_A) \hookleftarrow B$$
$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \mapsfrom b$$
are spectrum preserving morphisms of finite type k-algebras. The bijection
$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$
so obtained is the bijection determined by the given Morita equivalence.

<u>Remark.</u> If $f\colon A \to B$ is a spectrum preserving morphism of finite type k-algebras, then the resulting bijection
$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$
is a homeomorphism. For a proof of this assertion see [BN, Theorem 3, p.342]. Consequently, if A, B are two unital finite type k-algebras, and
$$_AV_B \qquad _BW_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B$$
is a Morita equivalence, then the resulting bijection
$$\mathrm{Prim}(B) \longleftrightarrow \mathrm{Prim}(A)$$
is a homeomorphism.

Definition 7.3. An ideal I in a k-algebra A is a *k-ideal* if $\omega a \in I \ \forall\, (\omega, a) \in k \times I$.

<u>Remark.</u> Any primitive ideal in a k-algebra A is a k-ideal.

Given A, B two finite type k-algebras, let $f\colon A \to B$ be a morphism of k-algebras.

Definition 7.4. f is *spectrum preserving with respect to filtrations* if there are k-ideals

$$0 = I_0 \subset I_1 \subset \cdots \subset I_{r-1} \subset I_r = A \qquad \text{in } A$$

and k ideals

$$0 = J_0 \subset J_1 \subset \cdots \subset J_{r-1} \subset J_r = B \qquad \text{in } B$$

with $f(I_j) \subset J_j$, $(j = 1, 2, \ldots, r)$ and $I_j/I_{j-1} \to J_j/J_{j-1}$, $(j = 1, 2, \ldots, r)$ is spectrum preserving.

The primitive ideal spaces of the subquotients I_j/I_{j-1} and J_j/J_{j-1} are the strata for stratifications of $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$. Each stratum of $\mathrm{Prim}(A)$ is mapped homeomorphically onto the corresponding stratum of $\mathrm{Prim}(B)$. However, the map

$$\mathrm{Prim}(A) \to \mathrm{Prim}(B)$$

might not be a homeomorphism.

8. Algebraic Variation of k-structure

Let A be a unital $\mathbb{C}$-algebra, and let

$$\Psi\colon k \to Z\left(A[t, t^{-1}]\right)$$

be a unital morphism of $\mathbb{C}$-algebras. Here t is an indeterminate,so $A[t, t^{-1}]$ is the algebra of Laurent polynomials with coefficients in A. As above Z denotes "center". For $\zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, $ev(\zeta)$ denotes the "evaluation at ζ" map:

$$ev(\zeta)\colon A[t, t^{-1}] \to A$$
$$\sum a_j t^j \mapsto \sum a_j \zeta^j$$

Consider the composition

$$k \xrightarrow{\Psi} Z\left(A[t, t^{-1}]\right) \xrightarrow{ev(\zeta)} Z(A).$$

Denote the unital k-algebra so obtained by A_ζ. $\forall \zeta \in \mathbb{C}^\times = \mathbb{C} - \{0\}$, the underlying $\mathbb{C}$-algebra of A_ζ is A.

$$(A_\zeta)_{\mathbb{C}} = A \qquad\qquad \forall \zeta \in \mathbb{C}^\times$$

Such a family $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, of unital k-algebras, will be referred to as *an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$*.

9. Stratified Equivalence

With k fixed, consider the collection of all finite type k-algebras. On this collection, *stratified equivalence* is, by definition, the equivalence relation generated by the two elementary steps :

Elementary Step 1. If there is a morphism of k-algebras $f : A \to B$ which is spectrum preserving with respect to filtrations, then $A \sim B$.

Elementary Step 2. If there is $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, such that each A_ζ is a unital finite type k-algebra, then for any $\zeta, \eta \in \mathbb{C}^\times, A_\zeta \sim A_\eta$.

Thus, two finite type k-algebras A, B are equivalent if and only if there is a finite sequence $A_0, A_1, A_2, \ldots, A_r$ of finite type k-algebras with $A_0 = A, A_r = B$, and for each $j = 0, 1, \ldots, r-1$ one of the following three possibilities is valid :

- a morphism of k-algebras $A_j \to A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- a morphism of k-algebras $A_j \leftarrow A_{j+1}$ is given which is spectrum preserving with respect to filtrations.
- $\{A_\zeta\}$, $\zeta \in \mathbb{C}^\times$, an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$, is given such that each A_ζ is a unital finite type k-algebra, and $\eta, \tau \in \mathbb{C}^\times$ have been chosen with $A_j = A_\eta, A_{j+1} = A_\tau$.

To give a stratified equivalence relating A and B, the finite sequence of elementary steps (including the filtrations) must be given. Once this has been done, a bijection of the primitive ideal spaces and an isomorphism of periodic cyclic homology [BN] are determined:

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B) \qquad \mathrm{HP}_*(A) \cong \mathrm{HP}_*(B)$$

Proposition 9.1. *If two unital finite type k-algebras A, B are Morita equivalent (as k-algebras) then they are stratified equivalent.*

$$A \underset{Morita}{\sim} B \Longrightarrow A \sim B$$

Proof. Let A, B two unital finite type k-algebras, and suppose given a Morita equivalence

$${}_AV_B \qquad {}_BW_A \qquad \alpha\colon V \otimes_B W \to A \qquad \beta\colon W \otimes_A V \to B$$

The linking algebra is

$$L({}_AV_B, {}_BW_A) := \begin{pmatrix} A & V \\ W & B \end{pmatrix}$$

The inclusions

$$A \hookrightarrow L({}_AV_B, {}_VW_A) \hookleftarrow B$$

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} \leftarrow\!\shortmid b$$

are spectrum preserving morphisms of finite type k-algebras. Hence A and B are stratified equivalent. □

According to the above, a Morita equivalence of A and B gives a homeomorphism

$$\mathrm{Prim}(A) \simeq \mathrm{Prim}(B)$$

However, the bijection

$$\mathrm{Prim}(A) \longleftrightarrow \mathrm{Prim}(B)$$

obtained from a stratified equivalence might not be a homeomorphism, as in the following example.

Example 9.2. *Let Y be a sub-variety of X. We will write $\mathcal{I}_Y$ for the ideal in $\mathcal{O}(X)$ determined by Y, so that*

$$\mathcal{I}_Y = \{\omega \in \mathcal{O}(X) \mid \omega(p) = 0 \quad \forall\, p \in Y\}\,.$$

Let A be the algebra of all 2×2 matrices whose diagonal entries are in $\mathcal{O}(X)$ and whose off-diagonal entries are in $\mathcal{I}_Y$. Addition and multiplication in A are matrix addition and matrix multiplication. As a k-module, A is the direct sum of $\mathcal{O}(X) \oplus \mathcal{O}(X)$ with $\mathcal{I}_Y \oplus \mathcal{I}_Y$.

$$A = \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{O}(X) \end{pmatrix}$$

Set $B = \mathcal{O}(X) \oplus \mathcal{O}(Y)$, so that B is the coordinate algebra of the disjoint union $X \sqcup Y$. We have $\mathcal{O}(Y) = \mathcal{O}(X)/\mathcal{I}_Y$. As a $k = \mathcal{O}(X)$-module, B is the direct sum $\mathcal{O}(X) \oplus (\mathcal{O}(X)/\mathcal{I}_Y)$.

Theorem 9.3. *The algebras A and B are stratified equivalent but not Morita equivalent:*

$$A \sim B \qquad A \underset{Morita}{\not\sim} B$$

Proof. Let $\mathrm{M}_2(\mathcal{O}(X)$ denote the algebra of all 2×2 matrices with entries in $\mathcal{O}(X)$. Consider the algebra morphisms

$$A \longrightarrow \mathrm{M}_2(\mathcal{O}(X) \oplus \mathcal{O}(Y)) \longleftarrow \mathcal{O}(X) \oplus \mathcal{O}(Y)$$

$$T \mapsto (T, t_{22}|Y) \qquad\qquad (T_\omega, \theta) \leftarrow\!\shortmid (\omega, \theta)$$

where

$$T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} \qquad T_\omega = \begin{pmatrix} \omega & 0 \\ 0 & 0 \end{pmatrix}$$

The filtration of A is given by

$$\{0\} \subset \begin{pmatrix} \mathcal{O}(X) & \mathcal{I}_Y \\ \mathcal{I}_Y & \mathcal{I}_Y \end{pmatrix} \subset A$$

and the filtration of $\mathrm{M}_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y)$ is given by

$$\{0\} \subset \mathrm{M}_2(\mathcal{O}(X) \oplus \{0\}) \subset \mathrm{M}_2(\mathcal{O}(X)) \oplus \mathcal{O}(Y).$$

The rightward pointing arrow is spectrum preserving with respect to the indicated filtrations. The leftward pointing arrow is spectrum preserving (no filtrations needed). We infer that

$$A \sim B.$$

Note that

$$\mathrm{Prim}(A) = X \text{ with each point of } Y \text{ replaced by two points}$$

and

$$\begin{aligned}\mathrm{Prim}(B) &= \mathrm{Prim}\big(\mathcal{O}(X) \oplus \mathcal{O}(Y)\big)\\ &= X \sqcup Y\end{aligned}$$

The spaces $\mathrm{Prim}(A)$ and $\mathrm{Prim}(B)$ are not homeomorphic, and therefore A is not Morita equivalent to B. □

Unlike Morita equivalence, stratified equivalence works well for finite type k-algebras whether or not the algebras are unital, e.g. A and $\mathrm{M}_n(A)$ are stratified equivalent even when A is not unital. See Proposition 8.5 below.

For any k-algebra A there is the evident isomorphism of k-algebras $\mathrm{M}_n(A) \cong A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. Hence, using this isomorphism, if W is a representation of A and U is a representation of $\mathrm{M}_n(\mathbb{C})$, then $W \otimes_{\mathbb{C}} U$ is a representation of $\mathrm{M}_n(A)$.

As above, $\mathrm{M}_{n,1}(\mathbb{C})$ denotes the $n \times 1$ matrices with entries in $\mathbb{C}$. Matrix multiplication gives the usual action of $\mathrm{M}_n(\mathbb{C})$ on $\mathrm{M}_{n,1}(\mathbb{C})$.

$$\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_{n,1}(\mathbb{C}) \longrightarrow \mathrm{M}_{n,1}(\mathbb{C})$$

This is the unique irreducible representation of $\mathrm{M}_n(\mathbb{C})$. For any k-algebra A, if W is a representation of A, then $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is a representation of $\mathrm{M}_n(A)$.

Lemma 9.4. *Let A be a finite type k-algebra and let n be a positive integer. Then:*
(i) If W is an irreducible representation of A, $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is an irreducible representation of $\mathrm{M}_n(A)$.
(ii) The resulting map $\mathrm{Irr}(A) \to \mathrm{Irr}(\mathrm{M}_n(A))$ is a bijection.

<u>Proof.</u> For (i), suppose given an irreducible representation W of A. Let J be the primitive ideal in A which is the null space of W. Then the null space of $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is $J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$.Consider the quotient algebra $A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})/J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C}) = (A/J) \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. This is isomorphic to $\mathrm{M}_{rn}(\mathbb{C})$ where $A/J \cong \mathrm{M}_r(\mathbb{C})$, and so $W \otimes_{\mathbb{C}} \mathrm{M}_{n,1}(\mathbb{C})$ is irreducible.

Proposition 9.5. *Let A be a finite type k-algebra and let n be a positive integer, then A and $\mathrm{M}_n(A)$ are stratified equivalent.*

Proof. Let $f\colon A \to \mathrm{M}_n(A)$ be the morphism of k-algebras which maps $a \in A$ to the diagonal matrix

$$\begin{bmatrix} a & 0 & \dots & 0 \\ 0 & a & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a \end{bmatrix}$$

It will suffice to prove that $f\colon A \to \mathrm{M}_n(A)$ is spectrum preserving.
Let J be an ideal in A. Denote by $J^\diamond$ the ideal in $\mathrm{M}_n(A)$ consisting of all $[a_{ij}] \in \mathrm{M}_n(A)$ such that each a_{ij} is in J. Equivalently, $\mathrm{M}_n(A)$ is $A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ and $J^\diamond = J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$. It will suffice to prove

(1) If J is a primitive ideal in A, then $J^\diamond$ is a primitive ideal in $\mathrm{M}_n(A)$.
(2) If L is any primitive ideal in $\mathrm{M}_n(A)$, then there is a primitive ideal J in A with $L = J^\diamond$.

For (1), J primitive $\Longrightarrow J^\diamond$ primitive, because the quotient algebra $\mathrm{M}_n(A)/J^\diamond$ is $(A/J) \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ which is (isomorphic to) $\mathrm{M}_{rn}(\mathbb{C})$ where $A/J \cong \mathrm{M}_r(\mathbb{C})$.

For (2), since $\mathbb{C}$ is commutative, the action of $\mathbb{C}$ on A can be viewed as both a left and right action. Matrix multiplication then gives a left and a right action of $\mathrm{M}_n(\mathbb{C})$ on $\mathrm{M}_n(A)$

$$\mathrm{M}_n(\mathbb{C}) \times \mathrm{M}_n(A) \to \mathrm{M}_n(A)$$
$$\mathrm{M}_n(A) \times \mathrm{M}_n(\mathbb{C}) \to \mathrm{M}_n(A)$$

for which the associativity rule

$$(\alpha\theta)\beta = \alpha(\theta\beta) \qquad \alpha, \beta \in \mathrm{M}_n(A) \quad \theta \in \mathrm{M}_n(\mathbb{C})$$

is valid.
If V is any representation of $\mathrm{M}_n(A)$, the associativity rule

$$(\alpha\theta)(\beta v) = \alpha[(\theta\beta)v] \qquad \alpha, \beta \in \mathrm{M}_n(A) \quad \theta \in \mathrm{M}_n(\mathbb{C}) \quad v \in V$$

is valid.

Now let V be an irreducible representation of $\mathrm{M}_n(A)$, with L as its null-space. Define a (left) action

$$\mathrm{M}_n(\mathbb{C}) \times V \to V$$

of $\mathrm{M}_n(\mathbb{C})$ on V by proceeding as in the proof of Lemma 4.2 (the "k-action for free" lemma) i.e. given $v \in V$, choose $v_1, v_2, \dots, v_r \in V$ and $\alpha_1, \alpha_2, \dots, \alpha_r \in \mathrm{M}_n(A)$ with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r$$

For $\theta \in \mathrm{M}_n(\mathbb{C})$, define θv by :

$$\theta v = (\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \dots + (\theta\alpha_r)v_r$$

The strict non-degeneracy, Lemmas 4.1 and 4.3, of V implies that θv is well-defined as follows. Suppose that $u_1, u_2, \dots, u_s \in V$ and $\beta_1, \beta_2, \dots, \beta_r \in \mathrm{M}_n(A)$ are chosen with

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_r v_r = \beta_1 u_1 + \beta_2 u_2 + \dots + \beta_s u_s$$

If α is any element of $\mathrm{M}_n(A)$, then
$\alpha[(\theta\alpha_1)v_1 + (\theta\alpha_2)v_2 + \dots + (\theta\alpha_r)v_r - (\theta\beta_1)u_1 - (\theta\beta_2)u_2 - \dots - (\theta\beta_s)u_s] =$

$(\alpha\theta)[\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_r v_r - \beta_1 u_1 - \beta_2 u_2 - \cdots - \beta_s u_s] = (\alpha\theta)[v - v] = 0$

Use $f\colon A \to \mathrm{M}_n(A)$ to make V into an A-module

$$av := f(a)v \qquad a \in A \quad v \in V$$

The actions of A and $\mathrm{M}_n(\mathbb{C})$ on V commute. Thus for each $\theta \in \mathrm{M}_n(\mathbb{C}), \theta V$ is a sub-A-module of V, where θV is the image of $v \mapsto \theta v$. Denote by E_{ij} the matrix in $\mathrm{M}_n(\mathbb{C})$ which has 1 for its (i,j) entry and zero for all its other entries. Then, as an A-module, V is the direct sum

$$V = E_{11}V \oplus E_{22}V \oplus \cdots \oplus E_{nn}V$$

Moreover, the action of E_{ij} on V maps $E_{jj}V$ isomorphically (as an A-module) onto $E_{ii}V$. Hence as an $\mathrm{M}_n(A) = A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ module, V is isomorphic to $(E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ — where $\mathbb{C}^n$ is the standard representation of $\mathrm{M}_n(\mathbb{C})$ i.e. is the unique irreducible representation of $\mathrm{M}_n(\mathbb{C})$.

$$V \cong (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$$

$E_{11}V$ is an irreducible A-module since if not $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ would not be an irreducible $A \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$-module.

If J is the null space (in A) of $E_{11}V$, then $J^{\diamond} = J \otimes_{\mathbb{C}} \mathrm{M}_n(\mathbb{C})$ is the null space of $V = (E_{11}V) \otimes_{\mathbb{C}} \mathbb{C}^n$ and this completes the proof. □

10. Permanence properties

Theorem 10.1. *Denote by Γ a finite group. Stratified equivalence persists under the formation of tensor products $A \otimes B$, and the formation of crossed products $A \rtimes \Gamma$.*

Proposition 10.2. *Given k_1-algebras $A \sim A'$ and k_2-algebras $B \sim B'$ then we have*

$$A \otimes B \sim A' \otimes B'$$

as $k_1 \otimes k_2$-algebras, all tensor products over $\mathbb{C}$.

Proof. We first prove that $A \otimes B \sim A' \otimes B$. Let $\{I_j\}$ be the filtration for A, and $\{I'_j\}$ be the filtration for A'. Then $\{I_j \otimes B\}$ (resp. $\{I'_j \otimes B\}$ will serve as filtrations for $A \otimes B$, (resp. $A' \otimes B$), in the sense that the maps

$$I_j/I_{j-1} \otimes B \to I'_j/I'_{j-1} \otimes B$$

are spectrum-preserving. A similar argument will prove that $A' \otimes B \sim A' \otimes B'$. Then we have

$$A \otimes B \sim A' \otimes B \sim A' \otimes B'$$

as required.

Let Ψ be a variation of k_1-structure for A:

$$k_1 \to Z(A[t, t^{-1}])$$

and let

$$\Psi_B : k_2 \to Z(B)$$

be the k_2-structure for B. Then

$$\Psi \otimes \Psi_B : k_1 \otimes k_2 \to Z(A[t, t^{-1}]) \otimes Z(B) \simeq Z(A \otimes B)[t, t^{-1}]$$

will be a variation of $k_1 \otimes k_2$-structure for $A \otimes B$. □

Lemma 10.3. *Assume that A and B both have Γ acting (as automorphisms of k-algebras), and we are given a Γ-equivariant morphism $\phi : A \to B$ which is spectrum preserving. Then $\phi \rtimes \Gamma : A \rtimes \Gamma \to B \rtimes \Gamma$ is a spectrum preserving morphism of k-algebras.*

Proof. According to Lemma A.1 in [ABPS], $\mathrm{Irr}(A \rtimes \Gamma), \mathrm{Irr}(B \rtimes \Gamma)$ are in bijection with the (twisted) extended quotients:

$$(\mathrm{Irr}(A)//\Gamma)_\natural \simeq \mathrm{Irr}(A \rtimes \Gamma)$$
$$(\mathrm{Irr}(B)//\Gamma)_\natural \simeq \mathrm{Irr}(B \rtimes \Gamma)$$

Since $\phi : A \to B$ is spectrum preserving, the 2-cocycles $\natural$ of Γ associated to $\mathrm{Irr}(A)$ and $\mathrm{Irr}(B)$ can be identified. Hence ϕ determines a bijection of the (twisted) extended quotients

$$(\mathrm{Irr}(A)//\Gamma)_\natural \longleftarrow (\mathrm{Irr}(B)//\Gamma)_\natural$$

Applying Lemma A.1 of [ABPS], this implies that $\phi \rtimes \Gamma$ is spectrum preserving.

□

Proposition 10.4. *If A and B both have Γ acting (as automorphisms of k-algebras), and we are given a Γ-equivariant stratified equivalence $A \sim B$, then we have*

$$A \rtimes \Gamma \sim B \rtimes \Gamma$$

Proof. Suppose now that we are given a Γ-equivariant morphism of k-algebras $\phi : A \to B$ which is, in a Γ-equivariant way, spectrum preserving with respect to filtrations, i.e. the filtrations of A and B are preserved by the action of Γ. This gives a map of k-algebras $\phi \rtimes \Gamma : A \rtimes \Gamma$ to $B \rtimes \Gamma$. We claim that this map is spectrum-preserving with respect to filtrations.

Using the filtrations, consider the map $I_j/I_{j-1} \to J_j/J_{j-1}$. This is a Γ-equivariant spectrum preserving map. Hence by Lemma 10.3 above, the map

$$(I_j/I_{j-1}) \rtimes \Gamma \to (J_j/J_{j-1}) \rtimes \Gamma$$

is spectrum preserving. So the filtrations of $A \rtimes \Gamma$ and $B \rtimes \Gamma$ given by $I_j \rtimes \Gamma$ and $J_j \rtimes \Gamma$ are the required filtrations.

Suppose given a Γ-equivariant variation of k-structure for A, i.e. a Γ-equivariant unital map of $\mathbb{C}$-algebras

$$k \longrightarrow Z(A[t, t^{-1}])$$

where the Γ-action on k is trivial. Due to the triviality of this action, k is mapped to

$$Z(A[t, t^{-1}])^\Gamma = (Z(A))^\Gamma[t, t^{-1}]$$

By the standard inclusion

$$(Z(A))^\Gamma \subset Z(A \rtimes \Gamma)$$

k is mapped to $Z(A \rtimes \Gamma)[t, t^{-1}]$:

$$k \longrightarrow Z(A \rtimes \Gamma)[t, t^{-1}]$$

which is the required variation of k-structure for the crossed product $A \rtimes \Gamma$. □

11. Affine Hecke Algebras

Let $(X, Y, R, R^\vee)$ be a root datum in the standard sense [L, p.73]. This root datum delivers the following items:

– a finite Weyl group W_0
– an extended affine Weyl group $W_0X := W_0 \ltimes X$
– for each $q \in \mathbb{C}^\times$, an affine Hecke algebra $\mathcal{H}_q$
– a complex torus $T := \mathrm{Hom}(X, \mathbb{C}^\times)$
– a complex variety $X := T/W_0$
– a canonical isomorphism $\mathcal{O}(X) \simeq Z(\mathcal{H}_q)$

The detailed construction of $\mathcal{H}_q$ is described in the article of Lusztig [L, p.74]. Set $k := \mathcal{O}(X)$. Then, for all $q \in \mathbb{C}^\times$, $\mathcal{H}_q$ is a unital finite type k-algebra. If $q = 1$, then $\mathcal{H}_1$ is the group algebra of the extended affine Weyl group:

$$\mathcal{H}_1 = \mathbb{C}[W_0X]$$

11.1. **Stratified equivalence for affine Hecke algebras.**

Theorem 11.1. *Except for q in a finite set of roots of unity, none of which is 1, $\mathcal{H}_q$ is stratified equivalent to $\mathcal{H}_1$:*

$$\mathcal{H}_q \sim \mathcal{H}_1.$$

Proof. Let J be Lusztig's asymptotic algebra [L, §8]. As a $\mathbb{C}$-vector space, J has a basis $\{t_v : v \in W_0X\}$, and there is a canonical structure of associative $\mathbb{C}$-algebra on J. The algebra $\mathcal{H}_q$ is viewed as a k-algebra via the canonical isomorphism

$$\mathcal{O}(T/W_0) \cong Z(\mathcal{H}_q).$$

Lusztig's map ϕ_q maps $Z(\mathcal{H}_q)$ to $Z(J)$ by Proposition 1 in [ABP] and thus determines a unique k-structure for J such that the map ϕ_q is a morphism of k-algebras. J with this k-structure will be denoted J_q.

Let

$$a : W_0X \to \mathbb{N} \cup \{0\}$$

be the weight function defined by Lusztig.

The algebra J has a unit element of the form $1 = \sum_{d \in \mathcal{D}}$ where $\mathcal{D}$ is a certain set of involutions in W_a. For any $q \in \mathbb{C}^\times$, the $\mathbb{C}$-linear map $\psi_q : \mathcal{H}_q \to J$ defined by

$$\phi_q(C_w) = \sum h_{w,d,v} t_v \tag{1}$$

is a $\mathbb{C}$-linear homomorphism preserving 1. The summation is over the set

$$\{d \in \mathcal{D}, v \in W_0X : a(v) = a(d)\}.$$

The elements C_w are defined in [L, p.82] and form a $\mathbb{C}$-basis of $\mathcal{H}_q$. The coefficients $h_{w,d,v}$ originate in the multiplication rule for the C_w, see [L, p.82].

The map ϕ_q is injective. Thus all algebras $\mathcal{H}_q$ with $q \in \mathbb{C}^\times$ appear as subalgebras of a single $\mathbb{C}$-algebra J.

Define

$$\begin{aligned} J_k &= \text{ideal generated by } \{t_v \in J : a(v) \leq k\} \\ I_k &= \phi_q^{-1}(J_k) \end{aligned}$$

We have the filtrations

$$\begin{aligned} 0 \subset I_1 \subset I_2 \subset \cdots \subset I_r &= \mathcal{H}_q \\ 0 \subset J_1 \subset J_2 \subset \cdots \subset J_r &= J \end{aligned} \tag{2}$$

We have

$$\begin{aligned} \mathrm{Irr}(J) &= \bigsqcup \mathrm{Irr}(J_k/J_{k-1}) \\ \mathrm{Irr}(\mathcal{H}_q) &= \bigsqcup \mathrm{Irr}(I_k/I_{k-1}) \end{aligned}$$

Let M be a simple $\mathcal{H}_q$-module (resp. simple J-module). The *weight* a_M of M is defined by Lusztig in [L, p.82]. Let $\mathrm{Irr}(\mathcal{H}_q)_k$ (resp. $\mathrm{Irr}(J)_k$) denote the set of simple $\mathcal{H}_q$-modules (resp. simple J-modules) of weight k. It follows from Lusztig's definition that we have

$$\begin{aligned} \mathrm{Irr}(J_k/J_{k-1}) &= \mathrm{Irr}(J)_k \\ \mathrm{Irr}(I_k/I_{k-1}) &= \mathrm{Irr}(\mathcal{H}_q)_k \end{aligned}$$

Consider the commutative diagram

$$\begin{tikzcd} \mathcal{H}_q \arrow[r, "\phi_q"] & J \\ I_k \arrow[u] \arrow[r, "\phi_q|I_k"] \arrow[d] & J_k \arrow[u] \arrow[d] \\ I_k/I_{k-1} \arrow[r, "(\phi_q)_k"'] & J_k/J_{k-1} \end{tikzcd}$$

The middle horizontal map is the restriction of ϕ_q to I_k. This map is defined by equation (1) but with summation over the set

$$\{d \in \mathcal{D}, v \in W_0X : a(v) = a(d) \leq k\}.$$

The bottom horizontal map is middle horizontal map after descent to the quotient I_k/I_{k-1}. This map is defined by equation (1) but with summation over the set

$$\{d \in \mathcal{D}, v \in W_0X : a(v) = a(d) = k\}.$$

We now apply one of Lusztig's main theorems, Theorem 8.1 in [L]: Assume that $q \in \mathbb{C}^\times$ is either 1 or is not a root of 1. Let M' be a simple J-module of weight $a_{M'}$. The pre-image $\phi_q^{-1}(M')$ will contain a unique subquotient M of maximal weight $a_M = a_{M'}$. Then the map $M' \mapsto M$ is a bijection from the set of simple J-modules (up to isomorphism) to the set of simple $\mathcal{H}_q$-modules (up to isomorphism).

We reconcile our construction with Lusztig's bijection by observing that our map

$$(\phi_q)_k : I_k/I_{k-1} \to J_k/J_{k-1}$$

determines a map

$$\mathrm{Irr}(J)_k \to \mathrm{Irr}(\mathcal{H}_q)_k \tag{3}$$

This map is weight-preserving, by construction, and coincides with the Lusztig bijection. The map (3) is spectrum-preserving by Lusztig's theorem, and so the original map ϕ_q is spectrum preserving with respect to the filtrations (2).

$\mathcal{H}_q$ is then stratified equivalent to $\mathcal{H}_1$ by the three elementary steps

$$\mathcal{H}_q \rightsquigarrow J_q \rightsquigarrow J_1 \rightsquigarrow \mathcal{H}_1.$$

The second elementary step (i.e. passing from J_q to J_1) is an algebraic variation of k-structure with parameter space $\mathbb{C}^\times$. The first elementary step uses Lusztig's map ϕ_q, and the third elementary step uses Lusztig's map ϕ_1. Hence (provided q is not in the exceptional set of roots of unity—none of which is 1) $\mathcal{H}_q$ is stratified equivalent to

$$\mathcal{H}_1 = \mathbb{C}[W_0X] = \mathcal{O}(T) \rtimes W_0$$

□

Remark 11.2. In Theorem 11.1, the condition on q can be replaced by the following more precise condition:

$$\sum_{w \in W_0} q^{\ell(w)} \neq 0$$

where $\ell : W_0X \to \mathbb{N}$ is the length function, see [Xi2, Theorem 3.2].

If $q \neq 1$ then $\mathcal{H}_q$ and $\mathcal{H}_1$ are not Morita equivalent as k-algebras, for the exceptional sets are not equal:

$$q \neq 1 \implies \mathfrak{E}(\mathcal{H}_q) \neq \mathfrak{E}(\mathcal{H}_1)$$

11.2. **The affine Hecke algebras of** $\mathrm{SL}_2(\mathbb{C})$**.** If the root datum

$$(X, Y, R, R^\vee)$$

arises from a connected complex reductive Lie group G, then we will write $\mathcal{H}_q(G) = \mathcal{H}_q$.

An interesting situation arises for $\mathcal{H}_q(\mathrm{SL}_2(\mathbb{C}))$. Let s, t be the simple reflections in the (extended) affine Weyl group W_0X. When $q + 1 \neq 0$ there is an isomorphism of $\mathbb{C}$-algebras between $\mathcal{H}_q$ and $\mathcal{H}_1$:

$$\mathcal{H}_q \simeq \mathcal{H}_1$$

$$T_s \mapsto \frac{q+1}{2}s + \frac{q-1}{2}, \qquad T_t \mapsto \frac{q+1}{2}t + \frac{q-1}{2}$$

On the other hand, when $q \neq 1$, $\mathcal{H}_q$ and $\mathcal{H}_1$ cannot be isomorphic as k-algebras, for the exceptional sets are different. Let T denote the standard maximal torus in $\mathrm{SL}_2(\mathbb{C})$:

$$\left\{ \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} : z \neq 0 \right\}$$

We have $W_0 = \mathbb{Z}/2\mathbb{Z}$, $k = \mathcal{O}(T/W_0)$ the algebra of $\mathbb{Z}/2\mathbb{Z}$-invariant Laurent polynomials in one variable. Note that the non-trivial element of W acts by

$$\begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \mapsto \begin{pmatrix} 1/z & 0 \\ 0 & z \end{pmatrix}$$

The quotient variety T/W_0 comprises unordered pairs $\{z_1, z_2\}$ of nonzero complex numbers which satisfy the equation $z_1 z_2 = 1$.

The exceptional sets are

$$\begin{aligned}\mathfrak{E}(\mathcal{H}_1) &= \{-1,-1\} \sqcup \{1,1\}\\ \mathfrak{E}(\mathcal{H}_q) &= \{-1,-1\} \sqcup \{q,1/q\}\end{aligned}$$

As q with $q \neq 1$ is deformed to 1, the point $\{-1,-1\}$ stays fixed, and the point $\{q,1/q\}$ moves to the point $\{1,1\}$.

11.3. **The affine Hecke algebras of** $\mathrm{SL}_3(\mathbb{C})$**.** The case of $\mathrm{SL}_3(\mathbb{C})$ is considered in §11.7 of Xi [Xi] where it is proved that $\mathcal{H}_q$ and $\mathcal{H}_1$ are not isomorphic as $\mathbb{C}$-algebras whenever $q \neq 1$. As k-algebras, $\mathcal{H}_q$ is not Morita equivalent to $\mathcal{H}_1$ whenever $q \neq 1$ because the exceptional set $\mathfrak{E}(\mathcal{H}_q)$ with $q \neq 1$ is not the same as $\mathfrak{E}(\mathcal{H}_1)$.

In contrast to this, except for a finite set of roots of unity (none of which is 1), according to Theorem 11.1, $\mathcal{H}_q$ is stratified equivalent to $\mathcal{H}_1$.

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